Computers and Mathematics with Applications 56 (2008) 3015–3022

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Four positive periodic solutions of a discrete time delayed predator–prey system with nonmonotonic functional response and harvesting[☆]

Daowei Hu, Zhengqiu Zhang [*]

College of Mathematics and Econometrics, Hunan University, Changsha, 410082, PR China

ARTICLE INFO

Article history:
Received 1 November 2007
Received in revised form 22 August 2008
Accepted 10 September 2008

Keywords:
Four positive periodic solutions
Predator-prey system
Continuation theorem of coincidence degree theory
Nonmonotonic functional response
Harvesting

ABSTRACT

In this paper, by employing the continuation theorem of coincidence degree theory, we establish an easily verifiable criteria for the existence of at least four positive periodic solutions for a discrete time delayed predator–prey system with nonmonotonic functional response and harvesting.

1. Introduction

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1]. Since so, it has attracted many scholars' attention, and many authors have studied this class of models in the literatures, see [2–11]. Generally speaking, the traditional Lotka–Volterra type predator–prey model with ratio-dependent functional response is described as follows:

$$\begin{aligned} x'(t) &= x[a - bx] - cyg(x/y), \\ y'(t) &= y[-d + fg(x/y)], \end{aligned} \tag{1.1}$$

where $x(t)$ and $y(t)$ stand for the densities of the prey and predator, respectively, a, c, d, f are the prey intrinsic growth rate, capture rate, death rate of the predator, the conversion rate, respectively, a/b gives the carrying capacity of the prey, $g(u)$ is the functional response function. Since realistic models require the inclusion of the effect of changing environment and delays, this motive us to consider the following nonautonomous ratio-dependent predator–prey system with delays:

$$\begin{cases} x'(t) = x(t)[a(t) - b(t)x(t)] - c(t)y(t)g\left(\dfrac{x(t)}{y(t)}\right), \\ y'(t) = y(t)\left[-d(t) + f(t)g\left(\dfrac{x(t-\tau(t))}{y(t-\tau(t))}\right)\right], \end{cases} \tag{1.2}$$

☆ The Project supported by NNSF of China (No:10271044), by Postdoctoral Fund of China (No:20060400267) and Postdoctoral Fund of Central South University in China.

* Corresponding author.
E-mail addresses: hudw83@163.com (D. Hu), z_q_zhang@sina.com.cn (Z. Zhang).

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doi:10.1016/j.camwa.2008.09.009

where $\tau(t)$ denotes the delays in the conversion of prey to predator. In particular, considering (1.2) with Monod–Haldane nonmonotonic functional response (i.e. $g(u) = \frac{u}{m^2+u^2}$, see [2]), we obtain the following delayed predator–prey system:

$$\begin{cases} x'(t) = x(t)\left[a(t) - b(t)x(t) - \dfrac{c(t)y^2(t)}{m^2y^2(t)+x^2(t)}\right], \\ y'(t) = y(t)\left[\dfrac{f(t)x(t-\tau(t))y(t-\tau(t))}{m^2y^2(t-\tau(t))+x^2(t-\tau(t))} - d(t)\right]. \end{cases}$$

To make the model more realistic, now we consider the harvesting rate of prey in the above model and get the following model:

$$\begin{cases} x'(t) = x(t)\left[a(t) - b(t)x(t) - \dfrac{c(t)y^2(t)}{m^2y^2(t)+x^2(t)}\right] - h(t), \\ y'(t) = y(t)\left[\dfrac{f(t)x(t-\tau(t))y(t-\tau(t))}{m^2y^2(t-\tau(t))+x^2(t-\tau(t))} - d(t)\right]. \end{cases} \tag{1.3}$$

In this paper, we will consider a discrete analogue of system (1.3). First, with the help of differential equations with piecewise constant argument(for details, see [3]), we get the following discrete analogue of system (1.3):

$$\begin{cases} x(k+1) = x(k)\exp\left\{a(k) - b(k)x(k) - \dfrac{c(k)y^2(k)}{m^2y^2(k)+x^2(k)} - \dfrac{h(k)}{x(k)}\right\}, \\ y(k+1) = y(k)\exp\left\{\dfrac{f(k)x(k-\tau(k))y(k-\tau(k))}{m^2y^2(k-\tau(k))+x^2(k-\tau(k))} - d(k)\right\}. \end{cases} \tag{1.4}$$

In recent years, the powerful and effective method of coincidence degree has been applied to study the existence of a periodic solution or multiple periodic solutions in delayed differential population models and a number of good results have been obtained, for papers on a periodic solution, see [4,5], on multiple periodic solutions, see [6,7]. Since so much progress has been made in delay differential models, one question arises naturally: can we apply this powerful method to study the existence of periodic solutions for discrete analogues of these models governed by difference equations? Motivated by this problem, recently many authors have studied discrete models in the literature, see [2,11,12]. However, there are few papers on multiple periodic solutions of discrete models. For system (1.4), to the best of our knowledge, there is no result on multiple periodic solutions in the literature. So, in this paper, our purpose is to study the existence of multiple positive periodic solutions for system (1.4) by applying the continuation theorem of coincidence degree theory. Therefore, we assume that all parameters in system (1.4) are positive ω-periodic sequences, ω is a fixed positive integer denoting the common period of all the parameters in system (1.4). Since the nonmonotonic functional response is more difficult to deal with, we will employ some new arguments in our discussion. Our main result is presented in Section 2.

2. Existence of four positive periodic solutions

For the reader's convenience, we first summarize a few concepts from the book by Gaines and Mawhin [13].

Let X and Z be real normed vector spaces. Let $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping and $N : X \times [0, 1] \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$, and $X = \operatorname{Ker} L \bigoplus \operatorname{Ker} P$, $Z = \operatorname{Im} L \bigoplus \operatorname{Im} Q$. It follows that $L/_{\operatorname{Dom} L\cap \operatorname{Ker} P:}(I - P)X \to \operatorname{Im} L$ is invertible and its inverse is denoted by Kp. If Ω is a bounded open subset of X, the mapping N is called L−compact on $\bar{\Omega} \times [0, 1]$, if $QN(\overline{\Omega} \times [0, 1])$ is bounded and $Kp(I - Q)N : \overline{\Omega} \times [0, 1] \to X$ is compact. Because $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

In the proof of our existence result, we need the following continuation theorem.

Lemma 2.1 (*Continuation Theorem, Gaines and Mawhin [13]*). *Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega} \times [0, 1]$. Suppose*

(a) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda N(x, \lambda)$ is such that $x \notin \partial\Omega \cap \operatorname{Dom} L$;*
(b) *$QN(x, 0) \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$;*
(c) $\deg\{JQN(x, 0), \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then the equation $Lx = N(x, 1)$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

For the sake of convenience, we introduce some notations

$$I_\omega = \{0, 1, \dots, \omega - 1\}, \qquad Z_0 = \{0, \pm 1, \pm 2, \dots, \pm n, \dots\},$$

$$\bar{g} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} g(k), \qquad g^L = \min_{k\in I_\omega} g(k), \qquad g^M = \max_{k\in I_\omega} g(k),$$

where $g(k)$ is an ω-periodic sequence of real numbers defined for $k \in Z_0$.

Now we introduce two assumptions:

(H_1) $a^L > \frac{c^M}{m^2} + 2\sqrt{b^M h^M}$,

(H_2) $\bar{f} > 2m\bar{d}e^{2(\bar{a}+\bar{d})\omega}$.

We also introduce twelve positive numbers:

$$l_\pm = \frac{a^L - \frac{c^M}{m^2} \pm \sqrt{\left(a^L - \frac{c^M}{m^2}\right)^2 - 4b^M h^M}}{2b^M}, \qquad p_\pm = \frac{a^M \pm \sqrt{a^{2M} - 4b^L h^L}}{2b^L},$$

$$i_\pm = \frac{\bar{f}e^{2(\bar{a}+\bar{d})\omega} \pm \sqrt{\bar{f}^2 e^{4(\bar{a}+\bar{d})\omega} - 4m^2\bar{d}^2}}{2\bar{d}}, \qquad x_\pm = \frac{\bar{a} \pm \sqrt{\bar{a}^2 - 4\bar{b}\bar{h}}}{2\bar{b}},$$

$$j_\pm = \frac{\bar{f} \pm \sqrt{\bar{f}^2 - 4m^2\bar{d}^2 e^{4(\bar{a}+\bar{d})\omega}}}{2\bar{d}e^{2(\bar{a}+\bar{d})\omega}}, \qquad u_\pm = \frac{\bar{f} \pm \sqrt{\bar{f}^2 - 4m^2\bar{d}^2}}{2\bar{d}}.$$

In order to apply coincidence theory to our study of system (1.4), we will state the following definitions and propositions. For details and proof, see [14].

Define $l^\omega = \left\{u = (u_1, u_2)^{\mathrm{T}} = \{(u_1(k), u_2(k))^{\mathrm{T}}\} : u_i(k+\omega) = u_i(k),\ k \in Z_0,\ i = 1, 2\right\}$. For $a = (a_1, a_2)^{\mathrm{T}} \in R^2$, define $|a| = \max\{a_1, a_2\}$. Let $\|u\| = \max_{k\in\omega} |u(k)|$, for $u \in l^\omega$. Equipped with above norm $\|\cdot\|$, l^ω is a finite- dimensional Banach space.

Let

$$l_0^\omega = \left\{u = \{(u_1(k), u_2(k))^{\mathrm{T}}\} \in l^\omega : \sum_{k=0}^{\omega-1} u_i(k) = 0, i = 1, 2\right\},$$

$$l_c^\omega = \{u = \{(u_1(k), u_2(k))^{\mathrm{T}}\} \in l^\omega : u_i(k) = u_i \in R, i = 1, 2\},$$

then it follows that l_0^ω and l_c^ω are both closed linear subspaces of l^ω and

$$l^\omega = l_0^\omega \bigoplus l_c^\omega, \qquad \dim\ l_c^\omega = 2.$$

Now we reach the position to state our main result in this paper.

Theorem 2.1. *Assume that* (H_1) *and* (H_2) *hold. Then system* (1.4) *has at least four positive* ω*-periodic solutions.*

Proof. Since we are concerned with positive periodic solutions of system (1.4), we make change of variables:

$$x(k) = e^{x_1(k)}, \qquad y(k) = e^{x_2(k)}. \tag{2.1}$$

Then system (1.4) becomes

$$\begin{cases} x_1(k+1) - x_1(k) = a(k) - b(k)e^{x_1(k)} - \dfrac{c(k)e^{2x_2(k)}}{m^2 e^{2x_2(k)} + e^{2x_1(k)}} - \dfrac{h(k)}{e^{x_1(k)}}, \\ x_2(k+1) - x_2(k) = -d(k) + \dfrac{f(k)\exp\{x_1(k-\tau(k)) + x_2(k-\tau(k))\}}{m^2\exp\{2x_2(k-\tau(k))\} + \exp\{2x_1(k-\tau(k))\}}. \end{cases} \tag{2.2}$$

We make change of variables in system (2.2) as follows:

$$u_1(k) = x_1(k), \qquad u_2(k) = x_1(k) - x_2(k). \tag{2.3}$$

Then system (2.2) becomes

$$\begin{cases} u_1(k+1) - u_1(k) = a(k) - b(k)e^{u_1(k)} - \dfrac{c(k)}{m^2 + e^{2u_2(k)}} - \dfrac{h(k)}{e^{u_1(k)}}, \\ u_2(k+1) - u_2(k) = a(k) + d(k) - b(k)e^{u_1(k)} - \dfrac{c(k)}{m^2 + e^{2u_2(k)}} - \dfrac{h(k)}{e^{u_1(k)}} - \dfrac{f(k)e^{u_2(k-\tau(k))}}{m^2 + e^{2u_2(k-\tau(k))}}. \end{cases} \tag{2.4}$$

It is easy to see that system (2.4) is equivalent to system (2.2), we prefer to study system (2.4) in the sequel because it is more convenient for our further discussion.

Now let us define $X = Z = l^\omega$, $\quad (Lu)(k) = u(k+1) - u(k)$ and

$$N(u,\lambda)(k) = \begin{pmatrix} a(k) - b(k)e^{u_1(k)} - \lambda\dfrac{c(k)}{m^2 + e^{2u_2(k)}} - \dfrac{h(k)}{e^{u_1(k)}} \\ a(k) + d(k) - b(k)e^{u_1(k)} - \lambda\dfrac{c(k)}{m^2 + e^{2u_2(k)}} - \dfrac{h(k)}{e^{u_1(k)}} - \dfrac{f(k)e^{u_2(k-\tau(k))}}{m^2 + e^{2u_2(k-\tau(k))}} \end{pmatrix}$$

for $u \in X$ and $k \in Z_0$. It is trivial to see that L is a bounded linear operators and

$$\operatorname{Ker} L = l_c^{\omega}, \qquad \operatorname{Im} L = l_0^{\omega},$$

as well as

$$\dim \operatorname{Ker} L = 2 = \operatorname{codim} \operatorname{Im} L;$$

then it follows that L is a Fredholm mapping of index zero. Define

$$Pu = \frac{1}{\omega}\sum_{k=0}^{\omega-1} u(k),\ u \in X, \qquad Qz = \frac{1}{\omega}\sum_{k=0}^{\omega-1} z(k),\ z \in Z.$$

It is not difficult to show that P and Q are continuous projects such that

$$\operatorname{Im} P = \ker L \quad \text{and} \quad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q).$$

Furthermore, the generalized inverse(to L)$K_p : \operatorname{Im} L \to \operatorname{Ker} P \bigcap \operatorname{Dom} L$ exists and is given by

$$K_p(z) = \sum_{s=0}^{\omega-1} z(s) - \frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega - s)z(s).$$

Obviously, QN and $K_p(I - Q)N$ are continuous. Since X is a finite-dimensional Banach space, it is not difficult to show that $\overline{K_p(I - Q)N(\bar{\Omega} \times [0, 1])}$ is compact for any bounded set $\bar{\Omega} \times [0, 1]$ by using Arzela–Ascoli theorem. Moreover, $QN(\bar{\Omega} \times [0, 1])$ is bounded. Thus, N is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

Corresponding to the operator equation $Lu = \lambda N(u, \lambda),\ \lambda \in (0, 1)$, we have

$$\begin{cases} u_1(k+1) - u_1(k) = \lambda\left[a(k) - b(k)e^{u_1(k)} - \lambda\dfrac{c(k)}{m^2 + e^{2u_2(k)}} - \dfrac{h(k)}{e^{u_1(k)}}\right], \\ u_2(k+1) - u_2(k) = \lambda\left[a(k) + d(k) - b(k)e^{u_1(k)} - \lambda\dfrac{c(k)}{m^2 + e^{2u_2(k)}} - \dfrac{h(k)}{e^{u_1(k)}} - \dfrac{f(k)e^{u_2(k-\tau(k))}}{m^2 + e^{2u_2(k-\tau(k))}}\right]. \end{cases} \tag{2.5}$$

Summing both sides of (2.5) from 0 to $\omega - 1$ gives

$$0 = \sum_{k=0}^{\omega-1}[u_1(k+1) - u_1(k)] = \lambda\sum_{k=0}^{\omega-1}\left[a(k) - b(k)e^{u_1(k)} - \lambda\frac{c(k)}{m^2 + e^{2u_2(k)}} - \frac{h(k)}{e^{u_1(k)}}\right],$$

and

$$0 = \sum_{k=0}^{\omega-1}[u_2(k+1) - u_2(k)] = \lambda\sum_{k=0}^{\omega-1}\left[a(k) + d(k) - b(k)e^{u_1(k)} - \lambda\frac{c(k)}{m^2 + e^{2u_2(k)}} - \frac{h(k)}{e^{u_1(k)}} - \frac{f(k)e^{u_2(k-\tau(k))}}{m^2 + e^{2u_2(k-\tau(k))}}\right].$$

From the above two equations, we get

$$\sum_{k=0}^{\omega-1} a(k) = \sum_{k=0}^{\omega-1}\left[b(k)e^{u_1(k)} + \lambda\frac{c(k)}{m^2 + e^{2u_2(k)}} + \frac{h(k)}{e^{u_1(k)}}\right], \tag{2.6}$$

and

$$\sum_{k=0}^{\omega-1} d(k) = \sum_{k=0}^{\omega-1}\frac{f(k)e^{u_2(k-\tau(k))}}{m^2 + e^{2u_2(k-\tau(k))}}. \tag{2.7}$$

From (2.5)–(2.7), it follows that

$$\sum_{k=0}^{\omega-1}|u_1(k+1) - u_1(k)| \le \sum_{k=0}^{\omega-1}\left[a(k) + b(k)e^{u_1(k)} + \lambda\frac{c(k)}{m^2 + e^{2u_2(k)}} + \frac{h(k)}{e^{u_1(k)}}\right] = 2\bar{a}\omega, \tag{2.8}$$

and

$$\begin{aligned} \sum_{k=0}^{\omega-1}|u_2(k+1) - u_2(k)| \le \sum_{k=0}^{\omega-1}\Bigg[&a(k) + d(k) + b(k)e^{u_1(k)} + \lambda\frac{c(k)}{m^2 + e^{2u_2(k)}} + \frac{h(k)}{e^{u_1(k)}} \\ &+ \frac{f(k)e^{u_2(k-\tau(k))}}{m^2 + e^{2u_2(k-\tau(k))}}\Bigg] = 2(\bar{a} + \bar{d})\omega. \end{aligned} \tag{2.9}$$

Since $(u_1, u_2)^{\mathrm{T}} \in X$, there exists $\xi_i, \eta_i \in I_\omega$ such that

$$u_i(\xi_i) = \min_{k\in I_\omega} u_i(k), \quad u_i(\eta_i) = \max_{k\in I_\omega} u_i(k),\ i = 1, 2. \tag{2.10}$$

(2.6) together with (2.10), implies that

$$\bar{a}\omega > \sum_{k=0}^{\omega-1} b(k)\mathrm{e}^{u_1(k)} \geq \mathrm{e}^{u_1(\xi_1)}\bar{b}\omega$$

and

$$\bar{a}\omega > \sum_{k=0}^{\omega-1} \frac{h(k)}{\mathrm{e}^{u_1(k)}} \geq \frac{\bar{h}\omega}{\mathrm{e}^{u_1(\eta_1)}},$$

that is

$$u_1(\xi_1) < \ln\frac{\bar{a}}{\bar{b}} \leq \ln\frac{a^M}{b^L}$$

and

$$u_1(\eta_1) > \ln\frac{\bar{h}}{\bar{a}} \geq \ln\frac{h^L}{a^M}.$$

Combining this with (2.8) gives for $k \in I_\omega$,

$$u_1(k) \geq u_1(\eta_1) - \sum_{k=0}^{\omega-1} |u_1(k+1) - u_1(k)| > \ln\frac{h^L}{a^M} - 2\bar{a}\omega := H_1 \tag{2.11}$$

and

$$u_1(k) \leq u_1(\xi_1) + \sum_{k=0}^{\omega-1} |u_1(k+1) - u_1(k)| < \ln\frac{a^M}{b^L} + 2\bar{a}\omega := H_2. \tag{2.12}$$

From the first equation of (2.5) and (2.10), we have

$$u_1(\xi_1+1) - u_1(\xi_1) = \lambda\left[a(\xi_1) - b(\xi_1)\mathrm{e}^{u_1(\xi_1)} - \lambda\frac{c(\xi_1)}{m^2 + \mathrm{e}^{2u_2(\xi_1)}} - \frac{h(\xi_1)}{\mathrm{e}^{u_1(\xi_1)}}\right] \geq 0,$$

and

$$u_1(\eta_1+1) - u_1(\eta_1) = \lambda\left[a(\eta_1) - b(\eta_1)\mathrm{e}^{u_1(\eta_1)} - \lambda\frac{c(\eta_1)}{m^2 + \mathrm{e}^{2u_2(\eta_1)}} - \frac{h(\eta_1)}{\mathrm{e}^{u_1(\eta_1)}}\right] \leq 0,$$

which implies that

$$b^L\mathrm{e}^{2u_1(\xi_1)} - a^M\mathrm{e}^{u_1(\xi_1)} + h^L < 0$$

and

$$b^M\mathrm{e}^{2u_1(\eta_1)} - \left(a^L - \frac{c^M}{m^2}\right)\mathrm{e}^{u_1(\eta_1)} + h^M > 0.$$

So we get

$$\ln p_- < u_1(\xi_1) < \ln p_+ \tag{2.13}$$

and

$$u_1(\eta_1) < \ln l_- \quad \text{or} \quad u_1(\eta_1) > \ln l_+. \tag{2.14}$$

It is easy to verify that

$$H_1 < \ln p_- < \ln l_- < \ln l_+ < \ln p_+ < H_2. \tag{2.15}$$

(2.7) together with (2.10) implies

$$\sum_{k=0}^{\omega-1} d(k) < \frac{\mathrm{e}^{u_2(\eta_2)}}{m^2}\sum_{k=0}^{\omega-1} f(k),$$

and

$$\sum_{k=0}^{\omega-1} d(k) < \frac{1}{e^{u_2(\xi_2)}} \sum_{k=0}^{\omega-1} f(k),$$

that is

$$u_2(\eta_2) > \ln \frac{m^2\bar{d}}{\bar{f}} \quad \text{and} \quad u_2(\xi_2) < \ln \frac{\bar{f}}{\bar{d}},$$

this combined with (2.9) gives for $k \in I_\omega$,

$$u_2(k) \geq u_2(\eta_2) - \sum_{k=0}^{\omega-1} |u_2(k+1) - u_2(k)| > \ln \frac{m^2\bar{d}}{\bar{f}} - 2(\bar{a} + \bar{d})\omega := H_3, \tag{2.16}$$

and

$$u_2(k) \leq u_2(\xi_2) + \sum_{k=0}^{\omega-1} |u_2(k+1) - u_2(k)| < \ln \frac{\bar{f}}{\bar{d}} + 2(\bar{a} + \bar{d})\omega := H_4. \tag{2.17}$$

From (2.7) and (2.10), we obtain

$$\sum_{k=0}^{\omega-1} d(k) < \frac{e^{u_2(\eta_2)}}{m^2 + e^{2u_2(\xi_2)}} \sum_{k=0}^{\omega-1} f(k),$$

and

$$\sum_{k=0}^{\omega-1} d(k) > \frac{e^{u_2(\xi_2)}}{m^2 + e^{2u_2(\eta_2)}} \sum_{k=0}^{\omega-1} f(k),$$

that is

$$u_2(\eta_2) > \ln \frac{\bar{d}}{\bar{f}}(m^2 + e^{2u_2(\xi_2)}),$$

and

$$u_2(\xi_2) > \ln \frac{\bar{d}}{\bar{f}}(m^2 + e^{2u_2(\eta_2)}).$$

From this and (2.9), we get

$$u_2(k) > \ln \frac{\bar{d}}{\bar{f}}(m^2 + e^{2u_2(\xi_2)}) - 2(\bar{a} + \bar{d})\omega,$$

and

$$u_2(k) < \ln \frac{\bar{d}}{\bar{f}}(m^2 + e^{2u_2(\eta_2)}) + 2(\bar{a} + \bar{d})\omega.$$

In particular, we have

$$u_2(\xi_2) > \ln \frac{\bar{d}}{\bar{f}}(m^2 + e^{2u_2(\xi_2)}) - 2(\bar{a} + \bar{d})\omega,$$

and

$$u_2(\eta_2) < \ln \frac{\bar{d}}{\bar{f}}(m^2 + e^{2u_2(\eta_2)}) + 2(\bar{a} + \bar{d})\omega.$$

This implies

$$\bar{d}e^{2u_2(\xi_2)} - \bar{f}e^{2(\bar{a}+\bar{d})\omega}e^{u_2(\xi_2)} + m^2\bar{d} < 0,$$

and

$$\bar{d}e^{2(\bar{a}+\bar{d})\omega}e^{2u_2(\eta_2)} - \bar{f}e^{u_2(\eta_2)} + m^2\bar{d}e^{2(\bar{a}+\bar{d})\omega} > 0.$$

Solving the above two inequalities gives

$$\ln i_- < u_2(\xi_2) < \ln i_+ \tag{2.18}$$

and

$$u_2(\eta_2) < \ln j_- \quad \text{or} \quad u_2(\eta_2) > \ln j_+. \tag{2.19}$$

It is easy to verify that

$$H_3 < \ln i_- < \ln j_- < \ln j_+ < \ln i_+ < H_4. \tag{2.20}$$

Obviously, $\ln i_\pm, \ln j_\pm, \ln l_\pm, \ln p_\pm, H_1, H_2, H_3$ and H_4 are independent of λ.

Now let us consider $QN(u, 0)$ with $u = (u_1, u_2)^{\mathrm{T}} \in R^2$. Note that

$$QN(u_1, u_2; 0) = \begin{pmatrix} \bar{a} - \bar{b}\mathrm{e}^{u_1} - \dfrac{\bar{h}}{\mathrm{e}^{u_1}} \\ \bar{a} + \bar{d} - \bar{b}\mathrm{e}^{u_1} - \dfrac{\bar{h}}{\mathrm{e}^{u_1}} - \dfrac{\bar{f}\mathrm{e}^{u_2}}{m^2 + \mathrm{e}^{2u_2}} \end{pmatrix}.$$

Since $\bar{a} > 2\sqrt{\bar{b}\bar{h}},\ \bar{f} > 2m\bar{d}$, we can show that $QN(u_1, u_2; 0)$ has four distinct solutions:

$$(u_1^1, u_2^1) = (\ln x_+, \ln u_+),\ (u_1^2, u_2^2) = (\ln x_+, \ln u_-), (u_1^3, u_2^3) = (\ln x_-, \ln u_+),\ (u_1^4, u_2^4) = (\ln x_-, \ln u_-).$$

It is easy to verify that

$$\ln l_+ < \ln x_+ < \ln p_+,\ \ln p_- < \ln x_- < \ln l_-;\ \ln j_+ < \ln u_+ < \ln i_+,\ \ln i_- < \ln u_- < \ln j_-.$$

Let

$$\Omega_1 = \left\{ u = (u_1, u_2)^{\mathrm{T}} \in X \,\middle|\, \begin{matrix} u_1(k) \in (\ln p_-, \ln l_-) \\ u_2(k) \in (\ln i_-, \ln j_-) \end{matrix} \right\},$$

$$\Omega_2 = \left\{ u = (u_1, u_2)^{\mathrm{T}} \in X \,\middle|\, \begin{matrix} \min\limits_{k\in I_\omega} u_1(k) \in (\ln p_-, \ln p_+) \\ \max\limits_{k\in I_\omega} u_1(k) \in (\ln l_+,\ H_2) \\ u_2(k) \in (\ln i_-, \ln j_-) \end{matrix} \right\},$$

$$\Omega_3 = \left\{ u = (u_1, u_2)^{\mathrm{T}} \in X \,\middle|\, \begin{matrix} u_1(k) \in (\ln p_-, \ln l_-) \\ \min\limits_{k\in I_\omega} u_2(k) \in (\ln i_-, \ln i_+) \\ \max\limits_{k\in I_\omega} u_2(k) \in (\ln j_+,\ H_4) \end{matrix} \right\},$$

and

$$\Omega_4 = \left\{ u = (u_1, u_2)^{\mathrm{T}} \in X \,\middle|\, \begin{matrix} \min\limits_{k\in I_\omega} u_1(k) \in (\ln p_-, \ln p_+) \\ \max\limits_{k\in I_\omega} u_1(k) \in (\ln l_+,\ H_2) \\ \min\limits_{k\in I_\omega} u_2(k) \in (\ln i_-, \ln i_+) \\ \max\limits_{k\in I_\omega} u_2(k) \in (\ln j_+,\ H_4) \end{matrix} \right\}.$$

It is easy to see that $(u_1^1, u_2^1) \in \Omega_4,\ (u_1^2, u_2^2) \in \Omega_2,\ (u_1^3, u_2^3) \in \Omega_3,\ (u_1^4, u_2^4) \in \Omega_1$ and Ω_i are open bounded subset of X. With the help of (2.13)–(2.15) and (2.18)–(2.20), it is not difficult to verify that $\Omega_i \bigcap \Omega_j = \emptyset$ and Ω_i satisfies condition (a) of Lemma 2.1 for $i, j = 1, 2, 3, 4, i \neq j$. Moreover, when $u \in \partial\Omega_i \bigcap \operatorname{Ker} L,\ i = 1, 2, 3, 4,\ QN(u, 0) \neq (0, 0)^{\mathrm{T}}$, so condition (b) of Lemma 2.1 holds.

Finally, we will show that condition (c) of Lemma 2.1 holds. By taking $J = I$ since $\operatorname{Ker} L = \operatorname{Im} Q$, a direct computation gives for $i = 1, 2, 3, 4$,

$$\begin{aligned} &\deg\left\{JQN(u, 0), \Omega_i \cap \operatorname{Ker} L, (0, 0)^{\mathrm{T}}\right\} \\ &\quad = \deg\left\{\left(\bar{a} - \bar{b}\mathrm{e}^{u_1} - \bar{h}\mathrm{e}^{-u_1}, \bar{a} + \bar{d} - \bar{b}\mathrm{e}^{u_1} - \bar{h}\mathrm{e}^{-u_1} - \frac{\bar{f}\mathrm{e}^{u_2}}{m^2 + \mathrm{e}^{2u_2}}\right)^{\mathrm{T}}, \Omega_i \cap \operatorname{Ker} L, (0, 0)^{\mathrm{T}}\right\} \\ &\quad = \operatorname{sign} \begin{vmatrix} -\bar{b}\mathrm{e}^{u_1^i} + \bar{h}\mathrm{e}^{-u_1^i} & 0 \\ -\bar{b}\mathrm{e}^{u_1^i} + \bar{h}\mathrm{e}^{-u_1^i} & \dfrac{\bar{f}\mathrm{e}^{u_2^i}(\mathrm{e}^{2u_2^i} - m^2)}{(\mathrm{e}^{2u_2^i} + m^2)^2} \end{vmatrix} \end{aligned}$$

$$= \operatorname{sign}\left[\left(-\bar{b}e^{u_1^i}+\bar{h}e^{-u_1^i}\right)\frac{\bar{f}e^{u_2^i}(e^{2u_2^i}-m^2)}{(e^{2u_2^i}+m^2)^2}\right]$$
$$= \operatorname{sign}\left[\left(\bar{a}-2\bar{b}e^{u_1^i}\right)(2\bar{d}e^{u_2^i}-\bar{f})\right].$$

Then

$$\begin{aligned}
\deg\left\{JQN(u,0),\,\Omega_1\cap \operatorname{Ker} L,\,(0,0)^{\mathrm{T}}\right\} &= \operatorname{sign}\left[\left(\bar{a}-2\bar{b}x_-\right)(2\bar{d}u_- -\bar{f})\right]\\
&= -1,\\
\deg\left\{JQN(u,0),\,\Omega_2\cap \operatorname{Ker} L,\,(0,0)^{\mathrm{T}}\right\} &= \operatorname{sign}\left[\left(\bar{a}-2\bar{b}x_+\right)(2\bar{d}u_- -\bar{f})\right]\\
&= 1,\\
\deg\left\{JQN(u,0),\,\Omega_3\cap \operatorname{Ker} L,\,(0,0)^{\mathrm{T}}\right\} &= \operatorname{sign}\left[\left(\bar{a}-2\bar{b}x_-\right)(2\bar{d}u_+ -\bar{f})\right]\\
&= 1,\\
\deg\left\{JQN(u,0),\,\Omega_4\cap \operatorname{Ker} L,\,(0,0)^{\mathrm{T}}\right\} &= \operatorname{sign}\left[\left(\bar{a}-2\bar{b}x_+\right)(2\bar{d}u_+ -\bar{f})\right]\\
&= -1.
\end{aligned}$$

So far we have proved that Ω_i satisfies all the assumptions in Lemma 2.1. Hence, system (2.4) has at least four ω-periodic solutions. According to the equivalence of system (2.4) and system (2.2), we conclude that system (2.2) has at least four different ω-periodic solutions. Thus, system (1.4) has at least four different positive ω-periodic solutions. This completes the proof. □

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