Discrete Mathematics 257 (2002) 165–168

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

On separable self-complementary graphs

Ken-ichi Kawarabayashi[a], Atsuhiro Nakamoto[b,*], Yoshiaki Oda[c], Katsuhiro Ota[a], Shinsei Tazawa[d], Mamoru Watanabe[e]

[a] *Department of Mathematics, Keio University, Yokohama 223-8522, Japan*
[b] *Department of Mathematics, Osaka Kyoiku University, 4-698-1 Asahigaoka, Kashiwara, Osaka 852-8582, Japan*
[c] *Department of Mathematics and Computer Science, Shimane University, Matsue 690-8504, Japan*
[d] *Department of Mathematics, Kinki University, Higashi-Osaka, 577-8502, Japan*
[e] *Department of Computer Science and Mathematics, Kurashiki University of Science and the Arts, Kurashiki 712-8505, Japan*

Received 23 April 2001; received in revised form 15 September 2001; accepted 8 October 2001

Abstract

In this paper, we describe the structure of separable self-complementary graphs. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Self-complementary graph; Cut vertex; Separable graph

1. Introduction

We consider only simple finite undirected graphs. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. A graph G is said to be *trivial* if G has just one vertex, and G is *empty* if G has no vertex. A graph G is said to be *separable* if G has a cut vertex. For a graph G, the *complement* of G is the graph denoted by $\bar{G}$ such that $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{uv\colon\ uv \notin E(G)\}$.

The following is well known and easy to see:

For every graph G, G itself or its complement $\bar{G}$ is connected.

We extend this observation to 2-connected graphs. That is, we would like to consider whether or not, for every graph G, G or $\bar{G}$ is 2-connected. If not, we want to char-

* Corresponding author. Present address: Department of Mathematics, Faculty of Education and Human Sciences, Yokohama National University, 79-2 Tokiwadai, Hodogaya-Ku, Yokohama 240-8501, Japan.

E-mail address: nakamoto@edhs.ynu.ac.jp (A. Nakamoto).

0012-365X/02/$ - see front matter © 2002 Elsevier Science B.V. All rights reserved.
PII: S0012-365X(02)00399-0

acterize such a graph. Note that by the above statement, if both G and $\bar{G}$ are not 2-connected, then at least one of G and $\bar{G}$ has a cut vertex. We first establish the following theorem.

Theorem 1. *Let G be a separable graph with a cut vertex v. Then the complement $\bar{G}$ of G is separable or disconnected if and only if (a) or (b) hold*:

(a) $\deg_G(v) \geqslant |V(G)| - 2$,
(b) *G has a vertex u of degree 1 such that $G - u$ has a spanning complete bipartite subgraph.*

A graph G is said to be *self-complementary* if G and $\bar{G}$ are isomorphic. Let G be a self-complementary graph with n vertices. For which $n \geqslant 2$ are there self-complementary graphs with precisely n vertices? Since $|E(G)| = |E(\bar{G})|$ and $|E(G)| + |E(\bar{G})| = n(n-1)/2$, the number $n(n-1)/2$ must be even, and hence we have $n \equiv 0, 1 \pmod 4$ [1]. Moreover, for every natural number $n \equiv 0, 1 \pmod 4$ with $n \geqslant 4$, we can construct self-complementary graphs, as follows.

Let H be a graph which is either empty, trivial, or self-complementary, and let $P_4 = v_1v_2v_3v_4$ be a 4-*path*, i.e., a path with exactly four vertices. Join each of v_2 and v_3 to all vertices of H. We call this operation a 4-*path addition*. The resulting graph with $|V(H)| + 4$ vertices can easily be checked to be self-complementary. Thus, for each $n \equiv 0, 1 \pmod 4$ with $n \geqslant 4$, we can inductively construct self-complementary graphs with precisely n vertices.

However, the self-complementary graphs obtained as above always have cut vertices and hence are separable. Our second result claims that the converse is also true.

Theorem 2. *Let G be a separable self-complementary graph with at least 4 vertices. Then, G can be obtained from a graph H by a 4-path addition, where H is either empty, trivial or self-complementary.*

Let H_1 and H_2 be two self-complementary graphs and let G_1 and G_2 be the two self-complementary graphs obtained from H_1 and H_2 by a 4-path addition, respectvely. Then, it is easy to see that H_1 and H_2 are isomorphic if and only if so are G_1 and G_2. Therefore, we can obtain the following corollary from Theorem 2. (Note that we do not regard a trivial graph as a self-complementary graph.)

Corollary 3. *The number of the separable self-complementary graphs with $n \geqslant 8$ vertices is the same as the number of the self-complementary graphs with $n - 4$ vertices.*

The number of the self-complementary graphs with n vertices has already been determined in [3]. Thus, by Corollary 3, we can obtain Table 1.

In general, we can construct many nonseparable self-complementary graphs by a method similar to a 4-path addition, as follows. Let B be a graph with $k \geqslant 2$ vertices and

Table 1
The number of separable self-complementary graphs with n vertices

The number of vertices	4	5	8	9	12	13	16	17
Self-comp.	1	2	10	36	720	5600	703 760	11 220 000
Separable self-comp.	1	1	1	2	10	36	720	5600
Nonseparable self-comp.	0	1	9	34	710	5564	703 040	11 214 400

let H be a self-complementary graph. Let $B_1 = B_4 = B$ and $B_2 = B_3 = \bar{B}$. Join all vertices of B_i and all vertices of B_{i+1} for $i = 1, 2, 3$, and join all vertices of B_j and all vertices of H for $j = 2, 3$. Then the resulting graph is k-connected and self-complementary.

2. Proof of theorems

Proof of Theorem 1. The sufficiency is obvious and hence we prove the necessity. Let v be a cut vertex of G and let $B_1, \ldots, B_k$ be the components of $G - v$. Then $\overline{G - v}$ has a spanning complete k-partite subgraph. Here, if $\deg_G(v) \geqslant |V(G)| - 2$, then v has degree at most 1 in $\bar{G}$. This is case (a).

We consider other cases. Since we may assume $\deg_{\bar{G}}(v) \geqslant 2$ in this case, $\bar{G}$ is 2-connected if and only if $\overline{G - v}$ is 2-connected. Thus, we check the 2-connectivity of $\overline{G - v}$. If $k \geqslant 3$, then $\overline{G - v}$ includes a spanning complete k-partite subgraph and hence $\overline{G - v}$ is 2-connected. Thus, we have $k = 2$. If $|V(B_1)| \geqslant 2$ and $|V(B_2)| \geqslant 2$, then $\overline{G - v}$ has $K_{|V(B_1)|,|V(B_2)|}$ as a spanning subgraph, and hence it is 2-connected.

Thus, we may suppose that $|B_1| = 1$ and put $B_1 = \{u\}$. Since $\deg_{\bar{G}}(v) \geqslant 2$ and since u is adjacent to all vertices $x \in V(G) - \{u, v\}$ in $\bar{G}$, $\bar{G}$ is not 2-connected if and only if $\overline{G - u}$ is not connected. Clearly, $\overline{G - u}$ is not connected if and only if $G - u$ has a spanning complete bipartite subgraph. This is case (b). □

Proof of Theorem 2. Let G be a separable self-complementary graph with $n \geqslant 4$ vertices. Then, by Theorem 1, G has a vertex u of degree 1. Since G is self-complementary, G has a vertex u' of degree $n - 2$. Let $\varphi : G \to \bar{G}$ be an isomorphism.

We first claim that G has a vertex v of degree 1 other than u. If u is a unique vertex of degree 1 in G, then $\varphi(u) = u'$ and $\varphi(u') = u$. If u and u' are adjacent in one of G and $\bar{G}$, say G, then $\varphi(u)(= u')$ and $\varphi(u')(= u)$ are not adjacent in $\bar{G}$, which contradicts that φ is an isomorphism. Thus, G has at least two distinct vertices u and v of degree 1, and hence G also have distinct vertices $u' = \varphi(u)$ and $v' = \varphi(v)$ of degree $n - 2$ in G.

Let u_1 and v_1 be the unique neighbors of u and v in G, respectively. Let $A = V(G) - \{u, v, u_1, v_1\}$. If $A = \emptyset$, then $V(G) = \{u, v, u_1, v_1\}$, and hence G itself is a 4-path with $\{u_1, v_1\} = \{u', v'\}$. Therefore, the theorem holds when $n = 4$.

So we may suppose that $A \neq \emptyset$. Observe that each vertex $x \in A$ can have at most $n - 3$ neighbors in G since u and v have degree 1 in G. Thus, u_1 and v_1 must be distinct and $\{u_1, v_1\} = \{u', v'\}$. Without loss of generality, we may put $u_1 = u'$ and

$v_1 = v'$. Moreover, u_1 is adjacent to all vertices of G except v, and v_1 is adjacent to all vertices in G except u.

Now we can find the 4-path $uu'v'v$ such that $\deg_G(u) = \deg_G(v) = 1$ and $\deg_G(u') = \deg_G(v') = n-2$, and other vertices $x \in A$ have degree $2 \leqslant \deg_G(x) \leqslant n-3$. So, φ maps $\{u, u', v', v\}$ to $\{u, u', v', v\}$. Since the 4-path induced by $\{u, u'v', v\}$ is self-complementary and since each of u' and v' is adjacent to all vertices in A, the subgraph $\langle A \rangle$ induced by A in G must be either trivial or self-complementary. Thus, the theorem holds for $n \geqslant 5$. $\square$

References

[1] F. Harary, E.M. Palmer, Enumeration of locally restricted digraphs, Canad. J. Math. 18 (1966) 853–860.
[2] K.R. Parthasarathy, Sridharan, Enumeration of self-complementary graphs and digraphs, J. Math. Phys. Soc. Madras 3 (1969) 410–414.
[3] R.C. Read, On the number of self-complementary graphs and digraphs, J. London Math. Soc. 38 (1963) 99–104.