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A DECISION PROCEDURE ON PARTIALLY COMMUTATIVE FREE MONOIDS

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In this paper we evaluate the complexity of an algorithm for deciding whether a partially commutative free monoid has an infinite number of square-free elements.

Let A be a finite *alphabet* and A^* the *free monoid* over A. If θ denotes a reflexive and symmetric relation on A we say that the letters $a, b \in A$ commute if and only if $(a,b) \in \theta$; θ is also called *commutation relation*.

We denote by $\sim$ the congruence generated by the set

$$\{(ab, ba) \mid (a,b) \in \theta\}$$

and by $M(A,\theta)$ the quotient monoid $A^*/\sim$. $M(A,\theta)$ is also called the *partially commutative free-monoid* over A relative to θ. An element of $M(A,\theta)$ is called *trace*. If $\theta = \emptyset$, then $M(A,\theta)$ is equal to the free monoid A^*; if $\theta = A \times A$, then $M(A,\theta)$ is the free commutative monoid over A.

Partially commutative free monoids have been considered first by Cartier and Foata [3] to deal with some combinatorial problems related to the *rearrangements of words*. More recently several authors have reconsidered these objects for problems of *parallel computation* and *concurrency processes* (cf. [1], and references therein). However the study of combinatorial and algebraic properties of the traces (a trace is a congruence class mod. $\sim$ of a word) and of trace-languages of a free monoid is interesting by itself. Some papers have been recently published on *recognizability* and *rationality* of trace-languages and on some combinatorial problems on the traces [2, 4, 5].

A trace $m \in M(A,\theta)$ is called *square-free* if and only if $m \neq rs^2t$, with $r, t \in M(A,\theta)$ and $s \in M(A,\theta) \setminus \{1\}$; in other terms a square-free element of $M(A,\theta)$ is a congruence class mod. $\sim$ which contains only square-free words of A^*. We shall denote by $L_2(M(A,\theta))$ the set of all square-free traces of $M(A,\theta)$.

In [2] has been proved the following theorem which gives an algorithm to decide whether the set $L_2(M(A,\theta))$ is infinite:

Theorem 1. *Let $M(A,\theta)$ be a partially commutative free monoid over A relative to θ. $L_2(M(A,\theta))$ is infinite if and only if at least one of the following conditions is verified:*

(i) *There exist three distinct letters $a,b,c\in A$ such that*

$$(a,b),(b,c),(c,a)\notin\theta.$$

(ii) *There exist four distinct letters $a,b,c,d\in A$ such that*

$$(a,b),(a,c),(a,d)\notin\theta.$$

(iii) *There exist four distinct letters $a,b,c,d\in A$ such that*

$$(a,b),(b,d),(d,c),(c,a)\notin\theta.$$

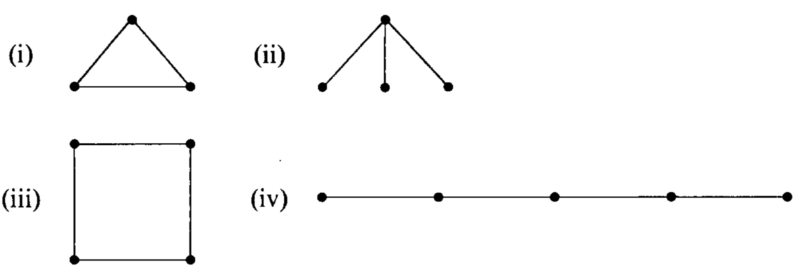
(iv) *There exist five distinct letters $a,b,c,d,e\in A$ such that*

$$(a,b),(b,c),(c,d),(d,e)\notin\theta.$$

Given the finite alphabet A and the commutation relation θ one can consider the graph $G=(N_A, V_\theta)$ where each node represents a letter of the alphabet A and each arc joins two nodes if and only if the corresponding letters commute. However, for our purposes it is more convenient to refer ourselves to the complementary graph $\bar{G}=(N_A, V_{\bar{\theta}})$ where $\bar{\theta}$ is the complementary relation of θ. In fact Theorem 1 is equivalent to the following:

Proposition 1. *$L_2(M(A,\theta))$ is infinite if and only if in the graph $\bar{G}$ there exists at least a node of degree >2 or a node of degree 2 having as adjacent nodes still nodes of degree 2.*

Proof. In terms of the relation $\bar{\theta}$ Theorem 1 can be restated as follows: $L_2(M(A,\theta))$ is infinite if and only if $\bar{G}$ contains at least one of the following subgraphs:



In the case (ii) one has a node of $\bar{G}$ of degree >2; otherwise there will exist a node of degree 2 having two adjacent nodes of degree 2 (cases (i), (iii) and (iv)).

The sufficiency is derived from the fact that if in $\bar{G}$ there is a node of degree >2, then we are in the case (ii) of the previous figure. If there exists a node a of degree 2 having two adjacent nodes b and c of degree 2, then the only possibilities which may occur are those of the cases (i), (iii), and (iv), according to the cases in which, respectively, (1) b and c are adjacent nodes, (2) b and c are adjacent to a same node $\neq a$, (3) b and c are adjacent to two distinct nodes $\neq a$. □

Hence the verification of Theorem 1 is reconducted to a simple property of the graph $\bar{G}$. From the previous result one has that the structure of data which is more convenient to describe $\bar{G}$ is the set of the *adjacency-lists* of the nodes of $\bar{G}$. The following holds.

Proposition 2. *Given a finite alphabet A and the set of adjacency-lists of the nodes of the graph $\bar{G}=(N_A, V_{\bar{\theta}})$ one can decide whether $L_2(M(A,\theta))$ is infinite by an algorithm whose cost (= number of elementary operations) is a linear function of* $n=\text{card}(A)$.

Proof. We denumerate by $1,2,\ldots$ the nodes of N_A and starting from the set of the adjacency-lists of $\bar{G}$ we construct an $n\times 3$ matrix C, where $n=\text{card}(A)$, in which the h-th row is in correspondence to the h-th node of $\bar{G}$. In the first column is reported the degree of the node h. If there is a node of degree >2, then one would conclude for the success of the procedure (i.e. $L_2(M(A,\theta))$ is infinite). One can then suppose $h\leq 2$. In the next two columns are reported the numbers representing the nodes adjacent to the node h. If the degree of h is 1, we set $c_{h.3}=0$; if h has no adjacent nodes, then we set $c_{h.2}=c_{h.3}=0$.

The construction of C requires at most $2n$ *reading-operations* (i.e. the scanning of the adjacency-lists of $\bar{G}$) and $3n$ *writing-operations* (the filling of the matrix C).

Thus a simple procedure for deciding whether $L_2(M(A,\theta))$ is infinite can be described as follows:

Let us consider the h-th, $1\leq h\leq n$, node of the graph $\bar{G}$. One reads $c_{h.1}$. If $c_{h.1}<2$, i.e. h is a node of degree <2, then one goes to analyze the $(h+1)$-th row of C. Let us suppose that $c_{h.1}=2$. We denote by $p=c_{h.2}$ and $q=c_{h.3}$ the adjacent nodes of h. If $c_{p.1}=2$ and $c_{q.1}=2$, then there exists a node of degree 2 having two adjacent nodes p and q of degree 2 so that $L_2(M(A,\theta))$ is infinite; otherwise one goes to examine the $(h+1)$-th row and this up to the n-th row.

It is obvious that in the worst case an upper bound, certainly not optimal, to the number of elementary operations (= readings of the elements of C) required to decide whether $L_2(M(A,\theta))$ is infinite is given by $5n$. A better estimation of this upper bound, whose proof we shall not report here, is given by $3n+2$. In any case the total number of elementary operations required to decide whether $L_2(M(A,\theta))$ is infinite is upper-limited by a linear function of n. □

References

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