Strongly clean triangular matrix rings over local rings

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Abstract

An element of a ring is called strongly clean if it can be written as the sum of a unit and an idempotent that commute. A ring is called strongly clean if each of its elements is strongly clean. In this paper, we investigate conditions on a local ring $R$ that imply that $T_n(R)$ is a strongly clean ring. It is shown that this is the case for commutative local rings $R$, as well as for a host of other classes of local rings. An example of a local ring $A$ for which $T_2(A)$ is not strongly clean is also given.

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1. Introduction

An element in a ring is clean if it is the sum of a unit and an idempotent. A ring is clean if each of its elements is clean. It is clear that a product of rings is clean if and only if each component is clean and that any homomorphic image of a clean ring is clean. The notion was introduced by Nicholson in [14] as a sufficient condition for a ring to have the exchange property. Camillo and Yu [3] further proved that for rings

$$\text{semiperfect} \implies \text{clean} \implies \text{exchange}$$

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with none of the implications reversible. A ring \( R \) with radical \( J \) is clean if and only if \( R/J \) is clean and idempotents lift modulo \( J(R) \) \([3,14]\). Camillo and Khurana \([1]\) proved that for rings

\[
\text{unit regular } \implies \text{ clean}
\]

although this implication does not hold at an element-wise level (e.g. \([9]\)).

Camillo, Khurana, Lam, Nicholson and Zhou \([2]\) define a clean module as one whose endomorphism ring is clean. Prior to this, Han and Nicholson \([8]\) proved that if \( M_1 \) and \( M_2 \) are clean modules then \( M_1 \oplus M_2 \) is clean. \( \tilde{\text{O}} \) Searcóid \([20]\) showed that a vector space is a clean module (slightly generalized in \([16]\)), and this result is capped by Camillo, Khurana, Lam, Nicholson and Zhou who proved in \([2]\) that continuous modules are clean. Thus clean rings and modules abound, and the condition fits pleasantly with many well-established notions.

Nicholson \([15]\) also defined the concept of strong cleanness. An element of a ring is strongly clean if it is the sum of a unit and an idempotent which commute. Again, a ring is called strongly clean if each of its elements is strongly clean, and a module is called strongly clean if its endomorphism ring is strongly clean. Local rings are strongly clean; conversely it follows from Nicholson’s characterization of exchange rings \([14]\) and the basic properties of local rings (see, for example, \([10, \text{Section 19}]\)), that an exchange ring with only trivial idempotents must be local. This motivates our study of local rings; they are precisely the clean rings with only trivial idempotents and, as such, provide a natural starting point for our investigation. Nicholson proved that strongly \( \pi \)-regular rings are strongly clean. (A ring \( R \) is strongly \( \pi \)-regular if all chains of the forms \( R \supseteq aR \supseteq a^2R \supseteq \cdots \) and \( R \supseteq Ra \supseteq Ra^2 \supseteq \cdots \) terminate.) Basic results on abelian regular rings (see \([7, \text{Chapter 3}]\)) and strongly \( \pi \)-regular rings (see \([4]\) or \([11, \text{Exercise 23.5}]\)) yield that abelian regular rings and right (or left) perfect rings are strongly clean. In particular right (or left) artinian rings are strongly clean.

The main goal of this paper is to extend the known classes of strongly clean rings by examining triangular matrix rings and incidence rings (of posets) over local rings. In Section 2 we prove a lemma that will be used throughout the paper and quickly prove that all upper-triangular matrices over “bleached” local rings (of which commutative local rings are examples) are strongly clean. In Section 3 we formulate more subtle conditions on a local ring \( R \) that suffice to imply that \( \mathbb{T}_n(R) \) is strongly clean. We also prove a converse to this theorem under the condition that \( R \) is an \( h \)-ring (defined in Section 3). In Section 4 we give examples that demonstrate the non-triviality of the previous results. A natural generalization of the ring of upper triangular matrices is the incidence ring of a locally finite partially ordered set, and in Section 5, we extend some of our results to incidence rings.

Throughout this paper, \( J(R) \), \( U(R) \), and \( Z(R) \) will denote, respectively, the Jacobson radical, the group of units of \( R \), and the center of \( R \).

2. Basic results for triangular matrix rings

In this section we will prove a lemma from which we will quickly deduce the strong cleanness of \( \mathbb{T}_n(R) \) for a large class of local rings \( R \). The lemma will also enable us to study the strong cleanness of both triangular matrix rings and incidence rings over local rings in more detail.

Throughout this section \( \mathbb{T}_n(R) \) will denote the ring of \( n \times n \) upper triangular matrices over \( R \) and, given a matrix \( A \), \( A_{ij} \) will denote the \((i, j)\)th entry of \( A \).

The following elementary lemma will be used repeatedly.
Lemma 1. Let $E, A, B \in T_n(R)$.

1. If $E^2 = E$ then $(E_{ii})^2 = E_{ii}$ for $i = 1, \ldots, n$.
2. $A \in T_n(R)$ is invertible if and only if $A_{ii} \in U(R)$ for $i = 1, \ldots, n$.
3. $B \in J(T_n(R))$ if and only if $B_{ii} \in J(R)$ for $i = 1, \ldots, n$.

Proof. These are well-known and straightforward calculations. 

From this, the following is also elementary:

Lemma 2. Let $R$ be a ring.

1. $T_n(R)$ is clean if and only if $R$ is clean.
2. If $T_n(R)$ is strongly clean, then $R$ is strongly clean.

Note that for a local ring $R$, $R = J(R) \cup U(R)$, and $R$ has only trivial idempotents. Here are some useful definitions.

Definition 3. Given $e^2 = e \in R$ and $a \in R$, $a$ is $e$-clean if $a - e$ is a unit and strongly $e$-clean if, in addition, $a$ and $e$ commute.

It should be noted that this definition is different from that given in [9].

Lemma 4. Let $a \in R$ and $e^2 = e \in R$. Then $a$ is (strongly) $e$-clean if and only if $1 - a$ is (strongly) $(1 - e)$-clean.

Proof. This is an easy calculation.

Lemma 5 (Block multiplication). Let $A, E \in T_n(R)$. If $A = \begin{pmatrix} A_1 & C \\ B & A_2 \end{pmatrix}$ and $E = \begin{pmatrix} E_1 & Z \\ F & E_2 \end{pmatrix}$ are compatible block decompositions, then

1. $E^2 = E$ if and only if $E_1^2 = E_1$, $E_2^2 = E_2$, and $E_1 Z + Z E_2 = Z$.
2. $AE = EA$ if and only if $A_1 E_1 = E_1 A_1$, $A_2 E_2 = E_2 A_2$, and $A_1 Z + C E_2 = E_1 C + Z A_2$.

Proof. This is another easy calculation.

The following lemma will be used repeatedly.

Lemma 6 (Workhorse lemma). Let $R$ be a local ring, $n \geq 2$ and $A, E \in T_n(R)$. Suppose that for all $(i, j) \neq (1, n)$, $(E^2)_{ij} = E_{ij}$ and $(AE - EA)_{ij} = 0$. Suppose that

$$A = \begin{pmatrix} a & \alpha & c \\ B & \beta \\ b \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} e & \gamma & z \\ F & \delta \\ f \end{pmatrix}$$

where $B, F \in T_{n-2}(R)$, $a, b, c, e, f, z \in R$, $\alpha, \gamma \in M_{1,n-2}(R)$ and $\beta, \delta \in M_{n-2,1}(R)$. Note that $e$ and $f$ are idempotents—hence either 0 or 1. Then
(1) If \( e = f = 1 \), then \( E^2 = E \) if and only if \( z = -\gamma \delta \), and in this case, \( AE = EA \).

(2) If \( e = f = 0 \), then \( E^2 = E \) if and only if \( z = \gamma \delta \), and in this case, \( AE = EA \).

(3) If \( e = 1 \) and \( f = 0 \), then \( E^2 = E \). Further, \( AE = EA \) if and only if \( z \) satisfies the equation

\[ az - zb = \gamma \beta - \alpha \delta + c. \]

(4) If \( e = 0 \) and \( f = 1 \), then \( E^2 = E \). Further, \( AE = EA \) if and only if \( z \) satisfies the equation

\[ az - zb = \gamma \beta - \alpha \delta - c. \]

**Proof.** We shall prove each statement in turn.

(1) Here \( e = f = 1 \). It is immediate that \( E^2 = E \) if and only if

\[ z = -\gamma \delta. \quad (1) \]

The hypotheses applied to the given blocks imply that

\[ \gamma F = 0, \quad (2) \]
\[ F\delta = 0, \quad (3) \]
\[ a\gamma + \alpha F = \alpha + \gamma B, \quad (4) \]
\[ B\delta + \beta = F\beta + \delta b. \quad (5) \]

From this we calculate that

\[
(EA)_{1n} = c + \gamma \beta - \gamma \delta b \quad \text{from (1)}
= c + \gamma (F\beta - B\delta) \quad \text{from (5)}
= c - \gamma B\delta \quad \text{from (2)},
\]

and that

\[
(AE)_{1n} = c - a\gamma \delta + \alpha \delta \quad \text{from (1)}
= c + (\alpha F - \gamma B)\delta \quad \text{from (4)}
= c - \gamma B\delta \quad \text{from (3)},
\]

which is all we need to ensure that \( EA = AE \).

(2) This time \( e = f = 0 \). Performing a similar calculation (or applying the previous argument to \( I - E \)) gives \( E^2 = E \) if and only if \( z = \gamma \delta \), in which case we again get \( AE = EA \) for free.

(3) In this case \( e = 1 \) and \( f = 0 \). The hypotheses give \( \gamma F = 0 \) and \( F\delta = \delta \), from which we deduce that

\[ \gamma \delta = \gamma F\delta = 0. \]

Thus, for any \( z \in R \),

\[
(E^2)_{1n} = z + \gamma \delta = z = E_{1n},
\]

and so \( E^2 = E \). We also see that \( AE = EA \) if and only if

\[ az + \alpha \delta = c + \gamma \beta + zb, \]
or equivalently, if and only if
\[ az - zb = \gamma \beta - \alpha \delta + c. \]
The latter is precisely the condition we seek.

(4) Here, \( e = 0 \) and \( f = 1 \); so we may apply the above argument to \( I - E \). Again any value of \( z \) gives \( E^2 = E \). This time, \( AE = EA \) if and only if
\[ az - zb = \gamma \beta - \alpha \delta - c, \]
which again is what we want. \( \square \)

Lemma 6 allows us to characterize the idempotent matrices in \( T_n(R) \) that commute with a given \( n \times n \) upper-triangular matrix. For \( a \in R \), \( l_a \) and \( r_a \) will denote, respectively, the abelian group endomorphisms of \( R \) given by left and right multiplication by \( a \).

**Lemma 7.** Let \( R \) be a local ring and suppose that \( A \in T_n(R) \). Write \( A \) as \( (a_{ij}) \). Then for any set \( \{e_{ii}\}_{i=1}^n \) of idempotents in \( R \), such that \( e_{ii} = e_{jj} \) whenever \( l_{a_{ii}} - r_{a_{jj}} \) is not a surjective abelian group endomorphism of \( R \), there exists an idempotent \( E \in T_n(R) \) such that \( AE = EA \) and \( E_{ii} = e_{ii} \) for every \( i \in \{1, \ldots, n\} \).

**Proof.** The proof will proceed by induction on \( n \), making extensive use of Lemma 6. The \( n = 1 \) case is easily proved. We will construct the idempotent \( E = (e_{ij}) \) inductively, setting \( E_{ii} = e_{ii} \). Now assume that for some \( k \geq 0 \), we have defined \( e_{ij} \) for all \( i \leq j \leq i + k \) so that for all \( i = 1, \ldots, n - k \),
\[
\begin{pmatrix}
a_{ii} & \cdots & a_{i,i+k} \\
& \ddots & \vdots \\
a_{i+k,i+k}
\end{pmatrix}
\]
commutes with
\[
\begin{pmatrix}
e_{ii} & \cdots & e_{i,i+k} \\
& \ddots & \vdots \\
e_{i+k,i+k}
\end{pmatrix},
\]
and the latter matrix is idempotent.

By induction, for each \( i = 1, \ldots, n - k - 1 \), the hypotheses of Lemma 6 apply to
\[
A' = \begin{pmatrix} a_{ii} & \cdots & a_{i,i+k+1} \\
& \ddots & \vdots \\
a_{i+k+1,i+k+1} \end{pmatrix}\quad \text{and} \quad E' = \begin{pmatrix} e_{ii} & \cdots & e_{i,i+k+1} \\
& \ddots & \vdots \\
e_{i+k+1,i+k+1} \end{pmatrix}.
\]
If \( e_{ii} = e_{i+k+1,i+k+1} \), then by Lemma 6, we can define \( e_{i,i+k+1} \) so that \( A' \) commutes with \( E' \), and \( E' \) is idempotent. If \( e_{ii} \neq e_{i+k+1,i+k+1} \), then by hypothesis \( l_{a_{ii}} - r_{a_{i+k+1,i+k+1}} \) is a surjective
map from $R$ to $R$, and hence Lemma 6 again ensures that we can define $e_{i,i+k+1}$ so that $A'$ commutes with $E'$, and $E'$ is idempotent. This completes the inductive step.

In particular, when $k = n - 1$, we deduce that $A$ commutes with $E$ and that $E_{ii} = e_{ii}$. □

The equational conditions in Lemma 6 and their subsequent use in Lemma 7 motivate the following.

**Definition 8.** A local ring $R$ is **bleached** if, for any $j \in J(R)$ and any $u \in U(R)$, the abelian group endomorphisms

$$l_u - r_j \quad \text{and} \quad l_j - r_u$$

of $R$ are surjective.

We can now prove the strong cleanness of certain triangular matrix rings. Indeed, the following theorem is now easy.

**Theorem 9.** Let $R$ be a bleached local ring. Then $T_n(R)$ is strongly clean.

**Proof.** Let $A \in T_n(R)$. Define the collection $\{e_{ii}\}_{i=1}^n$ of idempotents of $R$ by $e_{ii} = 1$ if $A_{ii} \in J(R)$ and $e_{ii} = 0$ if $A_{ii} \in U(R)$. Since $R$ is bleached, the collection $\{e_{ii}\}$ satisfies the hypotheses of Lemma 7, and we may therefore construct an idempotent $E \in T_n(R)$ such that $AE = EA$ and $E_{ii} = e_{ii}$ for every $i \in \{1, \ldots, n\}$. In addition, Lemma 1 shows that $A - E$ is invertible, since we have chosen the elements $E_{ii}$ so that $(A - E)_{ii} \in U(R)$ for every $i$. Thus $A$ is strongly $E$-clean. □

**Corollary 10.** Let $R$ be a commutative local ring. Then $T_n(R)$ is strongly clean.

**Proof.** All commutative local rings are bleached (see Example 13 below). □

**Remark 11.** Over a commutative local ring $R$, the maps $l_u - r_j$ and $l_j - r_u$ in Definition 8 are injective as well as surjective. Thus, given $A \in T_n(R)$, there is a unique idempotent $E$ that has the diagonal constructed in Theorem 9 and such that $AE = EA$. A ring $R$ is uniquely (strongly) clean if for all $a \in R$ there is a unique $e^2 = e \in R$ such that $a$ is (strongly) $e$-clean (see [17]). If $R$ is uniquely clean and $A - E$ is invertible then $E$ must have the diagonal constructed in Theorem 9. The following theorem of Wang and Chen is now immediate.

**Corollary 12.** [19] Let $R$ be a uniquely clean commutative local ring. Then $T_n(R)$ is a uniquely strongly clean ring.

**Example 13.** Below are some examples of bleached local rings. Such rings will be called uniquely bleached if the appropriate maps are injective as well as surjective.

1. Every commutative local ring is uniquely bleached.
2. Division rings are uniquely bleached.
(3) Any local ring $R$ for which $J(R)$ is a nil ideal is a uniquely bleached ring. In fact, if $u$ is a unit and $j^k = 0$, then the maps

$$
\varphi = l_{u-1} + l_{u-2}r_j + \cdots + l_{u-k}r_{jk-1}
$$

and

$$
\psi = -(r_{u-1} + r_{u-2}l_j + \cdots + r_{u-k}l_{jk-1})
$$

are the inverses, respectively, of $l_u - r_j$ and $l_j - r_u$.

(4) More generally, if $R$ is a local ring for which some power of each element of $J(R)$ is central in $R$, then $R$ is uniquely bleached. Using $\varphi$ and $\psi$ from the previous example, if $u$ is a unit and $j^k$ is central, then $(l_u - r_j)\varphi = l_1 - l_{u-k}r_{jk} = l_{1-u-k}j^k$ and $(l_j - r_u)\psi = r_1 - r_{u-k}l_{jk} = l_{1-u-k}j^k$ each represent multiplication by a unit of $R$ and, as such, are both clearly invertible.

(5) Similarly, if some power of each element of $U(R)$ is central in $R$, then $R$ is uniquely bleached.

(6) Power series rings over bleached local rings are bleached local rings. The verification proceeds by a straightforward induction.

(7) Similarly, any skew power series ring $R[x; \sigma]$, where $R$ is a bleached local ring and $\sigma : R \to R$ is a ring automorphism, is a bleached local ring.

For a further generalization of parts (3)–(5) of the previous example, see part (8) of Proposition 16. We will examine skew power series in more detail in Section 4, when we construct examples of local rings that are not bleached.

### 3. Necessary and sufficient conditions

In the proof of Theorem 9, we constructed the main diagonal of the idempotent $E$ using the rather crude method of defining $E_{ii}$ to be 1 if $A_{ii} \in J(R)$ and 0 otherwise. A closer study leads to the following observation.

Let $R$ be a local ring, and let $r$ be an element of $R$. If $r \in J(R)$, then 1 is the only idempotent of $R$ with respect to which $r$ is clean. On the other hand, if $r \in 1 - J(R)$, then 0 is the only idempotent of $R$ with respect to which $r$ is clean. For all other $r$ (if any), both $r$ and $1 - r$ are units in $R$. In fact, this observation is exactly what underlies the following observation of Nicholson about $2 \times 2$ upper-triangular matrices.

**Theorem 14.** [15] If $A$ and $B$ are local rings and $V = AV_B$ is an $(A, B)$-bimodule, then the following are equivalent.

1. The triangular matrix ring $(A\ V_B)$ is strongly clean.
2. If $1 - a \in J(A)$, $b \in J(B)$ and $v \in V$, then there exists $x \in V$ such that $ax - xb = v$.

Since $AV_B$ is a bimodule in Theorem 14, $l_a$ and $r_b$ define abelian group endomorphisms of $V$. Condition (2) of Theorem 14 can be restated as:

1. If $1 - a \in J(A)$ and $b \in J(B)$, then $l_a - r_b$ is a surjective abelian group endomorphism of $V$. 
Maps of the form $l_a - r_b$ play a central role in Lemmas 6, 7 and Theorem 14. The following definition is then natural.

**Definition 15.** Let $R$ be a local ring, and let $A \subseteq R$. Define the set $\text{Bl}(A)$ as follows:

$$\text{Bl}(A) = \{ b \in R \mid l_b - r_a \text{ and } l_a - r_b \text{ are surjective on } R \forall a \in A \}.$$ 

In the case where the set $A$ consists only of a single element $a$, we will write $\text{Bl}(a)$ instead of $\text{Bl}([a])$.

We first record the basic properties of the operator $\text{Bl}$ in the following proposition.

**Proposition 16.** Let $R$ be a local ring, and let $A$ be a subset of $R$. Then the following are true:

1. $A \subseteq \text{Bl}(B)$ if and only if $B \subseteq \text{Bl}(A)$.
2. If $A \subseteq B$, then $\text{Bl}(B) \subseteq \text{Bl}(A)$.
3. $A \subseteq \text{Bl}^2(A)$.
4. $\text{Bl}^3(A) = \text{Bl}(A)$.
5. For any central element $c \in R$, $\text{Bl}(c - A) = c - \text{Bl}(A)$. In particular, $\text{Bl}(1 - A) = 1 - \text{Bl}(A)$.
6. For any $u \in U(R)$, $\text{Bl}(u^{-1}A) = \text{Bl}(A) = u^{-1}(\text{Bl}(A))u$.
7. For any central unit $u \in R$, $\text{Bl}(uA) = u^{-1}\text{Bl}(A)$. In particular, $\text{Bl}(-A) = -\text{Bl}(A)$.
8. If $p(t) \in Z(R)[t]$, then for any $a, b \in R$, $p(a) \in \text{Bl}(p(b)) \Rightarrow a \in \text{Bl}(b)$.

**Proof.** Statements (1), (2), and (3) are straightforward consequences of Definition 15.

To prove (4), observe that $\text{Bl}(A) \subseteq \text{Bl}^2(\text{Bl}(A)) = \text{Bl}^3(A)$ by (3) and that $\text{Bl}^3(A) = \text{Bl}(\text{Bl}^2(A)) \subseteq \text{Bl}(A)$ by (3) and (2). As an aside, notice that we have shown that $R$ with $\text{Bl}$ is a Galois connection.

For the proof of (5), let $c$ be central, and let $b \in \text{Bl}(c - A)$. Then, for every $a \in A$,

$$l_{c-b} - r_a = -(l_{b-c} - r_a) = -(l_b - r_{c-a})$$

and

$$l_a - r_{c-b} = -(l_{a-c} - r_b) = -(l_{c-a} - r_b).$$

This shows that $b \in \text{Bl}(c - A)$ if and only if $c - b \in \text{Bl}(A)$, which is enough to prove (5).

To prove (6), suppose that $a, b \in R$ and that $u \in R$ is a unit. Then $l_b - r_a$ is surjective if and only if $l_{u^{-1}} \circ (l_b - r_a) \circ l_u = l_{u^{-1}bu} - r_a$ is surjective. Similarly, $l_a - r_b$ is surjective if and only if $r_u \circ (l_a - r_b) \circ r_{u^{-1}} = l_a - r_{u^{-1}bu}$ is surjective. Then, taking $b \in A$, we see that $a \in \text{Bl}(A)$ if and only if $b \in u^{-1}\text{Bl}(A)u$. On the other hand, taking $a \in A$, we see that $b \in \text{Bl}(A)$ if and only if $b \in \text{Bl}(u^{-1}A)u$.

To prove (7), note that, for a central unit $u$, $l_b - r_{ua}$ is surjective if and only if $l_{u^{-1}}(l_b - r_{ua}) = l_{u^{-1}b} - r_a$ is surjective.

To prove (8), let $\varphi_n = \sum_{i=0}^{n-1} l_a^{n-i} r_{b_i} \in \text{End} Z(R)$, and for $c \in Z(R)$, we will denote by $c$ the map $l_c = r_c \in Z(\text{End} Z(R))$. Note that

$$\varphi_n \circ (l_a - r_b) = l_a^n - r_b^n = (l_a - r_b) \circ \varphi_n.$$
In particular, \( \varphi_0 = 0 \) and \( \varphi_0 \circ (l_a - r_b) = l_1 - r_1 \). Write \( p(t) = c_n t^n + \cdots + c_1 t + c_0 \), where \( c_i \in Z(R) \). Then,

\[
l_p(a) - r_p(b) = (l_a - r_b) \circ (c_n \varphi_n + \cdots + c_1 \varphi_1 + c_0 \varphi_0).
\]

Therefore, if \( l_p(a) - r_p(b) \) is surjective, the same is true for \( l_a - r_b \) (if a composition of functions is surjective, then the last one applied is surjective). Interchanging the roles of \( a \) and \( b \) in the computation yields the result. □

We thank George Bergman for kindly pointing out the proof of (6) in Proposition 16.

Before continuing, one should remark that the facts contained in Theorem 9 and Theorem 14 can be recast, as follows below, using the language of Definition 15. For Theorem 14, we have specialized to the case \( A = B = V = R \) and also made use of the easily verified fact that \( l_a - r_b \) is surjective for every \( a \in 1 - J(R) \) and every \( b \in J(R) \) if and only if \( J(R) \subseteq Bl(1 - J(R)) \).

**Theorem 17** (Restatement of Theorem 9). Let \( R \) be a local ring for which \( U(R) = Bl(J(R)) \). Then \( \mathbb{T}_n(R) \) is strongly clean.

**Theorem 18** (Restatement of Theorem 14). Let \( R \) be a local ring. Then \( \mathbb{T}_2(R) \) is strongly clean if and only if \( J(R) \subseteq Bl(1 - J(R)) \), if and only if \( 1 - J(R) \subseteq Bl(J) \).

In the proof of the Theorem 9, we showed that an upper triangular matrix \( A \) over a bleached local ring \( R \) is strongly clean by explicitly constructing the corresponding idempotent \( E \) one diagonal at a time. Looking at the proofs of Lemma 7 and Theorem 9, we see that the choice of \( E_{ii} \) is governed both by the need to make \( A_{ii} - E_{ii} \) invertible and the need to have \( E_{ii} = E_{jj} \) if \( A_{ii} \notin Bl(A_{jj}) \).

With this in mind, the following sets are constructed to generalize the role played by \( J(R) \) in Theorem 9.

**Definition 19.** Let \( R \) be a local ring. Define the sets \( J_i(R) \) recursively as follows:

\[
J_0(R) = \{0\}, \quad J_{i+1}(R) = Bl(1 - J_i(R)) \setminus Bl(J_i(R)).
\]

In what follows, when the local ring \( R \) is understood, we shall write \( J_i \) instead of \( J_i(R) \).

Notice that \( J_1 = Bl(1) \setminus Bl(0) = J(R) \). The following proposition illustrates which properties of \( J(R) \) are shared by the sets \( J_i \).

**Proposition 20.** Let \( R \) be a local ring. Then the following are true for every integer \( n \geq 1 \):

1. \( J_n \cap (1 - J_n) = \emptyset \).
2. If \( J_{n-1} \subseteq J_n \), then \( J_n \cap Bl(J_n) = \emptyset \).
3. If \( J_{n-1} \subseteq J_n \), then the following are equivalent:
   a. \( J_n \subseteq Bl(1 - J_n) \),
   b. \( J_n \subseteq J_{n+1} \),
   c. \( 1 - J_n \subseteq Bl(J_n) \),
   d. \( 1 - J_n \subseteq (1 - J_{n+1}) \).
(4) If \( J_{n-1} \subseteq J_n \) and \( \text{Bl}(J_n) \cup \text{Bl}(1 - J_n) = R \), then \( J_n \subseteq \text{Bl}(1 - J_n) \).
(5) If \( J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n \) and \( \text{Bl}(J_n) \cup \text{Bl}(1 - J_n) = R \), then \( \text{Bl}(J_i) \cup \text{Bl}(1 - J_i) = R \) for every \( i \leq n \).

**Proof.** We shall prove each statement in turn.

1. By part (5) of Proposition 16, \( 1 - J_n = \text{Bl}(J_{n-1}) \setminus \text{Bl}(1 - J_{n-1}) \). By Definition 19, \( J_n = \text{Bl}(1 - J_{n-1}) \setminus \text{Bl}(J_{n-1}) \). This shows clearly that \( J_n \cap (1 - J_n) = \emptyset \).

2. Suppose that \( a \in J_n \). By definition of \( J_n \), \( a \in \text{Bl}(1 - J_{n-1}) \) but \( a \notin \text{Bl}(J_{n-1}) \). Since \( J_{n-1} \subseteq J_n \), part (2) of Proposition 16 shows that \( \text{Bl}(J_n) \subseteq \text{Bl}(J_{n-1}) \). This means that \( a \notin \text{Bl}(J_n) \), and thus that \( J_n \cap \text{Bl}(J_n) = \emptyset \).

3. We will show that (a) is equivalent to (b); the equivalence of these conditions to (c) and (d) follows by part (5) of Proposition 16. To show (a) implies (b), let \( a \in J_n \). By hypothesis, \( a \in \text{Bl}(1 - J_n) \), and by part (2), \( a \notin \text{Bl}(J_n) \). Thus \( a \in \text{Bl}(1 - J_n) \setminus \text{Bl}(J_n) = J_{n+1} \). The converse holds by definition of the set \( J_{n+1} \).

4. Suppose that \( a \in J_n \). We already know from part (2) that \( J_n \cap \text{Bl}(J_n) = \emptyset \), so the fact that \( \text{Bl}(J_n) \cup \text{Bl}(1 - J_n) = R \) gives us that \( J_n \subseteq \text{Bl}(1 - J_n) \).

5. Suppose that \( i \leq n \). Then \( J_i \subseteq J_n \) implies by part (2) of Proposition 16 that \( \text{Bl}(J_n) \subseteq \text{Bl}(J_i) \) and \( \text{Bl}(1 - J_n) \subseteq \text{Bl}(1 - J_i) \). Then \( \text{Bl}(J_i) \cup \text{Bl}(1 - J_i) \) contains \( \text{Bl}(J_n) \cup \text{Bl}(1 - J_n) = R \), and the proposition is proved. \( \square \)

Note that the sets \( 1 - J_i \) will also figure prominently in later results. Part (5) of Proposition 16 guarantees that the facts about the sets \( 1 - J_i \) will exactly parallel those for the sets \( J_i \).

The following lemma will also prove useful.

**Lemma 21.** Let \( R \) be a local ring. Then, for any \( n \geq 0 \), \( \text{Bl}(J_n) \cup \text{Bl}(1 - J_n) = R \) if and only if \( \text{Bl}(J_n) = R \setminus J_{n+1} \).

**Proof.** To prove the forward implication, suppose that \( a \notin \text{Bl}(J_n) \). Since \( \text{Bl}(J_n) \cup \text{Bl}(1 - J_n) = R \), we have that \( a \in \text{Bl}(1 - J_n) \). This shows that \( a \in J_{n+1} \). On the other hand, the definition of \( J_{n+1} \) implies that \( J_{n+1} \cap \text{Bl}(J_n) = \emptyset \), so \( \text{Bl}(J_n) = R \setminus J_{n+1} \).

Conversely, suppose that \( \text{Bl}(J_n) = R \setminus J_{n+1} \). Since \( J_{n+1} \subseteq \text{Bl}(1 - J_n) \), any \( a \in R \) is contained either in \( \text{Bl}(J_n) \) or in \( \text{Bl}(1 - J_n) \). \( \square \)

The sets \( J_i \) can be used to give more precise conditions for when \( \mathbb{T}_n(R) \) is a strongly clean ring (for fixed \( n \)). The main theorem is the following:

**Theorem 22.** Let \( R \) be a local ring. Define the conditions \( \mathcal{B}_n \) as follows. Set \( n = 2k \) if \( n \) is even, and \( n = 2k + 1 \) if \( n \) is odd,

\[
(B_{2k+1}): \begin{cases} 
J_0 \subseteq J_1 \subseteq \cdots \subseteq J_k, \quad \text{and} \\
\text{Bl}(J_k) \cup \text{Bl}(1 - J_k) = R,
\end{cases}
\]

\[
(B_{2k}): \begin{cases} 
J_0 \subseteq J_1 \subseteq \cdots \subseteq J_{k+1}, \quad \text{and} \\
\text{Bl}(J_{k-1}) \cup \text{Bl}(1 - J_{k-1}) = R.
\end{cases}
\]

If \( R \) satisfies condition \( \mathcal{B}_n \), then \( \mathbb{T}_n(R) \) is strongly clean.
Proof. The proof of this theorem will depend heavily on Lemma 7. If \( A \in \mathbb{T}_n(R) \), we will construct an appropriate idempotent \( E \) starting with the main diagonal. In general, we shall set \( E_{ii} = 1 \) if \( a_{ii} \in J_k \) and \( E_{ii} = 0 \) if \( a_{ii} \in 1 - J_k \). By the results proved about the sets \( J_i \) (and by analogy, about the sets \( 1 - J_i \)), this will be enough to show that \( A_{ii} - E_{ii} \) is a unit for all \( i \) and, through use of Lemma 7, that \( A E = EA \).

We will start with the case where \( n \) is odd. Suppose that \( A \) is a matrix in \( \mathbb{T}_n(R) \). Write \( n = 2k + 1 \), and let \( a_{11}, \ldots, a_{nn} \) be the diagonal entries of \( A \). By part (1) of Proposition 20, the sets \( J_k, 1 - J_k \) and \( R \setminus (J_k \cup (1 - J_k)) \) are mutually disjoint. We may therefore unambiguously sort the elements \( a_{11}, \ldots, a_{nn} \) into these three sets. For \( i \in \{1, \ldots, k\} \), set \( K_i = J_i \setminus J_{i-1} \), and note that \( 1 - K_i = (1 - J_i) \setminus (1 - J_{i-1}) \). It is clear that the sets \( K_1, \ldots, K_k \) and \( 1 - K_1, \ldots, 1 - K_k \) are pairwise disjoint, and that each is a subset of \( J_k \setminus (1 - J_k) \). We then consider two main cases.

Case I. The set \( R \setminus (J_k \cup (1 - J_k)) \) contains 2 or more of the diagonal entries \( a_{ii} \).

Since the \( 2k \) sets \( K_1, \ldots, K_k \), \( 1 - K_1, \ldots, 1 - K_k \) are all pairwise disjoint, at least one of them must fail to contain one of the remaining \( (2k - 1) \) or fewer elements \( a_{ii} \) (note that some of the \( a_{ii} \) may be 0 or 1, in which case they do not belong to any of the \( K_j \) or \( 1 - K_j \)). There is therefore a least positive integer \( 1 \leq l \leq k \) such that suppose without loss of generality (replacing \( A \) by \( 1 - A \), if necessary, cf. Lemma 4) that \( K_l \) contains no \( a_{ii} \). Define the elements \( e_{ii} \) as follows. Let \( e_{ii} = 1 \) if \( a_{ii} \in J_{l-1} \), and let \( e_{ii} = 0 \) if \( a_{ii} \in R \setminus J_l \), recalling that, by assumption \( K_l = J_l \setminus J_{l-1} \) contains no element \( a_{ii} \).

Note first that, if \( a_{ii} \in J(R) = J_l \), then \( e_{ii} = 1 \) and \( a_{ii} - e_{ii} \) is a unit, and if \( a_{ii} \in 1 - J(R) = 1 - J_1 \), then \( e_{ii} = 0 \) and \( a_{ii} - e_{ii} \) is again a unit. If \( a_{ii} \) is contained neither in \( J_1 \) nor in \( 1 - J_1 \), then \( a_{ii} - e_{ii} \) is a unit whether \( e_{ii} \) is 1 or 0. We claim that we can now apply Lemma 7 to this collection \( \{e_{ii}\}_{i=1}^n \) to construct an idempotent \( E \) such that \( A E = EA \) and \( E_{ii} = e_{ii} \) for every \( i \). This will exhibit the strong cleanliness of \( A \), since we have already demonstrated that \( A_{ii} - E_{ii} \) is invertible for every \( i \). If \( e_{ii} \neq e_{jj} \), then \( a_{ii} \in J_{l-1} \) and \( a_{jj} \in R \setminus J_l \) (or vice versa). Using \( B_{2k+1} \), part (5) of Proposition 20 and Lemma 21, we have \( B_l(J_{l-1}) = R \setminus J_l \). The idempotents \( e_{ii} \) thus satisfy the hypotheses of Lemma 7, showing the existence of an idempotent \( E \) such that \( A - E \) is invertible and \( A E = EA \).

Case II. The set \( R \setminus (J_k \cup (1 - J_k)) \) contains 1 or 0 of the diagonal elements \( a_{ii} \).

In this case, we set \( e_{ii} = 1 \) if \( a_{ii} \in J_k \) and \( e_{ii} = 0 \) if \( a_{ii} \in 1 - J_k \). If there is an \( i' \) for which \( a_{ii'} \) is in neither \( J_k \) nor \( 1 - J_k \), then set \( e_{i'i'} = 1 \) if \( a_{i'i'} \in B_l(1 - J_k) \) and \( e_{i'i'} = 0 \) otherwise (note that \( B_{2k+1} \) implies that \( B_l(J_k) \cup B_l(1 - J_k) = R \)). As before, one may check that \( a_{ii} - e_{ii} \) is a unit for all \( i \). Further, \( B_{2k+1} \) and part (4) of Proposition 20 imply that \( J_k \subseteq B_l(1 - J_k) \). Our choice of idempotents \( e_{ii} \) therefore guarantees that \( a_{ii} \in B_l(a_{jj}) \) if \( e_{ii} \neq e_{jj} \). As in Case I, Lemma 7 provides an idempotent \( E \) such that \( A - E \) is invertible and \( AE = EA \).

If \( n \) is even, the proof is similar. In this case, write \( n = 2k \), and let \( a_{11}, \ldots, a_{nn} \) be the diagonal entries of \( A \). We again investigate two cases and proceed as before.

Case I. The set \( R \setminus (J_k \cup (1 - J_k)) \) contains 1 or more of the diagonal elements \( a_{ii} \).

As in the \( n = 2k + 1 \) case, we may assume without loss of generality that there exists a least positive integer \( 1 \leq l \leq k \) such that \( J_l \setminus J_{l-1} \) contains none of the elements \( a_{ii} \). Set \( e_{ii} = 1 \) if \( a_{ii} \in J_{l-1} \) and \( e_{ii} = 0 \) if \( a_{ii} \in R \setminus J_l \). Once again, \( a_{ii} - e_{ii} \) is a unit for every \( i \). If \( e_{ii} \neq e_{jj} \), then \( a_{ii} \in J_{l-1} \) and \( a_{jj} \in R \setminus J_l \) (or vice versa). Part (5) of Proposition 20, \( B_{2k} \), and Lemma 21 imply
that \( \text{Bl}(J_{i-1}) = R \setminus J_i \). The idempotents \( e_{ii} \) thus satisfy the hypotheses of Lemma 7. As before, we find an idempotent \( E \) such that \( A \) is strongly \( E \)-clean.

**Case II.** The set \( R \setminus (J_k \cup (1 - J_k)) \) contains none of the diagonal elements \( a_{ii} \).

In this case, set \( e_{ii} = 1 \) if \( a_i \in J_k \) and \( e_{ii} = 0 \) if \( a_{ii} \in 1 - J_k \). It is again easy to verify that \( a_{ii} - e_{ii} \) is a unit for every \( i \). Further, by part (3) of Proposition 20, \( B_{2k} \) implies that \( J_k \subseteq \text{Bl}(1 - J_k) \) and therefore that \( a_{ii} \in \text{Bl}(a_{jj}) \) if \( e_{ii} \neq e_{jj} \). Lemma 7 again applies, and we find an idempotent \( E \) such that \( A \) is strongly \( E \)-clean. \( \square \)

**Remark 23.** It is straightforward to see that if \( \mathbb{T}_n(R) \) is strongly clean, then \( \mathbb{T}_{n-1}(R) \) is strongly clean. Compatible with this, \( B_n \Rightarrow B_{n-1} \). Indeed, it is immediate that \( B_{2k+2} \Rightarrow B_{2k+1} \). By parts (3)–(5) of Proposition 20, it is also true that \( B_{2k+1} \Rightarrow B_{2k} \).

The following corollary will prove useful in verifying the strong cleanness of triangular matrix rings.

**Corollary 24.** If \( R \) is a local ring such that \( \text{Bl}(J_i) \cup \text{Bl}(1 - J_i) = R \) for every \( i \geq 1 \), then \( \mathbb{T}_n(R) \) is strongly clean for every \( n \).

**Proof.** We will show by induction that \( J_i \subseteq J_{i+1} \) for every \( i \geq 0 \). This will suffice, by Theorem 22, to show that \( \mathbb{T}_n(R) \) is strongly clean for all \( n \). For \( i = 0 \), it is clear by Definition 19 that \( J_0 \subseteq J_1 \). Suppose now that the statement is true for \( i = k \). Since \( \text{Bl}(J_{k+1}) \cup \text{Bl}(1 - J_{k+1}) = R \) and \( J_k \subseteq J_{k+1} \), parts (3)–(4) of Proposition 20 show that \( J_{k+1} \subseteq J_{k+2} \), completing the induction. \( \square \)

Note that, although Theorem 22 offers a more detailed analysis of when the ring \( \mathbb{T}_k(R) \) is strongly clean for a particular \( k \), it, \textit{a priori}, still does not offer conditions that are necessary as well as being sufficient. However, as we shall see later, the application of a certain finiteness condition on the local ring \( R \) will allow us to obtain a converse to Theorem 22. Note, however, that Theorem 18 already shows (without additional hypotheses) that \( \mathbb{T}_2(R) \) is strongly clean if and only if the local ring \( R \) satisfies condition \( B_2 \) (by part (3) of Proposition 20).

In the meantime, however, let us reap the benefits of our hard labor.

**Lemma 25.** Let \( R \) be a local ring. Then \( R \) is bleached if and only if \( J_1 = J_2 \) and \( \text{Bl}(J_1) \cup \text{Bl}(1 - J_1) = R \).

**Proof.** Let \( R \) be bleached. This means that \( \text{Bl}(J_1) \) is equal to \( U(R) \). We then have \( \text{Bl}(J_1) = R \setminus J_1 \) and \( \text{Bl}(1 - J_1) = R \setminus (1 - J_1) \). It is then easy to see by Definition 19 that \( J_1 = J_2 \) and that \( \text{Bl}(J) \cup \text{Bl}(1 - J) = R \).

On the other hand, suppose that \( R \) satisfies the latter conditions. We wish to show that \( u \in \text{Bl}(J_1) \) for every unit \( u \) in \( R \). By Lemma 21, \( R \setminus J_2 = \text{Bl}(J_1) \). However, \( J_1 = J_2 \) by hypothesis, so \( R \) is bleached. \( \square \)

Note that Lemma 25 and Theorem 22 allow us to see again that if \( R \) is a bleached local ring, then \( \mathbb{T}_n(R) \) is strongly clean for all \( n \).

**Proposition 26.** Let \( R \) be a local ring for which \( \mathbb{T}_2(R) \) is strongly clean, and denote by \( \pi : R \to R / J(R) \) the natural quotient map. Assume \( \alpha \notin J(R) \), and that \( \pi(\alpha) \) is algebraic over \( \pi(Z(R)) \). Then \( \alpha \in \text{Bl}(J(R)) \).
Proof. Since $\pi(\alpha)$ is algebraic over $\pi(Z(R))$, there exists a polynomial $p(t) = c_n t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + c_0 \in Z(R)[t]$ such that $p(\alpha) \in J(R)$. Since $\pi(\alpha) \neq 0$, we may assume that $c_0 \neq 0$ (reducing degree otherwise), and we may further assume that $c_0 = 1$. Let $j \in J$. Note that $p(\alpha) \in J(R)$, and $p(j) \in 1 - J(R)$. By assumption, $\mathbb{T}_2(R)$ is strongly clean, which is equivalent to $J(R) \subseteq Bl(1 - J(R))$ by Theorem 18. Therefore, $p(\alpha) \in J \subseteq Bl(1 - J(R)) \subseteq Bl(p(j))$. Thus, by part (8) of Proposition 16, $\alpha \in Bl(j)$. We conclude that $\alpha \in Bl(J(R))$, as desired. □

Theorem 27. If the division ring $R/J(R)$ is an algebraic extension of its prime field (either $\mathbb{Q}$ or $\mathbb{Z}_p$ for some $p$), then the following are equivalent:

1. $R$ is bleached,
2. $\mathbb{T}_n(R)$ is strongly clean for all $n \in \mathbb{N}$,
3. $\mathbb{T}_n(R)$ is strongly clean for some $n > 1$,
4. $\mathbb{T}_2(R)$ is strongly clean.

In particular, the above are all equivalent whenever $R/J(R)$ is finite.

Proof. The implication (1) $\Rightarrow$ (2) holds by Theorem 9, the implication (2) $\Rightarrow$ (3) is an obvious weakening, and the implication (3) $\Rightarrow$ (4) is easy (a similar proof to Lemma 2) since $\mathbb{T}_n(R)$ can be viewed as

$$
\begin{pmatrix}
\mathbb{T}_2(R) & M_{2,(n-2)}(R) \\
\mathbb{T}_{n-2}(R)
\end{pmatrix}.
$$

The only implication which needs further proof is (4) $\Rightarrow$ (1). Let $\pi : R \rightarrow R/J(R)$ denote the natural map. Consider the (central) subring $S$ generated by $1 \in R$. If $R/J(R)$ is an algebraic extension of $\mathbb{Z}_p$, then $\pi(S) = \mathbb{Z}_p \subseteq R/J(R)$. If $R/J(R)$ is an algebraic extension of $\mathbb{Q}$, then it is clear that for $0 \neq n \in \mathbb{Z}$, $n \cdot 1 \neq J(R)$. Therefore, $\mathbb{Q}$ embeds in $R$ as a subring of $Z(R)$. In either case, by Proposition 26, every element of $U(R) = R \setminus J(R)$ is in Bl($J(R)$), and hence $R$ is bleached. □

We can also use the $J_i$s to give another sufficient condition for $\mathbb{T}_n(R)$ to be strongly clean.

Corollary 28. Let $R$ be a local ring. Suppose that $Bl(J_1) \cup Bl(1 - J_1) = R$ and that $Bl^2(J_1) \cup Bl^2(1 - J_1) = R$. Then $\mathbb{T}_n(R)$ is strongly clean for all $n$.

Proof. We shall show, by induction on $k$, that $Bl(J_k) \cup Bl(1 - J_k) = R$ and that $J_{k-1} \subseteq J_k$ for every $k$. By Theorem 22, this suffices to show that $\mathbb{T}_n(R)$ is strongly clean for all $n$. If $k = 1$, we are done by hypothesis. Suppose now that $Bl(J_m) \cup Bl(1 - J_m) = R$ and that $J_{m-1} \subseteq J_m$. Parts (3) and (4) of Proposition 20 imply that $J_m \subseteq J_{m+1}$. In order to show that $Bl(J_{m+1}) \cup Bl(1 - J_{m+1}) = R$, notice that $J_{m+1} \subseteq Bl(1 - J_m) \subseteq Bl(1 - J_1)$. Part (2) of Proposition 16 shows that $Bl^2(1 - J_1) \subseteq Bl(J_{m+1})$. Similarly, $Bl^2(J_1) \subseteq Bl(1 - J_{m+1})$. The hypothesis that $Bl^2(J_1) \cup Bl^2(1 - J_1) = R$ then implies that $Bl(J_{m+1}) \cup Bl(1 - J_{m+1}) = R$. This completes the induction and the proof. □

While the above-stated condition is sufficient, it is by no means necessary. For example, the ring $\mathbb{Z}_3(3)$ of integers localized at the prime ideal $3$ is a commutative local (and thus bleached)
ring. However, it can be checked that \( \mathbb{Z}(3) \) does not satisfy the hypotheses of Corollary 28. On the other hand, one can check that any bleached uniquely clean local ring satisfies the hypotheses of Corollary 28.

We now turn our attention to proving a converse to Theorem 22.

**Lemma 29.** Suppose that \( k \geq 1 \), \( J_0 \subseteq J_1 \subseteq \cdots \subseteq J_k \), and that \( Bl(J_{k-1}) \cup Bl(1 - J_{k-1}) = R \).

If \( a_k \in J_k \setminus J_{k-1} \) and \( b_k \in (1 - J_k) \setminus (1 - J_{k-1}) \), then there exist elements \( a_1, \ldots, a_{k-1} \) and \( b_1, \ldots, b_{k-1} \), with \( a_i \in J_i \setminus J_{i-1} \) and \( b_i \in (1 - J_i) \setminus (1 - J_{i-1}) \), such that, for each \( i \in \{1, \ldots, k - 1\} \), \( a_i \notin Bl(a_{i+1}) \) and \( b_i \notin Bl(b_{i+1}) \).

**Proof.** We first remark that, by the recursive nature of Definition 19, \( J_k \neq J_{k-1} \) implies that \( J_i \setminus J_{i-1} \) and \( (1 - J_i) \setminus (1 - J_{i-1}) \) are non-empty for every \( i \in \{1, \ldots, k\} \).

We will construct the \( a_i \) in reverse order, starting with the given \( a_k \); the \( b_i \) are constructed similarly. Suppose that \( a_k, \ldots, a_1 \) have been found as prescribed. Since \( a_{i+1} \in J_i + J_{i+1} = Bl(1 - J_i) \setminus Bl(J_i) \) by Definition 19, there must exist \( a_i \in J_i \) such that \( a_{i+1} \notin Bl(a_i) \). We now claim that \( a_i \notin J_{i-1} \). Part (5) of Proposition 20 implies that \( Bl(J_{i-1}) \cup Bl(1 - J_{i-1}) = R \), and Lemma 21 then implies that \( R \setminus J_i = Bl(J_{i-1}) \). However, we know by hypothesis that \( a_{i+1} \notin J_i \) and that \( a_{i+1} \notin Bl(a_i) \). This means that \( a_i \notin J_{i-1} \). This \( a_i \) is then the one we seek. \( \square \)

**Lemma 30.** Suppose that \( J_0 \subseteq J_1 \subseteq \cdots \subseteq J_k \) and \( Bl(J_{k-1}) \cup Bl(1 - J_{k-1}) = R \). For each \( i \in \{1, \ldots, k\} \), set \( K_i = J_i \setminus J_{i-1} \). Then the following are true.

1. Whenever \( i, j \in \{1, \ldots, k\} \) with \( |i - j| \geq 2 \), \( K_i \subseteq Bl(K_j) \) (equivalently \( (1 - K_i) \subseteq Bl(1 - K_j) \)).
2. \( K_i \subseteq Bl(1 - K_j) \) for all \( i, j \in \{1, \ldots, k - 1\} \).
3. If, in addition, \( J_k \subseteq J_{k+1} \), then \( K_i \subseteq Bl(1 - K_j) \) for all \( i, j \in \{1, \ldots, k\} \).

**Proof.** We shall prove each part in turn.

1. Suppose without loss of generality that \( i - j \geq 2 \). By part (5) of Proposition 20, \( Bl(J_j) \subseteq Bl(1 - J_j) = R \), and by Lemma 21, \( Bl(J_j) = R \setminus J_{j+1} \). Since \( K_j \subseteq J_j \), part (2) of Proposition 16 implies that \( Bl(J_j) \subseteq Bl(K_j) \). On the other hand, \( i - j \geq 2 \) implies that \( J_{j+1} \subseteq J_{j-1} \). Together with the definition of \( K_i \), we have \( K_i \subseteq R \setminus J_{i-1} \subseteq R \setminus J_{j+1} \). These inclusions show that \( K_i \subseteq Bl(K_j) \). The proof that \( (1 - K_i) \subseteq Bl(1 - K_j) \) proceeds similarly.

2. Part (3) of Proposition 20 implies that \( J_{k-1} \subseteq Bl(1 - J_{k-1}) \). Since \( K_i \subseteq J_{k-1} \) and \( (1 - K_j) \subseteq (1 - J_{k-1}) \), \( K_i \subseteq Bl(1 - K_j) \) for any \( i, j \in \{1, \ldots, k - 1\} \).

3. In this case, part (3) of Proposition 20 implies that \( J_k \subseteq Bl(1 - J_k) \). Thus, as before, \( K_i \subseteq Bl(1 - K_j) \) for any \( i, j \in \{1, \ldots, k\} \). \( \square \)

**Definition 31.** A ring \( R \) will be called an \( h \)-ring if whenever \( a, b \in R \) with \( la - rb \) surjective on \( R \), \( la - rb \) is injective as well.

Note that the map \( la - rb \) is a \( Z(R) \)-module homomorphism. Thus, if \( R_{Z(R)} \) is a Hopfian module (i.e. every surjective endomorphism is an isomorphism), then if \( la - rb \) is surjective, it must be injective as well. Thus, the condition that \( R \) is an \( h \)-ring is a weakening of the condition that \( R_{Z(R)} \) is a Hopfian module, and hence should be regarded as a finiteness condition on \( R \).
Lemma 32. Suppose that the local ring $R$ is an h-ring, and suppose that $(c_1,\ldots,c_n)$ is an ordered tuple of elements of $R$ such that, for $i \neq j$, $c_i \in \text{Bl}(c_j)$ if and only if $|i - j| > 1$. Then there exists an $n \times n$ upper-triangular matrix $C$ whose diagonal entries are a permutation of \{c_1,\ldots,c_n\} and with the property that the only idempotent upper-triangular matrices that commute with $C$ are $I$ and $0$.

Proof. Notice first that, by hypothesis, for each $i \in \{1,\ldots,n-1\}$, either $l_{c_{i+1}} - r_{c_i}$ or $l_{c_{i+1}} - r_{c_{i+2}}$ is not surjective. Order the elements $c_i$ by the following process. Start with the 1-element list $c_1$. For each $i \in \{1,\ldots,n-1\}$, if $l_{c_{i+1}} - r_{c_i}$ is not surjective, place $c_{i+1}$ at the far left of the list. Otherwise, place it at the far right. Note that, in this ordering, if $c_{i+1}$ is not adjacent to $c_i$, then the elements that occur between $c_{i+1}$ and $c_i$ are exactly some permutation of the elements $c_1,\ldots,c_{i-1}$.

Construct the matrix $C \in \mathbb{T}_n(R)$ in the following fashion. Begin by placing the elements $c_i$ along the diagonal of $C$ according to the new ordering. For each $i \in \{1,\ldots,n-1\}$, choose $y_i \in R$ to be outside of the image of $l_{c_{i+1}} - r_{c_i}$ if $l_{c_{i+1}} - r_{c_i}$ is not surjective; otherwise choose $y_i$ to be outside of the image of $l_{c_i} - r_{c_{i+1}}$. If $c_{i+1}$ is to the left of $(c_i)$, place $y_i$ in the same row of $C$ as $c_{i+1}$ and the same column as $c_i$. Otherwise place $y_i$ in the same column as $c_{i+1}$ and the same row as $c_i$. All other entries of $C$ will be zero. It is clear that $C \in \mathbb{T}_n(R)$.

We will now prove that the only idempotents of $\mathbb{T}_n(R)$ with which $C$ commutes are the trivial ones. Suppose that $E$ is a non-trivial idempotent such that $EC = CE$. For each $k$, define $C_k$ to be the smallest square block in $C$ that contains $c_{k-1}$ and $c_k$, and define $E_k$ to be the corresponding submatrix of $E$. We further define $e_i$ to be the diagonal entry of $E$ whose position corresponds to that of $c_i$ in $C$. Let $m$ be the smallest index for which $e_m \neq e_{m-1}$.

We now investigate two main cases to procure a contradiction. If $c_m$ is adjacent to $c_{m-1}$, then we are already at a contradiction, for, by Lemma 6, the commuting of $C$ and $E$ requires that we be able to solve an equation of the form $c_m z - z c_{m-1} = \pm y_{m-1}$, if $c_m$ is to the left of $c_{m-1}$, or $c_{m-1} z - z c_m = \pm y_{m-1}$ otherwise. This, however, is impossible by the choice of $y_{m-1}$.

On the other hand, if $c_m$ is not adjacent to $c_{m-1}$, then, by our ordering choice, it must be that they are at the $(1,1)$ and $(m,m)$ positions in $C_m$. In this case, we know that all of the diagonal elements of $E_m$ are the same, except at one end. In order to fix ideas (the other cases follow similarly), we shall assume that $c_m$ is to the left of $c_{m-1}$ and that the $(1,1)$ entry of $E_m$ is 1, the other diagonal entries being 0. Since $E$ is idempotent, the strictly upper-triangular part of $E_m$ must consists entirely of zeroes, except in the first row. By construction, the first row of $C_m$ consists entirely of zeroes, except possibly at the $(1,1)$ and $(1,m)$ positions. We now focus on the first row of $E_m$ in order to establish a contradiction.

Let $j$ be the smallest index greater than 1 such that the $(1,j)$ entry of $E_m$ is not zero; call this non-zero entry $z$. We then use Lemma 6 to evaluate the $(1,j)$ entry of the relation $C_m E_m b - E_m C_m = 0$. If $j \neq m$, $z$ must satisfy

$$c_m z - z c_i = 0$$

where $c_i \ (i < m - 1)$ is one of the diagonal entries of $C$. Since $c_i \in \text{Bl}(e_m)$ if $i < m - 1$ and since $R$ is an h-ring, this equation has the unique solution $z = 0$, which contradicts the choice of $z$ as a non-zero element.

On the other hand, if $j = m$, then Lemma 6 shows that $C_m E_m - E_m C_m = 0$ is equivalent to

\( c_m z - z c_{m-1} = y_{m-1}, \)

which is impossible by construction.

We thereby achieve a contradiction and must concede that the idempotent \( E \) is trivial. \( \square \)

We can now state a converse to Theorem 22.

**Theorem 33.** Suppose that \( R \) is a local \( h \)-ring. If \( \mathbb{T}_n(R) \) is strongly clean, then \( R \) satisfies the condition \( B_n \) (defined in Theorem 22).

**Proof.** Note that the theorem is true trivially for \( n = 1 \) and true for \( n = 2 \) by Nicholson’s result (Theorem 18). If the theorem is false, let \( m \) be the least integer for which \( \mathbb{T}_m(R) \) is strongly clean but for which \( R \) does not satisfy \( B_m \).

We have two cases.

**Case I.** \( m = 2k + 1 \) is odd.

We are given that \( \mathbb{T}_{2k+1}(R) \) is strongly clean, and that the theorem holds for \( i < 2k + 1 \). Since \( \mathbb{T}_{2k+1}(R) \) is strongly clean, it is easy to see that \( \mathbb{T}_{2k}(R) \) is strongly clean. By assumption, \( R \) satisfies \( B_{2k} \), and so \( J_0 \subseteq \cdots \subseteq J_{k+1} \) and \( \text{Bl}(J_{k-1}) \cup \text{Bl}(1 - J_{k-1}) = R \). Thus, if \( R \) does not satisfy \( B_{2k+1} \), then \( \text{Bl}(J_k) \cup \text{Bl}(1 - J_k) \neq R \). In this case, pick \( x \in R \) such that \( x \notin \text{Bl}(J_k) \cup \text{Bl}(1 - J_k) \). Then there exist \( a_k \in J_k \) and \( b_k \in 1 - J_k \) such that \( x \notin \text{Bl}(a_k) \) and \( x \notin \text{Bl}(b_k) \).

We claim, however, that \( x \in \text{Bl}(J_{k-1}) \cap \text{Bl}(1 - J_{k-1}) \). By \( B_{2k} \), \( \text{Bl}(J_{k-1}) \cap \text{Bl}(1 - J_{k-1}) = R \).

By the definition of \( J_k \), this means that

\[
R = J_k \cup (1 - J_k) \cup (\text{Bl}(J_{k-1}) \cap \text{Bl}(1 - J_{k-1})).
\]

Since \( x \notin \text{Bl}(J_k) \cup \text{Bl}(1 - J_k) \) and \( J_k \subseteq \text{Bl}(1 - J_k) \) (by \( B_{2k} \) and part (3) of Proposition 20) and therefore also \((1 - J_k) \subseteq \text{Bl}(J_k)\), (6) implies that \( x \in \text{Bl}(J_{k-1}) \cap \text{Bl}(1 - J_{k-1}) \). Since \( x \notin \text{Bl}(a_k) \) and \( x \notin \text{Bl}(b_k) \), we have \( a_k \notin J_{k-1} \) and \( b_k \notin (1 - J_{k-1}) \).

Knowing that \( a_k \in J_k \setminus J_{k-1} \) and \( b_k \in (1 - J_k) \setminus (1 - J_{k-1}) \), we can apply Lemma 29 to find \( a_{k-1}, \ldots, a_1 \) and \( b_{k-1}, \ldots, b_1 \) such that \( a_i \in J_i \setminus J_{i-1} \) and \( b_i \in (1 - J_i) \setminus (1 - J_{i-1}) \) and such that, for every \( i \in \{1, \ldots, k - 1\} \), \( a_i \notin \text{Bl}(a_{i+1}) \) and \( b_i \notin \text{Bl}(b_{i+1}) \). Further, since \( x \) is contained in \( \text{Bl}(J_{k-1}) \cap \text{Bl}(1 - J_{k-1}) \) but not in \( \text{Bl}(J_k) \cup \text{Bl}(1 - J_k) \),

\[
a_i, b_i \in \text{Bl}(x) \quad \text{if and only if} \quad i \neq k.
\]

Equation (7), together with parts (1) and (3) of Lemma 30 (whose use is justified by \( B_{2k} \)) show that the hypotheses of Lemma 32 are satisfied by the tuple \( (a_1, \ldots, a_k, x, b_k, \ldots, b_1) \). Lemma 32 now guarantees the existence of a matrix \( C \in \mathbb{T}_m(R) \) containing the elements \( \{a_1, \ldots, a_k, x, b_k, \ldots, b_1\} \) on the diagonal and such that \( C \) commutes with only the trivial idempotents. However, since \( a_1 \in J(R) \) and \( b_1 \in (1 - J(R)) \), neither \( C \) nor \( C - I \) is a unit. Thus \( C \) is not strongly clean, contradicting the strong cleanness of \( \mathbb{T}_m(R) \).

**Case II.** \( m = 2k \) is even.
We are given that $T_{2k}(R)$ is strongly clean. In this case, since $T_{2k-1}(R)$ is also strongly clean, we know by assumption that $R$ satisfies $B_{2k-1}$, i.e. that $J_0 \subseteq \cdots \subseteq J_k$, and $Bl(J_{k-1}) \cup Bl(1 - J_{k-1}) = R$. Since $J_{k-2} \subseteq J_{k-1}$ and $Bl(J_{k-1}) \cup Bl(1 - J_{k-1}) = R$, parts (3) and (4) of Proposition 20 imply that $J_{k-2} \subseteq Bl(1 - J_{k-1})$ and that $J_{k-1} \subseteq J_k$. Thus, if $R$ does not satisfy $B_{2k}$, it must be the case that $J_k \not\subseteq J_{k+1}$; equivalently, $J_k \not\subseteq Bl(1 - J_k)$. Pick $a_k \in J_k$ and $b_k \in 1 - J_k$ such that $a_k \notin Bl(b_k)$.

We claim that $a_k \in J_k \setminus J_{k-1}$ and that $b_k \in (1 - J_k) \setminus (1 - J_{k-1})$. By definition, $J_k = Bl(1 - J_{k-1}) \setminus Bl(J_{k-1})$. Since $a_k \in J_k$, $a_k \in Bl(1 - J_{k-1})$, implying (by the choice of $a_k$ and $b_k$) that $b_k \notin (1 - J_{k-1})$. Similarly, $a_k \notin J_{k-1}$.

As before, we use Lemma 29 to find elements $a_{k-1}, \ldots, a_1$ and $b_{k-1}, \ldots, b_1$ such that $a_i \in J_i \setminus J_{i-1}$ and $b_i \in (1 - J_i) \setminus (1 - J_{i-1})$ and such that, for every $i \in \{1, \ldots, k - 1\}$, $a_i \notin Bl(a_{i+1})$ and $b_i \notin Bl(b_{i+1})$. We claim that we can apply Lemma 32 to the tuple $(a_1, \ldots, a_k, b_k, \ldots, b_1)$.

In order to do this, we must show that the tuple $(a_1, \ldots, a_k, b_k, \ldots, b_1)$ satisfies the hypothesis of Lemma 32. Part (1) of Lemma 30 shows that $a_i \in Bl(a_j)$ and $b_i \in Bl(b_j)$ if $|i - j| \geq 2$. Additionally, since $J_k = Bl(1 - J_{k-1}) \setminus Bl(J_{k-1}) \subseteq Bl(1 - J_{k-1})$ and $a_k \in J_k$, we have that $a_k \in Bl(b_j)$ for every $i \in \{1, \ldots, k - 1\}$. Similarly, $b_k \in Bl(a_i)$ for every $i \in \{1, \ldots, k - 1\}$. Finally, part (2) of Lemma 30 shows that $a_i \in Bl(b_j)$ and $b_i \in Bl(a_j)$ for any $i, j \in \{1, \ldots, k - 1\}$. Lemma 32 can thus be applied to the tuple $(a_1, \ldots, a_k, b_k, \ldots, b_1)$ to construct a matrix $C$ with the elements $(a_1, \ldots, a_k, b_k, \ldots, b_1)$ on the diagonal such that $C$ commutes with no non-trivial idempotent. But neither $C$ nor $C - I$ is invertible, since $a_1 \in J_1$ and $b_1 \in 1 - J_1$. This contradicts the assumption that $T_m(R)$ is strongly clean and proves the theorem. □

**Remark 34.** Although there exist division rings that are not $h$-rings (see, for example, [12]), such examples have no bearing on Theorem 33 since division rings are already known to be bleached.

**Remark 35.** Theorem 33 can actually be proved in a slightly more general setting. In [6], Ghorbani and Haghany define a module $M$ to be $gh$ (generalized Hopfian) if the kernel of any surjective endomorphism of $M$ is a small submodule of $M$. This property is shown to be equivalent to $f^{-1}(X)$ being a small submodule of $M$ for any small submodule $X$ of $M$ and any surjective endomorphism $f$ of $M$. In a similar vein, one of the present authors (T. Dorsey) defines a ring $R$ to be a $gh$-ring if, for every $a, b \in R$ such that $l_a - r_b$ is surjective on $R_{Z(R)}$ and for every small submodule $X_{Z(R)} \subseteq S R_{Z(R)}$, the preimage of $X$ under $l_a - r_b$ is small in $R_{Z(R)}$. In [5], Dorsey shows that Theorem 33 remains true when $h$-ring is replaced by $gh$-ring.

### 4. Examples

Having given many examples of bleached local rings at the end of Section 2, we now turn our attention to the construction of certain examples of non-bleached rings. Our main tool in this section will be the construction of skew power series rings.

Let $R$ be a local ring, and let $\sigma : R \rightarrow R$ be a ring endomorphism. Denote by $S = R[x; \sigma]$ the ring of left skew power series over $R$. Elements of $S$ are power series in $x$ with coefficients in $R$ written on the left, subject to the relation $xr = \sigma(r)x$ for all $r \in R$. It is well known (see [10, p. 283] for details; the author’s restriction to ring automorphisms may be ignored) that $S$ is a local ring with radical $J(S) = J(R) + Sx$. In what follows, if $f$ is an element of $S$, then $f_i$ will denote the coefficient of $x^i$ in $f$.

We begin with a lemma.
Lemma 36. Let \( R \) be a commutative local ring, and let \( \sigma \) and \( S \) be defined as above. Suppose further that \( f = \sum_i f_i x^i \) and \( g = \sum_j g_j x^j \) are elements of \( S \). Then \( f \in \text{Bl}(g) \) if and only if \( f_0 - \sigma^i(g_0) \) and \( \sigma^i(f_0) - g_0 \) are units of \( R \) for all integers \( i \geq 0 \).

Proof. This proof is an easy verification. Suppose first that \( f_0 - \sigma^i(g_0) \) is a unit for every \( i \geq 0 \). If \( b \) is any element of \( S \), then solving the equation \( f z - zg = b \) for \( z \in S \) amounts to inductively solving the equations

\[
\begin{align*}
f_0 z_m - z_m \sigma^m(g_0) + \sum_{k=1}^m (f_k z_{m-k} - z_{m-k} \sigma^{m-k}(g_k)) &= b_m
\end{align*}
\]

over \( R \) for every \( m \geq 0 \).

In fact, under the given hypotheses, we can solve the above equations uniquely. If \( m = 0 \), then \( z_0 = (f_0 - g_0)^{-1} b_0 \). Having then inductively solved uniquely for \( z_0, \ldots, z_{m-1} \), we may clearly find a unique solution for \( z_m \), since \( f_0 - \sigma^m(g_0) \) is a unit and \( R \) is commutative. Likewise, one may find a unique \( z \) that solves \( zf - zg = b \) if \( \sigma^j(f_0) - g_0 \) is a unit for every \( j \geq 0 \).

On the other hand, suppose that \( i \) is minimal such that \( f_0 - \sigma^i(g_0) \) is not a unit. We claim that the equation \( f z - zg = x^i \) has no solution \( z \in S \). Solving inductively as before, we see that \( z_0 = z_1 = \cdots = z_{i-1} = 0 \). The equation \( f_0 z_i - z_i \sigma^i(g_0) = 1 \) then has no solution since \( f_0 - \sigma^i(g_0) \) is not a unit. Once again, the case where \( \sigma^j(f_0) - g_0 \) is a unit is handled analogously. \( \square \)

Remark 37. This lemma can be stated and proved in a much wider context. A suitable generalization of the above lemma then holds in the case that (the possibly non-commutative ring) \( R \) is an \( h \)-ring. One can then further look more specifically at the class (containing the commutative rings) of local rings for which \( l_x - r_y \) is surjective if and only if \( x - y \) is a unit. Although one can prove similar results in these cases, we will not need this level of generality in what follows. See, for instance, [5].

We can now give our first example of a non-bleached local ring.

Lemma 38. Let \( R \) be a commutative local ring and let \( \sigma \) be a ring endomorphism such that there is a non-unit \( r \in R \) and a unit \( u \in R \) with \( \sigma(r) = u \). Then the skew left power series ring \( R[x; \sigma] \) is a non-bleached local ring.

Proof. Define \( f, g \in R[x; \sigma] \) by \( f = f_0 = u \) and \( g = g_0 = r \). Then \( f \) is a unit in \( R[x; \sigma] \) and \( g \) is in \( J(R[x; \sigma]) \). However, it is easy to see that \( f_0 - \sigma(g_0) = 0 \) is not a unit, so \( f \notin \text{Bl}(g) \) by Lemma 36. We conclude that \( R[x; \sigma] \) is not bleached. \( \square \)

Example 39. Let \( k \) be a field, and let \( R = k[t_1, t_2, \ldots]_{(t_1)} \) be a ring of polynomials in countably many indeterminates, localized at the prime ideal \( (t_1) \). Let \( \sigma \) be the map that is the identity on \( k \) and which satisfies \( \sigma(t_1) = t_{i+1} \). It is easy to see that this extends to the localization. Since \( \sigma \) takes the non-unit \( t_1 \) to the unit \( t_2 \), Lemma 38 shows that the local ring \( R[x; \sigma] \) is not bleached.

Although the ring \( R[x; \sigma] \) of Example 39 is not bleached, we can still show that, for any commutative local ring \( R \), \( \mathbb{T}_n(R[x; \sigma]) \) is strongly clean for every \( n \) by calculating the sets \( J_i \) and appealing to Theorem 22.
Proof. The proof of statement (1) is an easy induction. As for (2), this is easy to see for $i = 1$ since a unit cannot map via $\sigma$ to a non-unit. Suppose inductively that it is true for $i = k - 1$. Let $\sigma(a) \in \Psi_k$. Then $\sigma(a) = r + \sigma^j(s)$ for some $r \in J(R)$, $s \in \Psi_{k-1}$ and $j \geq 0$. If $j = 0$, then $\sigma(a) \in J(R) + \Psi_{k-1} = \Psi_{k-1} \subseteq \Psi_k$ and we are done. If, instead, $j > 0$, then $\sigma(a - \sigma^{-1}(s)) \in J(R)$, and so $a - \sigma^{-1}(s) \in J(R)$, which again shows that $a \in \Psi_k$.

The proof of (3) proceeds also by induction on $i$, being easy for $i = 1$. Assume that the statement is true for $i = k - 1$. We shall suppose that $\Psi_k \cap (1 - \Psi_k) \neq \emptyset$ and establish a contradiction. If $a \in \Psi_k \cap (1 - \Psi_k)$, then there exist $r_1, r_2 \in J(R)$, $s_1, s_2 \in \Psi_{k-1}$ and $l, m \geq 0$ such that $r_1 + \sigma^l(s_1) = a = 1 - r_2 + \sigma^m(s_2)$. Without loss of generality, we may assume that $m \geq l$. Rearranging gives $\sigma^l(s_1 - \sigma^m(s_2)) = 1 - r_1 - r_2 \in 1 - J(R)$. However, $1 - J(R) = 1 - \Psi_1$ is invariant under $\sigma^{-1}$ (as $\Psi_1$ is), and so we are left with $s_1 - \sigma^m(s_2) \in 1 - J(R)$. This shows that $\sigma^m(s_2) \in 1 - J(R) - \Psi_{k-1} = 1 - \Psi_{k-1}$, and so $s_2 \in 1 - \Psi_{k-1}$, which is a contradiction. \hfill $\Box$

The theorem now follows easily.

Theorem 41. Let $R$ be a commutative local ring and let $S = R[x; \sigma]$ be a left skew power series ring over $R$. Then $T_n(S)$ is strongly clean for every $n$.

Proof. Define the sets $J_i = J_i(S)$ for the ring $S$ as in Definition 19.

For each $i \geq 0$, we will show inductively that $Bl(J_i) \cup Bl(1 - J_i) = S$ and that $J_{i+1} = \Psi_{i+1} + Sx$. Using Corollary 24, this suffices to show that $T_n(S)$ is strongly clean for every $n$.

The case $i = 0$ is trivial. We always have $Bl(J_0) \cup Bl(1 - J_0) = S$, and $J_1 = J(S) = J(R) + Sx = \Psi_1 + Sx$. Supposing that the statement is true for $i = k - 1$, we now prove it for $i = k$. We begin by computing $Bl(J_k)$. By appealing to Lemma 36, we see that this can be written as $\Phi_{k+1} + Sx$ for some subset $\Phi_{k+1}$ of $R$.

We now claim that $\Psi_{k+1} = R \setminus \Phi_{k+1}$. Given this claim along with the fact that $\Psi_{k+1} \cap (1 - \Psi_{k+1}) = \emptyset$, we see that $Bl(J_k) \cup Bl(1 - J_k) = S$ and, by Lemma 21, that $J_{k+1} = \Psi_{k+1} + Sx$, thus completing the induction.

We now prove the claim. We have defined $\Phi_{k+1}$ (using Lemma 36) by the relation $Bl(J_k) = Bl(\Psi_k + Sx) = \Phi_{k+1} + Sx$. Using the definition of $\Psi_k$, we can write $\Phi_{k+1} = C_1 \cap C_2$ where

$$C_1 = \{ a \in R \mid a - \sigma^j(\Psi_k) \subseteq U(R) \text{ for every } j \geq 0 \}$$

and

$$C_2 = \{ a \in R \mid \sigma^j(a - \Psi_k) \subseteq U(R) \text{ for every } j \geq 0 \}.$$
If \( r \notin C_1 \), then there is some \( j \geq 0 \) and some \( s \in \Psi_k \) such that \( r - \sigma^j(s) \in J(R) \). By definition of \( \Psi_{k+1} \), this implies that \( r \in \Psi_{k+1} \). Therefore, \( \Psi_{k+1} = R \setminus C_1 \), and we need only show that \( R \setminus C_2 \subseteq \Psi_{k+1} \). If \( a \notin C_2 \), then \( (\sigma^j(a) - \Psi_k) \cap J(R) \neq \emptyset \) for some \( j \). This means that \( \sigma^j(a) \in J(R) + \Psi_k = \Psi_k \), which implies, by part (2) of Lemma 40, that \( a \) is in \( \Psi_k \). Thus \( a \in \Psi_{k+1} \) by part (1) of Lemma 40, proving the claim and the theorem. \( \square \)

We now construct an example of a local ring \( A \) for which \( \mathbb{T}_2(A) \) is not strongly clean.

**Definition 42.** Let \( R \) be a commutative local ring, and let \( \sigma, \tau \) be two commuting ring endomorphisms of \( R \). Define \( \tau A_\sigma \) to be the ring of all formal power series of the form

\[
\sum_{i,j \geq 0} y^i a_{ij} x^j
\]

with coefficients \( a_{ij} \) in \( R \), subject to \( xy = yx, ry = y\tau(r) \) and \( xs = \sigma(s)x \) for all \( r, s \in R \).

The ring \( \tau A_\sigma \) may be viewed as a right skew power series ring over a left skew power series ring over a local ring, and is thus easily seen to be local for any commuting endomorphisms \( \sigma \) and \( \tau \) of the local ring \( R \).

**Lemma 43.** Let \( R \) be a commutative local ring, and let \( \sigma, \tau \) and \( A = \tau A_\sigma \) be as in Definition 42. Suppose that \( f = \sum y^i f_{ij} x^j \) and \( g = \sum y^i g_{ij} x^j \) are elements of \( A \). Then \( f \in \text{Bl}(g) \) if and only if \( \tau^i(f_{00}) - \sigma^i(g_{00}) \) and \( \sigma^i(f_{00}) - \tau^i(g_{00}) \) are units of \( R \) for every \( i, j \geq 0 \).

**Proof.** Writing \( f_{ij} \) for the \( y^i, x^j \) coefficient of an element \( f \in A \), one can check that, if \( f, g, b \in S \), then solving the equation \( fz - zg = b \) for \( z \) amounts to solving the equations

\[
\sum_{i=0}^{m} \sum_{j=0}^{n} [\tau^i(f_{m-i,n-j})\sigma^{n-j}(z_{i,j}) - \tau^{m-i}(z_{i,j})\sigma^j(g_{m-i,n-j})] = b_{mn}
\]

for all \( m, n \geq 0 \).

As in the proof of Lemma 36, suppose that \( \tau^i(f_{00}) - \sigma^i(g_{00}) \) is a unit for every \( i, j \geq 0 \). If all \( z_{ij} \) are known for \( i + j < m + n \), then Eq. (8) may be solved uniquely (since \( \tau^m(f_{00}) - \sigma^n(g_{00}) \) is a unit) for \( z_{mn} \). Thus one may inductively solve uniquely for the coefficients of the \( z \in A \) that satisfies \( fz - zg = b \).

For the converse, suppose without loss of generality that \( \tau^m(f_{00}) - \sigma^n(g_{00}) \) is not a unit, and that this is chosen with the sum \( m + n \) minimal. We claim that there is no \( z \in A \) such that \( fz - zg = y^m x^n \). Again using Eq. (8), one may show inductively (inducting on the sum \( i + j \)), that \( z_{ij} = 0 \) whenever \( i + j < m + n \). Thus the equation for the \((m, n)\) term reduces to

\[
\tau^m(f_{00})z_{mn} - z_{mn}\sigma^n(g_{00}) = 1
\]

which has no solution \( z_{mn} \) since \( \tau^m(f_{00}) - \sigma^n(g_{00}) \) is not a unit. \( \square \)

**Corollary 44.** Let \( R \) be a commutative local ring, and let \( \sigma \) and \( \tau \) be commuting ring endomorphisms. Define \( A = \tau A_\sigma \) as in Definition 42. If there exist \( r \in J(R) \) and \( s \in 1 - J(R) \) and integers \( m, n \geq 0 \) such that \( \tau^m(r) - \sigma^n(s) \) is not a unit in \( R \), then \( \mathbb{T}_2(A) \) is not strongly clean.
Proof. By Theorem 14, $T_2(A)$ is strongly clean if and only if $J(A) \subseteq \text{Bl}(1 - J(A))$. Define elements $f, g \in A$ by setting $f = f_{00} = r$ and $g = g_{00} = s$. One can easily check that $f \in J(A)$ and that $g \in 1 - J(A)$. By hypothesis, there exist $m, n \geq 0$ such that $\tau^m(f_{00}) - \sigma^n(g_{00}) = \tau^m(r) - \sigma^n(s)$ is not a unit. Lemma 43 then implies that $f/\in \text{Bl}(g)$. But, since $f \in J(A)$ and $g \in 1 - J(A)$, Theorem 14 implies that $T_2(A)$ is not strongly clean.

Example 45. We may use an example similar to Example 39. Define $\sigma$ and $\tau$ on $k[x_1, x_2, \ldots]$ each to be the identity on $k$ and such that $\sigma(x_i) = x_i + 1$ and $\tau(x_i) = x_i + 1 + 1$. These extend to the localization. We use these data to construct a left–right skew power series ring $A = \sigma A\tau$. Letting $r = x_1 \in J(R)$ and $s = 1 + x_1 \in 1 - J(R)$, we see that $\tau(r) - \sigma(s) = 0$. By Corollary 44, $T_2(A)$ is not strongly clean. Note, however, that $T_2(A)$ is semiperfect, giving another example answering Question 5 of [15] in the negative, the first such example having been given in [19].

5. Incidence rings

Incidence rings form a natural generalization of full and triangular matrix rings, and hence are an appropriate context in which to extend our study of clean properties. In the literature, they are often studied over a commutative ground ring in which case they are called incidence algebras.

Let $X$ be a locally finite preordered set: that is, $X$ is equipped with a reflexive, transitive relation $\leq$; and for all $x, y \in X$, the interval $[x, y] = \{ z \in X \mid x \leq z \leq y \}$ is finite. Let $R$ be a ring.

The incidence ring of $X$ over $R$ is the set

$$I(X, R) = \{ f : X \times X \to R \mid f(x, y) = 0 \text{ if } x \not\leq y \}$$

with addition and multiplication defined by

$$(f + g)(x, y) = f(x, y) + g(x, y),$$

$$(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y).$$

It is routine to check that $I(X, R)$ is a ring with

$$0_{I(X, R)}(x, y) = 0 \quad \forall x, y \in X,$$

$$1_{I(X, R)}(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Here are some examples:

Example 46. If $X$ is trivially preordered ($x \leq y$ if and only if $x = y$), then $I(X, R) \cong \prod_{x \in X} R$.

Example 47. If $X$ is completely preordered ($x \leq y$ for all $x, y \in X$) and $X$ contains $n$ elements ($X$ is necessarily finite), then $I(X, R) \cong M_n(R)$.

Example 48. If $X = \{1, \ldots, n\}$ with the usual ordering, then $I(X, R) \cong \mathbb{T}_n(R)$.
Incidence rings contain some naturally occurring substructures. If $Y$ is a subset of $X$ with the induced preordering, then $I(Y, R)$ is an incidence ring contained in $I(X, R)$. $I(Y, R)$ is not a subring of $I(X, R)$ (unless $Y = X$) since the element $\delta_Y$, defined by $\delta_Y(x, y) = 1$ if $x = y \in Y$ and 0 otherwise, is the identity of $I(Y, R)$. Clearly there is an isomorphism $\delta_Y I(X, R) \delta_Y \cong I(Y, R)$, so we may view $I(Y, R)$ as a Peirce corner of $I(X, R)$.

The isomorphism above may be viewed as induced by a natural map

$$I(X, R) \rightarrow I(Y, R)$$

given by $f \mapsto f|_Y$. Under certain conditions this map is well-behaved.

Recall that $Y \subseteq X$ is convex if given any $x, y \in Y$, the interval $[x, y] \subseteq Y$. An important example of a convex set is $Y = [x, y]$.

**Lemma 49.** If $Y$ is a convex subset of $X$ then the map $f \mapsto f|_Y$ is a ring homomorphism $I(X, R) \rightarrow I(Y, R)$.

**Proof.** We need only check multiplication:

$$f|_Y g|_Y(x, y) = \sum_{x \leq z \leq y} f|_Y(x, z)g|_Y(z, y)$$

$$= \sum_{x \leq z \leq y} f(x, z)g(z, y)$$

$$= (fg)|_Y(x, y). \quad \square$$

**Remark 50.** Lemma 49 tells us that given $x, y \in X$, and $f, g \in I(X, R)$, it makes no difference whether we evaluate $fg(x, y)$ in $I(X, R)$ or in $I([x, y], R)$; that is, we can evaluate $fg$ at $(x, y)$ “locally.”

For the balance of this section we will assume that $X$ is a locally finite partially ordered set (i.e. $\leq$ is antisymmetric). We will abbreviate “partially ordered set” to “poset.” Before formulating any new results, we collect some basic facts about incidence rings of posets, omitting the (straightforward) proofs. An accessible reference for these is [18]—for our purposes, the authors’ standing assumption that $R$ is commutative can safely be ignored.

**Lemma 51.** Let $R$ be a ring, $X$ a poset and $f \in I(X, R)$.

1. If $f^2 = f$ then $f(x, x)^2 = f(x, x)$ for all $x \in X$.
2. $f \in U(I(X, R))$ if and only if $f(x, x) \in U(R)$ for all $x \in X$.
3. $f \in J(I(X, R))$ if and only if $f(x, x) \in J(R)$ for all $x \in X$.

**Lemma 52.** Let $X = \{x_1, \ldots, x_n\}$ be a (finite) poset. We can index $X$ so that $x_i \leq x_j$ implies $i \leq j$.

**Lemma 53.** If $X = \{x_1, \ldots, x_n\}$ is a poset, then $I(X, R)$ can be embedded as a subring of $T_n(R)$. 
This representation of the incidence ring of a finite poset enables us to reuse the calculations of Lemma 6 to produce:

**Lemma 54 (Workhorse for incidence rings).** Let $R$ be a local ring and $X$ a locally finite poset. Let $f, g \in I(X, R)$. Let $x, y \in X$ such that $x > y$. Suppose that for all $x', y' \in [x, y]$ with $(x', y') \neq (x, y)$, we have $g^2(x', y') = g(x', y')$, $(fg - gf)(x', y') = 0$. Suppose that we write

\[
[f\rceil_{x,y} = \begin{pmatrix} a & \alpha & c \\ B & \beta \\ b \end{pmatrix}, \quad [g\rceil_{x,y} = \begin{pmatrix} e & \gamma & z \\ B & \delta \\ f \end{pmatrix}.
\]

Then

1. If $e = f = 1$, there exists a unique $z \in R$ such that $g^2(x, y) = g(x, y)$; and for this $z$, $(fg - gf)(x, y) = 0$.
2. If $e = f = 0$, there exists a unique $z \in R$ such that $g^2(x, y) = g(x, y)$; and for this $z$, $(fg - gf)(x, y) = 0$.
3. If $e = 1$ and $f = 0$, then $g^2(x, y) = g(x, y)$; and $(fg - gf)(x, y) = 0$ if and only if $z$ satisfies the equation $az - zb = \gamma \beta - \alpha \delta + c$.
4. If $e = 0$ and $f = 1$, $g^2(x, y) = g(x, y)$; and $(fg - gf)(x, y) = 0$ if and only if $z$ satisfies the equation $az - zb = \gamma \beta - \alpha \delta - c$.

**Theorem 55.** Let $R$ be a bleached local ring and $X$ a locally finite poset. Then $I(X, R)$ is strongly clean.

**Proof.** Let $f \in I(X, R)$. If $|[x, y]| = 1$ (i.e. if $x = y$), set $g(x, x) = 1$ if $f(x, x) \in J(R)$ and $g(x, x) = 0$ if $f(x, x) \in U(R)$. This ensures that for all $x \in X$, $(f - g)(x, x) \in U(R)$ and hence that $f - g$ is a unit.

Note also that if $|[x, y]| = 0$ or 1 then $g^2(x, y) = g(x, y)$ and $(fg - gf)(x, y) = 0$.

We will now define $g$ by induction on $|[x, y]|$ so that $g^2(x, y) = g(x, y)$ and $(fg - gf)(x, y) = 0$. The base cases have already been dealt with.

Assume that the inductive hypothesis holds for all $x', y' \in X$ with $|[x', y']| < n$ and that $|[x, y]| = n$. As in the proof of Lemma 7, the inductive hypothesis enables us to apply Lemma 54 and the hypothesis that $R$ is bleached ensures that all necessary equations can be solved. Hence we can define $g(x, y)$ so that $g^2(x, y) = g(x, y)$ and $(fg - gf)(x, y) = 0$ as required. \(\square\)

In Section 4 we saw that the condition that $R$ is bleached is stronger than needed to ensure that each $\mathbb{T}_n(R)$ is strongly clean; weaker conditions are given in Theorem 22. Since these conditions give sharper hypotheses under which Lemma 6 may be applied inductively, they apply to incidence rings too. The reader should refer to Section 3 for the definition and important properties of the operator $Bl$ and the sets $J_k$. The following theorem is a generalization of Corollary 24.

**Theorem 56.** Let $I(X, R)$ be an incidence ring of a locally finite poset over a local ring $R$. If $J_k \subset J_{k+1}$ and $R = Bl(J_k) \cup Bl(1 - J_k)$ for all $k \in \{0, 1, \ldots\}$, then $I(X, R)$ is strongly clean.

**Proof.** Let $f \in I(X, R)$. We define an idempotent $g$ as follows. If $f(x, x) \in J_k$ for some $k$, set $g(x, x) = 1$. Otherwise, set $g(x, x) = 0$. This ensures that $f - g$ is a unit. We then define $g(x, y)$
so that \( g^2(x, y) = g(x, y) \) and \( (fg - gf)(x, y) = 0 \) by induction on \( n = ||x, y|| \). Note that we have already taken care of the base cases \( n = 0 \) and \( n = 1 \). Now assume that we have appropriately defined \( g(x', y') \) for all \( (x', y') \) with \( ||x', y'|| < n \). If \( x, y \in X \) satisfy \( ||x, y|| = n \) then our inductive hypothesis ensures that Lemma 54 applies. If \( g(x, x) = g(y, y) \), then Lemma 54 gives us a suitable value for \( g(x, y) \). If \( g(x, x) \neq g(y, y) \), we may, without loss of generality, assume that \( g(x, x) = 1 \) and \( g(y, y) = 0 \). (If not, apply Lemma 54 to \( 1 - g \).) Then for some \( k \), \( f(x, x) \in J_k \) and \( f(y, y) \notin J_{k+1} \). Hence by Lemma 21, \( f(y, y) \in \text{Bl}(f(x, x)) \), and by Lemma 54 we can define \( g(x, y) \) as needed. This completes our induction.

**Remark 57.** The hypotheses of Theorem 56 represent the conjunction over \( n \) of the conditions \( B_n \) of Theorem 22. If \( X \) is finite, then we do not need all of these hypotheses to hold. An upper bound is given by \( |X| \). However this moves us further away from the original aim of finding necessary and sufficient conditions for \( I(X, R) \) to be strongly clean. It does not appear to be possible simply to impose the conditions up to the length (the supremum of the sizes of maximal chains) of \( |X| \), even if \( X \) is a finite distributive lattice.

6. Conclusion

In this paper, we have studied the strong cleanness of upper triangular matrix rings over local rings by studying the solutions of equations of the form

\[
ax - zb = c.
\]

Equations of this type (e.g. the metro equation) have been studied by others (e.g. [12,13]) for various classes of rings.

Regardless, several unanswered questions remain. The most notable of these concern missing examples.

**Problem 58.**

(a) For each \( n > 1 \), give an example of a local ring \( R \) such that \( T_n(R) \) is strongly clean, but \( T_{n+1}(R) \) is not strongly clean.

(b) For each \( n > 1 \) give a local ring \( R \) which satisfies \( B_n \) but which does not satisfy \( B_{n+1} \).

Example 45 provides a solution to parts (a) and (b) of Problem 58 for \( n = 1 \). By Theorem 22 and Theorem 33, parts (a) and (b) of Problem 58 are equivalent for local \( h \)-rings. Note that an answer to Problem 58 must necessarily avoid the hypotheses of Theorem 27.

It would also be of interest to pursue Problem 58 for rings which are not necessarily local. Theorem 22 shows that if \( R \) satisfies condition \( B_n \), then \( T_n(R) \) is strongly clean. On the other hand, the following problem remains open.

**Problem 59.**

(a) For \( n > 2 \), find a local ring \( R \) such that \( T_n(R) \) is strongly clean but such that \( R \) does not satisfy condition \( B_n \).

(b) For each \( n > 2 \), determine general conditions on a local ring \( R \) which imply that \( B_n \) is necessary and sufficient for \( T_n(R) \) to be strongly clean.
Recall that, by Theorem 33 and the second remark following, one must look outside the class of $gh$-rings for answers to Problem 59.

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