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Semigroups of Operators on the Space of Generalized Functions Exp $\mathscr{A}'$*

E. PAP AND S. PILIPOVIĆ

Institute of Mathematics, 21000 Novi Sad, Ilije Djuričića 4, Yugoslavia

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The inductive limit of spaces $\exp_p \mathscr{A}'$, $p \in N$ (Pilipović, *SIAM J. Math. Anal.* **17** (1986), 477–484) whose elements have unique orthonormal series expansions with exponential growth rate of the corresponding coefficients is to be studied in the first part of the paper. Then, we determine some semigroups of operators on this space. This enables us to solve some classes of infinite order partial differential equations.
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1. INTRODUCTION

Zemanian [7] has developed the theory of a wide class of spaces of generalized functions, denoted by $\mathscr{A}'$. The main point in his theory is the orthogonal expansions of generalized functions. Following this line Pilipović [5] has introduced and investigated spaces $\exp_p \mathscr{A}'$ ($p = 1, 2, \ldots$) which generalize Zemanian's spaces.

In this paper we shall investigate the inductive limit of such spaces, denoted by Exp $\mathscr{A}'$. We shall use the orthogonal expansions of elements from this space which enables us to develop a theory of some semigroups of operators on this space.

Using this theory we are able to solve a class of Cauchy problems in the space Exp $\mathscr{A}'$, which in a special case reduces to a partial differential equation of infinite order.

2. SPACES Exp $\mathscr{A}$ AND Exp $\mathscr{A}'$

We shall use notations from [5] and [7]. Let I be an open interval of the real line R and let $L^2(I)$ be the space of square integrable functions with

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the usual norm. We denote by $\mathscr{R}$ a linear differential selfadjoint operator of the form

$$\mathscr{R} = \theta_0 D^{n_1} \theta_1 D^{n_2} \cdots D^{n_v} \theta_v$$

such that

$$\mathscr{R} = \bar{\theta}_v (-D)^{n_v} \cdots (-D)^{n_2} \bar{\theta}_1 (-D)^{n_1} \bar{\theta}_0,$$

where $D = d/dx$, $\{n_k\}_{k=1}^{v}$ are non-negative integers, $\{\theta_k\}_{k=0}^{v}$ are smooth functions on I without zeros and $\bar{\theta}_k$ are complex conjugate of θ_k, $k = 0, 1, ..., v$. We suppose that there exist a sequence of real numbers $\{\lambda_n\}_{n=0}^{\infty}$ and a sequence of smooth functions $\{\psi_n\}_{n=0}^{\infty}$ such that $\mathscr{R}\psi_n = \lambda_n \psi_n$. Furthermore, we suppose that the sequence $\{|\lambda_n|\}_{n=0}^{\infty}$ monotonically tends to infinity and that $\{\psi_n\}$ forms an orthonormal base of $L^2(I)$.

If $\{\alpha_n\}$ is a sequence of complex numbers different from zero, we denote by $S(\alpha_n)$ and $S^x(\alpha_n)$ Köthe sequence spaces defined as

$$\{a_n\}_{n=0}^{\infty} \in S(\alpha_n) \qquad \text{iff for every } k \in N_0,$$

$$\sum_{n=0}^{\infty} |a_n|^2 \, |\alpha_n|^{2k} < \infty;$$

$$\{b_n\}_{n=0}^{\infty} \in S^x(\alpha_n) \qquad \text{iff for some} \quad k \in N_0,$$

$$\sum_{n=0}^{\infty} |b_n|^2 \, |\alpha_n|^{-2k} < \infty \quad (N_0 = N \cup \{0\}).$$

Zemanian proved in [7, Chap. 9] that there exist bijections between the spaces $\mathscr{A}$ and $\mathscr{A}'$ and $S(\tilde{\lambda}_n)$ and $S^x(\tilde{\lambda}_n)$, respectively, where $\tilde{\lambda}_n = |\lambda_n|$ if $\lambda_n \neq 0$ and $\tilde{\lambda}_n = 1$ if $\lambda_n = 0$, $n = 0, 1, ...$

We denote by $\exp_p \mathscr{A}$ for $p \in N$ [5] a subspace of $L^2(I)$ defined as

$$\phi \overset{2}{=} \sum_{n=0}^{\infty} a_n \psi_n \in \exp_p \mathscr{A} \qquad \text{iff} \quad \{a_n\} \in S(\exp_p \tilde{\lambda}_n),$$

where

$$\exp_p \tilde{\lambda}_n := \underbrace{\exp(\exp \cdots (\exp}_{p} \tilde{\lambda}_n))$$

and $\overset{2}{=}$ means the equality in the sense of L^2-norm.

The dual space of $\exp_p \mathscr{A}$ is denoted by $\exp_p \mathscr{A}'$. It is proved in [5] that if $f \in \exp_p \mathscr{A}'$, then

$$f = \sum_{n=0}^{\infty} b_n \psi_n, \qquad b_n = \langle f, \bar{\psi}_n \rangle, \quad n = 0, 1, ... \tag{1}$$

and $\{b_n\} \in S^x(\exp_p \tilde{\lambda}_n)$, where the series converges weakly in $\exp_p \mathscr{A}'$ (the weak and strong convergence are coincident in $\exp_p \mathscr{A}'$).

Conversely, if $\{b_n\} \in S^x(\exp_p \tilde{\lambda}_n)$, then with the series in (1) a unique element from $\exp_p \mathscr{A}'$ is defined.

We denote by $\exp_{p,k} \mathscr{A}$, $k \in N_0$, a subspace of $L^2(I)$ defined as

$$\phi = \sum_{n=0}^{\infty} a_n \psi_n \in \exp_{p,k} \mathscr{A}$$

$$\text{iff} \quad \sum_{n=0}^{\infty} |a_n|^2 \exp 2k(\underbrace{\exp \cdots (\exp}_{p-1} \tilde{\lambda}_n)) < \infty$$

(for $p=1$, $\sum_{n=0}^{\infty} |a_n|^2 \exp(2k\tilde{\lambda}_n) < \infty$). Obviously, in the set-theoretical sense

$$\exp_p \mathscr{A} = \varprojlim_{k \to \infty} \exp_{p,k} \mathscr{A}.$$

PROPOSITION 1. (i) *Spaces* $\exp_{p,k} \mathscr{A}$, $k \in N_0$, *are* (B)-*spaces.*

(ii) *Inclusion mappings*

$$i_k : \exp_{p,k+1} \mathscr{A} \to \exp_{p,k} \mathscr{A}, \qquad k \in N_0$$

are compact.

Proof. The first part of the proposition can be proved by standard arguments and the second one follows by the Kolmogorov theorem which gives necessary and sufficient conditions for the compactness in spaces of sequences.

Since the space E of elements of the form $\sum_{n=0}^{s} a_n \psi_n$, $s \in N_0$, a_n are arbitrary complex numbers, is a dense subspace of any space $\exp_{p,k} \mathscr{A}$, $k \in N_0$; the projective sequence

$$\{\exp_{p,k} \mathscr{A}\}_{k=0}^{\infty}$$

is reduced (see [3, p. 33]). Thus, Proposition 1 implies that

$$\exp_p \mathscr{A}' = (\varprojlim_{k \to \infty} \exp_{p,k} \mathscr{A})' = \varinjlim_{k \to \infty} \exp_{p,k} \mathscr{A}'$$

in the sense of strong topologies, where $\exp_{p,k} \mathscr{A}'$, $k \in N_0$, is a subspace of E' such that

$$f = \sum_{n=0}^{\infty} b_n \psi_n \in \exp_{p,k} \mathscr{A}' \qquad \text{iff for some } k \in N_0,$$

$$\sum_{n=0}^{\infty} |b_n|^2 \exp(-2k(\underbrace{\exp \cdots (\exp}_{p-1} \tilde{\lambda}_n)) < \infty.$$

PROPOSITION 2. (i) *The sequence*

$$\{\exp_p \mathscr{A}\}_{p=1}^{\infty}$$

is projective and reduced according to the inclusion mappings $\{i_p\}$

$$i_p\colon \quad \exp_{p+1} \mathscr{A} \to \exp_p \mathscr{A}$$

which are compact.

(ii) $\operatorname{Exp} \mathscr{A} := \varprojlim_{p\to\infty} \exp_p \mathscr{A}$ *is a Frechet–Schwartz-space.*

(iii) $\operatorname{Exp} \mathscr{A}' := (\varprojlim_{p\to\infty} \exp_p \mathscr{A})' = \varinjlim_{p\to\infty} \exp_p \mathscr{A}'$ *in the sense of strong topologies.*

(iv) *Let* $\{b_n; n \in N_0\}$, *be a sequence of complex numbers such that for some* $p \in N$ *and* $k \in N_0$,

$$\sum_{n=0}^{\infty} |b_n|^2 \exp(-2k\underbrace{(\exp \cdots (\exp}_{p-1} \tilde{\lambda}_n)) < \infty \tag{2}$$

Then, $\sum_{n=0}^{\infty} b_n \psi_n$ *converges weakly in* $\operatorname{Exp} \mathscr{A}'$ *to some element* f. *Conversely, if* $f \in \operatorname{Exp} \mathscr{A}'$, *then there are* $p \in N$ *and* $k \in N_0$ *such that for the complex numbers* $b_n = (f, \psi_n)$, $n \in N_0$, (2) *holds and*

$$f = \sum_{n=0}^{\infty} b_n \psi_n$$

in the sense of weak convergence in $\operatorname{Exp} \mathscr{A}'$ $((f, \psi_n) = \langle f, \bar{\psi}_n \rangle)$.

Proof. (i) Space E is a dense subspace of $\exp_p \mathscr{A}$, $p \in N$. Similarly as in Proposition 1 one can prove that i_p, $p \in N$, are compact.

(ii) It follows from [7, p. 103, 1.8].

(iii) It follows from (i).

(iv) It follows from (iii).

Proposition 2 directly implies

PROPOSITION 3. (i) $\operatorname{Exp} \mathscr{A} = \varprojlim_{p\to\infty} \exp_{p,p} \mathscr{A}$ *and the inclusion mappings*

$$i_p\colon \quad \exp_{(p+1),(p+1)} \mathscr{A} \to \exp_{p,p} \mathscr{A}, \qquad p \in N,$$

are compact.

(ii) $\operatorname{Exp} \mathscr{A}' = \varinjlim_{p\to\infty} \exp_{p,p} \mathscr{A}'$ *in the sense of strong topologies.*

We remark that Proposition 3(ii) and [5, Theorem 10] give another useful determination of the space $\operatorname{Exp} \mathscr{A}'$.

Proposition 3 implies that $\operatorname{Exp} \mathscr{A}$ is a Montel space which implies that $\operatorname{Exp} \mathscr{A}'$ is also a Montel space. Thus Proposition 4 directly follows:

PROPOSITION 4. (i) *The weak and strong sequential convergences in* $\operatorname{Exp} \mathscr{A}'$ *are equivalent.*

(ii) $\operatorname{Exp} \mathscr{A}$ *and* $\operatorname{Exp} \mathscr{A}'$ *are reflexive.*

We shall always suppose that the topology in $\operatorname{Exp} \mathscr{A}'$ is the strong dual topology.

Proposition 3 implies:

PROPOSITION 5. *A sequence* $\{f_n\}$ *from* $\operatorname{Exp} \mathscr{A}'$ *converges to some* $f \in \operatorname{Exp} \mathscr{A}'$ *iff for some* $p \in N$ *and* $k \in N_0$, $f_n \in \exp_{p,k} \mathscr{A}'$, $n \in N$, $f \in \exp_{p,k} \mathscr{A}'$ *and* $f_n \to f$ *in* $\exp_{p,k} \mathscr{A}'$.

For our further investigations the following proposition is useful.

PROPOSITION 6. *A linear operator* L: $\operatorname{Exp} \mathscr{A}' \to \operatorname{Exp} \mathscr{A}'$ *is continuous iff for every sequence* $\{f_n\}$ *from* $\operatorname{Exp} \mathscr{A}'$ *and* $f \in \operatorname{Exp} \mathscr{A}'$, *such that* $f_n \to f$ *in* $\operatorname{Exp} \mathscr{A}'$, *then* $Lf_n \to Lf$ *in* $\operatorname{Exp} \mathscr{A}'$.

Proof. Since L is continuous on $\operatorname{Exp} \mathscr{A}'$ iff it is continuous on any space $\exp_{p,p} \mathscr{A}'$, $p \in N$, the proof follows.

3. SEMIGROUPS OF OPERATORS ON $\operatorname{Exp} \mathscr{A}'$

We denote by $\{T_t; t \geqslant 0\}$ (for short, T_t) a one-parameter family of continuous linear operators from $\operatorname{Exp} \mathscr{A}$ into $\operatorname{Exp} \mathscr{A}$ for which the following conditions hold:

(a) $T_t T_s = T_{t+s}$ for any $t, s \geqslant 0$,

(b) $T_0 = I$ (the identity operator).

In this case we call T_t a semigroup. If for a semigroup T_t the following condition holds:

(c) $\lim_{t \to s} T_t x = T_s x$ for any $s \geqslant 0$ and any $x \in \operatorname{Exp} \mathscr{A}$,

then T_t is said to be a strong continuous semigroup.

The infinitesimal generator A of a semigroup T_t is defined by

$$Ax := \lim_{h \to 0^+} \frac{1}{h} (T_h - I)x$$

whenever the limit exists in $\operatorname{Exp} \mathscr{A}$. We denote the domain of A by $D(A)$.

Since Exp $\mathscr{A}$ is tonnelé we can obtain from [4, Sect. 1] important properties of a strong continuous semigroup on Exp $\mathscr{A}$.

Let $\{T_t; t \geqslant 0\}$ be a strong continuous semigroup and let $\{T_t^*; t \geqslant 0\}$ be defined by

$$\langle x, T_t^* x' \rangle := \langle T_t x, x' \rangle, \qquad x \in \operatorname{Exp} \mathscr{A},\ x' \in \operatorname{Exp} \mathscr{A}'.$$

As well, we define A^* by

$$\langle x, A^* x' \rangle := \langle Ax, x' \rangle, \qquad x \in D(A),\ x' \in D(A^*).$$

Since Exp $\mathscr{A}$ is reflexive and Exp $\mathscr{A}'$ is tonnelé we have by [4, Sect. 2] the following proposition which we formulate here for the sake of completeness.

PROPOSITION 7. *Let $\{T_t; t \geqslant 0\}$ be a strong continuous semigroup on* Exp $\mathscr{A}$.

(i) *The semigroup $\{T_t^*; t \geqslant 0\}$ is a strong continuous semigroup in* Exp $\mathscr{A}'$ *such that its infinitesimal generator is precisely A^*.*

(ii) *$D(A^*)$ is a dense subspace of* Exp $\mathscr{A}'$ *and A^* is closed operator.*

(iii) *If $x' \in D(A^*)$ then $T_t^* x' \in D(A^*)$ for any $t \geqslant 0$ and $T_t^* x'$ is continuously differentiable in t relative to the strong topology in* Exp $\mathscr{A}'$. *Moreover,*

$$\frac{d}{dt} T_t^* x' = A^* T_t^* x = T_t^* A^* x \qquad \text{for every } t \geqslant 0.$$

(iv) *An element $x' \in$* Exp $\mathscr{A}'$ *belongs to $D(A^*)$ and $A^* x' = y'$ iff*

$$T_t^* x' - x' = \int_0^t T_s^* y' \, ds \qquad \text{for any} \quad t \geqslant 0.$$

(v) *For every $x' \in$* Exp $\mathscr{A}'$

$$\int_a^b T_s^* x' \, ds \in D(A^*) \qquad (0 \leqslant a, b < \infty)$$

and

$$A^* \int_a^b T_s^* x' \, ds = T_b^* x' - T_a^* x'.$$

Now we shall characterize some special semigroups of operators on $\operatorname{Exp} \mathscr{A}'$.

PROPOSITION 8. *Let* $\{T_t^*; t \geqslant 0\}$ *be a semigroup of operators on* $\operatorname{Exp} \mathscr{A}'$ *such that for every* $n \in N_0$ *there exists a measurable function* $c_n(t)$, $t \in [0, \infty)$, *such that*

$$T_t^* \psi_n = c_n(t)\, \psi_n, \qquad t \geqslant 0, \tag{3}$$

where $\psi_n(x)$, $n \in N_0$, *are observed as elements from* $\operatorname{Exp} \mathscr{A}'$.

Then, there exist $S \subset N_0$ *and complex numbers* u_n, $n \in S$ *such that*

(S.1) *there are* $p_0 \in N$ *and* $K > 0$ *such that*

$$\operatorname{Re} u_n \leqslant K \exp_{p_0} \tilde{\lambda}_n, \qquad n \in S$$

and

$$c_n(t) = e^{u_n t}, \qquad n \in S.$$

Proof. From the properties of the semigroup T_t^* and (3) it follows that

$$c_n(t+s) = c_n(t)\, c_n(s), \qquad \text{for any} \quad t, s > 0$$

and $c_n(0) = 1$. Since $c_n(t)$, $n \in N_0$, are measurable functions it follows [1; 2, Theorem 4.17.3] that for some $S \subset N_0$ and some set of complex numbers $\{u_n; n \in S\}$,

$$c_n(t) = e^{tu_n}, \quad n \in S, \qquad \text{and} \qquad c_n(t) = \begin{cases} 1 & \text{for} \quad t = 0 \\ 0 & \text{for} \quad t \neq 0, \quad n \in N_0 \backslash S. \end{cases}$$

If the set S is finite then (S.1) trivially holds for some $p_0 \in N$ and $K > 0$. If S is infinite and (S.1) does not hold one can easily show that for any $t > 0$,

$$\sum_{n \in S} e^{t \operatorname{Re} u_n} \psi_n(x)$$

does not belong to $\operatorname{Exp} \mathscr{A}'$.

Remark 1. The preceding theorem is also true in the case when $c_n(t)$ are real bounded functions on some finite intervals $[a_n, b_n] \subset (0, \infty)$, $n \in N_0$ [1; 2, Theorem 4.17.2].

Remark 2. Let us suppose that in the preceding proposition instead of (3) we have

$$T_t^* \psi_n = c_n(t)\, \psi_{n+i(n)}, \qquad c_n(0) = 1, n \in N_0, \tag{3'}$$

where $i\colon N_0 \to N_0$ such that $i(n) \neq 0$ for every $n \in N_0$. Since ψ_n, $n \in N_0$, are linearly independent functions, (3′) has no solution.

Remark 3. If we suppose that in Proposition 8, $c_n(t)$, $n \in N_0$, are continuous functions on $[0, \infty)$ then

$$c_n(t) = e^{tu_n}, \qquad n \in N_0,\ t \in [0, \infty)$$

and u_n, $n \in N_0$, satisfies (S.1).

We introduce the following condition for a semigroup $\{T_t^*; t \geqslant 0\}$ on $\operatorname{Exp} \mathscr{A}'$,

$$T_t^* \psi_n = c_n(t) \psi_n,\ n \in N_0,\ t \geqslant 0,\ c_n(t) \text{ are continuous functions on } [0, \infty),\ n \in N_0. \tag{3''}$$

PROPOSITION 9. *Semigroup of operators T_t^* for which (3″) holds is a strong continuous semigroup of operators. Particularly all assertions from Proposition 7 hold for this semigroup.*

Proof. We have only to prove that

$$\lim_{t_s \to t_0} T_{t_s}^* f = T_{t_0}^* f,$$

where $\{t_s\}$ is a sequence from $[0, \infty)$ which converges to $t_0 \in [0, \infty)$ $(t_s \neq t_0)$, and $f \in \operatorname{Exp} \mathscr{A}'$ is of the form

$$f = \sum_{n=0}^{\infty} b_n \psi_n. \tag{4}$$

There exists $p_1 \in N$ such that $f \in \exp_{p_1} \mathscr{A}'$, i.e.,

$$\sum_{n=0}^{\infty} |b_n|^2 (\exp_{p_1} \tilde{\lambda}_n)^{-2k} < \infty \tag{5}$$

for a suitable $k \in N_0$.

Let $t_0 \geqslant 0$. We have to prove that

$$(T_{t_s}^* - T_{t_0}^*) f = \sum_{n=0}^{\infty} (e^{t_s u_n} - e^{t_0 u_n})\, b_n \psi_n,$$

converges to zero in some $\exp_p \mathscr{A}'$ if $t_s \to t_0$, $s \to \infty$.

Let $p \geqslant \max\{p_1 + 1, p_0 + 2\}$. Since $t_s \to t_0$ there exists $C > \sup_s\{t_s\} + t_0$.

We have

$$\sum_{n=0}^{\infty} |b_n|^2 (e^{t_s u_n} - e^{t_0 u_n})^2 (\exp_p \tilde{\lambda}_n)^{-2k}$$
$$\leqslant \sum_{n=0}^{r_0-1} |b_n|^2 (e^{t_s u_n} - e^{t_s u_n})^2 (\exp_p \tilde{\lambda}_n)^{-2k}$$
$$+ \sum_{n=r_0}^{\infty} |b_n|^2 e^{2Cu_n} (\exp_p \tilde{\lambda}_n)^{-2k}.$$

As (5) holds with p instead of p_1, there exists r_0 such that the second term on the right side is smaller than $\varepsilon/2$. Now we choose s_0 such that for $s > s_0$ the first term on the right side is smaller than $\varepsilon/2$ as well. Thus we proved the assertion.

Remark 4. If we suppose that conditions of Proposition 8 hold for T_t^* then one can easily prove that this semigroup is continuous for $t > 0$. Also this semigroup is continuous for $t \geqslant 0$ on the domain $D \subset \operatorname{Exp} \mathscr{A}'$ defined by

$$f \in D \qquad \text{iff} \quad b_n = \langle f, \bar{\psi}_n \rangle = 0, n \in N_0 \backslash S$$

(S is defined in the proof of Proposition 8).

For the semigroup T_t^* defined by

$$T_t^* f = \sum_{n=0}^{\infty} b_n e^{t u_n} \psi_n, \qquad t \geqslant 0,$$

where f is of the form (4) and $\{u_n, n \in N_0\}$, satisfies (S.1), the infinitesimal generator A^* is

$$A^* f = \sum_{n=0}^{\infty} u_n b_n \psi_n, \qquad f \in D(A^*). \tag{6}$$

In fact, if the limit

$$\lim_{h \to 0^+} \frac{T_h^* - I}{h} f$$

exists, then the coefficients of $((T_h^* - I)/h) f$ converge to the corresponding coefficients of $A^* f$.

In the special case, when $\mathscr{A}'$ is the space determined by the operator $\mathscr{R} = iD^2$ on $I = (-\pi, \pi)$ with

$$\psi_n(x) = (1/\sqrt{2\pi}) \exp(inx) \qquad \text{and} \qquad \lambda_n = n, \quad n \in Z,$$

where Z is the set of all integers, we obtain the space of periodic distributions $\mathscr{P}'$ [7]. In this case, we have a quite natural characterization for a class of semigroups defined on Exp $\mathscr{P}'$.

PROPOSITION 10. *Let T_t^* be a semigroup of operators on* Exp $\mathscr{P}'$ *which commutes with the translation (i.e., if $T_t^* f(x) = f(x, t)$ then $T_t^* f(x+s) = f(x+s, t)$). Then $\{T_t^*, t > 0\}$ is of the form*

$$T_t^* f(x) = \sum_{-\infty}^{\infty} c_n(t)\, b_n\, e^{inx} \tag{7}$$

(f is of the form (4)), *where $c_n(t)$ are functions defined on $[0, \infty)$ such that*

$$c_n(0) = 1, \qquad c_n(t_1 + t_2) = c_n(t_1)\, c_n(t_2), \quad t_1, t_2 > 0. \tag{8}$$

Moreover, if for every $g \in$ Exp $\mathscr{P}'$, *$T_t^* g$ is weakly measurable, then there are complex numbers u_n, $n \in Z \backslash S$, for which* (S.1) *holds* (with $\lambda_n = n$) *and*

$$c_n(t) = e^{u_n t}, \quad n \in S, \qquad c_n(t) = \begin{cases} 0 & \text{for} \quad t > 0 \\ 1 & \text{for} \quad t = 0, \quad n \in Z \backslash S. \end{cases}$$

Particularly, T_t^* is a strong continuous semigroup of operators on Exp $\mathscr{P}'$ *if all the functions $c_n(t)$, $n \in Z$, are continuous on $[0, \infty)$.*

Proof. With

$$\langle T_t \phi, f \rangle := \langle \phi, T_t^* f \rangle, \qquad \phi \in \operatorname{Exp} \mathscr{P}, f \in \operatorname{Exp} \mathscr{P}'$$

a semigroup on Exp $\mathscr{P}$ is defined. We want to show that this semigroup commutes with the translation. Let

$$T_t \phi(x) = \phi(x, t) \qquad \text{and} \qquad T_t^* f = f(x, t).$$

For $s \in R$ and $t \geqslant 0$, we have

$$\begin{aligned} \langle T_t \phi(x+s), f(x) \rangle &= \langle \phi(x+s), f(x, t) \rangle = \langle \phi(x), f(x-s, t) \rangle \\ &= \langle \phi(x), T_t^* f(x-s) \rangle = \langle T_t \phi(x), f(x-s) \rangle \\ &= \langle \phi(x, t), f(x-s) \rangle = \langle \phi(x+s, t), f(x) \rangle. \end{aligned}$$

This implies that $T_t \phi(x+s) = \phi(x+s, t)$, $t \geqslant 0$. In the same way as in [2, Proof of Theorem 20.3.1] one can prove that if

$$\phi(x) = \sum_{n \in Z} a_n\, e^{inx} \in \operatorname{Exp} \mathscr{P},$$

then

$$T_t \phi = \sum_{n \in Z} v_n(t)\, a_n\, e^{inx},$$

where $v_n(t)$, $n \in Z$, are functions defined on $[0, \infty)$ such that

$$v_n(0) = 1, \qquad v_n(t_1 + t_2) = v_n(t_1)\, v(t_2), \quad t_1, t_2 \geqslant 0.$$

If we put $c_n(t) = \overline{v_n(t)}$ then T_t^* is of the form (7) such that (8) holds.

Since $T_t^* g$ is weakly measurable for any $g \in \operatorname{Exp} \mathscr{P}'$ we obtain that for every $n \in Z$, $c_n(t)$ is measurable function. It is known [1, 2] that in this case the solution of (7) is

$$c_n(t) = e^{tu_n} \qquad \text{for} \quad t \in [0, \infty)$$

or

$$c_n(t) = \begin{cases} 0 & \text{for} \quad t > 0 \\ 1 & \text{for} \quad t = 0, \quad n \in Z. \end{cases}$$

As in Proposition 8 we prove that (S.1) holds for $\{u_n; n \in Z\}$, and that T_t^* is a continuous semigroup on $\operatorname{Exp} \mathscr{P}'$ if all $c_n(t)$, $n \in Z$, are continuous.

PROPOSITION 11. *Let A^* be the infinitesimal generator of a semigroup $\{T_t^*; t \geqslant 0\}$ of the form*

$$T_t^* f = \sum_{n=0}^{\infty} e^{tu_n} b_n \psi_n, \qquad t \geqslant 0,$$

(f is of the form (4)) such that the following condition holds:

(S.2) *There exist $p_0 \in N$ and $S > 0$ such that*

$$|u_n| \leqslant S \exp_{p_0} \tilde{\lambda}_n, \qquad n \in N_0.$$

Then

$$\sum_{k=0}^{s} \frac{t^k A^{*k}}{k!} f \to T_t^* f \qquad as \quad s \to \infty$$

in the sense of convergence in $\operatorname{Exp} \mathscr{A}'$. *Thus we have $T_t^* = e^{tA^*}$, where*

$$e^{tA^*} f := \sum_{k=0}^{\infty} \frac{t^k A^{*k}}{k!} f.$$

Proof. Let f be of the form (4) and

$$\sum_n |b_n|^2 (\exp_p \tilde{\lambda}_n)^{-2p} < \infty \qquad \text{for some} \quad p \in N.$$

We have to prove that for some $k \in N$ and $t > 0$,

$$\sum_{n=0}^{\infty} \left| \sum_{i=0}^{s} \frac{t^i u_n^i}{i!} - e^{tu_n} \right|^2 |b_n|^2 (\exp_k \tilde{\lambda}_n)^{-2k} \to 0 \tag{9}$$

as $s \to \infty$.

The series

$$\sum_{n=0}^{\infty} e^{2t|u_n|} \, |b_n|^2 \, (\exp_k \tilde{\lambda}_n)^{-2k}$$

is convergent for $k \geqslant p + p_0 + 2tS + 1$. Thus for $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$\sum_{n=n_0+1}^{\infty} e^{2t|u_n|} \, |b_n|^2 (\exp_k \tilde{\lambda}_n)^{-2k} < \varepsilon.$$

Hence, we have for $k \geqslant p + p_0 + 2tS + 1$,

$$\sum_{n=0}^{\infty} \left| \sum_{i=0}^{s} \frac{t^i u_n^i}{i!} - e^{tu_n} \right|^2 |b_n|^2 (\exp_k \tilde{\lambda}_n)^{-2k}$$
$$\leqslant \sum_{n=0}^{n_0} \left| \sum_{i=0}^{s} \frac{t^i u_n^i}{i!} - e^{tu_n} \right|^2 |b_n|^2 (\exp_k \tilde{\lambda}_n)^{-2k} + \varepsilon.$$

Taking $s \to \infty$ we obtain (9).

Obviously, (S.2) $\Rightarrow$ (S.1). The following condition is of a practical use:

(S.3) (S.1) holds and $|\arg u_n| \leqslant (\pi/2) - \varepsilon$ for some $\varepsilon > 0$ and every $n \in N_0$.

From $|\operatorname{Im} u_n| \leqslant |\operatorname{Re} u_n| \cdot tg((\pi/2) - \varepsilon)$ we obtain that (S.3) implies (S.2).

If

$$A = \underbrace{\exp \cdots \exp}_{p-1} \mathscr{R}, \qquad p \geqslant 2 \tag{10}$$

($\mathscr{R}$ is defined in the first paragraph), then $e^{tA} = E_p^t$ where E_p^t is defined in [5].

4. Applications

Let A^* be an operator on $\operatorname{Exp} \mathscr{A}'$ defined by

$$A^* f = \sum_{n=0}^{\infty} u_n b_n \psi_n \qquad (f \text{ is of the form (4)}), \tag{11}$$

where the sequence $\{u_n, n \in N_0\}$, satisfies (S.2). Then using the preceding section we obtain that the Cauchy problem

$$\frac{\partial f(t, x)}{\partial t} = A^* f(t, x), \qquad f(0, x) = f_0(x) \in \operatorname{Exp} \mathscr{A}' \tag{12}$$

has the solution

$$T_t^* f_0 = \sum_{n=0}^{\infty} e^{u_n t} b_n \psi_n \qquad \left(\text{for } f_0 = \sum_{n=0}^{\infty} b_n \psi_n\right).$$

PROPOSITION 12. *If $u_n \neq 0$, $n = 0, 1, 2,\ldots$, then the solution $T_t^* f_0$ of (12) is unique on some interval $[0, \delta_0]$. If for some $n \in N_0$, $u_n = 0$ then the solution of (12) is not unique.*

Proof. If for example $u_0 = 0$, then solution of (12) is

$$T_t^* f_0 + m\psi_0$$

for arbitrary $m \in C$. This means that we have to prove only the first part of Proposition 12.

Let $f_1(t, x)$, $t \geqslant 0$ be another solution of (12) different from $T_t^* f_0$. Since for $t \to 0^+$, $f_1(t, x) \to f(0, x)$ in $\operatorname{Exp} \mathscr{A}'$ it follows that for appropriate $\delta > 0$ and $t \in [0, \delta]$, $f_1(t, x)$ belongs to some $\exp_p \mathscr{A}'$ and in $\exp_p \mathscr{A}'$ it converges to $f_0(x)$. Thus

$$f_1(t, x) = \sum_{n=0}^{\infty} b_n(t)\, \psi_n(x), \qquad t \in [0, \delta],$$

where $b_n(t) \to b_n$, $t \to 0^+$, $n = 0, 1,\ldots,$

$$\left(f_0 = \sum_{n=0}^{\infty} b_n \psi_n(x)\right).$$

For $(\partial/\partial t) f_1(t, x)$ we also have that for some $\delta_1 > 0$ and some $p_1 \in N$,

$$\frac{\partial}{\partial t} f_1(t, x) \in \exp_{p_1} \mathscr{A}'$$

for $t \in [0, \delta_1]$ and

$$\frac{\partial}{\partial t} f_1(t, x) = \sum_{n=0}^{\infty} c_n(t)\, \psi_n(x).$$

Thus for $t \in [0, \delta_0]$, where $\delta_0 = \min\{\delta. \delta_1\}$, we have

$$c_n(t) = \frac{\partial}{\partial t} b_n(t).$$

Putting this in (12) we obtain (for $t \in [0, \delta_0]$),

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial t} b_n(t)\, \psi_n(x) = \sum_{n=0}^{\infty} b_n(t)\, u_n \psi_n(x).$$

The system

$$\frac{\partial}{\partial t} b_n(t) = u_n b_n(t),$$

$b_n(0) = b_n$, $n = 0, 1,...,$ has the unique solution

$$b_n(t) = b_n\, e^{u_n t}, \qquad n = 0, 1,..., t \in [0, \delta_0].$$

This completes the proof.

We remark that if A is of the form (10), then (12) is a partial differential equation of infinite order.

We shall now examine the Cauchy problem

$$\frac{\partial f(t, x)}{\partial t} = A^* f(t, x) + g(x), \qquad u(0, x) = u_0(x) \in \operatorname{Exp} \mathscr{A}', \tag{13}$$

where $g \in \operatorname{Exp} \mathscr{A}'$. For solving this equation we need some properties of the integral of a semigroup $\{T_t^*, t \geqslant 0\}$.

PROPOSITION 13. *For every*

$$f = \sum_{n=0}^{\infty} a_n \psi_n \in \operatorname{Exp} \mathscr{A}'$$

and $0 \leqslant t < \infty$,

$$\int_0^t T_s^* f\, ds = \sum_{n \in W} \frac{e^{u_n t} - 1}{u_n} a_n \psi_n + \sum_{n \in N_0 \setminus W} t a_n \psi_n,$$

where W is the set of all non-negative integers n for which $u_n \neq 0$.

Proof. The assertation follows from the fact that for any $\phi \in \operatorname{Exp} \mathscr{A}$ the sequence

$$\left\langle \sum_{n=0}^{\nu} e^{s u_n} a_n \psi_n, \phi \right\rangle$$

converges uniformly on the interval $[0, t]$ $(t > 0)$ to

$$\left\langle \sum_{n=0}^{\infty} e^{s u_n} a_n \psi_n, \phi \right\rangle \qquad \text{as} \quad \nu \to \infty.$$

Obviously

$$\frac{d}{dt}\int_0^t T_s^* f\, ds = A^* \int_0^t T_s^* f\, ds = T_t^* f - f.$$

Now one can easily prove that the solution of (13) is

$$f(t, x) = T_t^* f_0(x) + \int_0^t T_s^* g(x)\, ds.$$

By Proposition 12 we get that this solution is unique if all u_n, $n \in N_0$ are different from zero.

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