JOURNAL OF DIFFERENTIAL EQUATIONS 116, 249–264 (1995)

# Real Polynomial Systems of Degree $n$ with $n+1$ Line Invariants

ROBERT E. KOOIJ

*Faculty of Technical Mathematics and Informatics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands*

Received December 6, 1991; revised May 14, 1992

In this paper we study the number of limit cycles for special classes of real polynomial systems of differential equations on the plane. It is shown that for polynomial systems of degree $n$ which possess $n+1$ line invariants both the relative positions of the $n+1$ line invariants and their types (i.e., real or complex) influence the number of limit cycles. © 1995 Academic Press, Inc.

## 0. INTRODUCTION

A real polynomial system of degree $n$ is a pair of autonomous differential equations on the plane

$$\begin{aligned} \frac{dx}{dt} &= \dot{x} = P(x, y) = \sum_{i+j=0}^{n} a_{ij} x^i y^j, \\ \frac{dy}{dt} &= \dot{y} = Q(x, y) = \sum_{i+j=0}^{n} b_{ij} x^i y^j, \end{aligned} \tag{0.1}$$

where at least one of the real polynomials $P$, $Q$ has degree $n$.

The most tempting problem associated with system (0.1) is finding the number of limit cycles. This is of course a part of the 16th problem proposed by Hilbert in 1900 [4].

It is well known that the number of limit cycles of system (0.1) is seriously restricted by the existence of algebraic invariant curves; see, for instance, Bautin [1], Rychkov [8], and Gasull [3]. Kooij [7], studied the cubic system (i.e., system (0.1) with $n=3$) that has four line invariants. In this paper the results of Kooij [7] are generalised, i.e, we study the real polynomial system of degree $n$ that has $n+1$ line invariants.



0022-0396/95 $6.00

## 1. Some Definitions and Lemmas

In this section some definitions and lemmas are stated, as well as the results from Kooij [7] that will be generalised.

Definition 1.1 (Christopher [2]). An algebraic invariant curve of system (0.1) is a set of points (considered over $\mathbb{C}^2$) satisfying an equation $C(x, y) = 0$ where $C$ is a polynomial in $x$ and $y$ such that $dC/dt = \dot{C} = C_x P + C_y Q = CL$, for some polynomial $L$ of degree $m \leqslant n-1$.

Lemma 1.1 (Ye Yanqian [10]). *If there exists a continuously differentiable function $\mu(x, y)$ in a simply connected region $G$ such that* $\operatorname{div}(\mu P, \mu Q) = 0$ *then system* (0.1) *has no limit cycles in $G$. If $G$ contains an antisaddle then it is a centre.*

*Remark.* The function $\mu(x, y)$ satisfying the conditions of Lemma 1.1 is called an integrating factor.

Lemma 1.2 (Ye Yanqian [10]). *If there exists a continuously differentiable function $B(x, y)$ in a simply connected region $G$ such that* $\operatorname{div}(BP, BQ)$ *has constant sign then system* (0.1) *has no limit cycles in the region $G$.*

*Remark.* Lemma 1.2 is known as Dulac's criterion. The function $B(x, y)$ satisfying the conditions of Lemma 1.2 is called a Dulac function.

Lemma 1.3. *A singularity that is part of a real line invariant cannot be surrounded by a closed orbit.*

*Proof.* A closed orbit surrounding a singularity that is part of a real line invariant would intersect this invariant and this would violate the uniqueness theorem.

Lemma 1.4. *Suppose that the lines $l_i = 0$, $i = 1, ..., n$, are such that not one pair of the lines is parallel and no more than two lines pass through the same point. Then any real polynomial system of degree $n$ for which the lines $l_i = 0$ are invariant can be written in the form*

$$\dot{x} = \left(-\sum_{i=1}^{n} \frac{A_i l_{iy}}{l_i} + b\right) \prod_{i=1}^{n} l_i,$$

$$\dot{y} = \left(\sum_{i=1}^{n} \frac{A_i l_{ix}}{l_i} + d\right) \prod_{i=1}^{n} l_i,$$

*where $l_{ix} = \partial l_i / \partial x$, $l_{iy} = \partial l_i / \partial y$, $b$, $d \in \mathbb{R}$ and $A_i$, $i = 1, ..., n$, is linear in $x$ and $y$.*

The proof of Lemma 1.4 follows from Thm. 3.7 of Christopher [2].

THEOREM 1.1 (Kooij [7]). *Suppose that the cubic system has four line invariants* $l_i = 0$, $i = 1, 2, 3, 4$.

(a) *the cubic system with four line invariants where not one pair is parallel, such that no more than two lines pass through the same point, has an integrating factor* $\mu(x, y) = 1/l_1 l_2 l_3 l_4$;

(b) *the cubic system with four line invariants where not one pair is parallel, such that exactly three lines pass through the same point, has a Dulac function* $B(x, y) = 1/l_1 l_2 l_3 l_4$;

(c) *the cubic system with four line invariants where exactly one pair is parallel, such that no more than two lines pass through the same point, has a Dulac function* $B(x, y) = 1/l_1 l_2 l_3 l_4$;

(d) *for the cubic system with four line invariants all passing through the same point, all singularities are part of* $l_1 l_2 l_3 l_4 = 0$.

In the next section Theorem 1.1 is generalised by studying the real polynomial system of degree $n$ with $n+1$ real line invariants. The following notation will be used: $E_n^m(k)$ denotes the real polynomial system of degree $n$ with $n+1$ real line invariants in $m$ directions, where it happens $k$ times that three lines pass through the same point. To elucidate this notation we refer to Fig. 1.1.

## 2. THE REAL POLYNOMIAL SYSTEM OF DEGREE $n$ WITH $n+1$ REAL LINE INVARIANTS

**2.1.** In this subsection we study systems of class $E_n^{n+1}(0)$, i.e., the real polynomial system of degree $n$ with $n+1$ real line invariants where not one pair is parallel, such that no more than two lines pass through the same point. Let the line invariants of $E_n^{n+1}(0)$ be $l_i = 0$, $i = 1, ..., n+1$.

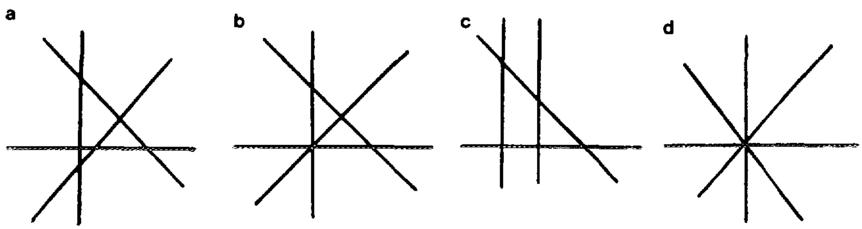


FIG. 1.1. The notation $E_n^m(k)$: (a) $E_3^4(0)$; (b) $E_3^4(1)$; (c) $E_3^3(0)$; (d) $E_3^4(2)$.

THEOREM 2.1. *The system* $E_n^{n+1}(0)$ *has an integrating factor* $\mu(x, y) = 1/(l_1 l_2 \cdot \cdots \cdot l_n l_{n+1})$.

COROLLARY. *The system* $E_n^{n+1}(0)$ *has no limit cycles. If it has an antisaddle* $S \notin l_1 l_2 \cdot \cdots \cdot l_n l_{n+1} = 0$ *then it is a centre.*

Without loss of generality it can be assumed that the $n+1$ line invariants are $l_i = 0$ and $x = 0$, where $l_i = \lambda_i x + \eta_i y + \mu_i$, $i = 1, ..., n$. Because for the $n$ lines $l_i$ not one pair is parallel and no more than two lines pass through the same point, it follows from Lemma 1.4 that system $E_n^{n+1}(0)$ has the form

$$\begin{aligned} \dot{x} &= \left( \sum_{i=1}^{n} \frac{-(p_i x + q_i y + r_i)\,\eta_i}{\lambda_i x + \eta_i y + \mu_i} + b \right) \prod_{i=1}^{n} l_i, \\ \dot{y} &= \left( \sum_{i=1}^{n} \frac{(p_i x + q_i y + r_i)\,\lambda_i}{\lambda_i x + \eta_i y + \mu_i} + d \right) \prod_{i=1}^{n} l_i, \end{aligned} \tag{2.1}$$

where

$$\left( \sum_{i=1}^{n} \frac{-(q_i y + r_i)\,\eta_i}{\eta_i y + \mu_i} + b \right) \prod_{i=1}^{n} (\eta_i y + \mu_i) = 0, \qquad \forall y \in \mathbb{R}, \tag{2.2}$$

because $x = 0$ is a line invariant of system (2.1).

Two more conditions have to be satisfied in order to make system (2.1) equivalent with system $E_n^{n+1}(0)$:

$$\eta_i \neq 0, \qquad i = 1, ..., n, \tag{2.3}$$

$$\mu_i \eta_j - \mu_j \eta_i \neq 0, \qquad i \neq j,\ i, j = 1, ..., n. \tag{2.4}$$

Note that for $\eta_i = 0$ the lines $x = 0$ and $l_i = 0$ are parallel and if $\mu_i \eta_j - \mu_j \eta_i = 0$ then $x = 0$, $l_i = 0$ and $l_j = 0$ all pass through the same point.

Next it will be shown that system (2.1), where (2.2), (2.3), and (2.4) are satisfied, has an integrating factor $\mu(x, y) = 1/(x l_1 l_2 \cdot \cdots \cdot l_n)$.

An elementary calculation shows that

$$\operatorname{div}(\mu P, \mu Q) = \frac{1}{x^2} \left( \sum_{i=1}^{n} \frac{q_i \lambda_i x + q_i \eta_i y + r_i \eta_i}{\lambda_i x + \eta_i y + \mu_i} - b \right).$$

Because

$$b = \sum_{i=1}^{n} \frac{(q_i y + r_i)\,\eta_i}{\eta_i y + \mu_i},$$

by (2.2), it follows that

$$x \operatorname{div}(\mu P, \mu Q) = \sum_{i=1}^{n} \frac{\lambda_i(q_i \mu_i - r_i \eta_i)}{(\lambda_i x + \eta_i y + \mu_i)(\eta_i y + \mu_i)}. \tag{2.5}$$

Consulting (2.2) with $y = 0$, one obtains

$$b = \sum_{i=1}^{n} \frac{r_i \eta_i}{\mu_i}, \tag{2.6}$$

where it can be assumed that $\mu_i \neq 0$ for if $\mu_i = 0$ then an appropriate translation will make $\mu_i \neq 0$.

From (2.2) and (2.6) it follows that

$$\sum_{i=1}^{n} \frac{t_i}{y + s_i} = 0, \qquad \forall y \in \mathbb{R}, \tag{2.7}$$

where $t_i = q_i - ((r_i \eta_i)/\mu_i)$ and $s_i = \mu_i/\eta_i$.

It is easy to see that because $s_i \neq s_j$ for $i \neq j$, by virtue of (2.4), (2.7) implies that $t_i = 0$ for $i = 1, ..., n$. Hence from (2.5) it shows that $\mu(x, y)$ is an integrating factor of system (2.1). This completes the proof of Theorem 2.1. The Corollary follows from Lemma 1.1.

**2.2.** In this subsection we study systems of class $E_n^{n+1}(1)$, i.e., the real polynomial system of degree $n$ with $n+1$ real line invariants where not one pair is parallel, such that there is only one point where more than two lines pass through. The number of lines passing through this point is three. Let the line invariants of $E_n^{n+1}(1)$ be $l_i = 0$, $i = 1, ..., n+1$.

THEOREM 2.2. *The system* $E_n^{n+1}(1)$ *has a Dulac function* $B(x, y) = 1/(l_1 l_2 \cdot \cdots \cdot l_n l_{n+1})$.

COROLLARY. *The system* $E_n^{n+1}(1)$ *has no limit cycles.*

Without loss of generality it can be assumed that the $n+1$ line invariants are $l_i = 0$ and $x = 0$, where $l_i = \lambda_i x + \eta_i y + \mu_i$, $i = 1, ..., n$. If for the $n$ lines $l_i$ not one pair is parallel and no more than two lines pass through the same point, Lemma 1.4 can be applied.

So the system $E_n^{n+1}(1)$ is also described by system (2.1) where (2.2) holds because $x = 0$ is a line invariant. Note that for system (2.1) to have a point with three lines passing through it, it is necessary that this happens at $x = 0$, so it can be assumed that

$$l_1 = \lambda_1 x + y + \mu_1, \qquad l_2 = \lambda_2 x + y + \mu_1. \tag{2.8}$$

In addition, two conditions have to be fulfilled in order to make system (2.1) equivalent with system $E_n^{n+1}(1)$:

$$\eta_i \neq 0, \qquad i=3, ..., n, \tag{2.9}$$

$$\mu_i \eta_j - \mu_j \eta_i \neq 0, \qquad i<j, (i, j) \neq (1, 2), i, j=1, ..., n. \tag{2.10}$$

Note that $\eta_1 = \eta_2 = 1$ and $\mu_1 = \mu_2$.

Next it will be shown that system (2.1) where (2.2), (2.8), (2.9), and (2.10) are satisfied, has a Dulac function $B(x, y) = 1/(xl_1 l_2 \cdot \cdots \cdot l_n)$.

Replacing $\mu$ by $B$ we again have to consider (2.5). Following (2.2) and (2.6), we obtain

$$\sum_{i=1}^{n} \frac{t_i}{y+s_i} = 0, \qquad \forall y \in \mathbb{R}, \tag{2.11}$$

where $t_i = q_i - ((r_i \eta_i)/\mu_i)$ and $s_i = \mu_i/\eta_i$.

Hence $s_1 = s_2$ by virtue of (2.8) and $s_i \neq s_j$, $i<j$, $(i, j) \neq (1, 2)$. Therefore it is easy to see that (2.11) implies that $t_1 + t_2 = 0$ and $t_i = 0$, $i=3, ..., n$. It follows from (2.5) that

$$x \operatorname{div}(\mathrm{BP}, \mathrm{BQ}) = \frac{(\lambda_1 - \lambda_2)(q_1 \mu_1 - r_1)}{(\lambda_1 x + y + \mu_1)(\lambda_2 x + y + \mu_1)},$$

and hence $B(x, y)$ is a Dulac function. This completes the proof of Theorem 2.2. The Corollary follows from Lemma 1.2.

**2.3.** In this subsection we study systems of class $E_n^n(0)$, i.e., the real polynomial system of degree $n$ with $n+1$ real line invariants where exactly one pair is parallel, such that no more than two lines pass through the same point. Let the line invariants of $E_n^n(0)$ be $l_i = 0$, $i=1, ..., n+1$.

THEOREM 2.3. *The system* $E_n^n(0)$ *has a Dulac function* $B(x, y) = 1/(l_1 l_2 \cdot \cdots \cdot l_n l_{n+1})$.

COROLLARY. *The system* $E_n^n(0)$ *has no limit cycles.*

Without loss of generality it can be assumed that the $n+1$ line invariants are $l_i = 0$ and $x=0$, where $l_i = \lambda_i x + \eta_i y + \mu_i$, $i=1, ..., n$. If for the $n$ lines $l_i$ not one pair is parallel and no more than two lines pass through the same point, Lemma 1.4 can be applied. So the system $E_n^n(0)$ is also described by system (2.1) where (2.2) holds because $x=0$ is a line invariant. Note that for system (2.1) to have a pair of parallel lines, it is necessary that $x=0$ is one of them, so it can be assumed that

$$l_1 = x + 1. \tag{2.12}$$

In addition, two conditions have to be fulfilled in order to make system (2.1) equivalent with system $E_n^n(0)$:

$$\eta_i \neq 0, \qquad i=2, ..., n, \tag{2.13}$$

$$\mu_i\eta_j - \mu_j\eta_i \neq 0, \qquad i\neq j,\ i,j=1, ..., n. \tag{2.14}$$

Note that $\eta_1 = 0$ and $\lambda_1 = \mu_1 = 1$.

Next it will be shown that system (2.1) where (2.2), (2.12), (2.13), and (2.14) are satisfied, has a Dulac function $B(x, y) = 1/(xl_1 l_2 \cdot \cdots \cdot l_n)$.

Replacing $\mu$ by $B$ we again have to consider (2.5). Following (2.2) and (2.6), we obtain

$$\sum_{i=2}^{n} \frac{t_i}{y+s_i} = 0, \qquad \forall y \in \mathbb{R}, \tag{2.15}$$

where $t_i = q_i - ((r_i\eta_i)/\mu_i)$ and $s_i = \mu_i/\eta_i$.

Hence $s_i \neq s_j$, $i \neq j$ by virtue of (2.14). Therefore it is easy to see that (2.15) implies that $t_i = 0$, $i = 2, ..., n$. It follows from (2.5) that

$$x\, \mathrm{div}(\mathrm{BP}, \mathrm{BQ}) = \frac{q_1}{x+1}$$

and hence $B(x, y)$ is a Dulac function. This completes the proof of Theorem 2.3. The Corollary follows from Lemma 1.2.

*Remark.* Theorem 2.3 also holds if the parallel lines coincide. This can be shown by assuming that $x = 0$ is a line invariant of multiplicity two. The proof is very similar to the previous one and is omitted.

**2.4.** In this subsection we will give examples of real polynomial systems of degree $n$ with $n+1$ real line invariants that have limit cycles. The systems are close to the ones studied in the previous three subsections, i.e., $E_n^{n+1}(2)$, $E_n^n(1)$, and $E_n^{n-1}(0)$.

THEOREM 2.4. *For all $n > 3$ ($n > 2$) there exist systems in $E_n^{n+1}(2)$ ($E_n^n(1)$ and $E_n^{n-1}(0)$) with at least one limit cycle.*

The system $E_n^{n+1}(2)$, with $n > 3$, has two types. Either there is one point with four lines passing through it or there are two points with three lines passing through them; see Fig. 2.1.

We will give examples of the two types of system $E_4^5(2)$ that exhibit the Andronov–Hopf bifurcation, generating a limit cycle.

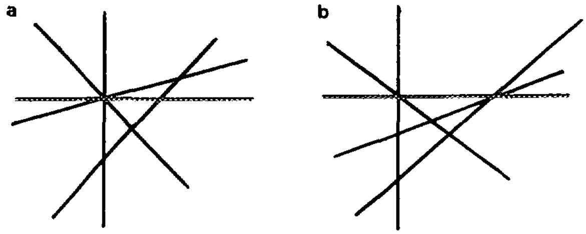


FIG. 2.1. $E_4^5(2)$: (a) type 1; (b) type 2.

Let be given the system

$$\begin{aligned}\dot{x} = {} & (x+1)((189-23\mu)x+(19\mu-297)y+(45\mu-375)x^2+(72\mu-96)xy \\ & +(57+21\mu)y^2+(81+13\mu)x^3+(21+33\mu)x^2y \\ & +(66+18\mu)xy^2+120y^3)=P, \qquad (2.16) \\ \dot{y} = {} & (y+1)((297-59\mu)x-(189+17\mu)y-(222+6\mu)x^2+120xy+120y^2 \\ & +(135-5\mu)x^3+120x^2y+120xy^2+(129-3\mu)y^3)=Q.\end{aligned}$$

The line invariants of system (2.16) are $l_1=x+1=0$, $l_2=y+1=0$, $l_3=x+y-1=0$, $l_4=2x+y+3=0$, and $l_5=3x+y+4=0$. Note that four lines pass through the point $(-1, -1)$.

For $\mu=0$ the origin of system (2.16) is an unstable first order weak focus and therefore system (2.16) has a limit cycle for $0<\mu\ll 1$.

Let be given the system

$$\begin{aligned}\dot{x} = {} & (x+1)((136\mu-12)x+(276\mu-30)y-(9+50\mu)x^2 \\ & +(18+64\mu)xy+(266\mu-27)y^2+(21+26\mu)x^3+(6-52\mu)x^2y \\ & +(15-26\mu)xy^2+(64\mu-6)y^3), \qquad (2.17) \\ \dot{y} = {} & 2(y+1)((15+190\mu)x+(6-24\mu)y+(9+10\mu)x^2 \\ & +(21+90\mu)xy+3y^2+12x^3+6xy^2+6\mu y^3).\end{aligned}$$

The line invariants of system (2.16) are $(x+1)(y+1)(y-x+2)\times(2y-x+3)(x+y+2)=0$. Note that three lines pass through the points $(-1, -1)$ and $(1, -1)$.

For $\mu=0$ the origin of system (2.17) is a stable first order weak focus and therefore system (2.17) has a limit cycle for $0<\mu\ll 1$.

Starting with system (2.16) we can construct a system in $E_n^{n+1}(2)$, with $n>4$, with at least one limit cycle. We fix the value of $\mu$ such that system (2.16) has at least one limit cycle. Next we choose $n-4$ real straight lines $l_i=\lambda_i x+\eta_i y+\mu_i=0$, $i=6, ..., n+1$, such that none of the lines $l_i=0$

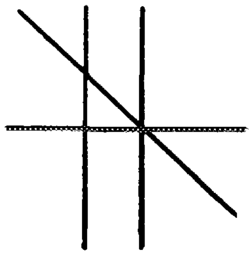

FIG. 2.2. $E_3^3(1)$.

intersect the limit cycle in system (2.16). Furthermore, it is assumed that not one pair of the lines $l_i = 0$, $i = 1, ..., n+1$, is parallel and in the only intersection point of the lines where more than two lines are passing through, exactly four lines intersect. Then the system

$$\dot{x} = P \prod_{i=6}^{n+1} l_i,$$

$$\dot{y} = Q \prod_{i=6}^{n+1} l_i,$$

is a system of type $E_n^{n+1}(2)$ with at least one limit cycle.

Kooij [6] gave an example of a cubic system with four real line invariants where one pair is parallel and three lines pass through the same point, see Fig. 2.2, with at least one limit cycle.

Starting with this system in $E_3^3(1)$ with at least one limit cycle, using the same construction as above, one can find systems in $E_n^n(1)$, with $n > 3$, with at least one limit cycle.

The system $E_n^{n-1}(0)$, with $n > 2$, has two types. Either there are two pairs of parallel lines or there are three lines parallel; see Fig. 2.3. Kooij [6] gave an example of a cubic system with two pairs of parallel real line invariants, with at least one limit cycle. This means that the system $E_n^{n-1}(0)$ of the first type can have limit cycles. We will give an example of the system $E_4^3(0)$ of the second type that exhibits the Andronov–Hopf bifurcation.

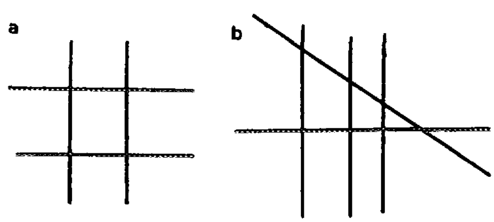


FIG. 2.3. $E_n^{n-1}(0)$: (a) $E_3^2(0)$, type 1; (b) $E_4^3(0)$, type 2.

Let be given the system

$$\begin{aligned} \dot{x} &= (x+1)(x-1)(x-2)(x+3y), \\ \dot{y} &= (y+1)(-(12+6\mu)\,x+(6\mu-2)\,y \\ &\quad +(16+3\mu)\,x^2-3\mu y^2+3xy^2-5x^3). \end{aligned} \tag{2.18}$$

The line invariants of system (2.18) are $(x+1)(x-1)(x-2)(y+1)(x+y-2)=0$. For $\mu=0$ the origin of system (2.18) is a stable first order weak focus and therefore system (2.18) has a limit cycle for $0<\mu\ll 1$.

Starting with system (2.18) with at least one limit cycle, using the same construction as above, one can find systems in $E_n^{n-1}(0)$, with $n>4$, with at least one limit cycle.

Theorem 2.4 is completely proved.

*Remark.* The stability of the origin of system (2.16), (2.17), and (2.18) for $\mu=0$, is determined with the aid of a program written in Macsyma; see Kertész and Kooij [5].

**2.5.** In this subsection we study systems of class $E_n^{n+1}(n-1)$, i.e., the real polynomial system of degree $n$ with $n+1$ real line invariants all passing through the same point. Let the line invariants of $E_n^{n+1}(n-1)$ be $l_i=0$, $i=1, ..., n+1$.

THEOREM 2.5. *All singularities of the system $E_n^{n+1}(n-1)$ are part of $l_1 l_2 \cdot \cdots \cdot l_n l_{n+1}=0$.*

COROLLARY. *The system $E_n^{n+1}(n-1)$ has no closed orbits.*

Without loss of generality it can be assumed that the $n+1$ line invariants are $x=0$, $y=0$, $y-\lambda_i x=0$, $i=1, ..., n-1$, with $\lambda_i \neq 0$. Then the system $E_n^{n+1}(n-1)$ has the form

$$\begin{aligned} \dot{x} &= x(P_0+P_1+\cdots+P_{n-1})=P, \\ \dot{y} &= y(Q_0+Q_1+\cdots+Q_{n-1})=Q, \end{aligned} \tag{2.19}$$

where $P_j$ and $Q_j$ are homogeneous polynomials in $x$ and $y$ of degree $j$.

Because $y-\lambda_i x=0$ are line invariants of system (2.19) it follows that

$$(Q-\lambda_i P)_{y=\lambda_i x}=0, \qquad \forall x\in\mathbb{R}, \quad i=1, ..., n-1. \tag{2.20}$$

It is easy to show that (2.20) is equivalent with

$$\sum_{j=0}^{n-1} (Q_j(1,\lambda_i)-P_j(1,\lambda_i))\,x^j=0, \qquad \forall x\in\mathbb{R}, \quad i=1, ..., n-1, \tag{2.21}$$

and therefore

$$Q_j(1, \lambda_i) = P_j(1, \lambda_i), \qquad \text{for} \quad i = 1, ..., n-1, \quad j = 0, ..., n-1. \tag{2.22}$$

For $j < n-1$ $P_j(1, z) - Q_j(1, z)$ is a polynomial in $z$ of degree less than $n-1$ that has $n-1$ roots (namely $\lambda_i$, $i = 1, ..., n-1$) and hence $P_j(1, z) = Q_j(1, z)$ for $j < n-1$ which implies $P_j(x, y) = Q_j(x, y)$ for $j < n-1$.

Thus the system $E_n^{n+1}(n-1)$ has the following form:

$$\begin{aligned} \dot{x} &= x(P_0 + P_1 + \cdots + P_{n-2} + P_{n-1}) = x\bar{P}, \\ \dot{y} &= y(P_0 + P_1 + \cdots + P_{n-2} + Q_{n-1}) = y\bar{Q}, \end{aligned} \tag{2.23}$$

Note that for $P_{n-1} = Q_{n-1}$, system (2.23) is essentially linear and has infinitely many line invariants.

Any singularity of system (2.23) not lying on $x = 0$ or $y = 0$ has to satisfy $\bar{P} = \bar{Q}$ and therefore $P_{n-1} - Q_{n-1} = 0$. From (2.22) with $j = n-1$, it follows that $P_{n-1} - Q_{n-1} = k \prod_{i=1}^{n-1} (y - \lambda_i x)$, $k \in \mathbb{R}$ and therefore all singularities of system (2.23) are part of a line invariant. This completes the proof of Theorem 2.5. The Corollary follows from Lemma 1.3.

A special subclass of the system $E_n^{n+1}(n-1)$ is formed by the homogeneous system. Let the line invariants of the homogeneous system be $l_i = 0$, $i = 1, ..., n+1$.

THEOREM 2.6. *The homogeneous system of degree n with $n+1$ real line invariants has an integrating factor* $\mu(x, y) = 1/(l_1 l_2 \cdot \cdots \cdot l_n l_{n+1})$.

Without loss of generality it can be assumed that the line invariants are $x = 0$, $y = 0$, $l_i = y - \lambda_i x = 0$, $i = 1, ..., n-1$. Then the homogeneous system reads

$$\begin{aligned} \dot{x} &= xP = x \sum_{i=0}^{n-1} a_i x^i y^{n-1-i}, \\ \dot{y} &= yQ = y \sum_{i=0}^{n-1} b_i x^i y^{n-1-i}. \end{aligned} \tag{2.24}$$

An elementary calculation shows that the condition that $l_i = y - \lambda_i x = 0$, $i = 1, ..., n-1$, are line invariants of system (2.24) implies $P(x, y) = Q(x, y) + (a_0 - b_0) \prod_{i=1}^{n-1} (y - \lambda_i x)$ and hence system (2.24) becomes

$$\begin{aligned} \dot{x} &= (a_0 - b_0)\, x \prod_{i=1}^{n-1} (y - \lambda_i x) + xQ = \bar{P}, \\ \dot{y} &= yQ \qquad\qquad\qquad\qquad\qquad\quad = \bar{Q}. \end{aligned} \tag{2.25}$$

Next we will show that $\mu(x, y) = 1/(xyl_1 l_2 \cdot \cdots \cdot l_{n-1})$ is an integrating factor for system (2.25).

It is easy to check that $\operatorname{div}(\mu\bar{P}, \mu\bar{Q}) = (xQ_x + yQ_y + (1-n)\,Q)\,\mu$. Finally, it can be deduced from $Q = \sum_{i=0}^{n-1} b_i x^i y^{n-1-i}$ that $xQ_x + yQ_y + (1-n)\,Q = 0$. This completes the proof of Theorem 2.6.

## 3. Complex Line Invariants, Some Definitions and Lemmas

In the previous section we have restricted our attention to real polynomial systems with real line invariants. However, it is well known that the real polynomial system can have complex line invariants. Obviously, complex line invariants of system (0.1) occur in conjugate pairs.

In order to extend the results of Section 2, let us state some definitions and lemmas.

Definition 3.1. Two lines $l_i = a_i x + b_i y + c_i = 0$, $a_i, b_i, c_i \in \mathbb{C}$, $i = 1, 2$, are called parallel if $a_1 b_2 - a_2 b_1 = 0$.

Lemma 3.1 (Suo Guangjian and Chen Yongshao [9]). *A pair of complex conjugated lines is either parallel or it has a real intersection point.*

Lemma 3.2 (Ye Yanqian [10]). *If there exists a continuously differentiable function $\mu(x, y)$ in a m-multiply connected region $G$ such that* $\operatorname{div}(\mu P, \mu Q) = 0$ *then system* (0.1) *has no limit cycles in $G$.*

*Remark.* The function $\mu(x, y)$ satisfying the conditions of Lemma 3.2 is called an integrating factor.

Lemma 3.3 (Ye Yanqian [10]). *If there exists a continuously differentiable function $B(x, y)$ in an m-multiply connected region $G$ such that* $\operatorname{div}(BP, BQ)$ *has constant sign then system* (0.1) *has at most $m-1$ limit cycles in $G$.*

*Remark.* The function $B(x, y)$ satisfying the conditions of Lemma 3.3 is called a Dulac function.

We will adopt the following notation: $\bar{E}_n^m(k)$, which denotes the real polynomial system of degree $n$ with $n+1$ line invariants (real or complex) in $m$ directions, where it happens $k$ times that three lines pass through the same point.

## 4. The Real Polynomial System of Degree $n$ with $n+1$ Real or Complex Line Invariants

**4.0.** The results in this section are an extention of Section 2 and of the results of Kooij [6] where real cubic systems with four line invariants (real or complex) are studied.

**4.1.** In this subsection we study systems of class $\bar{E}_n^{n+1}(0)$, i.e., the real polynomial system of degree $n$ with $n+1$ line invariants (real or complex) where not one pair is parallel, such that no more than two lines pass through the same point. Let the line invariants of $\bar{E}_n^{n+1}(0)$ be $l_i = 0$, $i = 1, ..., n+1$.

THEOREM 4.1. *The system $\bar{E}_n^{n+1}(0)$ has an integrating factor $\mu(x, y) = 1/(l_1 l_2 \cdot \cdots \cdot l_n l_{n+1})$.*

COROLLARY. *The system $\bar{E}_n^{n+1}(0)$ has no limit cycles.*

The proof of Theorem 4.1 is basically the same as the proof of Theorem 2.1. The corollary follows from Lemma 3.2. Note that for a pair of complex conjugated lines $l = ax + by + c = 0$, $\bar{l} = \bar{a}x + \bar{b}y + \bar{c} = 0$, the product $l\bar{l}$ is a quadratic polynomial in $x$ and $y$ with real coefficients. Therefore, if the system $\bar{E}_n^{n+1}(0)$ has $k$ pairs of complex conjugated lines, then, by Lemma 3.1, $\mu(x, y)$ has $k$ isolated singularities.

**4.2.** In this subsection we study systems of class $\bar{E}_n^{n+1}(1)$, i.e., the real polynomial system of degree $n$ with $n+1$ line invariants (real or complex) where not one pair is parallel, such that there is only one point where more than two lines pass through. The number of lines passing through this point is three. Let the line invariants of $\bar{E}_n^{n+1}(1)$ be $l_i = 0$, $i = 1, ..., n+1$.

THEOREM 4.2. *The system $\bar{E}_n^{n+1}(1)$ has a Dulac function $B(x, y) = 1/(l_1 l_2 \cdot \cdots \cdot l_n l_{n+1})$.*

COROLLARY 1. *The system $\bar{E}_n^{n+1}(1)$ that has k pairs of complex conjugated line invariants, such that none of the real lines pass through a real intersection point of the complex lines, has at most k limit cycles.*

COROLLARY 2. *The system $\bar{E}_n^{n+1}(1)$ that has k pairs of complex conjugated line invariants, such that one of the real lines passes through a real intersection point of the complex lines, has at most $k-1$ limit cycles.*

The proof of Theorem 4.2 is the same as the proof of Theorem 2.2. The corollaries follow from Lemma 3.3 because the Dulac function $B(x, y)$ of system $\bar{E}_n^{n+1}(1)$ described in Corollary 1 (resp. Corollary 2) has $k$ (resp. $k-1$) isolated singularities.

Next we will give an example of a system in $\bar{E}_4^5(1)$ that exhibits the Andronov–Hopf bifurcation, generating a limit cycle.

Let be given the system

$$\begin{aligned}\dot{x} = {} & -4\mu x - 8y - (8\mu+2)\,x^2 - 24xy + (4\mu-2)\,y^2 - 4\mu x^3 - (2\mu+23)\,x^2y \\ & + 9\mu xy^2 + (3-2\mu)\,y^3 + 2x^4 - (2\mu+7)\,x^3y \\ & + (5\mu+2)\,x^2y^2 + (3-2\mu)\,xy^3, \\ \dot{y} = {} & 8x - 4\mu y + 28x^2 - 12\mu xy + 4y^2 + 32x^3 - (8\mu+2)\,x^2y + 6xy^2 \\ & + (5\mu-2)\,y^3 + 12x^4 + 2x^3y + (2-2\mu)\,x^2y^2 + (5\mu+2)\,xy^3 - 2\mu x^4.\end{aligned} \tag{4.1}$$

The line invariants of system (4.1) are $(x+iy)(x-iy)(x+1)(2x-y+2)$ $(2x+y+2)=0$.

Note that three lines pass through the point $(-1, 0)$. For $\mu=0$ the origin of system (4.1) is an unstable first order weak focus and therefore system (4.1) has at least one limit cycle for $0<\mu\ll 1$. Because system (4.1) is of type $\bar{E}_4^5(1)$, this limit cycle is unique by Corollary 1 following Lemma 4.2.

**4.3.** In this subsection we study systems of class $\bar{E}_n^n(0)$, i.e., the real polynomial system of degree $n$ with $n+1$ line invariants (real or complex) where exactly one pair is parallel, such that no more than two lines pass through the same point. Let the line invariants of $\bar{E}_n^n(0)$ be $l_i=0$, $i=1, ..., n+1$.

THEOREM 4.3. *The system* $\bar{E}_n^n(0)$ *has a Dulac function* $B(x, y) = 1/(l_1 l_2 \cdot \cdots \cdot l_n l_{n+1})$.

COROLLARY 1. *The system* $\bar{E}_n^n(0)$ *that has k pairs of complex conjugated line invariants of which not one pair is parallel, has at most k limit cycles.*

COROLLARY 2. *The system* $\bar{E}_n^n(0)$ *that has k pairs of complex conjugated line invariants of which one pair is parallel, has at most* $k-1$ *limit cycles.*

The proof of Theorem 4.3 is basically the same as the proof of Theorem 2.3. The corollaries follow from Lemma 3.3 because the Dulac function $B(x, y)$ of system $\bar{E}_n^n(0)$ described in Corollary 1 (resp. Corollary 2) has $k$ (resp. $k-1$) isolated singularities.

Kooij [6] gave an example of a cubic system with two parallel line invariants (either real or complex) and one pair of intersecting complex conjugated line invariants, with at least one limit cycle. Because that system is of type $\bar{E}_3^3(0)$, the limit cycle is unique by Corollary 1 following Lemma 4.3.

**4.4.** In this subsection we study systems of class $\bar{E}_n^{n+1}(n-1)$, i.e., the real polynomial system of degree $n$ with $n+1$ line invariants (real or complex) all passing through the same point. Let the line invariants of $\bar{E}_n^{n+1}(n-1)$ be $l_i=0$, $i=1, ..., n+1$.

THEOREM 4.4. *All singularities of the system* $\bar{E}_n^{n+1}(n-1)$ *are part of* $l_1 l_2 \cdot \cdots \cdot l_n l_{n+1}=0$.

COROLLARY. *The system* $\bar{E}_n^{n+1}(n-1)$ *that has at least one real line invariant, has no closed orbits.*

The proof of Theorem 4.4 is basically the same as the proof of Theorem 2.5. The Corollary follows from Lemma 1.3.

*Remark.* If all line invariants of system $\bar{E}_n^{n+1}(n-1)$ are complex then a limit cycle can exist, see Kooij [6], where the existence and uniqueness of a limit cycle for the cubic system with four complex line invariants passing through one point is shown.

A special subclass of the system $\bar{E}_n^{n+1}(n-1)$ is formed by the homogeneous system of degree $n$. Obviously, such systems always have $n+1$ (real or complex) line invariants (multiplicity taken into account) passing through the same point. Let the line invariants of the homogeneous system be $l_i=0$, $i=1, ..., n+1$.

THEOREM 4.5. *The homogeneous system of degree n has an integrating factor* $\mu(x, y)=1/(l_1 l_2 \cdot \cdots \cdot l_n l_{n+1})$.

The proof of Theorem 4.5 is basically the same as the proof of Theorem 2.6.

## ACKNOWLEDGMENTS

The author wishes to thank A. Zegeling for his valuable suggestions and N. G. Lloyd and C. J. Christopher who suggested the use of Lemma 1.4, essential for most of the proofs in this paper.

## REFERENCES

1. N. N. BAUTIN, On periodic solutions of a system of differential equations, *Prikl. Mat. Mech.* **18** (1954), 128 [Russian].
2. C. J. CHRISTOPHER, Polynomial systems with invariant algebraic curves, preprint, University College of Wales, 1991.
3. A. GASULL, Polynomial systems with enough invariant algebraic curves, *in* "Dynamical Systems," pp. 73–79, Nankai Series in Pure Appl. Math. and Theor. Physics, Vol. 4, World Scientific, Singapore, 1993.

4. D. HILBERT, Mathematical problems, Mary Newton transl., *Bull. Amer. Math. Soc.* **8** (1902), 437–479.
5. V. KERTÉSZ AND R. E. KOOIJ, Degenerate Hopf bifurcation in two dimensions, *J. Nonlinear Anal. TMA* **17**, No. 3 (1991), 267–283.
6. R. E. KOOIJ, Limit cycles in polynomial systems, thesis, University of Technology, Delft, 1993.
7. R. E. KOOIJ, Cubic systems with four real line invariants, *Math. Proc. Camb. Phil. Soc.* (1995), to appear.
8. G. S. RYCHKOV, The limit cycles of the equation $u(x+1)\,du = (-x + ax^2 + bxu + cu + du^2)\,dx$, *Differencial'nye Uravnenjia* **8** (1972), 2257–2259 [Russian]; *Differential Equations* **8** (1972), 1748–1750 [English].
9. SUO GUANGJIAN AND CHEN YONGSHAO, The real quadratic system with two conjugate imaginary straight line solutions, *Ann. Differential Equations* **2**, No. 2 (1986), 197–207.
10. YE YANQIAN *et al.*, "Theory of Limit Cycles," Translations of Math. Monographs, Vol. 66, Amer. Math. Soc., Providence, RI, 1986.