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Local unitary cocycles of E_0-semigroups

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Abstract

This paper concerns the structure of the group of local unitary cocycles, also called the gauge group, of an E_0-semigroup. The gauge group of a spatial E_0-semigroup has a natural action on the set of units by operator multiplication. Arveson has characterized completely the gauge group of E_0-semigroups of type I, and as a consequence it is known that in this case the gauge group action is transitive. In fact, if the semigroup has index k, then the gauge group action is transitive on the set of $(k+1)$-tuples of appropriately normalized independent units. An action of the gauge group having this property is called $(k+1)$-fold transitive. We construct examples of E_0-semigroups of type II and index 1 which are not 2-fold transitive. These new examples also illustrate that an E_0-semigroup of type II_k need not be a tensor product of an E_0-semigroup of type II_0 and another of type I_k.

Keywords: CP-semigroup; E_0-semigroup; Units; Cocycles; Dilations

0. Introduction

An E_0-semigroup is a strongly continuous one-parameter semigroup of unit preserving $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H})$, the algebra of all bounded operators on a separable Hilbert space $\mathfrak{H}$. In the 1930s Wigner showed that a one-parameter group of $*$-automorphisms of $\mathfrak{B}(\mathfrak{H})$ is always given by the action of a one-parameter strongly continuous unitary group by conjugation. In particular, the classification of one-parameter automorphism groups of $\mathfrak{B}(\mathfrak{H})$ up to conjugacy

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can be reduced to the well-known multiplicity theory of Hahn–Hellinger of unbounded self-adjoint operators. In contrast, the classification theory of E_0-semigroups up to cocycle conjugacy (which is the appropriate equivalence relation in this context) has proved to be much richer and full of surprises (see for example [3,9–11,14,17,20,21]; we recommend Arveson's book [5] for an excellent introduction to the theory of E_0-semigroups).

One important cocycle conjugacy invariant of an E_0-semigroup is its gauge group (or more precisely the isomorphism class of the gauge group). Given an E_0-semigroup α, a cocycle C is a one-parameter strongly continuous family $\{C(t)\colon t \geqslant 0\}$ satisfying the cocycle identity $C(t+s) = C(t)\alpha_t(C(s))$ for $t, s \geqslant 0$. The cocycle C is said to be local if it satisfies the additional property that $C(t) \in \alpha_t(\mathfrak{B}(\mathfrak{H}))'$ for all $t \geqslant 0$. It is easy to verify that given two local cocycles C_1 and C_2, the expression $(C_1 \cdot C_2)(t) = C_1(t)C_2(t)$, for $t \geqslant 0$, defines another local cocycle. The set of unitary local cocycles is a group when endowed with this operation, and this group is called the gauge group of the E_0-semigroup α. In terms of the product system approach to the study of E_0-semigroups, the gauge group is canonically isomorphic to the group of automorphisms of the product system associated with α.

A unit for an E_0-semigroup α is a strongly continuous one-parameter semigroup of isometries $\{U(t)\colon t \geqslant 0\}$ which intertwines the E_0-semigroup: $\alpha_t(A)U(t) = U(t)A$ for all $t \geqslant 0$, $A \in \mathfrak{B}(\mathfrak{H})$. If an E_0-semigroup has at least one unit, it is called spatial. If it is spatial and it is generated by its units, it is called completely spatial or type I. All other spatial E_0-semigroups are called type II. Non-spatial E_0-semigroups, also called type III, have been proven to exist by Powers [14], and in fact it follows from work of Tsirelson [20] that there exists a continuum of pairwise non-cocycle conjugate examples. The type of an E_0-semigroup is also a cocycle conjugacy invariant. We will only consider spatial semigroups in this paper, although we should note that, to our knowledge, little is known about the gauge group of a non-spatial E_0-semigroup.

Our main goal in this work is to study the action of the gauge group of a spatial E_0-semigroup on the set of units. If U is a unit of α and C is a local unitary cocycle, then $U'(t) = C(t)U(t)$ is another unit of α, defining a natural action of the gauge group.

Completely spatial E_0-semigroups and their gauge groups are well understood. Every completely spatial E_0-semigroup is cocycle conjugate to a CAR/CCR flow, and they are completely classified by the index [3,15,19]. Furthermore, their gauge groups were completely characterized by Arveson [3,5]. One property which becomes apparent upon examining his characterization is that the gauge group of a completely spatial semigroup acts transitively on the set of units. In fact, even more can be gleaned from that characterization. If the completely spatial E_0-semigroup has index k, then any pair of $(k+1)$-tuples of appropriately normalized and independent units are related by an element of the gauge group. When the action of the gauge group of a spatial E_0-semigroup on its units has this property, we say that the action is $(k+1)$-fold transitive.

Alevras, Powers and Price [1] were the first to break ground on the study of the gauge group of E_0-semigroups of type II. In their work, they characterize all contractive local cocycles (not just unitary local cocycles) for a certain class of E_0-semigroups of type II and index zero. For semigroups of index zero, the set of units is essentially one-dimensional and the gauge group action is automatically (1-fold) transitive.

It is natural to inquire whether the action of the gauge group of a spatial E_0-semigroup of index k on the units is always $(k+1)$-fold transitive. In this paper we show that this is not the case, by constructing a class of E_0-semigroups of type II and index 1 whose gauge group action on the set of units is not 2-fold transitive.

It is possible, although we could not verify it, that within the class which we have constructed there could also be examples of E_0-semigroups whose gauge group action on the set of units is not transitive. While this article was in preparation, it came to our attention that non-transitive examples were obtained by Tsirelson [22], using different techniques. We do not know the exact relationship between his examples and our own. Nevertheless, in the last section we discuss some features which they have in common.

We also observe that our examples, as well as Tsirelson's [22], provide a direct answer to an old question. When Arveson [2] proved that the index is additive with respect to tensor products, it was natural to inquire whether type II_k semigroups can be decomposed as tensor products of type II_0 and I_k. Alas, that is not the case, as the E_0-semigroups which we construct are of type II_1 yet they are not tensor products of type II_0 and type I_1 semigroups.

Our approach involves a detailed analysis of the E_0-semigroups obtained via minimal dilation of certain CP-flows. Bhat [6] proved that CP-semigroups can be dilated to E_0-semigroups, and this result proved very useful for the construction and analysis of new examples of E_0-semigroups (a very incomplete list of work in this direction includes [4,8,12,13,16]). Bhat [7] has also found a one-to-one correspondence between the compressions of a CP-semigroup and the compressions of its minimal dilation. Pursuing this correspondence, Powers [18] subsequently carried out a study of a class of CP-semigroups, called CP-flows, and their minimal dilations to E_0-semigroups. In particular, several results were obtained in [18] for the analysis of the cocycle conjugacy of the minimal dilations of CP-flows, as well as their contractive local cocycles. We make full use of this favorable framework, which is in fact quite general, given that all spatial E_0-semigroups arise from the minimal dilation of an appropriate CP-flow (see [18]).

We now provide an outline of the contents of the following sections. In Section 1 we describe in detail the basic background and terminology, with an emphasis on the material related to [18]. In Section 2 we introduce the class of examples which will be of interest, and describe some of its key properties. In Section 3 we turn to the analysis of the local cocycles of the E_0-semigroups under consideration. Finally, in the last section we summarize our main results.

1. Background, notation and definitions

We begin with the definition of E_0-semigroups of $\mathfrak{B}(\mathfrak{H})$ the set of all bounded operators on a separable Hilbert space $\mathfrak{H}$. For a detailed discussion of E_0-semigroups we refer to Arveson's excellent book [5].

Definition 1.1. We say α is an E_0-*semigroup* of $\mathfrak{B}(\mathfrak{H})$ if the following conditions are satisfied:

(i) α_t is a $*$-endomorphism of $\mathfrak{B}(\mathfrak{H})$ for each $t \geqslant 0$.
(ii) α_0 is the identity endomorphism and $\alpha_t \circ \alpha_s = \alpha_{t+s}$ for all $s, t \geqslant 0$.
(iii) For each $\rho \in \mathfrak{B}(\mathfrak{H})_*$ (the predual of $\mathfrak{B}(\mathfrak{H})$) and $A \in \mathfrak{B}(\mathfrak{H})$ the function $\rho(\alpha_t(A))$ is a continuous function of t.
(iv) $\alpha_t(I) = I$ for each $t \geqslant 0$ (α_t preserves the unit).

The appropriate notions of when two E_0-semigroups are similar are conjugacy and cocycle conjugacy (which comes from Alain Connes' definition of outer conjugacy).

Definition 1.2. Suppose α and β are E_0-semigroups $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$. We say α and β are *conjugate*, denoted $\alpha \approx \beta$, if there is a $*$-isomorphism ϕ of $\mathfrak{B}(\mathfrak{H}_1)$ onto $\mathfrak{B}(\mathfrak{H}_2)$ so that $\phi \circ \alpha_t = \beta_t \circ \phi$ for all $t \geqslant 0$. We say α and β are *cocycle conjugate*, denoted $\alpha_t \sim \beta_t$, if α' and β are conjugate where α and α' differ by a unitary cocycle (i.e., there is a strongly continuous one-parameter family of unitaries $U(t)$ on $\mathfrak{B}(\mathfrak{H}_1)$ for $t \geqslant 0$ satisfying the cocycle condition $U(t)\alpha_t(U(s)) = U(t+s)$ for all $t, s \geqslant 0$ so that $\alpha'_t(A) = U(t)\alpha_t(A)U(t)^{-1}$ for all $A \in \mathfrak{B}(\mathfrak{H}_1)$ and $t \geqslant 0$).

An E_0-semigroup α_t is *spatial* if there is a semigroup of isometries $U(t)$ which intertwine so $U(t)A = \alpha_t(A)U(t)$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$. The property of being spatial is a cocycle conjugacy invariant.

An extremely useful and well-known result in the theory of C^*-algebras is the Gelfand–Segal construction of a cyclic $*$-representation of a C^*-algebra associated with a state of the C^*-algebra. In the study of E_0-semigroups there is a result in the same spirit which says that every semigroup of unital completely positive maps of $\mathfrak{B}(\mathfrak{K})$ can be dilated to an E_0-semigroup of $\mathfrak{B}(\mathfrak{H})$ where $\mathfrak{H}$ can be thought of as a larger Hilbert space containing $\mathfrak{K}$. We begin with a review of the properties of completely positive maps.

A linear map ϕ from a C^*-algebra $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ is *completely positive* if

$$\sum_{i,j=1}^{n} \big(f_i, \phi\big(A_i^* A_j\big) f_j\big) \geqslant 0$$

for $A_i \in \mathfrak{A}$, $f_i \in \mathfrak{H}$ for $i = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$. Stinespring's central result is that if $\mathfrak{A}$ has a unit and ϕ is a completely positive map from $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ then there is a $*$-representation π of $\mathfrak{A}$ on $\mathfrak{B}(\mathfrak{K})$ and an operator V from $\mathfrak{H}$ to $\mathfrak{K}$ so that $\phi(A) = V^*\pi(A)V$ for $A \in \mathfrak{A}$. And π is determined by ϕ up to unitary equivalence if the linear span of $\{\pi(A)Vf\}$ for $A \in \mathfrak{A}$ and $f \in \mathfrak{H}$ is dense in $\mathfrak{K}$.

Often we speak of one functional or map dominating another. We introduce a word for the functional or map that is dominated. The word is "*subordinate*." If A is an object which is positive with respect to some order structure we say B is a subordinate of A if B is the same kind of thing A is and B is positive and B is less than A. For example if we are speaking of the positive integers the subordinates of 4 are 4, 3, 2, 1. If A is a positive operator then the subordinates of A are operators B with $A \geqslant B \geqslant 0$. Suppose E is a projection. Are the subordinates of a projection E projections under E or the operators under E? The answer depends on the context.

A *CP-semigroup* of $\mathfrak{B}(\mathfrak{H})$ is a strongly continuous one-parameter semigroup of completely positive maps of $\mathfrak{B}(\mathfrak{H})$ into itself. We now state Bhat's theorem [6] for $\mathfrak{B}(\mathfrak{H})$.

Theorem 1.3. *Suppose α is a unital CP-semigroup of $\mathfrak{B}(\mathfrak{H})$. Then there is an E_0-semigroup α^d of $\mathfrak{B}(\mathfrak{H}_1)$ and an isometry W from $\mathfrak{H}$ to $\mathfrak{H}_1$ so that*

$$\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$$

and $\alpha_t(WW^*) \geqslant WW^*$ for $t > 0$ and if the projection $E = WW^*$ is minimal, which means the span of the vectors

$$\alpha_{t_1}^d(EA_1E)\alpha_{t_2}^d(EA_2E)\cdots\alpha_{t_n}^d(EA_nE)Wf$$

for $f \in \mathfrak{H}$, $A_i \in \mathfrak{B}(\mathfrak{H})$, $t_i \geqslant 0$ for $i = 1, 2, \ldots$ and $n = 1, 2, \ldots$ is dense in $\mathfrak{H}_1$, then α^d is determined up to conjugacy.

We use Arveson's definition of minimality which is easier to state and equivalent to Bhat's.

Suppose α is an E_0-semigroup of $\mathfrak{B}(\mathfrak{H})$. We characterize the subordinates of α, (i.e. the *CP*-semigroups β of $\mathfrak{B}(\mathfrak{H})$ so that the mapping $A \to \alpha_t(A) - \beta_t(A)$ is completely positive for all $t \geqslant 0$). The subordinates of α are given by positive local cocycles. A cocycle is a σ-weakly continuous one-parameter family of operators $C(t)$ satisfying the cocycle relation

$$C(t+s) = C(t)\alpha_t\bigl(C(s)\bigr)$$

for all $s, t \geqslant 0$. The cocycle $C(t)$ is local if $C(t) \in \alpha_t(\mathfrak{B}(\mathfrak{H}))'$ for all $t > 0$. The local cocycles and their order structure are a cocycle conjugacy invariant. As first shown by Bhat [6] there is an order isomorphism from the subordinates of a unital *CP*-semigroup of $\mathfrak{B}(\mathfrak{H})$ to the subordinates of its minimal dilation to an E_0-semigroup of $\mathfrak{B}(\mathfrak{H}_1)$ We use the notation of [18].

Theorem 1.4. *Suppose α is a unital CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ and α^d is the minimal dilation of α to an E_0-semigroup of $\mathfrak{B}(\mathfrak{H}_1)$ and W is an isometry from $\mathfrak{H}$ to $\mathfrak{H}_1$ so that WW^* is a minimal projection for α^d and*

$$\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$. Then there is an order isomorphism from the subordinates of α to the subordinates of α^d given as follows. Suppose γ is a subordinate of α^d and $C(t) = \gamma_t(I)$ for $t \geqslant 0$ is the local cocycle associated with γ then the subordinate β of α under this isomorphism is given by

$$\beta_t(A) = W^*C(t)\alpha_t^d(WAW^*)W$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$.

In this paper we will frequently make use of corners. This is a trick introduced by A. Connes.

Definition 1.5. Suppose α and β are *CP*-semigroups of $\mathfrak{B}(\mathfrak{H})$ and $\mathfrak{B}(\mathfrak{K})$. Then γ is a *corner from α to β* if Θ given by

$$\Theta_t\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \alpha_t(A) & \gamma_t(B) \\ \gamma_t^*(C) & \beta_t(D) \end{bmatrix}$$

for $t \geqslant 0$ and $A \in \mathfrak{B}(\mathfrak{H})$, $D \in \mathfrak{B}(\mathfrak{K})$, B a linear operator from $\mathfrak{K}$ to $\mathfrak{H}$ and C a linear operator from $\mathfrak{H}$ to $\mathfrak{K}$ is a *CP*-semigroup of $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{K})$.

Suppose γ is a corner from α to β and Θ is the *CP*-semigroup of $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{K})$ defined above. Suppose Θ' is a subordinate of Θ of the form

$$\Theta'_t\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \alpha'_t(A) & \gamma_t(B) \\ \gamma_t^*(C) & \beta'_t(D) \end{bmatrix}$$

for $t \geqslant 0$ for A, B, C and D as stated above. We say γ is *maximal* if for every subordinate Θ' of the above form we have $\alpha' = \alpha$. We say γ is *hyper-maximal* if for every subordinate Θ' of the above form we have $\alpha' = \alpha$ and $\beta' = \beta$.

We state Theorem 3.13 of [18] which shows how to determine when two *CP*-semigroups dilate to cocycle conjugate E_0-semigroups.

Theorem 1.6. *Suppose α and β are unital CP-semigroups of $\mathfrak{B}(\mathfrak{H})$ and $\mathfrak{B}(\mathfrak{K})$ and α^d and β^d are the minimal dilations of α and β to E_0-semigroups. Then α^d and β^d are cocycle conjugate if and only if there is a hyper-maximal corner γ from α to β.*

If α is a unital *CP*-semigroup and α^d is its minimal dilation to an E_0-semigroup then the corners from α to α come from contractive local cocycles. The following theorem follows from [18, Theorem 3.16 and Corollary 3.17].

Theorem 1.7. *Suppose α is a unital CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ and α^d is its minimal dilation to an E_0-semigroup α^d of $\mathfrak{B}(\mathfrak{H}_1)$. The relation between α and α^d is given by*

$$\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$ where W is an isometry from $\mathfrak{H}$ to $\mathfrak{H}_1$ and α^d is minimal over the range of W. Suppose γ is a corner from α to α. Then there is a unique contractive local cocycle C for α^d so that

$$\gamma_t(A) = W^*C(t)\alpha_t^d(WAW^*)W$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$. Conversely, if C is a contractive local cocycle for α^d then γ given above is a corner from α to α.

Furthermore, $C(t)$ is an isometry for all $t \geqslant 0$ if and only if γ is maximal and $C(t)$ is unitary for all $t \geqslant 0$ if and only if γ is hyper-maximal.

Also in [18, Theorem 3.16] there is a similar theorem for matrices of corners.

Theorem 1.8. *Suppose α is a unital CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ and α^d is its minimal dilation to an E_0-semigroup α^d of $\mathfrak{B}(\mathfrak{H}_1)$. The relation between α and α^d is given by*

$$\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$ where W is an isometry from $\mathfrak{H}$ to $\mathfrak{H}_1$ and α^d is minimal over the range of W.

Suppose n is a positive integer and Θ is positive $(n \times n)$-matrix of corners from α to α. Then there is a unique positive $(n \times n)$-matrix C of contractive local cocycles C_{ij} for α^d for $i, j = 1, \dots, n$ so that

$$\theta_t^{(ij)}(A) = W^*C_{ij}(t)\alpha_t^d(WAW^*)W$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$. Conversely, if C is a positive $(n \times n)$-matrix of contractive local cocycles for α^d then the matrix Θ whose coefficients $\theta^{(ij)}$ are given above is a positive $(n \times n)$-matrix of corners from α to α.

Next we define *CP*-flows. We believe these are the simplest objects which can be dilated to produce all spatial E_0-semigroups. *CP*-flows are studied extensively in [18].

Definition 1.9. Suppose $\mathfrak{K}$ is a separable Hilbert space and $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ and $U(t)$ is right translation of $\mathfrak{H}$ by $t \geqslant 0$. Specifically, we may realize $\mathfrak{H}$ as the space of $\mathfrak{K}$-valued Lebesgue measurable functions with inner product

$$(f,g) = \int_0^\infty \big(f(x), g(x)\big)\,dx$$

for $f, g \in \mathfrak{H}$. The action of $U(t)$ on an element $f \in \mathfrak{H}$ is given by $(U(t)f)(x) = f(x-t)$ for $x \in [t,\infty)$ and $(U(t)f)(x) = 0$ for $x \in [0,t)$. A semigroup α is a *CP-flow over* $\mathfrak{K}$ if α is a *CP*-semigroup of $\mathfrak{B}(\mathfrak{H})$ which is intertwined by the translation semigroup $U(t)$, i.e., $U(t)A = \alpha_t(A)U(t)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$.

Henceforth, unless stated explicitly otherwise, we will arrange our notation so that our *CP*-flows will be *CP*-flows over $\mathfrak{K}$ and acting on $B(\mathfrak{H})$, where $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$, and $U(t)$ will denote the translation semigroup on $\mathfrak{H}$.

In [18, Theorem 4.0A] it is shown that every spatial E_0-semigroup is cocycle conjugate to an E_0-semigroup which is also a *CP*-flow.

We introduce notation for describing *CP*-flows. Let $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ and $U(t)$ be translation by t. Let

$$E(t) = I - U(t)U(t)^* \quad \text{and} \quad E(a,b) = U(a)U(a)^* - U(b)U(b)^*$$

for $t \in [0,\infty)$ and $0 \leqslant a < b < \infty$. We will also write $E(t,\infty) = U(t)U(t)^*$. Let $d = d/dx$ be the differential operator of differentiation with the boundary condition $f(0) = 0$. More precisely, the domain $\mathfrak{D}(d)$ is all $f \in \mathfrak{H}$ of the form

$$f(x) = \int_0^x g(t)\,dt$$

with $g \in \mathfrak{H}$. The hermitian adjoint d^* is $-d/dx$ with no boundary condition at $x = 0$, that is to say, the domain $\mathfrak{D}(d^*)$ consists of the linear span of $\mathfrak{D}(d)$ and the functions $g(x) = e^{-x}k$ with $k \in \mathfrak{K}$. In summary, we can represent elements $f \in \mathfrak{D}(d^*)$ as $f = f_0 + f_+$ where $f_0 \in \mathfrak{D}(d)$ and $f_+(x) = f(0)e^{-x}$. Thus, the space $\mathfrak{D}(d^*)$ has a natural semi-definite inner product given by $\langle f, g\rangle = (f(0), g(0))$ which induces a (definite) inner product on $\mathfrak{D}(d^*)$ mod $\mathfrak{D}(d)$. This leads to a natural identification $\mathfrak{D}(d^*)/\mathfrak{D}(d) \simeq \mathfrak{K}$ via the map $[f] \mapsto f(0)$.

Suppose α is a *CP*-flow over $\mathfrak{K}$ and $A \in \mathfrak{B}(\mathfrak{H})$. Then, for $t > 0$, one computes

$$\alpha_t(A) = U(t)AU(t)^* + E(t)\alpha_t(A)E(t) = U(t)AU(t)^* + B$$

for all $t \geqslant 0$. Then B commutes with $E(s)$ for all $s \in [0,t]$, so B is of the form

$$(Bf)(x) = b(x)f(x)$$

and for $t > x \geqslant 0$, $b(x) \in \mathfrak{B}(\mathfrak{K})$ depends σ-strongly on A. We now define the boundary representation, π_0. Let δ be the generator of α. Then for $A \in \mathfrak{D}(\delta)$ we have $A\mathfrak{D}(d) \subset \mathfrak{D}(d)$ and $A\mathfrak{D}(d^*) \subset \mathfrak{D}(d^*)$ so A acts on $\mathfrak{D}(d^*)$ mod $\mathfrak{D}(d)$. In terms of the identification $\mathfrak{D}(d^*)/\mathfrak{D}(d) \simeq \mathfrak{K}$

discussed in the previous paragraph, it follows that if $f \in \mathfrak{D}(d^*)$, then $(Af)(0)$ only depends on $f(0)$. We call this mapping from $\pi_0 : \mathfrak{D}(\delta) \to \mathfrak{B}(\mathfrak{K})$, given by

$$\pi_0(A)\big(f(0)\big) = (Af)(0),$$

the *boundary representation*. Note π_0 tells you what flows in from the origin. The boundary representation need not be σ-weakly continuous and even when it is it may not tell the whole story. If π is a σ-weakly continuous completely positive contraction of $\mathfrak{B}(\mathfrak{K} \otimes L^2(0,\infty))$ into $\mathfrak{B}(\mathfrak{K})$ then there is a minimal *CP*-flow with that boundary representation and if that flow is unital then the E_0-semigroup induced by the flow is completely spatial (type I_n) where n is the rank of π. For a detailed discussion of these properties of the boundary representation, we refer the reader to [18].

We now define the generalized boundary representation. The resolvent R_α for α is given by

$$R_\alpha(A) = \int_0^\infty e^{-t}\alpha_t(A)\,dt.$$

Next we introduce some notation. If ϕ is a σ-weakly continuous mapping from $\mathfrak{B}(\mathfrak{H})$ to $\mathfrak{B}(\mathfrak{K})$ we define $\hat{\phi}$ is the predual map from $\mathfrak{B}(\mathfrak{K})_*$ to $\mathfrak{B}(\mathfrak{H})_*$ so we have $\rho(\phi(A)) = (\hat{\phi}\rho)(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $\rho \in \mathfrak{B}(\mathfrak{K})_*$. We define the mapping Γ as

$$\Gamma(A) = \int_0^\infty e^{-t}U(t)AU(t)^*\,dt$$

for $A \in \mathfrak{B}(\mathfrak{H})$. Note $R_\alpha - \Gamma$ is completely positive which we denote by writing $R_\alpha - \Gamma \geqslant 0$. Note Γ is the resolvent of a *CP*-flow with boundary representation $\pi_0 = 0$.

We need one more bit of notation. We define $\Lambda : \mathfrak{B}(\mathfrak{K}) \to \mathfrak{B}(\mathfrak{H})$ for $A \in \mathfrak{B}(\mathfrak{K})$ we define $\Lambda(A)$ by

$$\big(\Lambda(A)f\big) = e^{-x}Af(x).$$

We define $\Lambda = \Lambda(I)$. Note $\Gamma(I) = I - \Lambda$.

Now we present the main formula.

$$\hat{R}_\alpha(\rho) = \hat{\Gamma}\big(\omega(\hat{\Lambda}\rho) + \rho\big)$$

for $\rho \in \mathfrak{B}(\mathfrak{H})_*$ and $\eta \to \omega(\eta)$ is the boundary weight map and $\omega(\eta)$ is the boundary weight associated with η. A boundary weight is a particular example of a T-weight which we define presently.

Definition 1.10. Suppose $T \in \mathfrak{B}(\mathfrak{H})$ is a positive strictly contractive operator (i.e. $0 \leqslant T \leqslant I$ and $\|Tf\| < 1$ for $\|f\| \leqslant 1$ so one is not an eigenvalue for T). We denote by $\mathfrak{A}(\mathfrak{H}, T)$ the linear space

$$\mathfrak{A}(\mathfrak{H}, T) = (I-T)^{\frac{1}{2}}\mathfrak{B}(\mathfrak{H})(I-T)^{\frac{1}{2}}$$

and by $\mathfrak{A}(\mathfrak{H}, T)_*$ the linear functionals ρ on $\mathfrak{A}(\mathfrak{H}, T)$ of the form

$$\rho\big((I-T)^{\frac{1}{2}}A(I-T)^{\frac{1}{2}}\big) = \eta(A)$$

for $A \in \mathfrak{B}(\mathfrak{H})$ with $\eta \in \mathfrak{B}(\mathfrak{H})_*$. We call such functionals T-weights. The T-norm of a T-weight ρ denoted $\|\rho\|_T$ is the norm of η. If ρ is a T-weight and $\|\rho\|_T \leqslant 1$ we say ρ is T-contractive.

Suppose $T \in \mathfrak{B}(\mathfrak{H})$ is a positive strictly contractive operator and $P(\lambda)$ is the spectral resolution of T so

$$T = \int_0^1 \lambda \, dP(\lambda).$$

If $\rho \in \mathfrak{A}(\mathfrak{H}, T)_*$ then ρ restricted to $P(\lambda)\mathfrak{B}(\mathfrak{H})P(\lambda)$ is normal for all $\lambda > 0$.

Consider now the case when $T_1 \geqslant T_2 \geqslant 0$ and T_1 is strictly contractive so T_2 is strictly contractive. Define S on the range $\sqrt{I - T_2}$ by the relation

$$S(I-T_2)^{\frac{1}{2}} f = (I-T_1)^{\frac{1}{2}} f$$

for $f \in \mathfrak{H}$. Note

$$\big\|S(I-T_2)^{\frac{1}{2}} f\big\|^2 = \big(f, (I-T_1)f\big) \leqslant \big(f, (I-T_2)f\big) = \big\|(I-T_2)^{\frac{1}{2}} f\big\|^2$$

for $f \in \mathfrak{H}$. Then S is a contractive map on the range of $\sqrt{I-T_2}$ which is dense in $\mathfrak{H}$ so S has a unique bounded extension to a contraction defined on all of $\mathfrak{H}$. We also denote this operator by S. We note S is a contraction which satisfies the operator equation

$$S(I-T_2)^{\frac{1}{2}} = (I-T_1)^{\frac{1}{2}} \quad \text{so } (I-T_2)^{\frac{1}{2}} S^* A S (I-T_2)^{\frac{1}{2}} = (I-T_1)^{\frac{1}{2}} A (-T_1)^{\frac{1}{2}}$$

for $A \in \mathfrak{B}(\mathfrak{H})$ so it follows that $\mathfrak{A}(\mathfrak{H}, T_1) \subset \mathfrak{A}(\mathfrak{H}, T_2)$. We show that $\mathfrak{A}(\mathfrak{H}, T_2)_* \subset \mathfrak{A}(\mathfrak{H}, T_1)_*$. Suppose $\rho \in \mathfrak{A}(\mathfrak{H}, T_2)_*$ which means

$$\rho\big((I-T_2)^{\frac{1}{2}}A(I-T_2)^{\frac{1}{2}}\big) = \eta(A)$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ where $\eta \in \mathfrak{B}(\mathfrak{H})_*$. Then we have

$$\rho\big((I-T_1)^{\frac{1}{2}}A(I-T_1)^{\frac{1}{2}}\big) = \rho\big((I-T_2)^{\frac{1}{2}}S^*AS(I-T_2)^{\frac{1}{2}}\big) = \eta(S^*AS)$$

for $A \in \mathfrak{B}(\mathfrak{H})$. So we see $\rho \in \mathfrak{A}(\mathfrak{H}, T_1)$ and since S is a contraction we have $\|\rho\|_{T_1} \leqslant \|\rho\|_{T_2}$.

Note a 0-weight is just a normal functional.

We caution the reader that the T-weights we consider are not normal weights. A normal weight on a von Neumann algebra has the property that if $0 \leqslant A_1 \leqslant A_2 \leqslant \cdots$ is a increasing sequence of operators which converge strongly to A then $\omega(A)$ is the limit of the $\omega(A_k)$, where we allow $+\infty$ as a possible limit. Let $\mathfrak{H} = L^2(0,\infty)$ and let ω be the Λ-weight given by

$$\omega\big((I-\Lambda)^{\frac{1}{2}}A(I-\Lambda)^{\frac{1}{2}}\big) = (h, Ah)$$

for $A \in \mathfrak{B}(\mathfrak{H})$ where $h(x) = x^{-\frac{1}{2}s}(1-e^{-x})^{\frac{1}{2}}$ for $s \in (1,2)$. For each $n = 1, 2, 3, \ldots,$ let $\mathfrak{M}_n$ be the set of functions g in $\mathfrak{H}$ with support in $[1/n, \infty)$ and

$$\int_{1/n}^{\infty} x^{-s/2} g(x)\, dx = 0.$$

Let P_n be the orthogonal projection onto $\mathfrak{M}_n$, and consider $B_n = (I - \Lambda)^{-1/2} P_n$. Observe that B_n is as bounded operator, and moreover $A_n = B_n B_n^*$ satisfies $(I-\Lambda)^{1/2} A_n (I-\Lambda)^{1/2} = P_n$, for $n = 1, 2, \ldots.$ Note that $\omega(P_n) = (A_n h, h) = 0$ for $n = 1, 2, \ldots$ and $P_n \to I$ as $n \to \infty$, but it is not true that $\omega(I) = 0$. If we were to assign $\omega(I)$ a value it is $+\infty$ since ω is positive and unbounded. Although T-weights are not in general normal weights we do not think of them as pathological like non-normal bounded functionals since T-weights are normal when scaled down by $\sqrt{I-T}$.

In the particular case when $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$ when we speak of *boundary weights* we mean the following. Let Λ be the operator corresponding to multiplication by e^{-x}. Then the boundary algebra $\mathfrak{A}(\mathfrak{H})$ is

$$\mathfrak{A}(\mathfrak{H}) = \mathfrak{A}(\mathfrak{H}, \Lambda) = (I-\Lambda)^{\frac{1}{2}} \mathfrak{B}(\mathfrak{H})(I-\Lambda)^{\frac{1}{2}}$$

and the boundary weights denoted by $\mathfrak{A}(\mathfrak{H})_*$ are

$$\mathfrak{A}(\mathfrak{H})_* = \mathfrak{A}(\mathfrak{H}, \Lambda)_*.$$

If ω is a boundary weight we say ω is weight contractive if $\|\omega\|_\Lambda \leqslant 1$ and if ω is a positive boundary weight we say ω is normalized if $\|\omega\|_\Lambda = 1$. If ω is a boundary weight and we say ω is *bounded* we mean ω is bounded as a functional on $\mathfrak{B}(\mathfrak{H})$ (i.e. there exists $k > 0$ such that $|\omega(A)| \leqslant k\|A\|$ for all $A \in \mathfrak{A}(\mathfrak{H})$).

The mapping $\rho \to \omega(\rho)$ defined for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ is a *boundary weight map* if this mapping is a linear mapping of $\mathfrak{B}(\mathfrak{K})_*$ into boundary weights on $\mathfrak{A}(\mathfrak{H})$ and this mapping is completely bounded with the norm on $\mathfrak{B}(\mathfrak{K})_*$ being the usual norm and the norm on the boundary weights being the boundary weight norm. A boundary weight map is positive if it is completely positive. A boundary weight map ω is unital if $\omega(\rho)(I-\Lambda) = \rho(I)$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$.

Maintaining the notation of the above definition we observe that $U(t)AU(t)^* \in \mathfrak{A}(\mathfrak{H})$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$. Recall the mapping Γ defined above. Since Γ is completely positive and $\Gamma(I) = I - \Lambda$, so $\Gamma(I) \in \mathfrak{A}(\mathfrak{H})$, it follows that $\Gamma(A) \in \mathfrak{A}(\mathfrak{H})$ for all $A \in \mathfrak{B}(\mathfrak{H})$. For more details see the discussion after Definition 4.16 in [18].

Every *CP*-flow is given by a boundary weight map $\rho \to \omega(\rho)$. As we have mentioned the map is completely positive. There is a further complicated positivity condition. The condition says if you construct an approximation to the boundary representation π_t, then π_t is completely positive.

We describe the connection between boundary weight and boundary representation. One can construct a boundary weight map so that the boundary representation is a given σ-weakly continuous completely positive contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Suppose π is a σ-weakly continuous completely positive contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Let

$$\omega = \hat{\pi} + \hat{\pi}\hat{\Lambda}\hat{\pi} + \hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi} + \hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi} + \cdots.$$

This converges as a weight (i.e. the above series converges on the boundary algebra $\mathfrak{A}(\mathfrak{H})$) and this is the boundary weight map of a *CP*-flow. We call this the minimal *CP*-flow derived from π. Formally $\omega = \hat{\pi}(I - \hat{\Lambda}\hat{\pi})^{-1}$ and solving for π we have

$$\hat{\pi} = \omega(I + \hat{\Lambda}\omega)^{-1}.$$

If a boundary weight associated with a *CP*-flow is bounded the boundary representation is well defined as stated in the next theorem (see [18, Theorem 4.27]).

Theorem 1.11. *Suppose α is a CP-flow over $\mathfrak{K}$ and $\rho \to \omega(\rho)$ is the associated boundary weight map. Suppose $\|\omega(\rho)\| < \infty$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ so $\omega(\rho) \in \mathfrak{B}(\mathfrak{H})_*$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Then the mapping $\rho \to \rho + \hat{\Lambda}\omega(\rho)$ is invertible i.e. $(I + \hat{\Lambda}\omega)^{-1}$ exists and $\hat{\pi}$ given by*

$$\hat{\pi} = \omega(I + \hat{\Lambda}\omega)^{-1}$$

is a completely positive contraction from $\mathfrak{B}(\mathfrak{K})_*$ to $\mathfrak{B}(\mathfrak{H})_*$. There is a unique CP-flow derived from π and its boundary weight map is given by

$$\omega = \hat{\pi} + \hat{\pi}\hat{\Lambda}\hat{\pi} + \hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi} + \hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi} + \cdots.$$

So when $\omega(\rho)$ is bounded for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ we have

$$\omega = \hat{\pi}(I - \hat{\Lambda}\hat{\pi})^{-1} \quad \textit{and} \quad \hat{\pi} = \omega(I + \hat{\Lambda}\omega)^{-1}.$$

Now we introduce a bit of notation. Suppose ω is a boundary weight and $t > 0$. We denote by $\omega|_t$ the functional given by $\omega|_t(A) = \omega(E(t,\infty)AE(t,\infty))$ for $A \in \mathfrak{B}(\mathfrak{H})$. Note $\omega|_t(\rho) \in \mathfrak{B}(\mathfrak{H})_*$, i.e. $\omega|_t(\rho)$ is a bounded σ-weakly continuous functional. We use the same notation for operators. If $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$ then we denote $A|_t$ the operator $A|_t = E(t,\infty)AE(t,\infty)$. Note for ω a boundary weight and $A \in \mathfrak{B}(\mathfrak{H})$ then $\omega|_t(A) = \omega(A|_t)$.

From [18, Theorems 4.23 and 4.27 and Lemma 4.34] we have the following theorem.

Theorem 1.12. *Suppose $\rho \to \omega(\rho)$ is the boundary weight map of a CP-flow over $\mathfrak{K}$. Then for each $t > 0$ we have $\rho \to \omega|_t(\rho)$ is the boundary weight map of a CP-flow over $\mathfrak{K}$. Suppose $\rho \to \omega(\rho)$ is a completely positive mapping of $\mathfrak{B}(\mathfrak{K})$ into boundary weights on $\mathfrak{B}(\mathfrak{H})$ satisfying $\omega(\rho)(I - \Lambda) \leqslant \rho(I)$ for ρ positive. Suppose*

$$\hat{\pi}_t^{\#} = \omega|_t\big(I + \hat{\Lambda}\omega|_t\big)^{-1}$$

is a completely positive contraction of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{B}(\mathfrak{H})_*$ for each $t > 0$. Then $\rho \to \omega(\rho)$ is the boundary weight map of a CP-flow over $\mathfrak{K}$.

Furthermore, the mapping $\hat{\pi}_t^{\#}$ defined above has the property that if $\phi_t(A) = \pi_t^{\#}(E(s,\infty)AE(s,\infty))$ for $0 < t \leqslant s < \infty$ and $A \in \mathfrak{B}(\mathfrak{H})$ then ϕ_t is increasing in t in the sense complete positivity (i.e., the mapping $A \to \phi_t(A) - \phi_r(A)$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $0 < t < r \leqslant s$ is completely positive).

Definition 1.13. If $\rho \to \omega(\rho)$ is a mapping of $\mathfrak{B}(\mathfrak{K})_*$ into boundary weights on $\mathfrak{B}(\mathfrak{H})$ so that $\hat{\pi}_t^{\#}$ defined above is completely positive for each $t > 0$ we say this map is *q-positive*. The family

$\pi_t^\#$ of completely positive σ-weakly continuous contractions of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ is called the *generalized boundary representation*.

We remark that in checking that the $\pi_t^\#$ are completely positive it is only necessary to check for small t. If the mapping $\pi_t^\#$ is completely positive then $\pi_s^\#$ is completely positive for all $s \geqslant t$. Next we give the order relation for the generalized boundary representation(see [18, Theorem 4.20]).

Theorem 1.14. *If α and β are CP-flows over $\mathfrak{K}$ then β is a subordinate of α $(\alpha \geqslant \beta)$ if and only if $\pi_t^\# \geqslant \phi_t^\#$ for all $t > 0$ where $\pi_t^\#$ and $\phi_t^\#$ are the generalized boundary representations of α and β. Also we have if $\pi_t^\# \geqslant \phi_t^\#$ then $\pi_s^\# \geqslant \phi_s^\#$ for all $s \geqslant t$ so one only has to check for a sequence $\{t_n\}$ tending to zero.*

In Theorem 1.11 we used the phrase "α is derived from π." The next theorem (see [18, Theorem 4.24]) and definition will make this more precise. We need a bit of notation which is given in the next definition.

Definition 1.15. Let Q_0 be the map from $\mathfrak{K}$ to $\mathfrak{H}$ given by $(Q_0k)(x) = e^{-\sqrt{x}}k$. And let Φ be the mapping of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{B}(\mathfrak{H})_*$ given by $\Phi(\rho)(A) = \rho(Q_0^*AQ_0)$ for all $A \in \mathfrak{B}(\mathfrak{H})$.

Note that $\Phi(\rho)(U(t)AU(t)^*) = e^{-t}\Phi(\rho)(A)$, $\Phi(\rho)(\Gamma(A)) = \frac{1}{2}\Phi(\rho)(A)$ for $t \geqslant 0$ and $\Phi(\rho)(\Lambda(C)) = \frac{1}{2}\rho(C)$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$, $A \in \mathfrak{B}(\mathfrak{H})$ and $C \in \mathfrak{B}(\mathfrak{K})$.

Theorem 1.16. *Suppose $\rho \to \omega(\rho)$ defines a CP-flow over $\mathfrak{K}$ as described in Definition* 1.9 *and δ is the generator of α (i.e., δ is the derivative of α_t at $t = 0$). Suppose π is a completely positive normal contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Then the following are equivalent*:

(i) *$\Phi(\rho) \in \mathfrak{D}(\hat{\delta})$ and $\hat{\delta}(\Phi(\rho)) = \hat{\pi}(\rho) - \Phi(\rho)$ for each $\rho \in \mathfrak{B}(\mathfrak{K})_*$.*
(ii) *$\omega(\rho - \hat{\Lambda}(\hat{\pi}(\rho))) = \hat{\pi}(\rho)$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$.*
(iii) *$\pi(A) = \pi_0(A)$ for all $A \in \mathfrak{D}(\delta)$ where π_0 is the boundary representation of α.*

Definition 1.17. We say a *CP*-flow α over $\mathfrak{K}$ is derived from the completely positive normal contraction π of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ if it satisfies one and, therefore, all the conditions of Theorem 1.16.

As mentioned earlier for each such π there is a *CP*-flow α derived from π and the next theorem (see [18, Theorem 4.26]) gives a condition for uniqueness.

Theorem 1.18. *Suppose π is a completely positive σ-weakly continuous linear contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Then for each $\rho \in \mathfrak{B}(\mathfrak{K})_*$ the sum*

$$\omega(\rho) = \hat{\pi}(\rho) + \hat{\pi}\big(\hat{\Lambda}\big(\hat{\pi}(\rho)\big)\big) + \hat{\pi}\big(\hat{\Lambda}\big(\hat{\pi}\big(\hat{\Lambda}\big(\hat{\pi}(\rho)\big)\big)\big)\big) + \cdots$$

converges as a weight on $\mathfrak{A}(\mathfrak{H})$ and the mapping $\rho \to \omega(\rho)$ is the boundary weight map of a CP-flow α which is derived from π. Furthermore, this α is the minimal CP-flow derived from π in that if $\rho \to \eta(\rho)$ is the boundary weight map of a second CP-semigroup derived from π then $\omega(\rho) \leqslant \eta(\rho)$ for all positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Moreover, if $(\pi \circ \Lambda)^n(I) \to 0$ weakly as $n \to \infty$ then α defined above is unique (i.e. α is the only CP-flow derived from π).

We remark that we believe that this theorem can be strengthened with the stronger conclusion being that α is a flow subordinate of any *CP*-flow derived from π. In the examples we construct in this paper we will show that the stronger result holds.

So far most of the results in this section are proved in [18]. The next two theorems are new.

Theorem 1.19. *Suppose α is a CP-flow over $\mathfrak{K}$ derived from π as described in Definition* 1.17 *and β is CP-flow subordinate to α, so the mapping $A \to \alpha_t(A) - \beta_t(A)$ for $A \in \mathfrak{B}(\mathfrak{H})$ is completely positive for all $t \geqslant 0$. Then there is a unique completely positive normal contraction ϕ of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ which is subordinate to π so that β is derived from ϕ.*

Proof. Assume the hypothesis of the theorem and suppose δ_α and δ_β are the generators of α and β, respectively. Let $\gamma_t(A) = U(t)AU(t)^*$ for $t \geqslant 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. Since β is a subordinate of α and β is intertwined by $U(t)$ we have the maps $t \to \alpha_t(A) - \beta_t(A)$ and $t \to \beta_t(A) - \gamma_t(A)$ and $A \in \mathfrak{B}(\mathfrak{H})$ are completely positive for all $t \geqslant 0$. Suppose $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $\rho \geqslant 0$. Then we have

$$\begin{aligned}\vartheta_t &= t^{-1}\big(\hat{\alpha}_t\big(\Phi(\rho)\big) - \Phi(\rho)\big) + \Phi(\rho) \geqslant t^{-1}\big(\hat{\beta}_t\big(\Phi(\rho)\big) - \Phi(\rho)\big) + \Phi(\rho)\\ &= \nu_t \geqslant t^{-1}\big(\hat{\gamma}_t\big(\Phi(\rho)\big) - \Phi(\rho)\big) + \Phi(\rho) = t^{-1}\big(e^{-t} - 1 + t\big)\Phi(\rho)\end{aligned}$$

for $t > 0$ where the two equal signs are definitions of ϑ_t and ν_t. Since α is derived from π we have

$$\vartheta_t = t^{-1}\big(\hat{\alpha}_t\big(\Phi(\rho)\big) - \Phi(\rho)\big) + \Phi(\rho) \to \hat{\delta}\big(\Phi(\rho)\big) + \Phi(\rho) = \hat{\pi}(\rho) = \vartheta_0$$

as $t \to 0^+$ and the convergence is in norm. Since $\vartheta_0 = \hat{\pi}(\rho) \in \mathfrak{B}(\mathfrak{H})_*$ there is a positive trace class operator Ω_0 so that

$$\vartheta_0(A) = \hat{\pi}(\rho)(A) = \operatorname{tr}(A\Omega_0)$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and for every $\rho_1 \in \mathfrak{B}(\mathfrak{H})_*$ with $\vartheta_0 \geqslant \rho_1 \geqslant 0$ there is an $X \in \mathfrak{B}(\mathfrak{H})$ with $0 \leqslant X \leqslant I$ so that

$$\rho_1(A) = \operatorname{tr}\big(A\Omega_0^{\frac{1}{2}}X\Omega_0^{\frac{1}{2}}\big)$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and conversely if $X \in \mathfrak{B}(\mathfrak{H})$ and $0 \leqslant X \leqslant I$ then ρ_1 defined above is in $\mathfrak{B}(\mathfrak{H})_*$ and $0 \leqslant \rho_1 \leqslant \vartheta_0$. (If we require the null space of X contains $\text{Range}(\Omega_0)^\perp$ then X is uniquely determined by ρ_1.) Suppose $t > 0$ and Ω_t is the unique positive trace class operator so that

$$\vartheta_t(A) = \big(t^{-1}\big(\hat{\alpha}_t\big(\Phi(\rho)\big) - \Phi(\rho)\big) + \Phi(\rho)\big)(A) = \operatorname{tr}(A\Omega_t)$$

for $A \in \mathfrak{B}(\mathfrak{H})$. From the inequality above we have $\vartheta_t \geqslant \nu_t \geqslant 0$ so there is an operator $X_t \in \mathfrak{B}(\mathfrak{H})$ with $0 \leqslant X_t \leqslant I$ so that

$$\nu_t(A) = \big(t^{-1}\big(\hat{\beta}_t\big(\Phi(\rho)\big) - \Phi(\rho)\big) + \Phi(\rho)\big)(A) = \operatorname{tr}\big(A\Omega_t^{\frac{1}{2}}X_t\Omega_t^{\frac{1}{2}}\big)$$

for $A \in \mathfrak{B}(\mathfrak{H})$. We will require the null space of X_t contains $\mathrm{Range}(\Omega_t)^\perp$ so X_t is uniquely determined. Now let

$$\eta_t(A) = \mathrm{tr}\big(A\Omega_0^{\frac{1}{2}} X_t \Omega_0^{\frac{1}{2}}\big)$$

for $A \in \mathfrak{B}(\mathfrak{H})$. Now we have

$$\|\nu_t - \eta_t\| = \sup\big\{\mathrm{Re}\big(\mathrm{tr}\big(A\big(\Omega_t^{\frac{1}{2}} X_t \Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}} X_t \Omega_0^{\frac{1}{2}}\big)\big)\big)\colon A \in \mathfrak{B}(\mathfrak{H}),\ \|A\| \leqslant 1\big\}.$$

We have $|\mathrm{tr}(AB)| \leqslant \|A\|_{HS}\|B\|_{HS}$ for $A, B \in \mathfrak{B}(\mathfrak{H})$ with $\|A\|_{HS} = \mathrm{tr}(A^*A)^{\frac{1}{2}}$ the Hilbert–Schmidt norm. Then for $A \in \mathfrak{B}(\mathfrak{H})$ with $\|A\| \leqslant 1$,

$$\begin{aligned}
\big|\mathrm{tr}\big(A\big(\Omega_t^{\frac{1}{2}} X_t \Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}} X_t \Omega_0^{\frac{1}{2}}\big)\big)\big| &= \big|\mathrm{tr}\big(A\big(\Omega_t^{\frac{1}{2}} X_t \Omega_t^{\frac{1}{2}} - \Omega_t^{\frac{1}{2}} X_t \Omega_0^{\frac{1}{2}} + \Omega_t^{\frac{1}{2}} X_t \Omega_0^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}} X_t \Omega_0^{\frac{1}{2}}\big)\big)\big| \\
&\leqslant \big|\mathrm{tr}\big(A\Omega_t^{\frac{1}{2}} X_t\big(\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}\big)\big)\big| + \big|\mathrm{tr}\big(X_t\Omega_0^{\frac{1}{2}} A\big(\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}\big)\big)\big| \\
&\leqslant \big(\big\|A\Omega_t^{\frac{1}{2}} X_t\big\|_{HS} + \big\|A\Omega_0^{\frac{1}{2}} X_t\big\|_{HS}\big)\big\|\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}\big\|_{HS} \\
&\leqslant \big(\mathrm{tr}(\Omega_t)^{\frac{1}{2}} + \mathrm{tr}(\Omega_0)^{\frac{1}{2}}\big)\big\|\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}\big\|_{HS}
\end{aligned}$$

and, hence, it follows that

$$\|\nu_t - \eta_t\| \leqslant \big(\|\nu_t\|^{\frac{1}{2}} + \|\vartheta_0\|^{\frac{1}{2}}\big)\big\|\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}\big\|_{HS} \leqslant \big(\|\vartheta_t\|^{\frac{1}{2}} + \|\vartheta_0\|^{\frac{1}{2}}\big)\big\|\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}\big\|_{HS}.$$

Now if UT is the polar decomposition of $\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}$ so $U^*(\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}) = |\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}| = ((\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}})^2)^{\frac{1}{2}}$ we have

$$\begin{aligned}
\|\vartheta_t - \vartheta_0\| &\geqslant \big|\vartheta_t(U^*) - \vartheta_0(U^*)\big| = \big|\mathrm{tr}\big(U^*(\Omega_t - \Omega_0)\big)\big| \\
&= \big|\mathrm{tr}\big(U^*\big(\big(\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}\big)\big(\Omega_t^{\frac{1}{2}} + \Omega_0^{\frac{1}{2}}\big) + \big(\Omega_t^{\frac{1}{2}} + \Omega_0^{\frac{1}{2}}\big)\big(\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}\big)\big)\big)\big| \\
&= \mathrm{tr}\big(\big|\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}\big|\big(\Omega_t^{\frac{1}{2}} + \Omega_0^{\frac{1}{2}}\big)\big) \geqslant \mathrm{tr}\big(\big|\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}\big|^2\big) \\
&= \mathrm{tr}\big(\big(\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}\big)^2\big) = \big\|\Omega_t^{\frac{1}{2}} - \Omega_0^{\frac{1}{2}}\big\|_{HS}^2.
\end{aligned}$$

Hence, we have

$$\|\nu_t - \eta_t\| \leqslant \big(\|\vartheta_t\|^{\frac{1}{2}} + \|\vartheta_0\|^{\frac{1}{2}}\big)\|\vartheta_t - \vartheta_0\|^{\frac{1}{2}}$$

for all $t > 0$. Note $\vartheta_0 \geqslant \eta_t \geqslant 0$ for each $t > 0$. Note the set S of $\eta \in \mathfrak{B}(\mathfrak{H})_*$ with $\vartheta_0 \geqslant \eta \geqslant 0$ is compact. This may be seen as follows. For every $\epsilon_1 > 0$ there is a finite rank $\xi \in \mathfrak{B}(\mathfrak{H})_*$ so that $0 \leqslant \xi \leqslant \vartheta_0$ and $\|\xi - \vartheta_0\| < \epsilon_1$. Given $\epsilon > 0$ we can by choosing ϵ_1 small enough insure that for every $\eta \in S$ there is a positive $\eta' \leqslant \xi$ with $\|\eta' - \eta\| < \epsilon$. Hence, for every $\epsilon > 0$ there is a finite-dimensional compact subset of S so that the ϵ-neighborhoods of this set cover S, thus for every $\epsilon > 0$ there is a cover of S with a finite numbers of open balls of radius ϵ and we obtain that

S is totally bounded. Since S is complete (it is clearly closed) and totally bounded, it follows that it is compact. Since S is compact there is a sequence $t_n \to 0^+$ as $n \to \infty$ so that η_{t_n} converges to a limit η_0 in norm as $n \to \infty$. Since $\|\vartheta_t - \vartheta_0\| \to 0$ as $t \to 0^+$ it follows from the above estimate that $\nu_{t_n} \to \eta_0$ as $n \to \infty$. Hence, we have

$$\left\| t_n^{-1}\big(\hat{\beta}_{t_n}\big(\Phi(\rho)\big) - \Phi(\rho)\big) - \big(\eta_0 - \Phi(\rho)\big)\right\| \to 0$$

as $n \to \infty$. Now let

$$\mu_n = \frac{1}{t_n}\int_0^{t_n} \hat{\beta}_s\big(\Phi(\rho)\big)\, ds$$

for $n = 1, 2, \ldots$. Let δ_β be the generator of β. Note $\mu_n \in \mathfrak{D}(\hat{\delta}_\beta)$ and

$$\hat{\delta}_\beta(\mu_n) = t_n^{-1}\big(\hat{\beta}_{t_n}\big(\Phi(\rho)\big) - \Phi(\rho)\big) \to \eta_0 - \Phi(\rho)$$

as $n \to \infty$ where the convergence is in norm. Since $\mu_n \to \Phi(\rho)$ in norm as $n \to \infty$ it follows from the fact that $\hat{\delta}_\beta$ is closed that $\Phi(\rho) \in \mathfrak{D}(\hat{\delta}_\beta)$ and $\hat{\delta}_\beta(\Phi(\rho)) = \eta_0 - \Phi(\rho)$. Since ρ was an arbitrary positive element of $\mathfrak{B}(\mathfrak{H})_*$ it follows that $\Phi(\rho) \in \mathfrak{D}(\hat{\delta}_\beta)$ and $\hat{\delta}_\beta(\Phi(\rho)) + \Phi(\rho) = \hat{\phi}(\rho)$ where this equation defines ϕ. Since $\alpha \geqslant \beta \geqslant \gamma$ in the sense of complete positivity it follows that $\pi \geqslant \phi \geqslant 0$ is the sense of complete positivity. From Definition 1.17 it follows that β is derived from ϕ. The uniqueness of ϕ follows from the defining equation for ϕ. □

Theorem 1.20. *Suppose α is a CP-flow over $\mathfrak{K}$ and π is a normal completely positive contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ and suppose further that π is unital so $\pi(I) = I$. Suppose β is a CP-flow over $\mathfrak{K}$ derived from π and $\alpha \geqslant \beta$ (i.e. the mapping $A \to \alpha_t(A) - \beta_t(A)$ for $A \in \mathfrak{B}(\mathfrak{H})$ is completely positive for all $t \geqslant 0$). Then α is derived from π.*

Proof. Assume the hypothesis and notation of the theorem. Suppose $\rho \in \mathfrak{B}(\mathfrak{H})_*$ and ρ is positive. Then defining ϑ_t and ν_t as in the proof of the last theorem we have $\vartheta_t \geqslant \nu_t \geqslant 0$ for $t > 0$ and $\nu_t \to \hat{\pi}(\rho)$ in norm as $t \to 0^+$. Since $\vartheta_t - \nu_t \geqslant 0$ and $\alpha_t(I) \leqslant I$ and π is unital we have

$$\begin{aligned}\|\vartheta_t - \nu_t\| &= \vartheta_t(I) - \nu_t(I) = t^{-1}\Phi(\rho)\big(\alpha_t(I) - I\big) + \Phi(\rho)(I) - \nu_t(I) \\ &\leqslant \Phi(\rho)(I) - \nu_t(I) = \rho(I) - \nu_t(I) \to \rho(I) - \hat{\pi}(\rho)(I) = 0.\end{aligned}$$

Hence, $\vartheta_t \to \hat{\pi}(\rho)$ in norm as $t \to 0^+$. Since each $\rho \in \mathfrak{B}(\mathfrak{K})_*$ is the linear combination of at most four positive elements of $\mathfrak{B}(\mathfrak{K})_*$ we have

$$t^{-1}\big(\hat{\alpha}_t(\Phi(\rho)) - \Phi(\rho)\big) + \Phi(\rho) \to \hat{\pi}(\rho)$$

in norm as $t \to 0^+$. Thus, $\Phi(\rho) \in \mathfrak{D}(\hat{\delta})$ and $\hat{\delta}(\Phi(\rho)) + \Phi(\rho) = \hat{\pi}(\rho)$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Hence, α is derived from π. □

We will want to analyze the action of local cocycles on units. Suppose α is a unital *CP*-flow and α^d is the minimal dilation of α to an E_0-semigroup acting on $\mathfrak{B}(\mathfrak{H}_1)$ as described in Theorem 1.3. A *unit* for α^d is a one-parameter semigroup of isometries $V(t)$ which intertwine α^d

so ($V(t)A = \alpha_t^d(A)V(t)$ for all $A \in \mathfrak{B}(\mathfrak{H}_1)$ and $t \geqslant 0$). Units for α^d are in one to one correspondence with semigroups $S(t)$ acting on $\mathfrak{H}$ with the property that the semigroup $\Omega_t(A) = S(t)AS(t)^*$ is a trivially maximal subordinate of α (i.e., the mapping $A \to \alpha_t(A) - e^{st}\Omega_t(A)$ for $A \in \mathfrak{B}(\mathfrak{H})$ is completely positive for all $t \geqslant 0$ provided $s \leqslant 0$ and the mapping is not positive for $s > 0$ and $t > 0$). The next two theorems (see [18, Theorems 4.46, 4.50 and 4.51]) describe such semigroups and the connection between them and units for the dilated E_0-semigroup.

Theorem 1.21. *Suppose α is a CP-flow over $\mathfrak{K}$ and $S(t)$ is a strongly continuous one-parameter semigroup and $\Omega_t(A) = S(t)AS(t)^*$ for $t \geqslant 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ is a subordinate of α. Then $S(t)$ is a strongly continuous one-parameter semigroup of contractions with generator $-D$ where $\mathfrak{D}(D) = \{f \in \mathfrak{D}(d^*)\colon f(0) = Vf\}$ and $Df = -d^*f + cf$ where c is a complex number with non-negative real part and V is a linear operator from $\mathfrak{H}$ to $\mathfrak{K}$ with norm satisfying $\|V\|^2 \leqslant 2\,\mathrm{Re}(c)$. Furthermore, if $\pi(A) = (2\,\mathrm{Re}(c))^{-1}VAV^*$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and γ is the minimal CP-semigroup derived from π then α dominates γ. In the case $\mathrm{Re}(c) = 0$ we define $\pi = 0$.*

Conversely, if c is a complex number with $\mathrm{Re}(c) > 0$ and V is a linear operator from $\mathfrak{H}$ to $\mathfrak{K}$ with norm satisfying $\|V\|^2 \leqslant 2\,\mathrm{Re}(c)$ and if $\pi(A) = (2\,\mathrm{Re}(c))^{-1}VAV^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and γ is the minimal CP-semigroup derived from π and α dominates γ then if D is an operator with domain $\mathfrak{D}(D) = \{f \in \mathfrak{D}(d^*)\colon f(0) = Vf\}$ and $Df = -d^*f + cf$. Then $-D$ is the generator of a contraction semigroup $S(t)$ and if $\Omega_t(A) = S(t)AS(t)^*$ for $t \geqslant 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ and α dominates Ω.

Theorem 1.22. *Suppose α is a unital CP-flow over $\mathfrak{K}$ and α^d is the minimal dilation of α to an E_0-semigroup and suppose the relation between α and α^d is given by*

$$\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ (with $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$) and $t \geqslant 0$ where W is an isometry from $\mathfrak{H}$ to $\mathfrak{H}_1$ and WW^* is an increasing projection for α^d and α^d is minimal over the range of W. Then $\mathfrak{H}_1$ can be expressed as $\mathfrak{H}_1 = \mathfrak{K}_1 \otimes L^2(0,\infty)$ and α^d is a CP-flow over $\mathfrak{K}_1$ so that if $U(t)$ and $U_1(t)$ are right translation on $\mathfrak{H}$ and $\mathfrak{H}_1$ for α and α^d, respectively, then $U_1(t)W = WU(t)$ and $U_1(t)^*W = WU(t)^*$ for all $t \geqslant 0$. This means that W as a mapping of $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ into $\mathfrak{H}_1 = \mathfrak{K}_1 \otimes L^2(0,\infty)$ can be expressed in the form $W = W_1 \otimes I$ where W_1 is an isometry from $\mathfrak{K}$ into $\mathfrak{K}_1$.

Suppose $S(t)$ is a strongly continuous semigroup of contractions of $\mathfrak{H}$ and Ω given by $\Omega_t(A) = S(t)AS(t)^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$ is a subordinate of α. Further assume Ω is trivially maximal. Then there is a unique strongly continuous one-parameter semigroup of isometries $S_1(t)$ which intertwine α_t^d for each $t \geqslant 0$ and

$$S(t) = W^*S_1(t)W$$

for all $t \geqslant 0$.

Conversely, if $S_1(t)$ is a strongly continuous one-parameter semigroup of isometries which intertwine α_t^d for each $t \geqslant 0$ then if $S(t)$ is as defined in the equation above we have that $S(t)$ is a strongly continuous one-parameter semigroup of contractions so that Ω defined by $\Omega_t(A) = S(t)AS(t)^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$ is a subordinate of α which is trivially maximal.

We end this section with some notation and results which we will need in the next section. As we saw in Theorem 1.18 the boundary weight map of the minimal *CP*-flow derived from π is given by

$$\omega(\rho) = \hat{\pi}(\rho) + \hat{\pi}\big(\hat{\Lambda}\big(\hat{\pi}(\rho)\big)\big) + \hat{\pi}\big(\hat{\Lambda}\big(\hat{\pi}\big(\hat{\Lambda}\big(\hat{\pi}(\rho)\big)\big)\big)\big) + \cdots.$$

We introduce some notation. For $n \in \mathbb{N}$ we write $R_n(\phi)$ to denote finite sum and $R(\phi)$ to denote the infinite series

$$R_n(\phi) = I + \phi + \phi^2 + \cdots + \phi^n \quad \text{and} \quad R(\phi) = I + \phi + \phi^2 + \cdots.$$

Then the expression for ω above can be written

$$\omega = \hat{\pi} R(\hat{\Lambda}\hat{\pi}) = R(\hat{\pi}\hat{\Lambda})\hat{\pi}.$$

Formally, $R(\phi) = (I - \phi)^{-1}$, however the inverse in question may not exist. The sums above make sense in that the series

$$\omega(\rho)(A) = \rho\big(\pi(A)\big) + \rho\big(\pi\big(\Lambda\big(\pi(A)\big)\big)\big) + \cdots$$

converges absolutely for $A \in \mathfrak{A}(\mathfrak{H})$. This is seen by setting $A = I - \Lambda$ and assuming $\rho \in \mathfrak{B}(\mathfrak{K})_*$ is positive. As we saw in Theorem 1.18 the series above for ω defines the minimal *CP*-flow derived from π. We know from Theorem 1.12 that the truncated boundary weight map $\rho \to \omega|_t$ for $t > 0$ is the minimal *CP*-flow derived from the truncated boundary representation $\phi_t^{\#}$.

2. An almost type I *CP*-flow

In this section we study *CP*-flows derived from a particular strongly continuous $*$-representation π. Let $\mathfrak{K}$ be the infinite tensor product of $L^2(0,\infty)$ so $\mathfrak{K} = \bigotimes_{k=1}^{\infty} L^2(0,\infty)$ with the reference vector (see [23] for details of infinite tensor products of Hilbert spaces)

$$F_0 = k_1 \otimes k_2 \otimes \cdots$$

with

$$k_i(x) = \lambda_i e^{-\frac{1}{2}\lambda_i^2 x}$$

for $x \geqslant 0$ where $\lambda_i > 0$ for $i = 1, 2, \ldots$. The Hilbert space $\mathfrak{K}$ is spanned by product vectors of the form

$$F = f_1 \otimes f_2 \otimes \cdots$$

where

$$\sum_{i=1}^{\infty} \|f_i - k_i\|^2 < \infty. \tag{2.1}$$

The inner product between two such product vectors is given by

$$(F, G) = \prod_{i=1}^{\infty} (f_i, g_i).$$

We impose the following two conditions on the positive numbers λ_i:

$$\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{|\lambda_n - \lambda_{n+1}|^2}{\lambda_n^2 + \lambda_{n+1}^2} < \infty. \tag{2.2}$$

We note both these conditions are satisfied for $\lambda_n = n$ and the second condition is not satisfied for $\lambda_n = 2^n$. Let S_0 be the unitary mapping of $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$ into $\mathfrak{K}$ given by

$$S_0\big((f_1 \otimes f_2 \otimes \cdots) \otimes h\big) = h \otimes f_1 \otimes f_2 \otimes \cdots \tag{2.3}$$

and let $\pi(A) = S_0 A S_0^*$ and $\Delta = e^{-x} \otimes e^{-x} \otimes \cdots$ where e^{-x} is a shorthand for the operation of multiplication by e^{-x} on $L^2(0, \infty)$. The first sum condition insures that Δ is not zero and the second condition insures that S_0 is well defined. Note π is a normal $*$-representation of $\mathfrak{B}(\mathfrak{H})$ on $\mathfrak{B}(\mathfrak{K})$. Suppose n is a positive integer. We define $\mathfrak{K}_n$ as the tensor product of the Hilbert spaces $L^2(0, \infty)$ from $n+1$ on with the reference vector $F_{n0} = \bigotimes_{i=n+1}^{\infty} k_i$. Let S_n be the linear mapping from $\mathfrak{K}$ to $\mathfrak{K}_n$ which takes the product vector $F = \bigotimes_{i=1}^{\infty} f_i \in \mathfrak{K}$ to the product vector $S_n F = \bigotimes_{i=1}^{\infty} f_i \in \mathfrak{K}_n$. From the second sum condition above one finds S_n is well defined and one checks that S_n is unitary. We define $Q_n(A) = S_n A S_n^*$ for $A \in \mathfrak{B}(\mathfrak{K})$. Let $\mathfrak{K}'_n = \bigotimes_{i=1}^{n} L^2(0, \infty)$. We see that $\mathfrak{B}(\mathfrak{K}) = \mathfrak{B}(\mathfrak{K}'_n) \otimes \mathfrak{B}(\mathfrak{K}_n)$ and $Q_m(I_n \otimes Q_n(A)) = I_{n+m} \otimes Q_{n+m}(A)$ for $A \in \mathfrak{B}(\mathfrak{H})$ where I_k is the unit in $\mathfrak{B}(\mathfrak{K}'_k)$ for $n, m, k = 1, 2, \ldots$.

We have the formulae,

$$\begin{aligned}
\pi(A \otimes A_0) &= A_0 \otimes Q_1(A), \\
\Lambda(A) &= A \otimes e^{-x}, \\
\pi\big(\Lambda(A)\big) &= e^{-x} \otimes Q_1(A), \\
(\pi\Lambda)^n(A) &= e^{-x} \otimes e^{-x} \otimes \cdots \otimes e^{-x} \otimes Q_n(A)
\end{aligned}$$

for $A \in \mathfrak{B}(\mathfrak{K})$ and $n = 1, 2, \ldots$ where there are n factors of e^{-x} in the last equation. For $A = A_1 \otimes A_2 \otimes \cdots$ we write these formulae:

$$\begin{aligned}
\pi\big((A_1 \otimes A_2 \otimes \cdots) \otimes A_0\big) &= (A_0 \otimes A_1 \otimes A_2 \otimes \cdots), \\
\Lambda(A_1 \otimes A_2 \otimes \cdots) &= (A_1 \otimes A_2 \otimes \cdots) \otimes e^{-x}, \\
\pi\big(\Lambda(A_1 \otimes A_2 \otimes \cdots)\big) &= e^{-x} \otimes A_1 \otimes A_2 \otimes \cdots, \\
(\pi\Lambda)^n(A_1 \otimes A_1 \otimes \cdots) &= e^{-x} \otimes e^{-x} \otimes \cdots \otimes e^{-x} \otimes A_1 \otimes A_2 \otimes \cdots.
\end{aligned}$$

We first note that $(\pi\Lambda)^n(I)$ converges to Δ as $n \to \infty$. We have

$$(\pi\Lambda)^n(I) = e^{-x} \otimes e^{-x} \otimes \cdots \otimes e^{-x} \otimes I \otimes I \otimes \cdots$$

where there are n factors of e^{-x} and we see that $(\pi\Lambda)^n(I)$ forms a decreasing sequence of positive operators which must converge strongly to a limit which is

$$\Delta = e^{-x} \otimes e^{-x} \otimes \cdots.$$

As we have mentioned the first sum condition on the λ_n insures that Δ is not zero.

Next we note that if $\pi(\Lambda(A)) = A$ then A is a multiple of Δ (i.e. $A = c\Delta$ with $c \in \mathbb{C}$). In fact, we show first that it is enough to prove that if A is positive and $\pi(\Lambda(A)) = A$ then $A = \lambda\Delta$ with $\lambda \geqslant 0$. Note that if $\pi(\Lambda(A)) = A$ then if $A = A_1 + iA_2$ where A_1 and A_2 are hermitian then $\pi(\Lambda(A_i)) = A_i$ for $i = 1, 2$. So it is enough to show that if $A = A^*$ and $\pi(\Lambda(A)) = A$ then $A = \lambda\Delta$ with λ real. Next note that if $A \in \mathfrak{B}(\mathfrak{K})$ is hermitian and $\pi(\Lambda(A)) = A$ and $\|A\| = 1$ then $(\pi\Lambda)^n(I + A) \to \Delta + A$ as $n \to \infty$ and since $\Delta + A$ is the strong limit of positive operators we have $\Delta + A$ is positive. If $\Delta + A = \lambda\Delta$ it follows that A is a multiple of Δ. Hence, it is sufficient to show that if A is positive and $\pi(\Lambda(A)) = A$ then $A = \lambda\Delta$ with $\lambda \geqslant 0$. Suppose then that $A \in \mathfrak{B}(\mathfrak{K})$ is positive, $\|A\| = 1$ and $\pi(\Lambda(A)) = A$. Since $(\pi\Lambda)^n(I - A) \to \Delta - A \geqslant 0$ we have $0 \leqslant A \leqslant \Delta$. Recalling the reference vector F_0 we have

$$(F_0, \Delta F_0) = \big(k_1, e^{-x}k_1\big)\big(k_2, e^{-x}k_2\big)\cdots = \frac{\lambda_1^2}{1+\lambda_1^2} \cdot \frac{\lambda_2^2}{1+\lambda_2^2} \cdots.$$

Since $\Delta \geqslant A \geqslant 0$ we have $(F_0, AF_0) = c(F_0, \Delta F_0)$ with $c \in [0, 1]$. Now since $\pi(\Lambda(A)) = A$ it follows that

$$A = e^{-x} \otimes Q_1(A) = e^{-x} \otimes e^{-x} \otimes Q_2(A) = \cdots$$

and we have

$$\left(\bigotimes_{i=n+1}^{\infty} k_i, Q_n(A) \bigotimes_{i=n+1}^{\infty} k_i\right) = c\frac{\lambda_{n+1}^2}{1+\lambda_{n+1}^2} \cdot \frac{\lambda_{n+2}^2}{1+\lambda_{n+2}^2} \cdots$$

for $n = 1, 2, \ldots$. Now let

$$F = \bigotimes_{i=1}^{\infty} f_i \quad \text{and} \quad G = \bigotimes_{i=1}^{\infty} g_i$$

be product vectors so that $f_i = g_i = k_i$ for $i \geqslant m$. Then we see that

$$\begin{aligned}(F, AG) &= \big(f_1, e^{-x}g_1\big)\big(f_2, e^{-x}g_2\big)\cdots\big(f_m, e^{-x}g_m\big)c\frac{\lambda_{m+1}^2}{1+\lambda_{m+1}^2}\frac{\lambda_{m+2}^2}{1+\lambda_{m+2}^2}\cdots \\ &= c(F, \Delta G).\end{aligned}$$

Since such vectors F and G are dense in $\mathfrak{K}$ we have $A = c\Delta$. Then we have proved the following lemma.

Lemma 2.1. *Suppose π is the $*$-representation described above. Let Δ be as described above. Then $\Delta = \lim_{n\to\infty}(\pi\Lambda)^n(I)$. Furthermore, if $A \in \mathfrak{B}(\mathfrak{K})$ and $\pi(\Lambda(A)) = A$ then A is a multiple of Δ (i.e., $A = c\Delta$ with $c \in \mathbb{C}$).*

We will need a stronger characterization of this property which is provided by the following lemma.

Lemma 2.2. *Suppose $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $\rho(\Delta) = 0$. Then $\|(\hat{\Lambda}\hat{\pi})^n(\rho)\| \to 0$ as $n \to \infty$.*

Proof. Suppose $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $\rho(\Delta) = 0$. Suppose $\epsilon > 0$. Since ρ can be approximated arbitrarily well in norm by a finite sum of functionals ρ_i of the form $\rho_i(A) = (F_i, AG_i)$ with $F_i, G_i \in \mathfrak{K}$ for $i = 1, \dots, n$ and the vectors F_i and G_i can be approximated by vectors F_i' and G_i' which are finite sums of product vectors of the form $f_1 \otimes f_2 \otimes \cdots$ with $f_i = k_i$ for $i > m$ with m some large integer it follows that there is a functional η so that $\|\rho - \eta\| < \frac{1}{2}\epsilon(F_0, \Delta F_0)$ and

$$\eta(A) = \sum_{i=1}^{n} (F_i, AG_i)$$

and each of the vectors F_i and G_i is of the form

$$F \otimes \left(\bigotimes_{i=m+1}^{\infty} k_i \right)$$

(i.e. they consist of sums of product vectors with factors $f_i = k_i$ for $i > m$). Since we have

$$\left|\eta(\Delta)\right| = \left|\rho(\Delta) - \eta(\Delta)\right| \leqslant \|\rho - \eta\| < \frac{1}{2}\epsilon(F_0, \Delta F_0).$$

Now let $\mu(A) = \eta(A) - (F_0, AF_0)\eta(\Delta)(F_0, \Delta F_0)^{-1}$. Note

$$\|\rho - \mu\| \leqslant \|\rho - \eta\| + \|\eta - \mu\| \leqslant \frac{1}{2}\epsilon(F_0, \Delta F_0) + \frac{1}{2}\epsilon < \epsilon.$$

Since $(\hat{\Lambda}\hat{\pi})^k(\mu)(A) = \mu((\pi\Lambda)^k(A))$ and $(\pi\Lambda)^k(A)$ is of the form

$$(\pi\Lambda)^k(A) = e^{-x} \otimes e^{-x} \otimes \cdots \otimes e^{-x} \otimes Q_k(A)$$

where there are k factors of e^{-x}, and it follows from the form of μ that $\mu((\pi\Lambda)^k(A)) = 0$ for $k \geqslant m$. Hence, $\|(\hat{\Lambda}\hat{\pi})^k(\rho)\| < \epsilon$ for $k \geqslant m$ and we have $\|(\hat{\Lambda}\hat{\pi})^k(\rho)\| \to 0$ as $k \to \infty$. □

Let α^1 be the minimal *CP*-flow derived from π. If $\rho \to \omega^1(\rho)$ is the boundary weight map for α^1 then

$$\omega^1(\rho) = \hat{\pi}(\rho) + \hat{\pi}\hat{\Lambda}\hat{\pi}(\rho) + \hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi}(\rho) + \cdots = \hat{\pi}R(\hat{\Lambda}\hat{\pi})$$

where the shorthand $R(\psi) = I + \psi + \psi^2 + \cdots$ was introduced in the last section. We analyze *CP*-flows derived from π. We begin with the following observation.

Theorem 2.3. *Suppose α is a CP-flow derived from π and ω is the boundary weight map for α. Then ω is of the form*

$$\omega(\rho) = \omega^1(\rho) + \rho(\Delta)\xi$$

for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ where ω^1 is the boundary weight map for α^1 the minimal CP-flow derived from π and $\xi \in \mathfrak{A}(\mathfrak{H})_*$ is a positive boundary weight on $\mathfrak{A}(\mathfrak{H}) = (I - \Lambda)^{\frac{1}{2}}\mathfrak{B}(\mathfrak{H})(I - \Lambda)^{\frac{1}{2}}$ with $\xi(I - \Lambda) \leqslant 1$ and α is unital (i.e. $\alpha_t(I) = I$ for $t \geqslant 0$) if and only if $\xi(I - \Lambda) = 1$.

Proof. Assume the hypothesis and notation of the theorem. Since α is derived from π we have (by Theorem 1.16) that $\omega(\rho - \hat{\Lambda}\hat{\pi}(\rho)) = \hat{\pi}(\rho)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Suppose $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Let $\rho_n = \rho + \hat{\Lambda}\hat{\pi}(\rho) + \cdots + (\hat{\Lambda}\hat{\pi})^n(\rho)$. Then we have

$$\omega\big(\rho - (\hat{\Lambda}\hat{\pi})^{n+1}(\rho)\big) = \hat{\pi}(\rho) + \hat{\pi}\hat{\Lambda}\hat{\pi}(\rho) + \cdots + \hat{\pi}(\hat{\Lambda}\hat{\pi})^n(\rho).$$

Now suppose $\rho(\Delta) = 0$. Then by Lemma 2.2 we have $\|(\hat{\Lambda}\hat{\pi})^n\rho\| \to 0$ as $n \to \infty$ so we have taking the limit as $n \to \infty$ that $\omega(\rho) = \omega^1(\rho)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ with $\rho(\Delta) = 0$. Now suppose $\eta \in \mathfrak{B}(\mathfrak{K})_*$ is positive and $\eta(\Delta) = 1$. Then for arbitrary $\rho \in \mathfrak{B}(\mathfrak{K})_*$ we have

$$\omega(\rho) = \omega\big(\rho - \rho(\Delta)\eta\big) + \rho(\Delta)\omega(\eta) = \omega^1(\rho) + \rho(\Delta)\big(\omega(\eta) - \omega^1(\eta)\big).$$

Setting $\xi = \omega(\eta) - \omega^1(\eta)$ we have ω given in terms of ω^1 and ξ as stated in the theorem. Next we show ξ is a positive. Suppose $\rho \in \mathfrak{B}(\mathfrak{K})_*$ is positive and $\rho(\Delta) = 1$. Then we have

$$\omega\big((\hat{\Lambda}\hat{\pi})^n(\rho)\big) = \omega^1\big((\hat{\Lambda}\hat{\pi})^n\big)(\rho) + \xi$$

for each $n = 1, 2, \ldots$ and since $\omega^1((\hat{\Lambda}\hat{\pi})^n)(\rho) \to 0$ as a weight and since $(\hat{\Lambda}\hat{\pi})^n(\rho)$ is positive we have ξ is the limit of positive weights so ξ is positive. For $\rho \in \mathfrak{B}(\mathfrak{K})_*$ we have

$$\omega(\rho)(I - \Lambda) = \omega^1(\rho)(I - \Lambda) + \rho(\Delta)\xi(I - \Lambda)$$

and calculating $\omega^1(\rho)(I - \Lambda)$ we find

$$\begin{aligned}\omega^1(\rho)(I - \Lambda) &= \rho\big(\big(I - \pi(\Lambda)\big) + \big(\pi(\Lambda) - (\pi\Lambda)^2(\Lambda)\big) + \cdots\big)\\ &= \rho(I) - \rho(\Delta).\end{aligned}$$

Hence, we have $\omega(\rho)(I - \Lambda) = \rho(I) - \rho(\Delta)(1 - \xi(I - \Lambda))$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Since we have then inequality $\omega(\rho)(I - \Lambda) \leqslant \rho(I)$ for positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$ we find $\xi(I - \Lambda) \leqslant 1$ and $\omega(\rho)(I - \Lambda) = \rho(I)$ if and only if $\xi(I - \Lambda) = 1$. □

It follows from this result that if α is a unital *CP*-flow derived from π and α^d is its minimal dilation E_0-semigroup, then α^d is of type II and of index 1. This is because $\Delta \neq 0$, so the minimal *CP*-flow derived from π is not unital, and therefore it must be a proper subordinate. Now recall that since α is derived from π and π is σ-weakly continuous, we have that π is the normal spine of α (see [18, Definition 4.36 and Lemma 4.37]). Furthermore, by [18, Theorem 4.52], α^d is completely spatial if and only if α is the minimal *CP*-flow derived from its normal spine. It follows that α^d cannot be completely spatial. Finally, we observe that by [18, Theorem 4.49], the index of α^d is precisely the rank of the normal spine of α, and the rank of π is one.

We remark that it was shown in [18, Theorem 4.62] if ν is a positive element of $\mathfrak{B}(\mathfrak{H})_*$ with $\nu(I) \leqslant 1$ and ξ is of the form

$$\xi = \big(1 - \nu\big(\Lambda(\Delta)\big)\big)^{-1} R(\hat{\pi}\hat{\Lambda})\nu$$

then ω of the form given in Theorem 2.3 is the boundary weight map of a *CP*-flow α is unital if and only if $\nu(I) = 1$. In a subsequent paper we find necessary and sufficient conditions on ξ that ω as given in the statement of the above theorem is the boundary weight map of a *CP*-flow over $\mathfrak{K}$. If ξ satisfies these conditions we say ξ is *q-positive*. In a subsequent paper we show that the above formula for ξ can be generalized to positive $\Lambda(\Delta)$-weights with $\nu(I - \Lambda(\Delta)) \leqslant 1$. We also show that there are more general ξ. For this paper we simply note that there are plenty of q-positive ξ which yield unital *CP*-semigroups α.

3. Local flow cocycles

In this section we study the local flow cocycles associated with the *CP*-flows constructed in the previous section. Suppose α is a unital *CP*-flow and α^d is the minimal dilation of α to an E_0-semigroup. As we saw in Theorem 1.22 then the α^d is also a *CP*-flow over $\mathfrak{K}_1$ and the translation $U(t)$ on the Hilbert space $\mathfrak{H}$ on which α lives dilate to the translations $U^1(t)$ on the Hilbert space $\mathfrak{H}_1$ on which α^d lives. Recall $t \to C(t)$ is a local cocycle for α^d C is a cocycle and $C(t)$ commutes with $\alpha_t^d(\mathfrak{B}(\mathfrak{H}_1))$ for all $t \geqslant 0$. The cocycle C is a flow cocycle if $C(t)U^1(t) = U^1(t)$ for all $t \geqslant 0$. Just as each local unitary cocycle corresponds to a hyper-maximal corner from α to α, each local unitary flow cocycle for α^d the dilation of a *CP*-flow over $\mathfrak{K}$ corresponds to a hyper-maximal flow corner γ from α to α. Here a flow corner from α to α is a corner so that the matrix Θ in Definition 1.5 is a *CP*-flow over $\mathfrak{K} \oplus \mathfrak{K}$. Theorems 1.7 and 1.8 of Section 1 of this paper are valid if one replaces the word "*CP*-semigroup" with "*CP*-flow" and "cocycle" with "flow cocycle" (see [18, Theorem 4.54]). One ambiguity that occurs in speaking of flow corners is the following. When one says γ is a maximal flow corner do we mean γ is maximal as a flow corner or simply maximal as a corner. In [18, Lemma 4.55] it was shown that if α and β are *CP*-semigroups and γ is a flow corner from α to β then α and β are *CP*-flows. It then follows that the two notions of maximality are the same.

We mention one technical problem. Suppose α^d is the dilation of the *CP*-flow α and $t \to C(t)$ is a contractive local cocycle and $C(t)U^1(t) = \exp(-zt)U^1(t)$ for $t > 0$ where z is a complex number with positive real part. Let $C'(t) = \exp(zt)C(t)$. Then C' is a local flow cocycle, however, it is not clear that it is contractive so there may not be a flow corner associated with it. Fortunately, Theorem 4.61 in [18] shows that C' is contractive so there is a local flow corner associated with it. This means that every contractive local cocycle C is of the form $C(t) = \exp(-zt)C'(t)$ for $t \geqslant 0$ where C' is a flow cocycle and z is a complex number with non-negative real part.

Here we introduce some notation which we will use throughout this section. As in the last section π is the $*$-representation of $\mathfrak{B}(\mathfrak{H})$ on $\mathfrak{B}(\mathfrak{K})$ constructed in the last section. We denote by ξ a q-positive (usually unital) boundary weight and by $\alpha = \alpha^{\xi}$ the *CP*-flow derived from π associated with ξ as described in the last section. The boundary weight map for α is

$$\omega(\rho) = \omega^1(\rho) + \rho(\Delta)\xi$$

for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ where $\omega^1 = R(\hat{\pi}\hat{\Lambda})\hat{\pi}$. Recall that q-positive means that ω given above is the boundary weight of a *CP*-flow over $\mathfrak{K}$. As we mentioned in the last section the complete characterization of such ξ will be given in a subsequent paper but for now we simply remark there are many q-positive ξ as given in the previous section.

If z is a complex number with $|z| \leqslant 1$ we denote by

$$\omega^z = zR(z\hat{\pi}\hat{\Lambda})\hat{\pi} = z\hat{\pi}R(z\hat{\Lambda}\hat{\pi}) = z\hat{\pi} + z^2\hat{\pi}\hat{\Lambda}\hat{\pi} + z^3\hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi} + \cdots \tag{3.1}$$

where the sum converges as a boundary weight since the sum converges for $z = 1$ where all the terms are positive.

Next we introduce a family of one-parameter semigroups of isometries which intertwine α. For z any complex number we denote by U_z the one-parameter semigroup of isometries of $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$

$$U_z(t) = \exp(-tD_z) \quad \text{where } D_z = -d^* + \frac{1}{2}|z|^2 I$$

for $t \geqslant 0$ and d is the operation of differentiation defined in the last section and the domain $\mathfrak{D}(D_z) = \{f \in \mathfrak{D}(d^*)\colon\ f(0) = zS_0 f\}$ where S_0 is the unitary operator mapping $\mathfrak{H}$ into $\mathfrak{K}$ defined by (2.3) in the last section as

$$S_0\big((f_1 \otimes f_2 \otimes \cdots) \otimes f_0\big) = f_0 \otimes f_1 \otimes f_2 \otimes \cdots$$

for $f_i \in L^2(0,\infty)$ and the f_i satisfy condition (2.1) and S_0 defines π in that $\pi(A) = S_0 A S_0^*$ for $A \in \mathfrak{B}(\mathfrak{H})$. Note $U_0 = U$ the standard right translation and $D_0 = d$.

Suppose w and z are complex numbers. We show the U_z are a one-parameter family of isometries and the covariance $c(w,z)$ of U_w with U_z is given by

$$U_w(t)^* U_z(t) = \exp\big(c(w,z)t\big) I = \exp\left(\frac{1}{2}\big(2\overline{w}z - |w|^2 - |z|^2\big)t\right) I \tag{3.2}$$

for $t \geqslant 0$. For $f \in \mathfrak{D}(D_w)$ and $g \in \mathfrak{D}(D_z)$ we have

$$\begin{aligned}\frac{d}{dt}\big(U_w(t)f, U_z(t)g\big) &= \big(d^* U_w(t)f, U_z(t)g\big) + \big(U_w(t)f, d^* U_z(t)g\big) \\ &\quad - \frac{1}{2}\big(|w|^2 + |z|^2\big)\big(U_w(t)f, U_z(t)g\big)\end{aligned}$$

for $t \geqslant 0$. Now we have

$$\begin{aligned}\big(d^* U_w(t)f, U_z(t)g\big) + \big(U_w(t)f, d^* U_z(t)g\big) &= \big(\big(U_w(t)f\big)(0), \big(U_z(t)g\big)(0)\big) \\ &= \big(wS_0 U_w(t)f, zS_0 U_z(t)g\big) \\ &= \overline{w}z\big(U_w(t)f, U_z(t)g\big)\end{aligned}$$

for $t \geqslant 0$ where we have used the relation between $h(0)$ and $S_0 h$ for h in $\mathfrak{D}(D_z)$ or $\mathfrak{D}(D_w)$ and the fact that S_0 is an isometry. Hence we have

$$\frac{d}{dt}\big(U_w(t)f, U_z(t)g\big) = c(w,z)\big(U_w(t)f, U_z(t)g\big)$$

for $t \geqslant 0$ and since the domains $\mathfrak{D}(D_w)$ and $\mathfrak{D}(D_z)$ are dense in $\mathfrak{H}$ (see the argument in [18, Lemma 4.44]) Eq. (3.2) follows.

Next we note that S is a one-parameter semigroup, so Ω given by

$$\Omega_t(A) = S(t)^* A S(t)$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$ satisfies $\alpha \geqslant \Omega$ (meaning the mapping $A \to \alpha_t(A) - \Omega_t(A)$ is completely positive for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$) if and only if there are complex numbers y, z with $\mathrm{Re}(y) \geqslant 0$ so that

$$S(t) = e^{-yt} U_z(t)$$

for $t \geqslant 0$. This follows from Theorem 1.21 of Section 1 once one notes that the condition of the theorem is satisfied if and only if the mapping $A \to \pi(A) - (2\,\mathrm{Re}(c))^{-1} VAV^*$ is completely positive and since $\pi(A) = S_0 A S_0^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ this is the case if and only if V is an appropriate multiple of S_0.

Suppose $z \in \mathbb{C}$. We show U_z intertwines α. From the result just established we have the mapping

$$\beta_t(A) = \alpha_t(A) - U_z(t) A U_z(t)^*$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$ is completely positive. Suppose $t > 0$. Then since α is unital we have $\beta_t(I) = I - U_z(t)U_z(t)^*$. Since $U_z(t)$ is an isometry and β_t is positive we have

$$0 \leqslant \beta_t(A) \leqslant I - U_z(t)U_z(t)^*$$

for $A \in \mathfrak{B}(\mathfrak{H})$ with $0 \leqslant A \leqslant I$ and consequently

$$\beta_t(A) = \bigl(I - U_z(t)U_z(t)^*\bigr)\beta_t(A)\bigl(I - U_z(t)U_z(t)^*\bigr)$$

and by linearity this extends to all $A \in \mathfrak{B}(\mathfrak{H})$. Then we have

$$\alpha_t(A) = U_z(t) A U_z(t)^* + \bigl(I - U_z(t)U_z(t)^*\bigr)\alpha_t(A)\bigl(I - U_z(t)U_z(t)^*\bigr)$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. And multiplying the above equation on the right by $U_z(t)$ we obtain $U_z(t)A = \alpha_t(A)U_z(t)$ so U_z intertwines α. Summarizing our results to this point we have the mapping $A \to \alpha_t(A) - V(t)AV(t)^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$ is completely positive where V is a one-parameter semigroup of contractions then the $V(t)$ are in fact multiples of a semigroup U_z of isometries which intertwine α.

Now suppose α^d is the dilation of α to an E_0-semigroup on $\mathfrak{H}_1$ as described in Theorem 1.22. Then from Theorem 1.22 we see the mapping

$$W^* U_z^1(t) W = U_z(t) \tag{3.3}$$

for $t \geqslant 0$ and $z \in \mathbb{C}$ give us a bijection from the U_z to intertwining semigroups of isometries U_z^1 which intertwine α^d where the covariance for the U_z^1 is the same as the covariance for the U_z given in Eq. (3.2). Also every intertwining semigroup for α^d is of the form $V^1(t) = e^{-yt} U_z^1(t)$ with $y, z \in \mathbb{C}$.

Next we describe the action of local cocycles on the units U_z^1. One checks that if $t \to C(t)$ is a local cocycle for α^d then $C(t)U_z^1(t)$ is a intertwining semigroup for α^d. Now the action of the local unitary (respectively contractive) cocycles on the units U_z^1 restricts to an action of local unitary (respectively contractive) cocycles on γ, the type I part of α^d (see [5, Remark 10.4.2, p. 347]), which is the "maximal" type I E-semigroup subordinate to α^d (necessarily type I_1 in this case). Every E-semigroup is cocycle conjugate to an E_0-semigroup, hence the action at the level of γ must arise via cocycle conjugacy from an action of a subgroup of the gauge group (respectively semigroup of local contractive cocycles) of an E_0-semigroup of type I_1 acting on its set of units.

Local unitary cocycles generate automorphisms of the product systems associated with an E_0-semigroup and these have been computed in the type I case by Arveson in [3] (see also [5, Section 3.8]). Going one step further, Bhat [7] computed the positive contractive local cocycles of an E_0-semigroup of type I. The general contractive local flow cocycles for a CP-flow of type I are characterized in [1, Theorem 2.11]. Now we characterize the action of the contractive local cocycles on units. If C is a contractive local cocycle for α^d then there are complex numbers $a, b, c, y \in \mathbb{C}$ with $|a| \leqslant 1$ and $\mathrm{Re}(y) \geqslant 0$ so that the action of C on the units U_z^1 is given by

$$C(t)U_z^1(t) = \exp\biggl(t\biggl(-y - \frac{1}{2}|v+z|^2\bigl(1-|a|^2\bigr) + i\,\mathrm{Im}(\bar{c}z)\biggr)\biggr)U_{az+b}^1(t) \tag{3.4}$$

for $t \geqslant 0$ with

$$v = -\bigl(1-|a|^2\bigr)^{-1}(\bar{a}b + c)$$

and when $|a| = 1$ then numbers $a, b, c \in \mathbb{C}$ above satisfy the additional constraint $ac + b = 0$, so

$$C(t)U_z^1(t) = e^{-t(y + i\,\mathrm{Im}(a\bar{b}z))}U_{az+b}^1(t).$$

The action of C^* is obtained by making the replacements

$$a \to \bar{a}, \qquad b \leftrightarrow c \quad \text{and} \quad y \to \bar{y}.$$

In the case when $|a| = 1$ we parameterize C with complex numbers (y, a, b) not using c so the action of C^* in this case is given by

$$C(t)^*U_z^1(t) = e^{-t(\bar{y} - i\,\mathrm{Im}(\bar{b}z))}U_{\bar{a}(z-b)}^1(t).$$

If the cocycle is isometric then

$$|a| = 1, \qquad ac + b = 0, \quad \text{and} \quad \mathrm{Re}(y) = 0.$$

If the cocycle is a flow cocycle then $b = c = y = 0$ so the action of a flow cocycle on the units U_z^1 is given by

$$C(t)U_z^1(t) = e^{-\frac{1}{2}|z|^2(1-|a|^2)}U_{az}^1(t)$$

for $t \geqslant 0$ and $z \in \mathbb{C}$.

If C and C' are contractive local cocycles whose action on the units is characterized by the n-tuples (a, b, c, y) and (a', b', c', y') as describe above then the corresponding numbers for the product cocycle $t \to C(t)C'(t)$ are

$$\biggl(aa', ab' + b, \bar{a}'c + c', y + y' + i\,\mathrm{Im}(\bar{c}b') - \frac{1}{2}r\biggr)$$

where $r = 0$ if either $|a| = 1$ or $|a'| = 1$ and otherwise

$$r = \left(1 - |a'|^2\right)^{-1}|\bar{a}'b' + c'|^2 + \left(1 - |a|^2\right)^{-1}\left|b'\left(1 - |a|^2\right) - \bar{a}b - c\right|^2 \\ - \left(1 - |aa'|^2\right)^{-1}\left|\overline{aa}'(ab' + b) + \bar{a}'c + c'\right|^2$$

is a non-negative real function of (a, b, c, a', b', c'). Given the complexity of the function r above we wonder if there is a better parameterization of the action of the local cocycles on the units. If either of the local cocycles above is unitary the number r above is zero so the parameterization of contractive local cocycles is much more difficult than the parameterization of the unitary local cocycles.

We caution the reader that action of a local cocycle on the units U_z^1 does not completely determine the cocycle since in our case α^d is not completely spatial. In the next theorem we characterize the contractive local flow cocycles which as we have explained is equivalent to determining the flow corners from α to α. First we prove the following lemma.

Lemma 3.1. *Suppose ξ is a unital q-positive boundary weight on $\mathfrak{A}(\mathfrak{H})$ and α is the CP-flow over $\mathfrak{K}$ derived from π associated with ξ. Suppose γ is a flow corner from α to α which means that*

$$\Theta_t\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) = \begin{bmatrix} \alpha_t(A_{11}) & \gamma_t(A_{12}) \\ \gamma_t^*(A_{21}) & \alpha_t(A_{22}) \end{bmatrix}$$

for $t > 0$ and $A_{ij} \in \mathfrak{B}(\mathfrak{H})$ for $i, j = 1, 2$ is a CP-flow over $\mathfrak{K} \oplus \mathfrak{K}$. Then there is a complex number z with $|z| \leqslant 1$ so that Θ is derived from Π_z given by

$$\Pi_z\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) = \begin{bmatrix} \pi(A_{11}) & z\pi(A_{12}) \\ \bar{z}\pi(A_{21}) & \pi(A_{22}) \end{bmatrix}$$

for $A_{ij} \in \mathfrak{B}(\mathfrak{H})$ for $i, j = 1, 2$. Furthermore, for each $w \in \mathbb{C}$ we have

$$U_{zw}(t)A = e^{\frac{1}{2}t|w|^2(1-|z|^2)}\gamma_t(A)U_w(t)$$

and

$$U_w(t)A = e^{\frac{1}{2}t|w|^2(1-|z|^2)}\gamma_t^*(A)U_w(t)$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$.

Proof. Assume the hypothesis and notation of the theorem. Let α^d and Θ^d be the dilation of α and Θ to E_0-semigroups on $\mathfrak{H}_1$ and $\mathfrak{H}_1 \oplus \mathfrak{H}_1$ and the relation between the *CP*-flow and the dilated E_0-semigroup is as described in Section 1 so

$$\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$$

for $t \geqslant 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. We will show that there is $z \in \mathbb{C}$ with $|z| \leqslant 1$ so that Θ is derived from Π_z as defined above.

First note that $U(t) \oplus U(t)$ intertwines Θ. Using this we find the boundary representation of Θ is of the form

$$\Pi(A) = \Pi\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) = \begin{bmatrix} \pi(A_{11}) & \phi(A_{12}) \\ \phi^*(A_{21}) & \pi(A_{22}) \end{bmatrix}$$

for A in the domain of the generator of Θ. Since π is pure meaning the only subordinates of π are of the form $\lambda\pi$ with $0 \leqslant \lambda \leqslant 1$ and Π is completely positive it follows that $\phi = z\pi$ for some $z \in \mathbb{C}$ with $|z| \leqslant 1$. Note in general the boundary representation is the direct sum of a normal and a non-normal representation of the domain of the generator but in our case we are assured that there is no non-normal part because π is unital and therefore Π is normal. Thus the boundary representation of Θ is Π so Θ is derived from Π.

As we have seen since γ is a flow corner from α to α there is a unique contractive local flow cocycle C for α^d so that

$$\gamma_t(A) = W^*C(t)\alpha_t^d(WAW^*)W$$

for all $t \geqslant 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. Then as we have seen there is a number $y \in \mathbb{C}$ with $|y| \leqslant 1$ so that

$$C(t)U_w^1(t) = \exp\left(-\frac{1}{2}t|w|^2\left(1 - |y|^2\right)\right)U_{yw}^1(t)$$

for $t \geqslant 0$ and $w \in \mathbb{C}$. Then we have

$$\begin{aligned} \gamma_t(A)U_w(t) &= W^*C(t)\alpha_t^d(WAW^*)WU_w(t) \\ &= W^*C(t)\alpha_t^d(WAW^*)U_w^1(t)W \\ &= W^*C(t)U_w^1 WAW^*W \\ &= \exp\left(-\frac{1}{2}t|w|^2\left(1 - |y|^2\right)\right)W^*U_{yw}^1(t)WA \\ &= \exp\left(-\frac{1}{2}t|w|^2\left(1 - |y|^2\right)\right)U_{yw}(t)A \end{aligned}$$

for $t \geqslant 0$, $w \in \mathbb{C}$ and $A \in \mathfrak{B}(\mathfrak{H})$. Also since C is a local cocycle we have

$$\gamma_t^*(A) = W^*\alpha_t^d(WAW^*)C(t)^*W = W^*C(t)^*\alpha_t^d(WAW^*)W$$

so

$$\gamma_t^*(A)U_w(t) = \exp\left(-\frac{1}{2}t|w|^2\left(1 - |y|^2\right)\right)U_w(t)A$$

for $t \geqslant 0$, $w \in \mathbb{C}$ and $A \in \mathfrak{B}(\mathfrak{H})$. Hence, we have proved the lemma provided we can show $y = z$.

We show $y = z$. Let $d_2 = d \oplus d$ so d_2 is the ordinary differential operator d/dx on $\mathfrak{H} \oplus \mathfrak{H}$. We use capital letters F and G to denote elements of $\mathfrak{H} \oplus \mathfrak{H}$ and lower case letters f, g to denote elements of $\mathfrak{H}$. Recall that the boundary representation discussed in Section 1 for Θ is given by

$$\Pi_z(A)F(0) = (AF)(0)$$

for $F \in \mathfrak{D}(d_2^*)$ and $A \in \mathfrak{D}(\delta_2)$ where δ_2 is the generator of Θ. Suppose $w \in \mathbb{C}$ and $w \neq 0$. Now suppose $G = \{0, g\}$ and $g \in \mathfrak{D}(D_w)$ so $g \in \mathfrak{D}(d^*)$ and $g(0) = wS_0 g$. Suppose $A \in \mathfrak{D}(\delta_2)$ and $A_{ij} \in \mathfrak{B}(\mathfrak{H})$ are the matrix coefficients of A for $i = 1, 2$. Now from what we have shown we have

$$\gamma_t(A_{12})U_w(t)g = \exp\left(-\frac{1}{2}t|w|^2\left(1 - |y|^2\right)\right)U_{yw}(t)A_{12}g$$

for $t \geqslant 0$. Since $-D_w$ is the generator of U_w and $g \in \mathfrak{D}(D_w)$ we have $U_w(t)g$ is differentiable in t and since $A \in \mathfrak{D}(\delta_2)$ we have $\gamma_t(A_{12})$ is differentiable in t so the expression on the left-hand side of the above equation is differentiable in t. Hence, $U_{yw}(t)A_{12}g$ is differentiable in t so $Ag \in \mathfrak{D}(D_{yw})$ and we have $Ag \in \mathfrak{D}(d^*)$ and

$$\begin{aligned}(A_{12}g)(0) &= ywS_0A_{12}g = ywS_0A_{12}\left(w^{-1}S_0^*g(0)\right)\\ &= yS_0A_{12}S_0^*g(0) = y\pi(A_{12})g(0).\end{aligned}$$

Since $\Pi_z(A)F(0) = (AF)(0)$ we have

$$(A_{12}g)(0) = z\pi(A_{12})g(0)$$

and comparing the two equations we see $y = z$. □

Theorem 3.2. *Suppose ξ is a unital q-positive boundary weight on $\mathfrak{A}(\mathfrak{H})$ and α is the CP-flow over $\mathfrak{K}$ derived from π associated with ξ. Suppose γ is a flow corner from α to α which means that*

$$\Theta_t\left(\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\right) = \begin{bmatrix} \alpha_t(A_{11}) & \gamma_t(A_{12})\\ \gamma_t^*(A_{21}) & \alpha_t(A_{22})\end{bmatrix}$$

for $t > 0$ and $A_{ij} \in \mathfrak{B}(\mathfrak{H})$ for $i, j = 1, 2$ is a CP-flow over $\mathfrak{K} \oplus \mathfrak{K}$ and if Ω is the boundary weight map for Θ then Ω is of the form

$$\Omega\left(\begin{bmatrix} \rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{bmatrix}\right) = \begin{bmatrix} \omega(\rho_{11}) & \sigma(\rho_{12})\\ \sigma^*(\rho_{21}) & \omega(\rho_{22})\end{bmatrix}$$

for $\rho_{ij} \in \mathfrak{B}(\mathfrak{K})_*$ for $i, j = 1, 2$. Then there is a unique complex number z with $|z| \leqslant 1$ so if $z \neq 1$ then

$$\sigma(\rho) = \omega^z(\rho)$$

for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and if $z = 1$ then there is a boundary weight ξ' so that

$$\sigma(\rho) = \omega^1(\rho) + \rho(\Lambda)\xi'$$

for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$.

Proof. Assume the hypothesis and notation of the first paragraph of the theorem. Then from the previous lemma there is a unique $z \in \mathbb{C}$ with $|z| \leqslant 1$ so that Θ as given in the previous lemma is derived from Π_z. Since Θ is derived from Π_z we have repeating the argument of Theorem 2.3 that

$$\sigma\big(\rho - z^{n+1}(\hat{\Lambda}\hat{\pi})^{n+1}(\rho)\big) = z\hat{\pi}(\rho) + z^2\hat{\pi}\hat{\Lambda}\hat{\pi}(\rho) + \cdots + z^n\hat{\pi}(\hat{\Lambda}\hat{\pi})^n(\rho).$$

Suppose $\rho(\Delta) = 0$. Then we have from Lemma 2.2 that $\|(\hat{\Lambda}\hat{\pi})^n(\rho)\| \to 0$ as $n \to \infty$ so we have

$$\sigma(\rho) = z\hat{\pi}R(z\hat{\Lambda}\hat{\pi})(\rho).$$

Choose a positive ρ_1 so that $\rho_1(\Delta) = 1$ and we find

$$\begin{aligned}\sigma(\rho) &= \sigma\big(\rho - \rho(\Delta)\rho_1\big) + \rho(\Delta)\sigma(\rho_1)\\ &= z\hat{\pi}R(z\hat{\Lambda}\hat{\pi})(\rho) + \rho(\Delta)\big(\sigma(\rho_1) - z\hat{\pi}R(z\hat{\Lambda}\hat{\pi})(\rho_1)\big).\end{aligned}$$

Letting $\xi' = \sigma(\rho_1) - z\hat{\pi}R(z\hat{\Lambda}\hat{\pi})(\rho_1)$ we have

$$\sigma(\rho) = \omega^z(\rho) + \rho(\Delta)\xi'.$$

Now since σ is derived from $z\pi$ we have $\sigma(\rho - z\hat{\Lambda}\hat{\pi}(\rho)) = z\hat{\pi}(\rho)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and since ω^z is also derived from $z\pi$ we have the same equation is true for ω^z from which it follows that

$$\rho(\Delta)\xi' - z\hat{\Lambda}\hat{\pi}\rho(\Delta)\xi' = (1-z)\rho(\Delta)\xi' = 0$$

for $\rho \in \mathfrak{B}(\mathfrak{K})_*$. For $z \neq 1$ the only solution to this equation is $\xi' = 0$. Hence, if $z \neq 1$ we have

$$\sigma(\rho) = \omega^z(\rho) = z\hat{\pi}(\rho) + z^2\hat{\pi}\hat{\Lambda}\hat{\pi}(\rho) + z^3\hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi}(\rho) + \cdots$$

for $\rho \in \mathfrak{B}(\mathfrak{K})_*$. □

Theorem 3.3. *Suppose ξ is a unital q-positive boundary weight on $\mathfrak{A}(\mathfrak{H})$ and α is the CP-flow over $\mathfrak{K}$ derived from π associated with ξ. Suppose α^d is the minimal dilation of α to an E_0-semigroup of $\mathfrak{B}(\mathfrak{H}_1)$ as given in Theorem* 1.6 *so*

$$\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$. Then there is a bijection from the units U_z^1 of α^d onto the units of U_z of α given by

$$W^*U_z^1(t)W = U_z(t)$$

for $t \geqslant 0$ and $z \in \mathbb{C}$. Suppose $t \to C(t)$ is a local unitary local cocycle which fixes U_0^1 so that $C(t)U_0^1(t) = U_0^1(t)$ for $t \geqslant 0$. Then $C(t)U_z^1(t) = U_z^1(t)$ for $t \geqslant 0$ and all $z \in \mathbb{C}$. This means that the action of the local unitary cocycles on the units contains no rotations.

Proof. Assume the hypothesis and notation of the theorem. Let

$$\gamma_t(A) = W^* C(t)\alpha_t^d(WAW^*)W$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geqslant 0$. From Theorem 1.6 we have γ is a hyper-maximal flow corner from α to α. Since $C(t)$ is a unitary local cocycle which fixes U_0^1 we know from the general properties of such cocycles discussed before Lemma 3.1 there is a complex number y of modulus one so that $C(t)U_w^1(t) = U_{yw}^1(t)$ for all $t \geqslant 0$ and $w \in \mathbb{C}$. Since γ is a flow corner from α to α we know that there is a complex number z so that

$$U_{zw}(t)A = e^{\frac{1}{2}|w|^2(1-|z|^2)}\gamma_t(A)U_w(t)$$

for $w \in \mathbb{C}$, $t \geqslant 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. Then we have

$$\begin{aligned}
\gamma_t(A)U_w(t) &= W^* C(t)\alpha_t^d(WAW^*)WU_w(t) \\
&= W^* C(t)\alpha_t^d(WAW^*)U_w^1(t)W \\
&= W^* C(t)U_w^1(t)WAW^*W \\
&= W^* U_{yw}^1(t)WA = U_{yw}(t)A
\end{aligned}$$

for all $w \in \mathbb{C}$, $t \geqslant 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. Comparing the two equations we see $y = z$ so $|z| = 1$.

To complete the proof of the theorem all we need do is to show $z = 1$. Suppose $|z| = 1$ and $z \neq 1$. Now we apply Theorem 3.2. Let Θ be the *CP*-flow described in the theorem. Assuming the notation of Theorem 3.2 we have $\sigma(\rho) = \omega^z(\rho)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$. We show this implies that C is not unitary. From Theorem 1.6 we know C is a unitary cocycle if and only if γ is hyper-maximal. But γ is not hyper-maximal as can be seen as follows. Let Θ^1 be the *CP*-semigroup of $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H})$ given by

$$\Theta_t^1\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) = \begin{bmatrix} \eta_t(A_{11}) & \gamma_t(A_{12}) \\ \gamma_t^*(A_{21}) & \eta_t(A_{22}) \end{bmatrix}$$

for $t > 0$ and $A_{ij} \in \mathfrak{B}(\mathfrak{H})$ for $i, j = 1, 2$ where η is the minimal CP-flow derived from π. Note the boundary weight map Ω^1 for Θ^1 is of the form

$$\Omega^1\left(\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}\right) = \begin{bmatrix} \omega^1(\rho_{11}) & \omega^z(\rho_{12}) \\ \omega^{\bar{z}}(\rho_{21}) & \omega^1(\rho_{22}) \end{bmatrix}$$

for $\rho_{ij} \in \mathfrak{B}(\mathfrak{K})_*$ for $i, j = 1, 2$. Now we see γ is not hyper-maximal since $\alpha \geqslant \eta$ and if γ were hyper-maximal we would have $\alpha = \eta$. Hence, if $z \neq 1$ we have C is not unitary so the action of the local unitary cocycles does not contain the rotations. □

4. Conclusion

Here we present our conclusions. Suppose $\mathfrak{K}$ is a separable Hilbert space and $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$ and S is a unitary mapping from $\mathfrak{H}$ onto $\mathfrak{K}$ and $\pi(A) = SAS^*$ for $A \in \mathfrak{B}(\mathfrak{H})$. Note

π is an irreducible $*$-representation of $\mathfrak{B}(\mathfrak{H})$ on $\mathfrak{B}(\mathfrak{K})$. Suppose Λ is the mapping of $\mathfrak{B}(\mathfrak{K})$ into $\mathfrak{B}(\mathfrak{H})$ given by

$$\big(\Lambda(A)F\big)(x) = e^{-x}AF(x)$$

for $A \in \mathfrak{B}(\mathfrak{K})$ and $x \geqslant 0$ for all $\mathfrak{K}$-valued function $F \in \mathfrak{H}$. Let

$$\Delta = \lim_{n\to\infty} (\pi\Lambda)^n(I)$$

where the limit exists in the sense strong convergence since the terms are decreasing. Assume $\Delta \neq 0$. Assume further that for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ with $\rho(\Delta) = 0$ we have $\|(\hat{\Lambda}\hat{\pi})^n(\rho)\| \to 0$ as $n \to \infty$. It follows from this that if $\pi(\Lambda(A)) = A$ then $A = \lambda\Delta$.

Then there are unital CP-semigroups α of $\mathfrak{B}(\mathfrak{H})$ which are intertwined by the shifts U and there boundary weight maps are given by

$$\omega(\rho) = \omega^1(\rho) + \rho(\Delta)\xi$$

for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ where ω^1 is the boundary weight map for the minimal CP-flow derived from π and it is given by

$$\omega^1(\rho) = \hat{\pi}(\rho) + \hat{\pi}\hat{\Lambda}\hat{\pi}(\rho) + \hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi}(\rho) + \cdots = \hat{\pi}R(\hat{\Lambda}\hat{\pi})$$

and $\xi \in \mathfrak{A}(\mathfrak{H})_*$ is a positive boundary weight on

$$\mathfrak{A}(\mathfrak{H}) = (I - \Lambda)^{\frac{1}{2}}\mathfrak{B}(\mathfrak{H})(I - \Lambda)^{\frac{1}{2}}$$

with $\xi(I - \Lambda) \leqslant 1$ and α is unital (i.e. $\alpha_t(I) = I$ for $t \geqslant 0$) if and only if $\xi(I - \Lambda) = 1$. The boundary weight ξ satisfies certain positivity conditions which we analyze in a separate paper. It was shown in [18, Theorem 4.62] that if ν is a positive element of $\mathfrak{B}(\mathfrak{H})_*$ with $\nu(I) \leqslant 1$ and ξ is of the form

$$\xi = \big(1 - \nu\big(\Lambda(\Delta)\big)\big)^{-1}R(\hat{\pi}\hat{\Lambda})\nu$$

then ω as given above is the boundary weight map of a CP-flow α is unital if and only if $\nu(I) = 1$.

Then if α is such a unital CP-flow then α has a Bhat dilation to an E_0-semigroup α^d. This E_0-semigroup is of index one. The action of the local unitary cocycles on the units for this E_0-semigroup in not two-fold transitive. The Hilbert space for the dilation is of the form $\mathfrak{H}_1 = \mathfrak{K}_1 \otimes L^2(0,\infty)$ and if $U^1(t)$ is right translation by t on $\mathfrak{H}_1$ then U^1 is a unit for α^d meaning

$$U^1(t)A = \alpha_t^d(A)U^1(t)$$

for all $A \in \mathfrak{B}(\mathfrak{H}_1)$ and $t \geqslant 0$. If $C(t)$ is a unitary local cocycle for α^d so $C(t) \in \alpha_t^d(\mathfrak{B}(\mathfrak{H}_1))'$ for all $t \geqslant 0$ and the $C(t)$ are unitary operators satisfying the relation $C(t)\alpha_t^d(C(s)) = C(t+s)$ for $s, t \geqslant 0$ and if $C(t)U^1(t) = U^1(t)$ for $t \geqslant 0$ then $C(t) = I$ for all $t \geqslant 0$. This means the action of the gauge group on the units of α^d is a smaller group than for an E_0-semigroup type I_1. Also, this means α^d is not cocycle conjugate to the tensor product of a semigroup of type II_0 with a type I_1 for if this was the case the action of gauge group on the units would contain all the Euclidean transformations just as the action for an E_0-semigroup of type I_1. The same reasoning applies to the E_0-semigroups of type II_1 corresponding to the examples of product systems constructed by Tsirelson in [22].

The full action for the gauge group on a type I_1 is the Euclidean group whose action on $\mathbb{C}$ is given by $z \to az + b$ for $a, b \in \mathbb{C}$ and $|a| = 1$. In our examples we have the further restriction $a = 1$. Tsirelson has examples where there are the restrictions $a = 1$ and $\mathrm{Im}(b) = 0$. It is quite possible in our case there may be further restrictions. It may be that b lies on a one-dimensional line or even the further restriction $b = 0$. This would be interesting since it would give an example of an action which is rigid. That means that if $C(t)$ is a local unitary cocycle and U is a unit then

$$C(t)U(t) = e^{i\lambda t}U(t)$$

for $t \geqslant 0$.

We are somewhat embarrassed to report that in order to establish this result all that is required is to determine whether certain fairly simple first order differential equations with constant coefficients have a bounded solution or not. The equations are parameterized by the complex numbers (a, b) with $|a| = 1$. We have shown that if $a \neq 1$ the equations have no solution. If the equations never have solutions the action is rigid. If the equations have solutions when b lies on a one-dimensional line we are in the situation Tsirelson found and if the equations have a solution for all b then we are in the case where we have transitivity of the gauge group on the units but no two fold transitivity.

As the reader can probably guess the feature that makes these equations interesting and difficult is that they involve infinitely many variables. We will present them in a longer and more detailed paper.

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References

[1] A. Alevras, R.T. Powers, G.L. Price, Cocycles for one-parameter flows of $B(H)$, J. Funct. Anal. 230 (1) (2006) 1–64.
[2] W. Arveson, An addition formula for the index of semigroups of endomorphisms of $B(H)$, Pacific J. Math. 137 (1) (1989) 19–36.
[3] W. Arveson, Continuous analogues of Fock space, Mem. Amer. Math. Soc. 80 (409) (1989), iv+66 pp.
[4] W. Arveson, On the index and dilations of completely positive semigroups, Internat. J. Math. 10 (7) (1999) 791–823.
[5] W.B. Arveson, Noncommutative Dynamics and E-Semigroups, Springer Monogr. Math., Springer-Verlag, New York, 2003.
[6] B.V.R. Bhat, An index theory for quantum dynamical semigroups, Trans. Amer. Math. Soc. 348 (2) (1996) 561–583.
[7] B.V.R. Bhat, Cocycles of CCR flows, Mem. Amer. Math. Soc. 149 (709) (2001), xx+114 pp.
[8] B.V.R. Bhat, M. Skeide, Tensor product systems of Hilbert modules and dilations of completely positive semigroups, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 3 (4) (2000) 519–575.
[9] M. Izumi, Every sum system is divisible, eprint, arXiv: 0708.1591.

[10] M. Izumi, A perturbation problem for the shift semigroup, J. Funct. Anal. 251 (2) (2007) 498–545.
[11] M. Izumi, R. Srinivasan, Generalized CCR flows, eprint, arXiv: 0705.3280.
[12] D. Markiewicz, On the product system of a completely positive semigroup, J. Funct. Anal. 200 (1) (2003) 237–280.
[13] P.S. Muhly, B. Solel, Quantum Markov processes (correspondences and dilations), Internat. J. Math. 13 (8) (2002) 863–906.
[14] R.T. Powers, A nonspatial continuous semigroup of $*$-endomorphisms of $B(H)$, Publ. Res. Inst. Math. Sci. 23 (6) (1987) 1053–1069.
[15] R.T. Powers, An index theory for semigroups of $*$-endomorphisms of $B(H)$ and type II_1 factors, Canad. J. Math. 40 (1) (1988) 86–114.
[16] R.T. Powers, Induction of semigroups of endomorphisms of $\mathfrak{B}(\mathfrak{H})$ from completely positive semigroups of $(n \times n)$ matrix algebras, Internat. J. Math. 10 (7) (1999) 773–790.
[17] R.T. Powers, New examples of continuous spatial semigroups of $*$-endomorphisms of $B(H)$, Internat. J. Math. 10 (2) (1999) 215–288.
[18] R.T. Powers, Continuous spatial semigroups of completely positive maps of $B(H)$, New York J. Math. 9 (2003) 165–269 (electronic).
[19] R.T. Powers, G. Price, Continuous spatial semigroups of $*$-endomorphisms of $B(H)$, Trans. Amer. Math. Soc. 321 (1) (1990) 347–361.
[20] B. Tsirelson, From slightly coloured noises to unitless product systems, eprint, arXiv: math/0006165.
[21] B. Tsirelson, From random sets to continuous tensor products: Answers to three questions of W. Arveson, eprint, arXiv: math.FA/0001070, 2000.
[22] B. Tsirelson, On automorphisms of type II Arveson systems (probabilistic approach), eprint, arXiv: math. OA/0411062, 2004.
[23] J. von Neumann, On infinite direct products, Compos. Math. 6 (1939) 1–77.