Oscillation Criteria for Difference Equations with Unbounded Delay

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Abstract—Consider the delay difference equation

$$x_{n+1} - x_n + p_n x_{n+\tau(n)} = 0, \quad n = 0, 1, 2, \ldots,$$

where $\tau : \mathbb{N} \to \mathbb{Z}$ is nondecreasing, $\tau(n) < n$ for $n \in \mathbb{N}$ and $\lim_{n \to \infty} \tau(n) = \infty$, $(p_n)$ is a nonnegative sequence. Some oscillation criteria for this equation are obtained.

Keywords—Oscillations, Difference equations, Unbounded delay.

1. INTRODUCTION

Recently, there has been an increasing interest in the study of the oscillatory behavior of the solutions of the delay difference equation

$$\Delta x_n + p_n x_{n-\tau} = 0, \quad n = 0, 1, 2, \ldots,$$  

(1)

where $(p_n)$ is a real sequence, $\tau$ is a positive integer, and $\Delta x_n = x_{n+1} - x_n$. See, for example, [1-5].

As in [1], equation (1) can be a discrete analogue of the differential equation

$$x'(t) + p(t)x(t - \sigma) = 0,$$  

(2)

where $\sigma > 0$ is a constant. Equation (2) is a differential equation with a constant delay $\sigma$. In the applications [6], one sometimes considers delay differential equations with a variable delay of the form

$$x'(t) + p(t)x(\sigma(t)) = 0,$$  

(3)

where $\sigma \in C(R_+, R_+)$, $\sigma(t) < t$, and $\lim_{t \to \infty} \sigma(t) = \infty$. The oscillation of solutions of (3) has been investigated systematically in [7,8].

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In this note, we consider difference equations with a variable delay of the form

\[ x_{n+1} - x_n - p_n x_{\tau(n)} = 0, \quad n = 0, 1, \ldots \]  

(4)

Equation (4) is a discrete analogue of equation (3). Let \( N = \{0, 1, \ldots \} \) and \( Z = \{\ldots, -1, 0, 1, \ldots \} \). For equation (4), we always assume that the following hypotheses, designated by (H), hold:

(i) \( \tau : N \to Z \) is nondecreasing,
(ii) \( \tau(n) < n \) for \( n \in N \),
(iii) \( \lim_{n \to \infty} \tau(n) = \infty \), and
(iv) there exists a monotone sequence \( \{n_k\} \) such that \( \tau(n_k) = n_{k-1}, k = 1, 2, \ldots, \lim_{k \to \infty} n_k = \infty \), \( 0 \leq n_0 < n_1 < \cdots < n_k < \cdots \).

For example, we see that \( \tau(n) = n - \tau \), where \( \tau \) is a positive integer, and \( \tau(n) = \lfloor n/2 \rfloor \), where \( \lfloor \cdot \rfloor \) denotes the greatest integer \( m \leq n/2 \), satisfy condition (H). Clearly, the assumption (H) is similar to the corresponding assumptions for equation (3). Therefore, (4) includes difference equations with unbounded delay. To the best of our knowledge, there are no known oscillation criteria to cover difference equations with unbounded delay.

By a solution of (4), we mean a sequence \( \{x_n\} \) which is defined for \( n \geq \tau(0) \) and which satisfies (4) for \( n \geq 0 \). A solution \( \{x_n\} \) of equation (4) is said to be oscillatory if the terms \( x_n \) of the solution are neither eventually all positive nor eventually all negative. Otherwise, the solution is called nonoscillatory.

In this paper, we present some sufficient conditions under which all solutions of (4) oscillate.

For the sake of convenience, we introduce the following notations. Let

\[ \bar{\alpha}_n = \left( \frac{n - \tau(n)}{n - \tau(n) + 1} \right)^{n - \tau(n) + 1}, \quad n = 1, 2, \ldots, \]  

(5)

\[ \alpha_k = \max_{n_{k-1} \leq n \leq n_k} \{\bar{\alpha}_n\}, \quad k = 1, 2, \ldots, \]  

(6)

\[ \tau^0(n) = n, \quad \tau^m(n) = \tau(\tau^{m-1}(n)), \quad m = 1, 2, \ldots, \]  

\[ \tau_k = \min_{n_{k-1} \leq n \leq n_k} \{n - \tau(n)\}, \quad k = 1, 2, \ldots, \]  

(7)

and

\[ q_k = \min \left\{ \sum_{i=\tau(n)}^{n-1} p_i \mid n_{k-1} \leq n \leq n_k, \quad k \in N \right\}, \]  

(8)

where \( n_{k-1} = \tau(n_k) \).

Clearly, \( \tau^m(n) < \tau^{m-1}(n) \), for \( m = 1, 2, \ldots \). Since \( f(x) = (x/(1 + x))^{1+\varepsilon} \) is increasing in \( x \) on \((0, \infty)\), and \( f(x) \to 1/e \) as \( x \to \infty \), we can easily prove that

\[ \alpha_k = \left( \frac{\tau_k}{1 + \tau_k} \right)^{1+\tau_k}, \]  

(9)

and hence, \( \alpha_k < 1/e \).

2. LEMMAS

To obtain our main results, we need the following lemmas.

LEMMA 1. Assume that for some positive integer \( \bar{n} \), equation (4) has a solution \( \{x_n\} \) such that \( x_n > 0 \) on \( \{\tau^3(\bar{n}), \ldots, \bar{n} + 1\} \), \( p_n \geq 0 \) on \( \{\tau^2(\bar{n}), \ldots, \bar{n}\} \), and there exists a positive number \( B \) such that \( \sum_{i=\tau(n)}^{n} p_i \geq B > 0 \). Let

\[ N = \min_{\tau(n) \leq n \leq \bar{n}} \frac{x_{\tau(n)}}{x_n}, \]  

then \( N < 4/B^2 \).
Proof. By the assumption, there exists \( n^* \in \{ \tau(\bar{n}), \ldots, \bar{n} \} \) such that

\[
\sum_{i=\tau(n)}^{n^*} p_i \geq \frac{B}{2} \quad \text{and} \quad \sum_{i=n^*}^{n} p_i \geq \frac{B}{2}.
\]

Summing (4) and using the decreasing nature of \( \{ x_{\tau(n)} \} \) for \( \tau(\bar{n}) \leq i \leq \bar{n} \), we have

\[
x_{n+1} - x_{n^*} = - \sum_{i=n^*}^{n} p_i x_{\tau(i)} \leq -x_{\tau(n^*)} \sum_{i=n^*}^{n} p_i \leq -\frac{B}{2} x_{\tau(n)}.
\]

Similarly,

\[
x_{n^*+1} - x_{\tau(n)} = - \sum_{i=\tau(n)}^{n^*} p_i x_{\tau(i)} \leq -x_{\tau(n^*)} \sum_{i=\tau(n)}^{n^*} p_i \leq -\frac{B}{2} x_{\tau(n^*)}.
\]

Hence,

\[
x_{n^*} > \frac{B}{2} x_{\tau(n)} > \left( \frac{B}{2} \right)^2 x_{\tau(n^*)}.
\]

Thus, \( N \leq x_{\tau(n^*)}/x_{n^*} < 4/B^2 \). The proof is complete.

Lemma 2. Assume that \( \{ x_n \} \) is a solution of equation (4) such that for some positive integer \( \bar{n} \), \( x_n > 0 \) on \( \{ \tau(\bar{n}), \ldots, \bar{n} + 1 \} \), \( p_n \geq 0 \) on \( \{ \tau^2(\bar{n}), \ldots, \bar{n} \} \), and that

\[
\min_{\tau(n) \leq n \leq \bar{n}} \left\{ \sum_{i=\tau(n)}^{n-1} p_i \right\} = q_n > \alpha_n = \max_{\tau(n) \leq n \leq \bar{n}} \{ \alpha_n \} = \left( \frac{\tau_n}{\tau_n + 1} \right)^{r_{\tau(n)+1}}, \quad (10)
\]

where \( \tau_n = \min_{\tau(n) \leq n \leq \bar{n}} \{ n - \tau(n) \} \). Define \( M \) and \( N \) as follows:

\[
M = \min_{\tau^2(n) \leq n \leq \tau(n)} \frac{x_{\tau^2(n)}}{x_n} \quad \text{and} \quad N = \min_{\tau(n) \leq n \leq \bar{n}} \frac{x_{\tau(n)}}{x_n}.
\]

Then, \( N > 1 \) and

\[
N \geq \frac{1}{(1 - M q_n M^{-1})^q_n} \geq \frac{1}{(1 - \alpha_n M^{-1})^{r_n}} \geq M. \quad (11)
\]

Proof. From (4), if \( x_n > 0 \), then we have

\[
\frac{x_{n+1}}{x_n} = 1 - p_n \frac{x_{\tau(n)}}{x_n}.
\]

Hence, for \( \tau(\bar{n}) \leq n \leq \bar{n} \), by using the well-known inequality between the arithmetic and geometric mean, we obtain

\[
\frac{x_n}{x_{\tau(n)}} = \frac{n-1}{\tau(n)} \frac{x_{n+1}}{x_i} = \prod_{i=\tau(n)}^{n-1} \left( 1 - p_i \frac{x_{\tau(i)}}{x_i} \right) \leq \left( 1 - \frac{1}{n - \tau(n)} \sum_{i=\tau(n)}^{n-1} p_i \frac{x_{\tau(i)}}{x_i} \right)^{n-\tau(n)}. \quad (12)
\]

In the last inequality, \( i = \tau(n), \ldots, n - 1 \), where \( \tau(\bar{n}) \leq n \leq \bar{n} \), which implies that \( \tau^2(\bar{n}) \leq i \leq \bar{n} - 1 \). Thus, considering that the function \( (1 - b/x)^r \) is increasing in \( x \) on \([b, \infty)\), where \( b \) is a
positive constant, for $\tau(n) \leq n \leq \bar{n}$, we have from (12),

$$\frac{x_n}{x_{\tau(n)}} \leq \left(1 - \min(M, N)\frac{1}{n - \tau(n)} \sum_{i=\tau(n)}^{n-1} p_i\right)^{n - \tau(n)}$$

$$\leq \left(1 - \min(M, N)\frac{q_n}{n - \tau(n)}\right)^{n - \tau(n)}$$

$$\leq (1 - \min(M, N)q_n\tau_n^{-1})^{-\tau_n}$$

or

$$\frac{x_{\tau(n)}}{x_n} \geq (1 - \min(M, N)q_n\tau_n^{-1})^{-\tau_n} \geq (1 - \min(M, N)\alpha_n\tau_n^{-1})^{-\tau_n}. \quad (13)$$

Hence,

$$\frac{N}{1 - \min(M, N)q_n\tau_n^{-1} - \tau_n}. \quad (14)$$

$$N \geq (1 - \min(M, N)q_n\tau_n^{-1}) - \tau_n.$$

It is easy to see that

$$\max_{1/q_n \geq N > 0} \{N (1 - q_n N)^{-\tau_n}\} = \frac{\tau_n q_n}{q_n (1 + \tau_n)^{1+\tau_n}}. \quad (15)$$

If $\min(M, N) = N$, then (13) leads to

$$N \geq (1 - q_n\tau_n^{-1}N)^{-\tau_n} \geq q_n\tau_n^{-1}N\frac{(\tau_n + 1)\tau_n^{-1}}{\tau_n\tau_n} > N,$$

which is a contradiction. Hence, $\min(M, N) = M$. Then, from (13), we obtain

$$N \geq (1 - q_n\tau_n^{-1}M)^{-\tau_n} \geq (1 - \alpha_n\tau_n^{-1}M)^{-\tau_n} \geq M.$$

The proof is complete.

3. MAIN RESULTS

THEOREM 1. Assume that (H) holds, $\rho_n \geq 0$ and $q_n > \alpha_n$, for all large $n$, where $q_n$ and $\alpha_n$ are defined in Section 1. Furthermore, assume that

$$\sum_{i=1}^{\infty} (q_i - \alpha_i) = \infty. \quad (16)$$

Then every solution of (4) is oscillatory.

PROOF. Suppose to the contrary, that $\{x_n\}$ is an eventually positive solution of (4) with $x_n > 0$ for $n \geq T^3(n_k)$ and $\rho_n \geq 0$ for $n \geq T^3(n_k)$, where $k \geq 3$. It is easy to see that Lemma 2 is applicable. Define a sequence $\{N_i\}$ by

$$N_i = \min_{\tau(n+\iota) \leq n \leq n+\iota} \frac{x_{\tau(n)}}{x_n}, \quad i = 1, 2, \ldots. \quad (17)$$

Noting that $n_{k+i-1} = \tau(n_{k+i})$ and the definition of $g_{k+i}$, by (11) (set $N = N_i$ and $M = N_{i-1}$), we obtain

$$N_i \geq \frac{1}{(1 - q_{k+i}(\tau_{k+i} - 1)N_{i-1})^{\tau_{k+i}}} \geq \frac{1}{(1 - \alpha_{k+i}(\tau_{k+i} - 1)N_{i-1})^{\tau_{k+i}}} \geq N_{i-1}, \quad i = 1, 2, \ldots.$$
i.e., \( N_i \) is nondecreasing. By Lemma 1, \( \{N_i\} \) is bounded. Hence, \( \lim_{i \to \infty} N_i = N \) exists. In view of (11), (14), and (9), we have

\[
N_i \geq \frac{1}{(1 - q_{k+i} \tau_{k+i}^{-1} N_i^{-1})^{1+\tau}} \geq q_{k+i} \tau_{k+i}^{-1} N_i^{-1} \frac{(\tau_{k+i} + 1)^{\tau_{k+i} + 1}}{\tau_{k+i}} = q_{k+i} \frac{1}{\alpha_{k+i}}
\]

\[
= N_i - 1 \left( 1 + \frac{1}{\alpha_{k+i}} (q_{k+i} - \alpha_{k+i}) \right), \quad i = 1, 2, \ldots.
\]

By Lemma 2, \( N_i > 1 \). It is easy to see that \( \alpha_i < 1 \). Hence,

\[
N_i - N_{i-1} \geq N_i - 1 \alpha_{k+i}^{-1} (q_{k+i} - \alpha_{k+i}) \geq (q_{k+i} - \alpha_{k+i}),
\]

\[
N_{i+1} - N_i \geq N_i \alpha_{k+i+1}^{-1} (q_{k+i+1} - \alpha_{k+i+1}) \geq (q_{k+i+1} - \alpha_{k+i+1}),
\]

and hence,

\[
\bar{N} - N_i = \sum_{j=i}^{\infty} (N_{j+1} - N_j) \geq \sum_{j=i}^{\infty} (q_{k+j+1} - \alpha_{k+j+1}),
\]

which contradicts (15). The proof is complete.

For equation (1), \( \tau(n) = n - \tau, \alpha_i = (\tau/1 + \tau)^{1+\tau}, n_k = k\tau, \) and

\[
q_i = \min \left\{ \sum_{j=n}^{i-1} p_j | (i-1)\tau \leq n \leq i\tau \right\}.
\]

**Corollary 1.** Assume that \( p_n \geq 0, a_n > (\tau/(1+\tau))^{1+\tau} \) for all large \( n \), and

\[
\sum_{i=1}^{\infty} \left( q_i - \left( \frac{\tau}{\tau + 1} \right)^{\tau+1} \right) = \infty.
\]

Then every solution of (1) is oscillatory.

**Remark 1.** Condition (17) improves the condition [2]

\[
\lim_{n \to \infty} \frac{1}{\tau} \sum_{i=n-\tau}^{n-1} p_i > \frac{\tau}{(\tau + 1)^{\tau+1}},
\]

since (18) implies (17).

For the case \( \lim_{n \to \infty} (n - \tau(n)) = \infty \), we can derive the following explicit oscillation criterion from Theorem 1.

**Corollary 2.** Assume that \( (H) \) holds, \( p_n \geq 0, \lim_{n \to \infty} (n - \tau(n)) = \infty \), and

\[
\lim_{n \to \infty} \frac{1}{\tau} \sum_{i=n-\tau(n)}^{n-1} p_i > \frac{1}{\epsilon}.
\]

Then every solution of (4) is oscillatory.

**Proof.** From (19), there exist \( \epsilon > 0 \) and a positive integer \( k_1 \), such that

\[
\sum_{i=n-\tau(n)}^{n-1} p_i > \frac{1}{\epsilon} + \epsilon, \quad \text{for all } n \geq n_k.
\]
On the other hand, \((n - \tau(n))/(n - \tau(n) + 1)^n-\tau(n)+1\) is increasing for large \(n\), and

\[
\lim_{n \to \infty} \left( \frac{n - \tau(n)}{n - \tau(n) + 1} \right)^{n-\tau(n)+1} = \frac{1}{e}.
\]

(20)

Hence, for any small positive number \(\epsilon\), there exists a positive integer \(k_2\) such that

\[
\left( \frac{n - \tau(n)}{n - \tau(n) + 1} \right)^{n-\tau(n)+1} < \frac{1}{e} + \frac{\epsilon}{2},
\]

for all \(n \geq n_{k_2}\).

Let \(\bar{k} = \max(k_1, k_2)\). Then we have

\[
\sum_{i=\tau(n)}^{n-1} p_i > \left( \frac{\tau(m)}{\tau(m) + 1} \right)^{m-\tau(m)+1} + \frac{\epsilon}{2},
\]

for all \(n, m \geq n_{\bar{k}}\).

Hence,

\[
q_{k+1} > \alpha_{k+1} + \frac{\epsilon}{2} > \alpha_{k+1},
\]

for \(k \geq \bar{k}\), and

\[
\sum_{i=k+1}^{\infty} (q_i - \alpha_i) = \infty,
\]

i.e., all assumptions of Theorem 1 are satisfied. By Theorem 1, every solution of \((4)\) oscillates.

REMARK 2. If \(\lim_{n \to \infty} (n - \tau(n)) = \infty\), then by (20), (19) can be written to the form

\[
\lim_{n \to \infty} \sum_{i=\tau(n)}^{n-1} p_i > \lim_{n \to \infty} \left( \frac{n - \tau(n)}{n - \tau(n) + 1} \right)^{n-\tau(n)+1}.
\]

(21)

If \(\tau(n) = n - \tau\), (21) becomes (18). Hence, condition (19) is a generalization of (18) for the equation with unbounded delay.

REMARK 3. We note that (19) is not necessary for (15). That is, even with

\[
\liminf_{n \to \infty} \sum_{i=\tau(n)}^{n-1} p_i = e^{-1},
\]

(15) still is possible.

If (19) does not hold, then we have the following result.

THEOREM 2. Assume that \((H)\) holds, \(p_n \geq 0\) and \(\sum_{i=\tau(n)}^{n} p_i \geq B \in (0, 1)\), for all large \(n\). Furthermore, assume that

\[
\limsup_{n \to \infty} \sum_{i=\tau(n)}^{n-1} p_i > 1 - \left( \frac{B}{2} \right)^2.
\]

(22)

Then every solution of \((4)\) oscillates.

PROOF. Suppose to the contrary, that \(\{z_n\}\) is a positive solution of \((4)\). Then \(\Delta z_n \leq 0\) eventually. If (22) holds, summing (4), we get

\[
0 = z_n - z_{\tau(n)} + \sum_{i=\tau(n)}^{n-1} p_i z_{\tau(i)}
\]

\[
\geq z_n - z_{\tau(n)} + z_{\tau(n)} \sum_{i=\tau(n)}^{n-1} p_i
\]

\[
= z_{\tau(n)} \left( \sum_{i=\tau(n)}^{n-1} p_i - 1 + \frac{z_n}{z_{\tau(n)}} \right).
\]
Similar to Lemma 1 with slight revision we can prove that \( \frac{x_n}{x_{\tau(n)}} \geq (B/2)^2 \). Hence, by (22), the last inequality implies that

\[
0 \geq x_{\tau(n)} \left( \sum_{i=\tau(n)}^{n-1} p_i - 1 + \left( \frac{B}{2} \right)^2 \right) > 0,
\]

which is a contradiction. The proof is complete.

**Example 1.** Consider

\[
x_{n+1} - x_n + p_n x_{[n/2]} = 0, \quad n = 0, 1, \ldots,
\]

where \( p_n = 1/(n+1) \), \([n/2]\) denotes the greatest integer \( m \leq n/2 \), \( n = 0, 1, \ldots \). To the best of our knowledge, there is no known oscillation criterion to apply to (23). We see that, if \( n = 2k \), then

\[
\sum_{i=\tau(n)}^{n-1} p_i = \sum_{i=k}^{2k-1} \frac{1}{i+1} - \frac{k}{2k} - \frac{1}{2} > \frac{1}{e},
\]

and if \( n = 2k - 1 \), then

\[
\sum_{i=\tau(n)}^{n-1} p_i = \sum_{i=k-1}^{2k-2} \frac{1}{i+1} - \frac{k}{2k-1} > \frac{1}{2} > \frac{1}{e}.
\]

Hence,

\[
\liminf_{n \to \infty} \sum_{i=\tau(n)}^{n-1} p_i \geq \frac{1}{2} > \frac{1}{e},
\]

i.e., (19) holds. By Corollary 2, every solution of (23) oscillates.

**Example 2.** Consider the equation

\[
x_{n+1} - x_n + p_n x_{[n/2]} = 0, \quad n = 1, 2, \ldots,
\]

where

\[
p_n = \begin{cases} 
2 \left( 1 - \frac{1}{16e^2} \right) \frac{1}{n+1}, & n = 2k, \\
\frac{1}{e(n+1)}, & n = 2k - 1, \quad k = 1, 2, \ldots.
\end{cases}
\]

We see that, if \( n = 2k \), then

\[
\sum_{i=\tau(n)}^{n-1} p_i = 2 \left( 1 - \frac{1}{16e^2} \right) \sum_{i=k}^{2k-1} \frac{1}{i+1} > 2 \left( 1 - \frac{1}{16e^2} \right) \frac{k}{2k} = 1 - \frac{1}{16e^2},
\]

and if \( n = 2k - 1 \), then

\[
\frac{1}{e} > \sum_{i=\tau(n)}^{n-1} p_i = \frac{1}{e} \sum_{i=k-1}^{2k-2} \frac{1}{i+1} > \frac{k}{e(2k-1)} > \frac{1}{2e}.
\]

Hence, (19) does not hold. Corollary 2 cannot apply to (24).

On the other hand, \( \sum_{i=\tau(n)}^{n-1} p_i > B = 1/2e \) and \( \limsup_{n \to \infty} \sum_{i=\tau(n)}^{n-1} p_i \geq 1/16e^2 \). Therefore, all assumptions of Theorem 2 are satisfied. By Theorem 2, every solution of (24) oscillates.
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