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Second-order regular variation, convolution and the central limit theorem

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Abstract

Second-order regular variation is a refinement of the concept of regular variation which is useful for studying rates of convergence in extreme value theory and asymptotic normality of tail estimators. For a distribution tail $1-F$ which possesses second-order regular variation, we discuss how this property is inherited by $1-F^2$ and $1-F^{*2}$. We also discuss the relationship of central limit behavior of tail empirical processes, asymptotic normality of Hill's estimator and second-order regular variation. © 1997 Elsevier Science B.V.

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1. Introduction

In this paper we assume that the distribution function F is concentrated on $[0,\infty)$. The tail $1-F(x)$ is regularly varying with index $-\alpha$, $\alpha>0$ (written $1-F\in RV_{-\alpha}$) if

$$\lim_{t\to\infty}\frac{1-F(tx)}{1-F(t)}=x^{-\alpha},\quad x>0. \tag{1.1}$$

The distribution tail $1-F$ is second-order regularly varying with first-order parameter $-\alpha$ and second-order parameter ρ (written $1-F\in 2RV(-\alpha,\rho)$) if there exists a function $A(t)\to 0$, $t\to\infty$ which ultimately has constant sign such that the following refinement of (1.1) holds:

$$\lim_{t\to\infty}\frac{\frac{1-F(tx)}{1-F(t)}-x^{-\alpha}}{A(t)}=H(x):=cx^{-\alpha}\int_1^x u^{\rho-1}\mathrm{d}u,\quad x>0 \tag{1.2}$$

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for $c \neq 0$. Note that for $x > 0$

$$H(x) = \begin{cases} cx^{-\alpha} \log x, & \text{if } \rho = 0, \\ cx^{-\alpha} \frac{x^{\rho}-1}{\rho}, & \text{if } \rho < 0. \end{cases}$$

It is well known (Geluk and de Haan, 1987, Theorem 1.9) that if (1.2) holds with $H(x)$ not a multiple of $x^{-\alpha}$, then H satisfies the above representation, $|A| \in RV_{\rho}$ and no other choices of ρ are consistent with $A(t) \to 0$. Moreover convergence in (1.2) is uniform in x on compact intervals of $(0, \infty)$. See de Haan and Stadtmüller (1996) for a related discussion.

Second-order regular variation has proven very useful in establishing asymptotic normality of extreme value statistics and also for the study of rates of convergence to extreme value and stable distributions (de Haan and Resnick, 1996; de Haan and Peng, 1995a, b, c; Smith, 1982).

An example of the statistical uses of second-order regular variation is as follows: Suppose $Z_1, \dots, Z_n$ are iid random variables with common distribution F satisfying (1.1). A commonly used estimator of α^{-1} is Hill's estimator

$$H_{k,n} := \frac{1}{k} \sum_{i=1}^{k} \log \frac{Z_{(i)}}{Z_{(k+1)}},$$

where $Z_{(1)} > \cdots > Z_{(n)}$ are the order statistics of the sample. Under the assumption that the number of upper order statistics k used in the estimation satisfies $k \to \infty$, $k/n \to 0$,

$$H_{k,n} \xrightarrow{P} \alpha^{-1},$$

and under a von Mises condition there exists constants $\alpha_{k,n}^{-1}$ such that

$$\sqrt{k}(H_{k,n} - \alpha_{k,n}^{-1}) \Rightarrow N,$$

where N is a normal random variable (Mason, 1982; Hall, 1982; Csörgő et al., 1985; Häusler and Teugels, 1985; Dekkers and de Haan, 1989; Resnick and Stărică, 1995; Davis and Resnick, 1984; Csörgő and Mason, 1985). In order to construct confidence statements for the inference, one needs to replace $\alpha_{k,n}$ by α in the central limit theorem and for this one needs to know $\sqrt{k}(\alpha_{k,n}^{-1} - \alpha^{-1}) \to 0$ as $n \to \infty$. A convenient way to assure this is by assuming $1 - F \in 2RV(-\alpha, \rho)$. (See also Resnick and Stărică, 1997a, b; Kratz and Resnick, 1996.)

A related statistical problem assumes that one observes $X_1, \dots, X_n$ where $\{X_n\}$ is a stationary infinite order moving average process of the form

$$X_t = \sum_{j=0}^{\infty} c_j Z_{t-j}, \quad t = 0, \pm 1, \pm 2, \dots$$

where $\{Z_t\}$ are iid with common distribution satisfying (1.1) and (1.2). Resnick and Stărică (1995) have proven that the Hill estimator applied to $X_1, \dots, X_n$ is a consistent estimator of α^{-1}. In order to assess the performance mathematically of this estimator and to compare it to competing procedures, the asymptotic normality must be investigated. In order to do this successfully, one must understand how second-order regular variation behaves under convolution and this was the strongest motivation for the present investigation. This time series estimation problem is further discussed in Resnick and Stărică (1997b).

First-order behavior of regularly varying tails under convolution is fairly tame: If $1-F$ satisfies (1.1), then the convolution tail $1-F^{*2}$ satisfies

$$1-F^{*2}(t)\sim 2(1-F(t)), \quad (t\to\infty).$$

Feller (1971) has a straightforward and clear analytical proof and Resnick (1986, 1987) proves this probabilistically using point processes. However, second-order regularly varying tails behave in a much more complicated way.

In Section 2 we prepare the way by discussing behavior of distribution tails of maxima $Z_1 \vee Z_2$ of iid random variables having common distribution F satisfying (1.2). The behavior turns out to depend on how

$$\lim_{t\to\infty}\frac{1-F(t)}{A(t)}$$

behaves. Section 3 gives some results for convolution tails and Section 4 discusses a probabilistically equivalent statement to (1.2) involving the central limit theorem and shows that in a manner to be made precise in Theorem 4.2, asymptotic normality of Hill's estimator is equivalent to second-order regular variation.

We end this introduction with two examples.

Example 1.1 (*Log gamma distribution*). Suppose X_1, X_2 are iid with standard exponential density. The log gamma distribution is the distribution of $\exp\{X_1+X_2\}$. For $x>1$,

$$\begin{aligned} P[\exp\{X_1+X_2\}>x] &= P[X_1+X_2>\log x] \\ &= \exp\{-\log x\}+\exp\{-\log x\}\log x \\ &= x^{-1}(1+\log x) := 1-F(x). \end{aligned}$$

Thus for $x>1$

$$\begin{aligned} \frac{1-F(tx)}{1-F(t)}-x^{-1} &= x^{-1}\left(\frac{\log x}{1+\log t}\right) \\ &\sim x^{-1}\frac{\log x}{\log t} \end{aligned}$$

and thus

$$\lim_{t\to\infty}\frac{\frac{1-F(tx)}{1-F(t)}-x^{-1}}{1/\log t}=x^{-1}\log x,$$

and with $A(t)=1/\log t$ we have $\alpha=1, \rho=0$ and

$$\lim_{t\to\infty}\frac{1-F(t)}{A(t)}=0.$$

Example 1.2 (*Hall/Weiss class*). Suppose for $x \geqslant 1$, $\alpha > 0$, $\rho < 0$ that

$$1 - F(x) = \tfrac{1}{2} x^{-\alpha}(1 + x^{\rho}).$$

Then as $t \to \infty$

$$\frac{1 - F(tx)}{1 - F(t)} - x^{-\alpha} = x^{-\alpha}\left\{\frac{1 + (tx)^{\rho}}{1 + t^{\rho}} - 1\right\}$$
$$\sim x^{-\alpha}(x^{\rho} - 1)t^{\rho}$$

and so we may set $A(t) = \rho t^{\rho}$ and

$$\lim_{t\to\infty} \frac{1 - F(t)}{|A(t)|} = \begin{cases} 0 & \text{if } |\rho| < \alpha, \\ |\rho|^{-1} & \text{if } |\rho| = \alpha, \\ \infty & \text{if } |\rho| > \alpha. \end{cases}$$

2. Maxima

As preparation for further work, we begin with a two-dimensional result.

Theorem 2.1. *Let Z_1, Z_2 be non-negative iid random variables with common distribution F satisfying* (1.2). *Then for $x > 0, y > 0$*

$$\lim_{t\to\infty} \frac{\frac{P\{[Z_1 > tx] \cup [Z_2 > ty]\}}{1 - F(t)} - (x^{-\alpha} + y^{-\alpha})}{A(t)} = H(x) + H(y) - l(xy)^{-\alpha} \tag{2.1}$$

provided

$$\lim_{t\to\infty} \frac{1 - F(t)}{A(t)} = l, \quad |l| < \infty. \tag{2.2}$$

If $|l| = \infty$, then

$$\lim_{t\to\infty} \frac{\frac{P\{[Z_1 > tx] \cup [Z_2 > ty]\}}{1 - F(t)} - (x^{-\alpha} + y^{-\alpha})}{1 - F(t)} = -(xy)^{-\alpha}. \tag{2.3}$$

Proof. We have

$$P\{[Z_1 > tx] \cup [Z_2 > ty]\}$$
$$= 1 - F(tx)F(ty) = 1 - F(tx) + 1 - F(ty) - (1 - F(tx))(1 - F(ty))$$

and so

$$\frac{P\{[Z_1 > tx] \cup [Z_2 > ty]\}}{1 - F(t)} - (x^{-\alpha} + y^{-\alpha})$$
$$= \left(\frac{1 - F(tx)}{1 - F(t)} - x^{-\alpha}\right) + \left(\frac{1 - F(ty)}{1 - F(t)} - y^{-\alpha}\right) - \left(\frac{(1 - F(tx))(1 - F(ty))}{1 - F(t)}\right) \tag{2.4}$$

and the stated results follow. □

By letting $x = y$, we immediately get the corollary about tail behavior of distribution of maxima.

Corollary 2.2. *Let Z_1, Z_2 be non-negative iid random variables with common distribution F satisfying* (1.2). *Then for $x > 0$*

$$\lim_{t\to\infty} \frac{\frac{P[Z_1 \vee Z_2 > tx]}{1-F(t)} - 2x^{-\alpha}}{A(t)} = 2H(x) - lx^{-2\alpha} \tag{2.5}$$

if

$$\lim_{t\to\infty} \frac{1-F(t)}{A(t)} = l, \quad |l| < \infty, \tag{2.6}$$

and if $|l| = \infty$

$$\lim_{t\to\infty} \frac{\frac{P[Z_1 \vee Z_2 > tx]}{1-F(t)} - 2x^{-\alpha}}{1-F(t)} = -x^{-2\alpha}. \tag{2.7}$$

Remarks. (1) Changing normalizations in (2.5) yields

$$\lim_{t\to\infty} \frac{\frac{P[Z_1 \vee Z_2 > tx]}{P[Z_1 \vee Z_2 > t]} - x^{-\alpha}}{A(t)} = \left(1 + \frac{1}{2}\alpha l\right) H(x), \quad x > 0 \tag{2.8}$$

so that $P[Z_1 \vee Z_2 > t] \in 2RV(-\alpha, \rho)$. Note that for $l \neq 0$ we have $-\alpha = \rho$ since $1 - F(t) \sim lA(t)$. Similarly, modifying (2.7) yields

$$\lim_{t\to\infty} \frac{\frac{P[Z_1 \vee Z_2 > tx]}{P[Z_1 \vee Z_2 > t]} - x^{-\alpha}}{1-F(t)} = \frac{1}{2}x^{-\alpha}(1 - x^{-\alpha}), \quad x > 0 \tag{2.9}$$

so that in this case, $P[Z_1 \vee Z_2 > t] \in 2RV(-\alpha, -\alpha)$. Thus $P[Z_1 > t]$ and $P[Z_1 \vee Z_2 > t]$ may have different second-order parameters.

(2) Applying Theorem 2.1 with x replaced by x/c_1 and y replaced by y/c_2 where $x > 0$, $y > 0$, $c_i > 0$, $i = 1, 2$ yields

$$\frac{\frac{P\{[c_1 Z_1 > tx] \cup [c_2 Z_2 > ty]\}}{1-F(t)} - (c_1^\alpha x^{-\alpha} + c_2^\alpha y^{-\alpha})}{A(t)} = H(xc_1^{-1}) + H(yc_2^{-1}) - lc_1^\alpha c_2^\alpha (xy)^{-\alpha} \tag{2.10}$$

if (2.6) is satisfied. The second-order behavior of $P[c_1 Z_1 \vee c_2 Z_2 > t]$ under condition (2.5) is obtained by setting $x = y$ as follows:

$$\frac{\frac{P[c_1 Z_1 \vee c_2 Z_2 > tx]}{1-F(t)} - (c_1^\alpha + c_2^\alpha) x^{-\alpha}}{A(t)} = H(xc_1^{-1}) + H(xc_2^{-1}) - lc_1^\alpha c_2^\alpha x^{-2\alpha}, \quad x > 0, \tag{2.11}$$

extending (2.5). A similar modification of (2.7) holds.

The following proposition asserts that the relations (2.1) and (2.3) characterize second-order regular variation of the underlying distribution tail function. Moreover the normalizations $A(t)$ and $1 - F(t)$ are the only possible.

Proposition 2.3. *Define for $x>0$, $y>0$*

$$LHS(t,x,y) := \frac{P\{[Z_1>tx]\cup[Z_2>ty]\}}{1-F(t)} - (x^{-\alpha}+y^{-\alpha}).$$

Suppose further there exists a function $\psi(t)>0$ such that for all $x>0$, $y>0$

$$\lim_{t\to\infty} \frac{LHS(t,x,y)}{\psi(t)} = c(x,y) \tag{2.12}$$

where $\psi(t)\to 0$ $(t\to\infty)$ and the function $c(x,y)$ satisfies $|c(x,y)|<\infty$ for all x,y and $c(x,x)\neq a_1x^{-\alpha}+a_2x^{-2\alpha}$ for any choice of real a_1, a_2. Then $1-F$ satisfies (1.2) with $A(t)\sim c\psi(t)$ for some $c\neq 0$ and

$$\lim_{t\to\infty} \frac{1-F(t)}{|A(t)|} \in [0,\infty) \text{ exists.} \tag{2.13}$$

Proof. Since $\psi(t)\to 0$ and $|c(x,y)|<\infty$ it follows that

$$LHS(t,x,x) = 2\left(\frac{1-F(tx)}{1-F(t)} - x^{-\alpha}\right) - \frac{(1-F(tx))^2}{1-F(t)} \to 0 \quad (t\to\infty).$$

Obviously, for $x>1$ we have $(1-F(tx))^2/(1-F(t))\to 0$ as $t\to\infty$ and hence it follows that $1-F\in RV_{-\alpha}$. Observe that

$$\frac{LHS(t,1,1)}{\psi(t)} = \frac{-(1-F(t))}{\psi(t)} \to c(1,1). \tag{2.14}$$

Hence if $c(1,1)=0$ we have $1-F(t)=o(\psi(t))$ implying $(1-F(tx))^2/(1-F(t))=o(\psi(t))$, as $t\to\infty$ for $x>0$. As a consequence, for $x>0$

$$\frac{LHS(t,x,x)}{\psi(t)} = \frac{\frac{1-F(tx)}{1-F(t)} - x^{-\alpha}}{\psi(t)/2} + o(1) \to c(x,x) \quad \text{as} \quad t\to\infty.$$

Since $c(x,x)$ is not a multiple of $x^{-\alpha}$, it follows from Geluk and de Haan (1987, Theorem 1.9) that $c(x,x)=cH(x)$ with $c\neq 0$ and H as in (1.2) and hence $1-F\in 2RV(-\alpha,\rho)$. Therefore, there exists a function A such that (1.2) is satisfied and

$$\frac{LHS(t,x,x)}{\psi(t)} = (2H(x)+o(1))\frac{A(t)}{\psi(t)} - (x^{-2\alpha}+o(1))\frac{1-F(t)}{\psi(t)}$$
$$= (2H(x)+o(1))\frac{A(t)}{\psi(t)} + o(1).$$

Let $t\to\infty$ and we conclude

$$c(x,x) = 2H(x)\lim_{t\to\infty}\frac{A(t)}{\psi(t)},$$

and hence we get

$$\lim_{t\to\infty}\frac{A(t)}{\psi(t)} = \frac{c}{2}. \tag{2.15}$$

Thus, if $c(1,1)=0$, then $1-F(t)=o(A(t))$, $t\to\infty$.

In case $c(1,1)\neq 0$ we have $1-F(t)\sim -c(1,1)\psi(t)$, and therefore

$$\frac{LHS(t,x,x)}{\psi(t)}=\frac{\frac{1-F(tx)}{1-F(t)}-x^{-\alpha}}{\psi(t)/2}-\frac{(1-F(tx))^2}{(1-F(t))^2}\frac{1-F(t)}{\psi(t)}$$
$$=\frac{\frac{1-F(tx)}{1-F(t)}-x^{-\alpha}}{\psi(t)/2}+x^{-2\alpha}c(1,1)+o(1)\to c(x,x)\quad\text{as } t\to\infty.$$

It follows that for $x>0$

$$\frac{\frac{1-F(tx)}{1-F(t)}-x^{-\alpha}}{\psi(t)/2}\to c(x,x)-c(1,1)x^{-2\alpha}\quad(t\to\infty). \tag{2.16}$$

By assumption the right-hand side is not a multiple of $x^{-\alpha}$, hence (1.2) holds with some function $A(t)\sim c_1\psi(t)$ $(t\to\infty)$, where $c_1\neq 0$ is a constant. □

Remark. (1) In case $c(x,x)=cx^{-2\alpha}$ for some $c\neq 0$ it follows that $c=c(1,1)$ and

$$\left(\frac{1-F(tx)}{1-F(t)}-x^{-\alpha}\right)\Big/(\psi(t)/2)\to 0,\quad x>0$$

as $t\to\infty$. Hence $(1-F(tx))/(1-F(t))-x^{-\alpha}=o(1-F(t))$. Thus, in this case if F satisfies (1.2), then the limit in (2.13) is infinite.

3. Convolution

In the sequel we denote $1-F$ by $\bar F$. The results for convolution are more complex than for maxima. In order to prove the main result Theorem 3.2 we first need a lemma.

Lemma 3.1. *Suppose for $i=1,2$ that $F_i\in 2RV(-\alpha_i,\rho_i)$, i.e. F_i satisfies (1.2) with auxiliary function A_i. Further suppose*

$$\bar F_{i+2}(x)-k_i\bar F_i(x)=(d_i+o(1))A_i(x)\bar F_i(x)\quad(x\to\infty)$$

and

$$\bar F_i(x-b)-\bar F_i(x)=o(A_i(x)\bar F_i(x))\quad(x\to\infty),$$

where $k_i>0, b, d_i\in\mathbb{R}$. Then as $x\to\infty$

$$\overline{F_3*F_4}(x)-\bar F_3(x)-\bar F_4(x)$$
$$=k_1k_2(\overline{F_1*F_2}(x)-\bar F_1(x)-\bar F_2(x))+o\left(\sum_{i=1}^{2}A_i(x)\bar F_i(x)\right).$$

Proof. By assumption, for $\varepsilon>0$ there exists $a>0$ such that for $i=1,2$

$$\bar F_i(x)<\varepsilon,\quad \bar F_{i+2}(x)-k_i\bar F_i(x)\leqslant(d_i+\varepsilon)A_i(x)\bar F_i(x),\quad x>a.$$

It follows that for $x>a$

$$\begin{aligned}\overline{F_3 * F_4}(x) - \bar{F}_4(x) &= \int_0^x \bar{F}_3(x-u)\,\mathrm{d}F_4(u)\\ &\leqslant \int_0^{x-a} \bar{F}_3(x-u)\,\mathrm{d}F_4(u) + (\bar{F}_4(x-a) - \bar{F}_4(x))\\ &\leqslant k_1 \int_0^{x-a} \bar{F}_1(x-u)\,\mathrm{d}F_4(u) + (d_1+\varepsilon)\\ &\quad\times \int_0^{x-a} A_1(x-u)\bar{F}_1(x-u)\,\mathrm{d}F_4(u)\\ &\quad + k_2(\bar{F}_2(x-a) - \bar{F}_2(x)) + \mathrm{o}(A_2(x)\bar{F}_2(x))\\ &=: k_1 I_1 + (d_1+\varepsilon) I_2 + \mathrm{o}(A_2(x)\bar{F}_2(x)) \quad (x\to\infty).\end{aligned} \tag{3.1}$$

Now I_1 is estimated as follows.

$$\begin{aligned}I_1 &= \int_0^{x-a} \bar{F}_1(x-u)\,\mathrm{d}F_4(u) = \int_0^{x} \bar{F}_1(x-u)\,\mathrm{d}F_4(u) + \mathrm{o}(A_2(x)\bar{F}_2(x))\\ &= \int_0^{x-a} \bar{F}_4(x-u)\,\mathrm{d}F_1(u) + \bar{F}_1(x) - \bar{F}_4(x) + \mathrm{o}\left(\sum_{i=1}^{2} A_i(x)\bar{F}_i(x)\right)\\ &\leqslant k_2 \int_0^{x-a} \bar{F}_2(x-u)\,\mathrm{d}F_1(u) + (d_2+\varepsilon)\int_0^{x-a} A_2(x-u)\bar{F}_2(x-u)\,\mathrm{d}F_1(u)\\ &\quad + \bar{F}_1(x) - \bar{F}_4(x) + \mathrm{o}\left(\sum_{i=1}^{2} A_i(x)\bar{F}_i(x)\right)\\ &\leqslant k_2(\overline{F_1 * F_2}(x) - \bar{F}_1(x)) + (d_2+\varepsilon)\int_0^{x-a} A_2(x-u)\bar{F}_2(x-u)\,\mathrm{d}F_1(u)\\ &\quad + \bar{F}_1(x) - \bar{F}_4(x) + \mathrm{o}\left(\sum_{i=1}^{2} A_i(x)\bar{F}_i(x)\right) \quad (x\to\infty).\end{aligned} \tag{3.2}$$

Since $A_2 \in RV_{\rho_2}$ and for $i=1,2$ we have $\bar{F}_i \in RV_{-\alpha_i}$, we have

$$\int_0^{x-a} A_2(x-u)\bar{F}_2(x-u)\,\mathrm{d}F_1(u) \sim A_2(x)\bar{F}_2(x)$$

and since a lower inequality for I_1 can be proved similarly, combination with (3.2) gives

$$I_1 = k_2(\overline{F_1 * F_2}(x) - \bar{F}_1(x)) + \bar{F}_1(x) - \bar{F}_4(x) + d_2 A_2(x)\bar{F}_2(x) + \mathrm{o}\left(\sum_{i=1}^{2} A_i(x)\bar{F}_i(x)\right).$$

Substituting $\bar{F}_4(x) = k_2\bar{F}_2(x) + (d_2 + \mathrm{o}(1))A_2(x)\bar{F}_2(x)$ we find

$$I_1 = k_2(\overline{F_1 * F_2}(x) - \bar{F}_1(x)) + \bar{F}_1(x) - k_2\bar{F}_2(x) + \mathrm{o}\left(\sum_{i=1}^{2} A_i(x)\bar{F}_i(x)\right). \tag{3.3}$$

Similarly, regular variation of $A_1, \bar{F}_1$ and $\bar{F}_4$ implies

$$I_2 \sim A_1(x)\bar{F}_1(x)\ (x \to \infty). \tag{3.4}$$

Since a corresponding lower inequality for (3.1) can be proved similarly, combination of (3.1), (3.3) and (3.4) gives an expression for $\overline{F_3 * F_4}(x) - \bar{F}_4(x)$. Subtracting $\bar{F}_3(x) = k_1\bar{F}_1(x) + d_1A_1(x)\bar{F}_1(x) + o(A_1(x)\bar{F}_1(x))$ then gives the required result. □

Theorem 3.2. *Suppose Z_1, Z_2 are iid non-negative random variables with common distribution function F satisfying (1.2) and suppose $c_1 > 0$, $c_2 > 0$. There exist for each case two functions $\tilde{A}$ and $\tilde{H}$ such that*

$$\lim_{t\to\infty} \frac{\frac{P(c_1Z_1 + c_2Z_2 > tx)}{\bar{F}(t)} - (c_1^\alpha + c_2^\alpha)x^{-\alpha}}{\tilde{A}(t)} = \tilde{H}(x) \tag{3.5}$$

for $x > 0$. Define $\xi_\alpha = -\Gamma^2(1-\alpha)/\Gamma(1-2\alpha)$ for $\alpha < 1$. If

I. $\alpha < 1$

1. $\rho \leqslant -\alpha$ *and* $\bar{F}(t)/A(t) \to \infty$ *as* $t \to \infty$, *then* $\tilde{A}(t) = \bar{F}(t)$ *and* $\tilde{H}(x) = c_1^\alpha c_2^\alpha \xi_\alpha x^{-2\alpha}$.
2. $\rho \geqslant -\alpha$ *and* $\bar{F}(t)/A(t) \to l < \infty$ *as* $t \to \infty$, *then* $\tilde{A}(t) = A(t)$ *and* $\tilde{H}(x) = lc_1^\alpha c_2^\alpha \xi_\alpha x^{-2\alpha} + H(c_1^{-1}x) + H(c_2^{-1}x)$,

II. $\alpha = 1$ *and the mean* μ *is finite*

1. $\rho \leqslant -1$ *and* $tA(t) \to l < \infty$ *as* $t \to \infty$, *then* $\tilde{A}(t) = t^{-1}$ *and* $\tilde{H}(x) = 2\mu c_1c_2x^{-2} + l(H(c_1^{-1}x) + H(c_2^{-1}x))$,
2. $\rho \geqslant -1$ *and* $tA(t) \to \infty$ *as* $t \to \infty$, *then* $\tilde{A}(t) = A(t)$ *and* $\tilde{H}(x) = H(c_1^{-1}x) + H(c_2^{-1}x)$,

III. $\alpha > 1$

1. $\rho \leqslant -1$ *and* $tA(t) \to l < \infty$ *as* $t \to \infty$ *then* $\tilde{A}(t) = t^{-1}$ *and* $\tilde{H}(x) = \alpha\mu(c_1^\alpha c_2 + c_1c_2^\alpha)x^{-\alpha-1} + l(H(c_1^{-1}x) + H(c_2^{-1}x))$,
2. $\rho \geqslant -1$ *and* $tA(t) \to \infty$ *as* $t \to \infty$ *then* $\tilde{A}(t) = A(t)$ *and* $\tilde{H}(x) = H(c_1^{-1}x) + H(c_2^{-1}x)$.

Proof. The proof for I is based on the previous Lemma 3.1 and also Lemma 2.1 of Geluk (1996). It is known (de Haan, 1996) that there exists a distribution function F_0 with $\bar{F}_0 \in 2RV(-\alpha, \rho)$ and $F_0(0) = 0$ which has a differentiable density $f_0 \in RV(-\alpha - 1)$ such that

$$\bar{F}(t) - \bar{F}_0(t) = o(A(t)\bar{F}(t)) \quad (t \to \infty). \tag{3.6}$$

Let Z_1^*, Z_2^* be iid non-negative random variables with common distribution F_0. First we prove I.2. We intend to apply Lemma 3.1 to

$$F_1(t) = P(Z_1^* \leqslant t), \qquad F_2(t) = P(Z_2^* \leqslant t),$$
$$F_3(t) = P(c_1Z_1 \leqslant t), \qquad F_4(t) = P(c_2Z_2 \leqslant t).$$

In order to apply the lemma, we first verify its hypotheses. From (3.6) and the fact that F satisfies (1.2) it is clear that F_1 and F_2 satisfy (1.2) as well (with the same function A). Note that as $t\to\infty$ for $i=1,2$

$$\begin{aligned}\frac{P(c_iZ_i>t)-c_i^\alpha P(Z_i^*>t)}{A(t)\bar F_0(t)} &= \frac{\bar F(t/c_i)-c_i^\alpha \bar F_0(t)}{A(t)\bar F_0(t)}\\ &= \frac{\bar F(t/c_i)-\bar F_0(t/c_i)}{A(t/c_i)\bar F(t/c_i)}\frac{A(t/c_i)\bar F(t/c_i)}{A(t)\bar F_0(t)}\\ &\quad +\frac{\bar F_0(t/c_i)-c_i^\alpha \bar F_0(t)}{A(t)\bar F_0(t)}\\ &\to H(c_i^{-1}).\end{aligned}$$

So the first condition of Lemma 3.1 is verified with $k_1=c_1^\alpha$, $k_2=c_2^\alpha$, $d_1=H(c_1^{-1})$, $d_2=H(c_2^{-1})$. The second condition reads

$$\frac{\bar F_0(t-b)-\bar F_0(t)}{\bar F_0(t)A(t)}=\frac{\bar F_0(t(1-b/t))-(1-b/t)^{-\alpha}\bar F_0(t)}{\bar F_0(t)A(t)}+\frac{(1-b/t)^{-\alpha}-1}{A(t)}.$$

The first term tends to 0 for $t\to\infty$ due to the uniform convergence in the second-order condition (1.2). Since $(1-b/t)^{-\alpha}-1\sim\alpha b/t$ and $-\rho\leqslant\alpha<1$ the second term clarifies as

$$\frac{(1-b/t)^{-\alpha}-1}{A(t)}\sim\frac{\alpha b}{tA(t)}\to 0$$

as $t\to\infty$. Since the hypotheses of the Lemma are verified it follows that as $t\to\infty$

$$\frac{P(c_1Z_1+c_2Z_2>t)-P(c_1Z_1>t)-P(c_2Z_2>t)}{A(t)\bar F(t)}=c_1^\alpha c_2^\alpha\frac{\bar F_0^{*2}(t)-2\bar F_0(t)}{\bar F^2(t)}\frac{\bar F(t)}{A(t)}+o(1)$$

or

$$\frac{P(c_1Z_1+c_2Z_2>t)-P(c_1Z_1>t)-P(c_2Z_2>t)}{A(t)\bar F(t)}\to l\xi_\alpha c_1^\alpha c_2^\alpha \tag{3.7}$$

since

$$\frac{\bar F_0^{*2}(t)-2\bar F_0(t)}{\bar F^2(t)}\to\xi_\alpha \tag{3.8}$$

(Omey and Willekens, 1986) and $\bar F(t)/A(t)\to l$ from the assumptions of I.2. To finish the proof for this case one applies (3.7) with c_i replaced by c_i/x $(i=1,2)$ and adds

$$\frac{P(c_1Z_1>tx)-c_1^\alpha x^{-\alpha}\bar F(t)}{A(t)\bar F(t)}+\frac{P(c_2Z_2>tx)-c_2^\alpha x^{-\alpha}\bar F(t)}{A(t)\bar F(t)}\to H(c_1^{-1}x)+H(c_2^{-1}x). \tag{3.9}$$

The proof of I.1 follows the same path, the only difference being that instead of Lemma 3.1 we will employ Lemma 2.1 (Geluk, 1996). The choice of α from Lemma 2.1 is $\alpha=2$. Defining F_1 to F_4 as above it follows from (1.2) and (3.6) that the first condition in the lemma is verified with $k_1=c_1^\alpha$, $k_2=c_2^\alpha$, $d_1=0$, $d_2=0$. The second condition reads

$$\frac{\bar F_0(t-b)-\bar F_0(t)}{\bar F_0^2(t)}=\frac{\bar F_0(t(1-b/t))-(1-b/t)^{-\alpha}\bar F_0(t)}{\bar F_0(t)A(t)}\frac{A(t)}{\bar F_0(t)}+\frac{(1-b/t)^{-\alpha}-1}{\bar F_0(t)}.$$

The first term on the right-hand side tends to 0 since both factors tend to zero (by uniform convergence in (1.2) and by assumption, respectively). The second behaves like $\alpha b(t\bar F_0(t))^{-1}$. Since we are under the assumption that $\bar F\in RV_{-\alpha}$ with $\alpha<1$, we have that $t\bar F_0(t)\to\infty$ as $t\to\infty$. Therefore the hypotheses of Lemma 2.1 are verified. Thus combination of Lemma 2.1 with (3.8) above gives

$$\frac{P(c_1Z_1+c_2Z_2>t)-P(c_1Z_1>t)-P(c_2Z_2>t)}{\bar F^2(t)}\to\xi_\alpha c_1^\alpha c_2^\alpha. \tag{3.10}$$

Since in this case $A(t)=o(\bar F(t))$ (3.9) implies

$$\frac{P(c_1Z_1>tx)-c_1^\alpha x^{-\alpha}\bar F(t)}{A(t)\bar F(t)}\frac{A(t)}{\bar F(t)}+\frac{P(c_2Z_2>tx)-c_2^\alpha x^{-\alpha}\bar F(t)}{A(t)\bar F(t)}\frac{A(t)}{\bar F(t)}\to 0. \tag{3.11}$$

The proof of I.1 is finished if we replace c_i by c_i/x $(i=1,2)$ in (3.10) and add (3.11).

For cases II and III a different approach is needed. Decompose $P(c_1Z_1+c_2Z_2>t)$ as follows:

$$\begin{aligned}
&P(c_1Z_1+c_2Z_2>t)\\
&\quad=P(c_1Z_1+c_2Z_2>t,c_1Z_1\vee c_2Z_2>t)+P(c_1Z_1+c_2Z_2>t,c_1Z_1\vee c_2Z_2\leqslant t)\\
&\quad=P(c_1Z_1\vee c_2Z_2>t)+P(c_1Z_1+c_2Z_2>t,c_1Z_1\vee c_2Z_2\leqslant t,c_1Z_1\wedge c_2Z_2\leqslant t/2)\\
&\qquad+P(c_1Z_1+c_2Z_2>t,c_1Z_1\vee c_2Z_2\leqslant t,c_1Z_1\wedge c_2Z_2>t/2)\\
&\quad=P(c_1Z_1>t)+P(c_2Z_2>t)-P(c_1Z_1>t)P(c_2Z_2>t)\\
&\qquad+\int_0^{t/2}(\bar F_3(t-u)-\bar F_3(t))\,\mathrm{d}F_4(u)+\int_0^{t/2}(\bar F_4(t-u)-\bar F_4(t))\,\mathrm{d}F_3(u)\\
&\qquad+(\bar F_1(t/2)-\bar F_1(t))(\bar F_2(t/2)-\bar F_2(t)),
\end{aligned}$$

where F_3 and F_4 are defined as above. Therefore

$$\begin{aligned}
&(P(c_1Z_1+c_2Z_2>t)-P(c_1Z_1>t)-P(c_2Z_2>t))/(\tilde A(t)\bar F(t))\\
&\quad=\frac{A(t/c_2)}{\tilde A(t)}\frac{\bar F(t/c_2)}{\bar F(t)}\int_0^{t/2}\frac{\bar F((t/c_2)(1-u/t))-\bar F(t/c_2)}{\bar F(t/c_2)A(t/c_2)}\,\mathrm{d}F_3(u)\\
&\qquad+\frac{A(t/c_1)}{\tilde A(t)}\frac{\bar F(t/c_1)}{\bar F(t)}\int_0^{t/2}\frac{\bar F((t/c_1)(1-u/t))-\bar F(t/c_1)}{\bar F(t/c_1)A(t/c_1)}\,\mathrm{d}F_4(u)\\
&\qquad+\frac{\bar F(t)}{\tilde A(t)}\left(\frac{\bar F(t/(2c_1))}{\bar F(t)}-\frac{\bar F(t/c_1)}{\bar F(t)}\right)\left(\frac{\bar F(t/(2c_2))}{\bar F(t)}-\frac{\bar F(t/c_2)}{\bar F(t)}\right)\\
&\qquad-\frac{\bar F(t)}{\tilde A(t)}\frac{\bar F(t/c_1)\bar F(t/c_2)}{\bar F^2(t)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A(t)}{\tilde{A}(t)}\frac{A(t/c_2)}{A(t)}\frac{\bar{F}(t/c_2)}{\bar{F}(t)}\int_0^{t/2}\frac{\frac{\bar{F}((t/c_2)(1-u/t))}{\bar{F}(t/c_2)}-(1-u/t)^{-\alpha}}{A(t/c_2)}\,\mathrm{d}F_3(u)\\
&+\frac{1}{t\tilde{A}(t)}\frac{\bar{F}(t/c_2)}{\bar{F}(t)}\int_0^{t/2}t((1-u/t)^{-\alpha}-1)\,\mathrm{d}F_3(u)\\
&+\frac{A(t)}{\tilde{A}(t)}\frac{A(t/c_1)}{A(t)}\frac{\bar{F}(t/c_1)}{\bar{F}(t)}\int_0^{t/2}\frac{\frac{\bar{F}((t/c_1)(1-u/t))}{\bar{F}(t/c_1)}-(1-u/t)^{-\alpha}}{A(t/c_1)}\,\mathrm{d}F_4(u)\\
&+\frac{1}{t\tilde{A}(t)}\frac{\bar{F}(t/c_1)}{\bar{F}(t)}\int_0^{t/2}t((1-u/t)^{-\alpha}-1)\,\mathrm{d}F_4(u)\\
&+\frac{\bar{F}(t)}{\tilde{A}(t)}\left(\left(\frac{\bar{F}(t/(2c_1))}{\bar{F}(t)}-\frac{\bar{F}(t/c_1)}{\bar{F}(t)}\right)\left(\frac{\bar{F}(t/(2c_2))}{\bar{F}(t)}-\frac{\bar{F}(t/c_2)}{\bar{F}(t)}\right)\right.\\
&\left.-\frac{\bar{F}(t/c_1)\bar{F}(t/c_2)}{\bar{F}^2(t)}\right)\\
&=\frac{A(t)}{\tilde{A}(t)}\mathrm{I}+\frac{1}{t\tilde{A}(t)}\mathrm{II}+\frac{A(t)}{\tilde{A}(t)}\mathrm{III}+\frac{1}{t\tilde{A}(t)}\mathrm{IV}+\frac{\bar{F}(t)}{\tilde{A}(t)}\mathrm{V}.
\end{aligned}
$$

The expression of interest becomes

$$
\begin{aligned}
&\frac{P(c_1Z_1+c_2Z_2>t)-(c_1^\alpha+c_2^\alpha)\bar{F}(t)}{\bar{F}(t)\tilde{A}(t)}\\
&=\frac{P(c_1Z_1+c_2Z_2>t)-P(c_1Z_1>t)-P(c_2Z_2>t)}{\tilde{A}(t)\bar{F}(t)}\\
&+\frac{A(t)}{\tilde{A}(t)}\left(\frac{\frac{P(c_1Z_1>t)}{\bar{F}(t)}-c_1^\alpha}{A(t)}+\frac{\frac{P(c_2Z_2>t)}{\bar{F}(t)}-c_2^\alpha}{A(t)}\right)\\
&=\frac{A(t)}{\tilde{A}(t)}(\mathrm{I}+\mathrm{III}+\mathrm{VI})+\frac{1}{t\tilde{A}(t)}(\mathrm{II}+\mathrm{IV})+\frac{\bar{F}(t)}{\tilde{A}(t)}\mathrm{V},
\end{aligned}
$$

where VI denotes the expression between brackets in the middle term. The second-order variation assumption (1.2) implies that V and VI converge as $t\to\infty$. Under the assumption of finite mean we prove that I, II, III, IV also converge. The argument, based on Lebesgue's dominated convergence theorem follows. Due to symmetry we consider only I and II. Define

$$
G_t(u)=\frac{\bar{F}((t/c_2)u)-u^{-\alpha}\bar{F}(t/c_2)}{\bar{F}(t/c_2)A(t/c_2)}.
$$

Since (1.2) holds locally uniformly, it follows that for any $\varepsilon>0$, there exists a t_0 such that, for $t>t_0$ and all $x\in[\frac{1}{2},1]$

$$
H(x)-\varepsilon\leqslant G_t(x)\leqslant H(x)+\varepsilon.
$$

The limits of integration in I assure that $\frac{1}{2} \leqslant 1 - u/t \leqslant 1$ and therefore $G_t(1 - u/t)$ can be bounded as follows:

$$\begin{aligned} 2^{\alpha}\frac{2^{-\rho}-1}{\rho} - \varepsilon &\leqslant (1-u/t)^{-\alpha}\frac{(1-u/t)^{\rho}-1}{\rho} - \varepsilon \\ &\leqslant G_t(1-u/t) \leqslant (1-u/t)^{-\alpha}\frac{(1-u/t)^{\rho}-1}{\rho} + \varepsilon \\ &< \varepsilon. \end{aligned}$$

The previous bound together with the fact that $G_t(1 - u/t) \to 0$ as $t \to \infty$ implies by Lebesgue's dominated convergence theorem that $\mathrm{I} \to 0$ as $t \to \infty$. For II notice that as $t \to \infty$

$$t((1-u/t)^{-\alpha} - 1) \to \alpha u$$

and that

$$0 \leqslant t((1-u/t)^{-\alpha} - 1) \leqslant 2(2^{\alpha} - 1)u$$

for $0 \leqslant u \leqslant t/2$ since $s \mapsto ((1-s)^{-\alpha} - 1)/s$ is non-decreasing on $(0, 1)$. Therefore,

$$\mathrm{II} = \frac{\bar{F}(t/c_2)}{\bar{F}(t)} \int_0^{t/2} t((1-u/t)^{-\alpha} - 1)\,\mathrm{d}F_3(u) \to \alpha c_1 c_2^{\alpha} \mu.$$

To summarize one has

$$\begin{aligned} &\lim_{t\to\infty} \frac{P(c_1 Z_1 + c_2 Z_2 > t) - (c_1^{\alpha} + c_2^{\alpha})\bar{F}(t)}{\bar{F}(t)\tilde{A}(t)} \\ &\quad = (H(c_1^{-1}) + H(c_2^{-1})) \lim_{t\to\infty} \frac{A(t)}{\tilde{A}(t)} \\ &\qquad + \alpha\mu(c_1 c_2^{\alpha} + c_1^{\alpha} c_2) \lim_{t\to\infty} \frac{1}{t\tilde{A}(t)} + 2^{\alpha+1} c_1^{\alpha} c_2^{\alpha} (2^{\alpha-1} - 1)\frac{\bar{F}(t)}{\tilde{A}(t)}. \end{aligned}$$

Making specific choices of $\tilde{A}(t)$ one recovers the different limit functions specified in items II and III of the theorem. □

4. Central limit theorem

The first-order regular variation of distribution tails has an exact probabilistic equivalent in the weak convergence of associated point processes to a Poisson process limit. This has been a very useful tool in studying heavy tailed phenomena which are quite complicated functionals of iid random variables. See Resnick (1986, 1987). We present a probabilistic equivalent of second-order regular variation which is then applied to discuss the equivalence of second-order regular variation and asymptotic normality of Hill's estimator.

The connection between second-order regular variation and the central limit theorem stems from the following invariance principle.

Proposition 4.1. *Suppose $\{Z_n, n \geqslant 1\}$ are iid non-negative random variables with common distribution F whose tail is regularly varying so that (1.1) holds. Let $b(t)$ be the quantile function defined by*

$$b(t) = \left(\frac{1}{1-F}\right)^{\leftarrow}(t).$$

Let the tail empirical measure be

$$\nu_n(\cdot) = \frac{1}{k}\sum_{i=1}^{n} \varepsilon_{Z_i/b(n/k)}(\cdot)$$

so that $k\nu_n(A)$ is the cardinality of $\{i : Z_i/b(n/k) \in A\}$. Then if $k = k(n)$ satisfies $k \to \infty$ and $k/n \to 0$, we have

$$W_n(x) := \sqrt{k}(\nu_n((x,\infty]) - E\nu_n((x,\infty])) \Rightarrow W(x^{-\alpha}) \tag{4.1}$$

in $D((0,\infty])$, where $\{W(t), t \geqslant 0\}$ is a standard Brownian motion.

See Mason (1988), Einmahl (1990, 1992), Csörgő et al. (1986), de Haan and Resnick (1993), Resnick and Stărică (1997a).

Note that

$$E\nu_n((x,\infty])) = \frac{n}{k}\left(1 - F\left(b\left(\frac{n}{k}\right)x\right)\right).$$

Here is a characterization of second-order regular variation based on the central limit theorem. The setup in Proposition 4.1 is still in force.

Theorem 4.2. *Suppose $1 - F \in RV_{-\alpha}$. We have that $1 - F$ is second-order regularly varying iff for some $\theta \in [0, 1)$ there exists a function $U \in RV_\theta$ such that $U(t) \to \infty$ as $t \to \infty$ and there exists a function $g(x)$, $x \geqslant 1$ not identically zero such that with $k = [U(n)]$ we have for each $x > 0$*

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{n} \varepsilon_{Z_i/b(n/k)}(x,\infty] - x^{-\alpha}\right) \Rightarrow W(x^{-\alpha}) + g(x) \tag{4.2}$$

in $D(0,\infty)$. In this case, $1 - F \in 2RV(-\alpha, \rho)$ with

$$\theta = \frac{2|\rho|}{\alpha + 2|\rho|}, \qquad \rho = \frac{-\alpha\theta/2}{1-\theta},$$

and

$$\frac{\frac{1-F(tx)}{1-F(t)} - x^{-\alpha}}{A(t)} \to g(x) \tag{4.3}$$

where the function A is specified as follows. Define

$$h(t) = \frac{t}{U(t)} \in RV_{1-\theta}, \tag{4.4}$$

where $0 < 1 - \theta \leqslant 1$,

$$\chi(t) = U \circ h^{\leftarrow}(t) \in RV_{\frac{\theta}{1-\theta}} \tag{4.5}$$

and

$$A(b(t)) = \frac{1}{\sqrt{\chi(t)}} \in RV_{\frac{-\theta/2}{1-\theta}}. \tag{4.6}$$

Proof. Suppose first that $1 - F \in 2RV(-\alpha, \rho)$ and (1.2) holds. Then

$$A(t) \in RV_{\rho}, \qquad b(t) \in RV_{1/\alpha}$$

so that

$$A(b(t)) \in RV_{\rho/\alpha}.$$

Define

$$V(x) = \frac{\sqrt{x}}{A(b(x))} \in RV_{\frac{1}{2}+\frac{|\rho|}{\alpha}}$$

so that

$$V^{\leftarrow} \in RV_{\frac{2\alpha}{\alpha+2|\rho|}},$$

and set

$$U(t) = \frac{t}{V^{\leftarrow}(\sqrt{t})} \in RV_{\frac{2|\rho|}{\alpha+2|\rho|}}.$$

We may set $\theta = 2|\rho|/(\alpha + 2|\rho|)$ and then $0 \leqslant \theta < 1$. Thus $U(t)/t \to 0$ as $t \to \infty$. Furthermore we claim that $U(t) \to \infty$. If $|\rho| > 0$, this claim is obvious. If not, note $U(t) \to \infty$ iff $t^2/V^{\leftarrow}(t) \to \infty$ iff $V^2(t)/t \to \infty$ iff $1/A^2(b(t)) \to \infty$ which follows from the fact that $A(t) \to 0$.

Now we may set $k = [U(n)]$ confident that $k \to \infty$ and $k/n \to 0$. Also we observe that

$$\begin{aligned}
\sqrt{k}A(b(n/k)) &= \sqrt{n}\sqrt{k/n}A(b(n/k)) \\
&= \sqrt{n}\frac{A(b(n/[U(n)]))}{\sqrt{n/[U(n)]}} \\
&\sim \sqrt{n}\left(V\left(\frac{n}{U(n)}\right)\right)^{-1} \\
&= \sqrt{n}(V(V^{\leftarrow}(\sqrt{n})))^{-1} \\
&\sim 1
\end{aligned}$$

as $n \to \infty$. So it follows that

$$\begin{aligned}
&\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{n} \varepsilon_{Z_i/b(n/k)}(x,\infty] - x^{-\alpha}\right) \\
&= \sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{n} \varepsilon_{Z_i/b(n/k)}(x,\infty] - \frac{n}{k}(1 - F(b(n/k)x))\right)
\end{aligned}$$

$$+\sqrt{k}\left(\frac{n}{k}(1-F(b(n/k)x))-x^{-\alpha}\right)$$

$$=W_n(x^{-\alpha})+\sqrt{k}A(b(n/k))\left(\frac{\frac{n}{k}(1-F(b(n/k)x))-x^{-\alpha}}{A(b(n/k))}\right)$$

$$\Rightarrow W(x^{-\alpha})+1H(x),$$

so the desired result holds with $g=H$.

Conversely, suppose

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{n}\varepsilon_{Z_i/b(n/k)}(x,\infty]-x^{-\alpha}\right)\Rightarrow W(x^{-\alpha})+g(x).$$

Referring to Proposition 4.1, we conclude that

$$\frac{\frac{n}{k}(1-F(b(n/k)x))-x^{-\alpha}}{1/\sqrt{k}}\to g(x).$$

Define

$$A(b(n/k))=\frac{1}{\sqrt{U(n)}},\qquad \chi(t)=\frac{1}{A^2(b(t))}.$$

So

$$\chi\left(\frac{n}{U(n)}\right)\sim U(n),$$

where $U\in RV_\theta$, $1>\theta\geqslant 0$ and

$$h(t):=\frac{t}{U(t)}\in RV_{1-\theta},\qquad h^{\leftarrow}(t)\in RV_{1/(1-\theta)}.$$

It follows that

$$\chi(t)\sim U(h^{\leftarrow}(t))\in RV_{\frac{\theta}{1-\theta}}$$

so

$$A(b(t))\sim\frac{1}{\sqrt{\chi(t)}}\in RV_{\frac{-\theta/2}{1-\theta}}.$$

Therefore

$$\frac{\frac{n}{k}(1-F(b(n/k)x))-x^{-\alpha}}{A(b(n/k))}\to g(x)$$

and a standard argument (Geluk and de Haan, 1987) allows the conclusion that

$$\frac{\frac{1-F(tx)}{1-F(t)}-x^{-\alpha}}{A(t)}\to g(x)$$

and with

$$\rho=\frac{-\alpha\theta/2}{1-\theta},$$

we get $1-F\in 2RV(-\alpha,\rho)$ as claimed. □

Remark. Examining Theorem 4.2, one sees that (4.2) in fact holds in $D(0,\infty]$ and with $g = H$.

We now discuss the relationship between asymptotic normality of Hill's estimator and second-order regular variation.

Theorem 4.3. *Suppose* $1-F \in RV_{-\alpha}$ *and that the von Mises condition holds: F has a density F' satisfying*

$$\lim_{x\to\infty} \frac{xF'(x)}{1-F(x)} = \alpha.$$

Then $1-F$ *is second-order regularly varying iff for some* $\theta \in [0,1)$ *there exists a function* $U \in RV_0$ *such that* $U(t)\to\infty$ *as* $t\to\infty$ *and there exist non-zero constants* c *and* $\sigma > 0$ *such that with* $k = [U(n)]$ *we have*

$$\sqrt{k}(H_{k,n} - \alpha^{-1}) \Rightarrow N(c,\sigma^2). \tag{4.7}$$

Proof. Suppose $1-F \in 2RV(-\alpha,\rho)$ so that (1.2) holds. From Theorem 4.2, there exists $U \in RV_0$, $U(t)\to\infty$ and with $k=[U(n)]$ we have

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{n} \varepsilon_{Z_i/b(n/k)}(x,\infty] - x^{-\alpha}\right) \Rightarrow W(x^{-\alpha}) + H(x) \tag{4.8}$$

in $D(0,\infty]$. Applying Vervaat's lemma (Vervaat, 1972) to the convergence

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{n} \varepsilon_{Z_i/b(n/k)}(x^{-1/\alpha},\infty] - x\right) \Rightarrow W(x) + H(x^{-1/\alpha})$$

in $D[0,\infty)$, we get on taking inverses

$$\sqrt{k}\left(\left(\frac{1}{k}\sum_{i=1}^{n} \varepsilon_{Z_i/b(n/k)}((\cdot)^{-1/\alpha},\infty]\right)^{\leftarrow}(x) - x\right) \Rightarrow -(W(x) + H(x^{-1/\alpha})) \tag{4.9}$$

and thus

$$\sqrt{k}\left(\left(\frac{Z_{([kx])}}{b(n/k)}\right)^{-\alpha} - x\right) \Rightarrow -(W(x) + H(x^{-1/\alpha})) \tag{4.10}$$

in $D[0,\infty)$ and

$$\frac{Z_{([k])}}{b(n/k)} \Rightarrow 1 \tag{4.11}$$

in $\mathbb{R}_+$. In fact, (4.8), (4.10) and (4.11) hold jointly in $D(0,\infty] \times D[0,\infty) \times \mathbb{R}_+$. Applying composition of the third and the first components of this joint convergence yields

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{n} \varepsilon_{Z_i/Z_{(k+1)}}(x,\infty] - \left(\frac{Z_{(k+1)}x}{b(n/k)}\right)^{-\alpha}, \left(\frac{Z_{(k+1)}x}{b(n/k)}\right)^{-\alpha} - x^{-\alpha}\right)$$
$$\Rightarrow (W(x^{-\alpha}) + H(x), -(W(1) + H(1))x^{-\alpha})$$

in $D(0,\infty]\times\mathbb{R}$. Remembering that $H(1)=0$, we get by addition

$$\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{n}\varepsilon_{Z_i/Z_{(k+1)}}(x,\infty]-x^{-\alpha}\right)$$

$$=\sqrt{k}\left(\frac{1}{k}\sum_{i=1}^{n}\varepsilon_{Z_i/Z_{(k+1)}}(x,\infty]-\left(\frac{Z_{(k+1)}x}{b(n/k)}\right)^{-\alpha}\right)+\sqrt{k}\left(\left(\frac{Z_{(k+1)}x}{b(n/k)}\right)^{-\alpha}-x^{-\alpha}\right)$$

$$\Rightarrow W(x^{-\alpha})+H(x)-x^{-\alpha}W(1)$$

and so a continuous mapping argument (map $x(\cdot)\mapsto\int_1^\infty x(s)/s\,\mathrm{d}s$) yields (cf. Resnick and Stărică, 1997a)

$$\sqrt{k}\left(H_{k,n}-\frac{1}{\alpha}\right)\Rightarrow\int_1^\infty W(x^{-\alpha})\frac{\mathrm{d}x}{x}+\int_1^\infty H(x)\frac{\mathrm{d}x}{x}-\frac{W(1)}{\alpha}.$$

Note

$$\int_1^\infty H(x)\frac{\mathrm{d}x}{x}\neq 0$$

and so the limit is normal with non-zero mean as required.

Conversely, suppose (4.7) holds. From Davis and Resnick (1984) or Csörgő and Mason (1985) we have

$$\sqrt{k}\left(H_{k,n}-\frac{n}{k}\int_{b(n/k)}^\infty(1-F(s))\frac{\mathrm{d}s}{s}\right)\Rightarrow N\left(0,\frac{1}{\alpha^2}\right)$$

and the convergence to types theorem yields

$$\sqrt{k}\left(\frac{n}{k}\int_{b(n/k)}^\infty(1-F(s))\frac{\mathrm{d}s}{s}-\frac{1}{\alpha}\right)\to c\neq 0$$

from which it follows that

$$\frac{\int_t^\infty\frac{1-F(s)}{1-F(t)}\frac{\mathrm{d}s}{s}-\frac{1}{\alpha}}{A(t)}\to c$$

where $A(t)$ is defined as in (4.6). Second-order regular variation then follows from the following proposition and the proof of Theorem 4.3 is complete. □

The following result is the second-order version of Karamata's theorem. It is similar to the second remark following de Haan's (1996) Theorem 1.

Proposition 4.4. *Suppose F is a distribution concentrating on $[0,\infty)$. Then*

$$1-F\in 2RV(-\alpha,\rho)$$

iff there exists a function $A(t)$ satisfying $A>0$, $A(t)\to 0$ and $A\in RV_\rho$ for some $\rho\leqslant 0$ such that

$$\lim_{t\to\infty}\frac{\int_t^\infty\frac{1-F(x)}{1-F(t)}\frac{\mathrm{d}x}{x}-\frac{1}{\alpha}}{A(t)}\to c\neq 0,\tag{4.12}$$

where c is a non-zero constant.

Proof. Begin by assuming (4.12) and for specificity suppose that $c>0$. Then there exists a function $V \in RV_\rho$ such that

$$\frac{\int_t^\infty (1-F(x))\frac{dx}{x}}{1-F(t)} = \frac{1}{\alpha} + V(t).$$

Thus

$$-\left(\log \int_t^\infty (1-F(x))\frac{dx}{x}\right)' = \frac{(1-F(t))/t}{\int_t^\infty (1-F(x))\frac{dx}{x}} = \frac{t^{-1}}{\alpha^{-1}+V(t)}.$$

So integrating from 1 to x gives for some $k>0$

$$\int_x^\infty (1-F(s))\frac{ds}{s} = k \exp\left\{-\int_1^x \frac{1}{\alpha^{-1}+V(s)}\frac{ds}{s}\right\}$$

and therefore we get a representation for $1-F$, namely,

$$1-F(x) = \frac{k}{\alpha^{-1}+V(x)} \exp\left\{-\int_1^x \frac{1}{\alpha^{-1}+V(s)}\frac{ds}{s}\right\}.$$

We may now use this representation to prove the second-order regular variation. We have for $x>1$

$$\frac{\frac{1-F(tx)}{1-F(t)} - x^{-\alpha}}{V(t)} = \frac{\frac{\alpha^{-1}+V(t)}{\alpha^{-1}+V(tx)} \exp\{-\int_1^x \frac{1}{\alpha^{-1}+V(ts)}\frac{ds}{s}\} - x^{-\alpha}}{V(t)} \tag{4.13}$$

and writing

$$\frac{\alpha^{-1}+V(t)}{\alpha^{-1}+V(tx)} = 1 + \frac{V(t)-V(tx)}{\alpha^{-1}+V(tx)}$$

we get the second-order ratio in (4.13) equal to

$$= x^{-\alpha}\left[\frac{\exp[\alpha\int_1^x \frac{\alpha V(ts)}{1+V(ts)\alpha}\frac{ds}{s}] - 1}{V(t)} + \frac{V(t)-V(tx)}{V(t)(\alpha^{-1}+V(tx))}\exp\left[\alpha\int_1^x \frac{\alpha V(ts)}{1+V(ts)\alpha}\frac{ds}{s}\right]\right]$$

$$= x^{-\alpha}[\mathrm{I} + \mathrm{II}].$$

Since $V(t) \to 0$,

$$\int_1^x \frac{\alpha V(ts)}{1+\alpha V(ts)}\frac{ds}{s} \to 0$$

and therefore as $t \to \infty$

$$\mathrm{I} \sim \frac{\alpha\int_1^x \frac{\alpha V(ts)}{1+\alpha V(ts)}\frac{ds}{s}}{V(t)} \to \alpha^2 \int_1^x s^{\rho-1}\,ds = \alpha^2\left(\frac{x^\rho - 1}{\rho}\right),$$

and

$$\mathrm{II} \sim \alpha\left(1 - \frac{V(tx)}{V(t)}\right) \to \alpha(1-x^\rho).$$

So the limit of the second-order ratio is of the form

$$k'x^{-\alpha}\left(\frac{x^{\rho}-1}{\rho}\right)$$

as desired.

Conversely, suppose $1-F \in 2RV(-\alpha,\rho)$ so that (1.2) holds. Write

$$\frac{\int_t^{\infty} \frac{1-F(x)}{1-F(t)}\frac{\mathrm{d}x}{x} - \frac{1}{\alpha}}{A(t)} = \int_1^{\infty} \frac{\left(\frac{1-F(ts)}{1-F(t)} - s^{-\alpha}\right)}{A(t)}\frac{\mathrm{d}s}{s}.$$

The result follows by applying dominated convergence to the integral on the right. For $\rho = 0$, this step is justified by Theorem 1.20(ii) of Geluk and de Haan (1987) and for $\rho < 0$ the justification is Theorem 1.8 of Geluk and de Haan. □

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