Combinatorial triangulations of homology spheres

Bhaskar Bagchi\textsuperscript{a}, Basudeb Datta\textsuperscript{b}

\textsuperscript{a}Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore 560 059, India
\textsuperscript{b}Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India

Received 11 November 2003; accepted 20 June 2005
Available online 15 November 2005

Abstract

Let \( M \) be an \( n \)-vertex combinatorial triangulation of a \( \mathbb{Z}_2 \)-homology \( d \)-sphere. In this paper we prove that if \( n \leq d + 8 \) then \( M \) must be a combinatorial sphere. Further, if \( n = d + 9 \) and \( M \) is not a combinatorial sphere then \( M \) cannot admit any proper bistellar move. Existence of a 12-vertex triangulation of the lens space \( L(3,1) \) shows that the first result is sharp in dimension three.

In the course of the proof we also show that any \( \mathbb{Z}_2 \)-acyclic simplicial complex on \( \leq 7 \) vertices is necessarily collapsible. This result is best possible since there exist 8-vertex triangulations of the Dunce Hat which are not collapsible.

© 2005 Elsevier B.V. All rights reserved.

Keywords: Combinatorial spheres; pl manifolds; Collapsible simplicial complexes; Homology spheres

1. Introduction and results

All the simplicial complexes considered in this paper are finite. We say that a simplicial complex \( K \) triangulates a topological space \( X \) (or \( K \) is a triangulation of \( X \)) if \( X \) is homeomorphic to the geometric carrier \( |K| \) of \( K \).

The vertex set of a simplicial complex \( K \) is denoted by \( V(K) \). If \( K, L \) are two simplicial complexes, then a simplicial isomorphism from \( K \) to \( L \) is a bijection \( \pi : V(K) \rightarrow V(L) \) such that for \( \sigma \subseteq V(K) \), \( \sigma \) is a face of \( K \) if and only if \( \pi(\sigma) \) is a face of \( L \). The complexes \( K, L \) are called (simplicially) isomorphic when such an isomorphism exists. We identify two simplicial complexes if they are isomorphic.
A simplicial complex $K$ is called pure if all the maximal faces of $K$ have the same dimension. A maximal face in a pure simplicial complex is also called a facet.

If $\sigma$ is a face of a simplicial complex $K$ then the link of $\sigma$ in $K$, denoted by $\text{Lk}_K(\sigma)$ (or simply by $\text{Lk}(\sigma)$), is by definition the simplicial complex whose faces are the faces $\tau$ of $K$ such that $\tau$ is disjoint from $\sigma$ and $\sigma \cup \tau$ is a face of $K$.

A subcomplex $L$ of a simplicial complex $K$ is called an induced (or full) subcomplex of $K$ if $\sigma \in K$ and $\sigma \subseteq V(L)$ imply $\sigma \in L$. The induced subcomplex of $K$ on the vertex set $U$ is denoted by $K[U]$.

For a commutative ring $R$, a simplicial complex $K$ is called $R$-acyclic if $|K|$ is $R$-acyclic, i.e., $\tilde{H}_q(|K|, R) = 0$ for all $q \geq 0$ (where $\tilde{H}_q(|K|, R)$ denotes the reduced homology).

By a subdivision of a simplicial complex $K$ we mean a simplicial complex $K'$ together with a homeomorphism from $|K'|$ onto $|K|$ which is facewise linear. Two simplicial complexes $K$ and $L$ are called combinatorially equivalent (denoted by $K \approx L$) if they have isomorphic subdivisions. So, $K \approx L$ if and only if $|K|$ and $|L|$ are piecewise-linear (pl) homeomorphic (see [11]).

For a set $U$ with $d + 1$ elements, let $K$ be the simplicial complex whose faces are all the non-empty subsets of $U$. Then $K$ triangulates the $d$-dimensional closed unit ball. This complex is called the standard $d$-ball and is denoted by $A_{d+1}^d(U)$ or simply by $A_{d+1}^d$. A polyhedron is called a pl $d$-ball if it is pl homeomorphic to $|A_{d+1}^d|$. A simplicial complex $X$ is called a combinatorial $d$-ball if it is combinatorially equivalent to $A_{d+1}^d$. So, $X$ is a combinatorial $d$-ball if and only if $|X|$ is a pl $d$-ball.

For a set $V$ with $d + 2$ elements, let $S$ be the simplicial complex whose faces are all the non-empty proper subsets of $V$. Then $S$ triangulates the $d$-sphere. This complex is called the standard $d$-sphere and is denoted by $S_{d+2}^d(V)$ or simply by $S_{d+2}^d$. A polyhedron is called a pl $d$-sphere if it is pl homeomorphic to $|S_{d+2}^d|$. A simplicial complex $X$ is called a combinatorial $d$-sphere if it is combinatorially equivalent to $S_{d+2}^d$. So, $X$ is a combinatorial $d$-sphere if and only if $|X|$ is a pl $d$-sphere.

A simplicial complex $K$ is called a combinatorial $d$-manifold if the link of each vertex is a combinatorial $(d − 1)$-sphere. A simplicial complex $K$ is a combinatorial $d$-manifold if and only if $|K|$ is a closed pl $d$-manifold (see [11]).

If a triangulation $K$ of a space $X$ is a combinatorial manifold then $K$ is called a combinatorial triangulation of $X$. If $K$ is a triangulation of a 3-manifold then the link of a vertex is a triangulation of the 2-sphere and all triangulations of the 2-sphere are combinatorial 2-spheres. So, any triangulation of a 3-manifold is a combinatorial triangulation.

Let $\tau \subseteq \sigma$ be two faces of a simplicial complex $K$. We say that $\tau$ is a free face of $\sigma$ if $\sigma$ is the only face of $K$ which properly contains $\tau$. (It follows that $\dim(\sigma) − \dim(\tau) = 1$ and $\sigma$ is a maximal simplex in $K$.) If $\tau$ is a free face of $\sigma$ then $K' := K \setminus \{\tau, \sigma\}$ is a simplicial complex.

We say that there is an elementary collapse of $K$ to $K'$. We say $K$ collapses to $L$ and write $K \searrow L$ if there exists a sequence $K = K_0, K_1, \ldots, K_n = L$ of simplicial complexes such that there is an elementary collapse of $K_{i−1}$ to $K_i$ for $1 \leq i \leq n$ (see [3]). If $L$ consists of a 0-simplex (a point) we say that $K$ is collapsible and write $K \searrow 0$. Clearly, if $K \searrow L$ then $|K| \searrow |L|$ as polyhedra and hence $|K|$ and $|L|$ have the same homotopy type (see [11]).

So, if a simplicial complex $K$ is collapsible then $|K|$ is contractible and hence, in particular, $K$ is $\mathbb{Z}_2$-acyclic. Here we prove:
Theorem 1. If a $\mathbb{Z}_2$-acyclic simplicial complex has $\leq 7$ vertices then it is collapsible.

As an application of Theorem 1, we prove our main result—a recognition theorem for combinatorial spheres:

Theorem 2. Let $M$ be an $n$-vertex combinatorial triangulation of a $\mathbb{Z}_2$-homology $d$-sphere. Suppose $M$ has an $m$-vertex combinatorial $d$-ball as an induced subcomplex, where $n \leq m + 7$. Then $M$ is a combinatorial sphere.

In consequence we get the following.

Corollary 3. Let $M$ be an $n$-vertex combinatorial $d$-manifold. If $|M|$ is a $\mathbb{Z}_2$-homology sphere and $n \leq d + 8$ then $M$ is a combinatorial sphere.

Corollary 4. Let $M$ be a $(d + 9)$-vertex combinatorial triangulation of a $\mathbb{Z}_2$-homology $d$-sphere. If $M$ is not a combinatorial sphere then $M$ cannot admit any bistellar $i$-move for $i > 0$.

Since by the universal coefficient theorem any integral homology sphere is a $\mathbb{Z}_2$-homology sphere, Theorem 2, Corollary 3 and Corollary 4 remain true if we replace $\mathbb{Z}_2$-homology by integral homology in the hypothesis. In particular, we have:

Corollary 5. Let $M$ be an $n$-vertex combinatorial triangulation of an integral homology $d$-sphere.

(a) If $n \leq d + 8$ then $M$ is a combinatorial sphere.

(b) If $n = d + 9$ and $M$ is not a combinatorial sphere then $M$ cannot admit any bistellar $i$-move for $i > 0$.

Remark 1. Corollary 3 is clearly trivial for $d \leq 2$. In [5], Brehm and Kühnel proved that any $n$-vertex combinatorial $d$-manifold is a combinatorial $d$-sphere if $n < 3[d/2] + 3$ and it is either a combinatorial $d$-sphere or a cohomology projective plane if $n = 3d/2 + 3$. So, Corollary 3 has new content only for $3 \leq d \leq 8$.

Remark 2. Another result in [5] says that any $n$-vertex combinatorial $d$-manifold is simply connected for $n \leq 2d + 2$. Since a simply connected integral homology sphere is a sphere for $d \neq 3$, and since for $d \neq 4$ all combinatorial triangulations of $d$-spheres are combinatorial spheres, this result implies that all combinatorial triangulations of integral homology $d$-spheres ($d \neq 3, 4$) with $\leq 2d + 2$ vertices are combinatorial spheres. This is stronger than Corollary 5 (a) for $d \geq 6$. Thus Corollary 5(a) has new content only for $d = 3, 4, 5$.

Remark 3. In [8, p. 35], Lutz presented a 12-vertex combinatorial triangulation of the lens space $L(3, 1)$. (It is mentioned in [7, p. 79] that Brehm obtained a 12-vertex combinatorial triangulation of $L(3, 1)$ earlier.) Since $L(3, 1)$ is a $\mathbb{Z}_2$-homology 3-sphere ($H_1(L(3, 1), \mathbb{Z}) = \mathbb{Z}_3, H_2(L(3, 1), \mathbb{Z}) = 0$), Corollary 3 is sharp for $d = 3$. 
It follows from Corollary 3 that 12 is the least number of vertices required to triangulate \( L(3, 1) \). It follows from Corollary 4 that a 12-vertex combinatorial triangulation of \( L(3, 1) \) cannot admit any bistellar \( i \)-move for \( 1 \leq i \leq 3 \).

**Remark 4.** Recall that the Dunce Hat is the topological space obtained from the solid triangle \( abc \) by identifying the oriented edges \( \overrightarrow{ab}, \overrightarrow{bc} \) and \( \overrightarrow{ac} \). The following is a triangulation of the Dunce Hat using 8 vertices.

Since this example is contractible but not collapsible, it follows that the bound 7 in Theorem 1 is best possible.

**Remark 5.** Let \( H^3 \) be the non-orientable 3-manifold obtained from \( S^2 \times [0, 1] \) by identifying \((x, 0)\) with \((-x, 1)\). It follows from works of Walkup [14, Theorems 3, 4] that if \( K \) is a combinatorial 3-manifold and \(|K|\) is not homeomorphic to \( S^3 \), \( S^2 \times S^1 \) or \( H^3 \) then \( f_1(K) \geq 4f_0(K) + 8 \) and hence \( f_0(K) \geq 11 \). Thus if \( M \neq S^3 \) is a \( \mathbb{Z}_2 \)-homology 3-sphere then at least 11 vertices are needed for any combinatorial triangulation of \( M \). Now, Corollary 3 implies that at least 12 vertices are needed. In [4], Björner and Lutz have presented a 16-vertex combinatorial triangulation of the Poincaré homology 3-sphere.

In [2], we have shown that all combinatorial triangulations of \( S^4 \) with at most 10 vertices are combinatorial 4-spheres. Now, Corollary 3 implies that all combinatorial triangulations of \( S^4 \) with at most 12 vertices are combinatorial spheres. So, any combinatorial triangulation (if it exists) of \( S^4 \) which is not a combinatorial sphere requires at least 13 vertices.

**Remark 6.** The conclusion in Corollary 4 (namely, that certain combinatorial manifolds do not admit any proper bistellar move) appears to be a strong structural restriction. We owe to F. H. Lutz the information that the smallest known combinatorial sphere (other than a standard sphere) not admitting any proper bistellar move is a 16-vertex 3-sphere.

2. Preliminaries and definitions

For a simplicial complex \( K \), the maximum \( k \) such that \( K \) has a \( k \)-face is called the *dimension* of \( K \). An one-dimensional simplicial complex is called a *graph*. A simplicial complex \( K \) is called *connected* if \(|K|\) is connected.

For \( i = 1, 2, 3 \), the \( i \)-faces of a simplicial complex are also called the *edges*, *triangles* and *tetrahedra* of the complex, respectively. For a face \( \sigma \) in a simplicial complex \( K \), the number of vertices in \( \text{Lk}_K(\sigma) \) is called the *degree* of \( \sigma \) in \( K \) and is denoted by \( \text{deg}_K(\sigma) \).
If the number of $i$-simplices of a $d$-dimensional simplicial complex $K$ is $f_i(K)$, then the vector $f = (f_0, \ldots, f_d)$ is called the $f$-vector of $K$ and the number $\chi(K) := \sum_{i=0}^{d} (-1)^i f_i(K)$ is called the Euler characteristic of $K$. If $f_{k-1} = \binom{f_k}{k}$ then $K$ is called $k$-neighbourly.

For two simplicial complexes $K, L$ with disjoint vertex sets, the join $K \ast L$ is the simplicial complex $K \cup L \cup \{\sigma \cup \tau : \sigma \in K, \tau \in L\}$.

If $K$ is a $d$-dimensional simplicial complex then define the pure part of $K$ as the simplicial complex whose simplices are the subsimplices of the $d$-simplices of $K$.

A $d$-dimensional pure simplicial complex $K$ is called a weak pseudomanifold if each $(d-1)$-face is contained in exactly two facets of $K$. A $d$-dimensional weak pseudomanifold $K$ is called a pseudomanifold if for any pair $\tau, \sigma$ of facets, there exists a sequence $\tau = \tau_0, \ldots, \tau_n = \sigma$ of facets of $K$, such that $\tau_{i-1} \cap \tau_i$ is a $(d-1)$-simplex of $K$ for $1 \leq i \leq n$. In other words, a weak pseudomanifold is a pseudomanifold if and only if it does not have any weak pseudomanifold of the same dimension as a proper subcomplex. Clearly, any connected combinatorial manifold is a pseudomanifold.

For $n \geq 3$, the $n$-vertex combinatorial one-sphere ($n$-cycle) is the unique $n$-vertex one-dimensional pseudomanifold and is denoted by $S^1_1$.

A $d$-dimensional pure simplicial complex $K$ is called a weak pseudomanifold with boundary if each $(d-1)$-face is contained in 1 or 2 facets of $K$ and there exists a $(d-1)$-face of degree 1. The boundary $\partial K$ of $K$ is by definition the pure simplicial complex whose facets are the degree one $(d-1)$-faces of $K$.

A simplicial complex $K$ is called a combinatorial $d$-manifold with boundary if the link of each vertex is either a combinatorial $(d-1)$-sphere or a combinatorial $(d-1)$-ball and there exists a vertex whose link is a combinatorial $(d-1)$-ball. A simplicial complex $K$ is a combinatorial $d$-manifold with boundary if and only if $|K|$ is a compact pl $d$-manifold with non-empty boundary. Clearly, if $K$ is a combinatorial $d$-manifold with boundary then $\partial K \neq \emptyset$ and $\text{Lk}_{\partial K}(v) = \partial (\text{Lk}_K(v))$, for $v \in V(\partial K)$. Therefore, $\partial K$ is a combinatorial $(d-1)$-manifold. Clearly, if $K$ is a combinatorial $d$-ball ($d > 0$) then $K$ is a combinatorial $d$-manifold with boundary and $\partial K$ is a combinatorial $(d-1)$-sphere.

**Example 1.** Some weak pseudomanifolds on 6 or 7 vertices.
\( \Sigma_1, \ldots, \Sigma_5 \) are combinatorial spheres. \( \mathbb{R}P_6^2 \) triangulates the real projective plane. \( \gamma_1, \gamma_2 \) are the smallest examples of weak pseudomanifolds which are not pseudomanifolds.

The following results (which we need later) follow from the classification of all two-dimensional weak pseudomanifolds on \( \leq 7 \) vertices (e.g., see [1,6]).

**Proposition 2.1.** Let \( K \) be an \( n \)-vertex two-dimensional weak pseudomanifold. If \( n \leq 6 \) then \( K \) is isomorphic to \( S^2_4, S^3_3 \ast S^0_2, S^0_2 \ast S^0_2 \ast S^0_2, \mathbb{R}P_6^2 \) or \( \Sigma_1 \) above.

**Proposition 2.2.** Let \( K \) be a 7-vertex two-dimensional weak pseudomanifold. If the number of facets of \( K \) is \( \leq 10 \) then \( K \) is isomorphic to \( S^1_3 \ast S^0_2, \Sigma_2, \ldots, \Sigma_5, \gamma_1 \) or \( \gamma_2 \) above.

Let \( X \) be a pure simplicial complex of dimension \( d \geq 1 \). Let \( A \) be a set of size \( d + 2 \) such that \( A \) contains at least one and at most \( d + 1 \) facets of \( X \). (It follows that all except at most one element of \( A \) are vertices of \( X \).) Define the pure \( d \)-dimensional simplicial complex \( \kappa_A(X) \) as follows. The facets of \( \kappa_A(X) \) are (i) the facets of \( X \) not contained in \( A \) and (ii) the \( (d+1) \)-subsets of \( A \) which are not facets of \( X \). \( \kappa_A \) is said to be a generalized bistellar move. Clearly \( \kappa_A(\kappa_A(X)) = X \). Let \( \beta = \{ x \in A : A \setminus \{ x \} \in X \} \) and \( \alpha = A \setminus \beta \). Then \( \alpha \in X \) and \( \beta \in \kappa_A(X) \). The set \( \beta \) is called the core of \( \alpha \). If \( \alpha \) is a \((d-i)\)-simplex of \( X \) then \( \kappa_A \) is also called a generalized bistellar \( i \)-move. Observe that if \( d \) is even and \( \kappa_A \) is a generalized bistellar \((d/2)\)-move then \( f_d(\kappa_A(X)) = f_d(X) \).

Now suppose \( X \) is a weak pseudomanifold, and \( A, \alpha \) and \( \beta \) are as above. Notice that (a) either \( \alpha \) is a \( d \)-simplex in \( X \) or \( V(Lk_X(\alpha)) \supseteq \beta \) and (b) if \( \beta \in X \) then \( Lk_{\kappa_A(X)}(\beta) = Lk_X(\beta) \cup S^i_{i+1}(\alpha) \neq S^i_{i+1}(\alpha) \) (and therefore \( \kappa_A(X) \) is not a combinatorial manifold even if \( X \) is so). We shall say that \( \kappa_A \) is a bistellar move if (bs1) \( \beta \notin X \) and (bs2) either \( \alpha \) is a \( d \)-simplex in \( X \) or \( V(Lk_X(\alpha)) = \beta \) (and hence \( Lk_X(\alpha) \) is the standard sphere on the vertex set \( \beta \)). If \( 1 \leq i \leq d - 1 \) then a bistellar \( i \)-move is called a proper bistellar move. Observe that if \( X \) is a combinatorial \( d \)-manifold then (bs2) holds for any \((d+2)\)-subset \( A \). If a generalized bistellar move is not a bistellar move then it is called singular.

Two weak pseudomanifolds are called bistellar equivalent if there exists a finite sequence of bistellar moves leading from one to the other. Let \( \kappa_A \) be a proper bistellar move on \( X \). If \( X_1 \) is obtained from \( X \) by starring \([1] \) a new vertex in \( \alpha \) and \( X_2 \) is obtained from \( \kappa_A(X) \) by starring a new vertex in \( \beta \) then \( X_1 \) and \( X_2 \) are isomorphic. Thus, if \( X \) and \( Y \) are bistellar equivalent then \( X \approx Y \). In [10], Pachner proved the following: two combinatorial manifolds are bistellar equivalent if and only if they are combinatorially equivalent.

**Example 2.** Let the notations be as in Example 1.

(a) Let \( A = \{1, 2, 5, 6\} \subseteq V(\mathbb{R}P_6^2) \). Put \( R = \kappa_A(\mathbb{R}P_6^2) \). Then \( R \) is not a weak pseudomanifold. Observe that (bs1) is not satisfied here and hence \( \kappa_A \) is a singular bistellar move. Note that the automorphism group \( A_5 \) of \( \mathbb{R}P_6^2 \) is transitive on the 4-subsets of its vertex set. In consequence, all singular bistellar \( 1 \)-moves on \( \mathbb{R}P_6^2 \) yield isomorphic simplicial complexes.

(b) Let \( B = \{2, 3, 6, 7\} \subseteq V(\Sigma_2) \). Then \( \kappa_B(\Sigma_2) \) is the union of two spheres with one common edge 67. Here (bs1) is not satisfied.
Let $L \subseteq K$ be simplicial complexes. The \textit{simplicial neighbourhood} of $L$ in $K$ is the subcomplex $N(L, K)$ of $K$ whose maximal simplices are those maximal simplices of $K$ which intersect $V(L)$. Clearly, $N(L, K)$ is the smallest subcomplex of $K$ whose geometric carrier is a topological neighbourhood of $|L|$ in $|K|$. The induced subcomplex $C(L, K)$ on the vertex-set $V(K)\setminus V(L)$ is called the \textit{simplicial complement} of $L$ in $K$.

Suppose $P' \subseteq P$ are polyhedra and $P = P' \cup B$, where $B$ is a pl $k$-ball (for some $k \geq 1$). If $P' \cap B$ is a pl $(k-1)$-ball then we say that there is an \textit{elementary collapse} of $P$ to $P'$. We say that $P$ collapses to $Q$ and write $P \searrow Q$ if there exists a sequence $P = P_0, P_1, \ldots, P_n = Q$ of polyhedra such that there is an elementary collapse of $P_{i-1}$ to $P_i$ for $1 \leq i \leq n$. If $Q$ is a point we say that $P$ is collapsible and write $P \searrow 0$. For two simplicial complexes $K$ and $L$, if $K \searrow L$ then clearly $|K| \searrow |L|$. A \textit{regular neighbourhood} of a polyhedron $P$ in a pl $d$-manifold $M$ is a $d$-dimensional submanifold $W$ with boundary such that $W \searrow P$ and $W$ is a neighbourhood of $P$ in $M$. The following is a direct consequence of the Simplicial Neighbourhood Theorem \cite[Theorem 3.11]{11}.

**Proposition 2.3.** Let $K$ be a combinatorial $d$-manifold with boundary. Suppose $\partial K$ is an induced subcomplex of $K$. Let $L$ be the simplicial complement of $\partial K$ in $K$. Then $|K| \searrow |L|$.

**Proof.** Let $M$ be a pl d-manifold such that $|K|$ is in the interior of $M$ (we can always find such $M$, e.g., one such $M$ can be obtained from $|K| \sqcup (|\partial K| \times [0, 1])$ by identifying $(x, 0)$ with $x \in |\partial K|$).

Since $L = C(\partial K, K), |L| \subseteq |K| \setminus |\partial K|$ and hence $|K|$ is a neighbourhood of $|L|$ in $\text{int}(M)$. Again, since $L$ is the simplicial complement of $\partial K$ in $K$ and $\partial K$ is an induced subcomplex of $K, C(L, K) = \partial K$. Finally, since $\partial K$ is an induced subcomplex of dimension $d-1$, each $d$-simplex of $K$ intersects $V(L)$. This implies that $N(L, K) = K$.

Let $P = |L|$, $A = |K|$ and $J = \partial K$. Then $\partial A = |\partial K|$ and $N(L, K) := N(L, K) \cap C(L, K) = J$. Thus (i) $P$ is a compact polyhedron in the interior of the pl manifold $M$, (ii) $A$ is a neighbourhood of $P$ in $\text{int}(M)$, (iii) $A$ is a compact pl manifold with boundary and (iv) $(K, L, J)$ are triangulations of $(A, P, \partial A)$ where $L$ is an induced subcomplex of $K, K = N(L, K)$ and $J = \hat{N}(L, K)$. Then, by the Simplicial Neighbourhood Theorem, $A$ is a regular neighbourhood of $P$. Hence $A \searrow P$. \hfill $\square$

We need the following well-known results (see \cite[Lemma 1.10, Corollaries 3.13, 3.28]{11}) later.

**Proposition 2.4.** Let $B, D$ be pl $d$-balls and $h: \partial B \rightarrow \partial D$ a pl homeomorphism. Then $h$ extends to a pl homeomorphism $h_1: B \rightarrow D$. 
Proposition 2.5. Let $S$ be a $pl$ $d$-sphere. If $B \subseteq S$ is a $pl$ $d$-ball then the closure of $S \setminus B$ is a $pl$ $d$-ball.

Proposition 2.6. A collapsible $pl$ manifold with boundary is a $pl$ ball.

Question. Is it true that under the hypothesis of Proposition 2.3, we have $K \setminus L$?

3. $\mathbb{Z}_2$-acyclic simplicial complexes

In this section we prove Theorem 1.

Lemma 3.1. Let $X$ be a 7-vertex simplicial complex. Suppose (a) $X$ is $\mathbb{Z}_2$-acyclic, (b) $X$ is not collapsible, and (c) $X$ is minimal subject to (a) and (b) (i.e., $X$ has no proper subcomplex satisfying (a) and (b)). Then $X$ is pure of dimension $d = 2$ or $3$ and each $(d - 1)$-face of $X$ occurs in at least two facets.

Proof. Notice that, because of the minimality assumption, $X$ has no free face. Clearly, $\dim(X) \leq 5$, since otherwise $X$ is a combinatorial ball. Suppose $\dim(X) = 5$. By minimality, each 4-face of $X$ is in 0 or $\geq 2$ facets. Since $X$ has 7 vertices, it follows that each 4-face is in 0 or 2 facets. Therefore, the pure part $Y$ of $X$ is a 7-vertex five-dimensional weak pseudomanifold and hence $Y = S^5_7 \subseteq X$. Then $H_5(X, \mathbb{Z}_2) \neq 0$, a contradiction. Thus $\dim(X) \leq 4$.

Suppose, if possible, $\dim(X) = 4$. Let $Y$ be the pure part of $X$. Then, each 3-face of $Y$ occurs in at least two facets. If $\#(V(Y)) \leq 6$, then $Y = S^4_6$ and hence $H_4(X, \mathbb{Z}_2) \neq 0$, a contradiction. Thus $V(Y) = V(X)$ has size 7. Define a binary relation $\sim$ on $V(Y)$ by $y_1 \sim y_2$ if $V(Y) \setminus \{y_1, y_2\}$ is not a facet of $Y$. Since each 3-face of $Y$ is in at least two facets, it follows that $\sim$ is an equivalence relation with at least two equivalence classes. Therefore, either there is an equivalence class $W$ of size 6 or else we can write $V(Y) = V_1 \sqcup V_2$, where $V_1, V_2$ are unions of $\sim$-classes and $\#(V_1) \geq 2, \#(V_2) \geq 2$. In consequence $Y$ (and hence $X$) contains a 4-sphere as a subcomplex: the standard sphere on $W$ or the join of the standard spheres on $V_1$ and $V_2$. Therefore $H_4(X, \mathbb{Z}_2) \neq 0$, a contradiction. Thus, $\dim(X) \leq 3$.

If $\dim(X) = 1$ then $X$ is a $\mathbb{Z}_2$-acyclic connected graph and hence is a tree. But any tree has end vertices and hence is collapsible, a contradiction. So, $\dim(X) = 2$ or 3.

Since $H_0(X, \mathbb{Z}_2) = 0$, $X$ is connected. Since $X$ has no free vertex, it follows that each vertex of $X$ is in at least two edges.

Next we show that $X$ has no maximal edge. Suppose, on the contrary, $X$ has a maximal edge $e$. Then $Y := X \setminus \{e\}$ is a subcomplex of $X$. We claim that $Y$ is disconnected. If not, then there is a subcomplex $K = S^1_{11}$ of $X$ containing the edge $e$. The formal sum of the edges in $K$ is an 1-cycle over $\mathbb{Z}_2$ which is not a boundary since it involves the maximal edge $e$. Hence $H_1(X, \mathbb{Z}_2) \neq 0$, a contradiction. So, $Y$ is disconnected. Since each vertex of $X$ is in at least two edges, it follows that each component of $Y$ has at least 3 vertices. Since $X$ has seven vertices, it follows that some component of $Y$ has exactly three vertices and contains an $S^1_3$. If these three vertices span a 2-face then its edges are free in $X$, contradicting minimality. In the remaining case $X$ has an induced $S^1_3$ whose edges are maximal, contradicting $\mathbb{Z}_2$-acyclicity of $X$. 
In case \( \dim(X) = 2 \), this shows that \( X \) is pure. In case \( \dim(X) = 3 \), we proceed to show that \( X \) has no maximal 2-face, proving that it is pure in that case too.

Suppose, on the contrary, that \( \dim(X) = 3 \) and \( X \) has a maximal 2-face \( A = abc \). Let us say that an edge of \( X \) is good if it is in a tetrahedron of \( X \), and call it bad otherwise. First, suppose that all three edges in \( A \) are good. Since \( X \) has no free triangle, each vertex in the link of an edge has degree 0 or \( \geq 2 \) and hence there are at least three vertices of degree \( \geq 2 \) in the link of a good edge. Since \( A \) is maximal, it follows that the link of each of the three edges in \( A \) has \( \geq 3 \) vertices outside \( A \). Since, there are only four vertices outside \( A \), it follows from the pigeonhole principle that there is a common vertex \( x \) outside \( A \) which occurs in the link of all three edges in \( A \). Hence \( S_4^2(A \cup \{x\}) \) is a subcomplex of \( X \). The sum of the four triangles in this \( S_4^2 \) is a 2-cycle (with \( \mathbb{Z}_2 \) coefficients) which cannot be the boundary of a 3-chain since one of these triangles is maximal. Therefore, \( H_2(X, \mathbb{Z}_2) \neq 0 \), a contradiction. Thus, \( A \) contains at least one bad edge.

We claim that \( A \) cannot have more than one bad edges. Suppose, on the contrary, that \( ab \) and \( ac \) are bad edges in \( X \). Notice that (arguing as in the proof of the case \( \dim(X) = 4 \)), if a three-dimensional simplicial complex on \( \leq 6 \) vertices has \( \geq 2 \) tetrahedra through each triangle then it contains a combinatorial \( S^3 \). Therefore, the pure part \( Y \) of \( X \) must have seven vertices. In particular \( a \in Y \). Since \( ab \) and \( ac \) are bad edges, \( b, c \notin \text{Lk}_Y(a) \) and hence \( \text{deg}_Y(a) \leq 4 \). Therefore, \( \text{Lk}_Y(a) = S_4^2 \). Hence we can apply an improper bistellar move to \( Y \) to remove the vertex \( a \), yielding a 6-vertex three-dimensional simplicial complex \( \tilde{Y} \) with \( \geq 2 \) tetrahedra through each triangle. Hence \( \tilde{Y} \) has an \( S^3 \) as a subcomplex, so that \( H_3(Y, \mathbb{Z}_2) = H_3(\tilde{Y}, \mathbb{Z}_2) \neq 0 \). Therefore, \( H_3(X, \mathbb{Z}_2) \neq 0 \), a contradiction. Thus, \( A \) contains exactly one bad edge, say \( ab \). Hence \( ac \) and \( bc \) are good edges.

Since \( X \) has no free edge, there is a second triangle, say \( abd \), through \( ab \). Since \( ab \) is a bad edge, \( abd \) is maximal. By the above argument, \( ad \) and \( bd \) are good edges. If both \( acd \) and \( bcd \) are triangles of \( X \) then \( X \) has \( S_4^2(a, b, c, d) \) as a subcomplex, and at least one of the triangles of this \( S_4^2 \) is maximal in \( X \), yielding the contradiction \( H_2(X, \mathbb{Z}_2) \neq 0 \) as before. Therefore, without loss of generality, we may assume \( bcd \notin X \). Note that \( a \) is an isolated vertex in \( \text{Lk}_X(bc) \) and \( d \) does not occur in \( \text{Lk}_X(bc) \). Since \( bc \) is a good edge, it follows that all three vertices outside \( \{a, b, c, d\} \) (say \( x, y \) and \( z \)) occur in \( \text{Lk}_X(bc) \). Similarly, \( x, y, z \in \text{Lk}_X(bd) \). Again, the good edges \( ac \) and \( ad \) have at most one non-isolated vertex from \( \{a, b, c, d\} \) in their links, hence each of them has at least two of \( x, y, z \) in their links. Therefore, there is one vertex, say \( x \), which occurs in the link of all the four edges \( ac, bc, ad, bd \). Hence \( S_3^3(c, d) * S_4^1(a, b, x) \) is a subcomplex of \( X \). Since one of the triangles in this 2-sphere is maximal, it follows that \( H_2(X, \mathbb{Z}_2) \neq 0 \), a contradiction. Thus, \( X \) has no maximal triangles nor maximal edges, so \( X \) is pure.

Finally, the last assertion follows from purity and minimality of \( X \).  

**Lemma 3.2.** Let \( X \) be a 7-vertex two-dimensional \( \mathbb{Z}_2 \)-acyclic simplicial complex. Then \( X \) is collapsible.

**Proof.** Let \( X \) be a minimal counter example. Let \( f_i, 0 \leq i \leq 2 \), be the number of \( i \)-faces in \( X \). Since \( X \) is \( \mathbb{Z}_2 \)-acyclic, \( \chi(X) = 1 \). Thus, \( f_0 = 7 \) and \( f_1 = f_2 + 6 \).
For $i \geq 0$, let $e_i$ be the number of edges of degree $i$ in $X$. By Lemma 3.1, $e_i = 0$ for $i \leq 1$. Two-way counting yields

$$\sum_{i=2}^{5} e_i = f_1 = f_2 + 6, \quad \sum_{i=2}^{5} i e_i = 3 f_2.$$  

Hence

$$e_3 + 3e_5 \leq e_3 + 2e_4 + 3e_5 = f_2 - 12. \quad (1)$$

Let us say that an edge of $X$ is odd (respectively even) if it lies in an odd (respectively even) number of triangles. Note that each graph has an even number of vertices of odd degree. Applying this trivial observation to the vertex links of $X$, we conclude that each vertex of $X$ is in an even number of odd edges. Thus, the total number $e_3 + e_5$ of odd edges is $=0$ or $\geq 3$. If there is no odd edge then the sum of all the triangles gives a non-zero element of $H_2(X, \mathbb{Z}_2)$, a contradiction. So, $e_3 + e_5 \geq 3$. Combining this with (1), we get $f_2 \geq 15$ and hence $f_1 \geq 21 = \binom{7}{2}$. Hence $f_1 = 21, f_2 = 15, e_3 = 3, e_4 = e_5 = 0$.

Since each vertex is in an even number of odd edges, it follows that the three odd edges form a triangle $A$, which may or may not be in $X$.

If $A$ is in $X$, then the sum of the remaining triangles gives a non-zero element of $H_2(X, \mathbb{Z}_2)$, a contradiction. If $A$ is not in $X$ then (as each of the three edges in $A$ has three vertices in its link and there are four vertices outside $A$) by the pigeonhole principle there is a vertex $x \notin A$ such that $x$ occurs in the link of each of the three edges in $A$. Then the sum of all the triangles excepting the three triangles in $A \cup \{x\}$ gives a non-zero element of $H_2(X, \mathbb{Z}_2)$, a contradiction. \(\Box\)

**Lemma 3.3.** Let $U$ be a two-dimensional pure simplicial complex on $\leq 7$ vertices. Suppose the number of triangles in $U$ is $\leq 10$ and each edge of $U$ is in an even number of triangles. Then either $U$ is the union of two combinatorial spheres (on 4 or 5 vertices) with no common triangle, or $U$ is isomorphic to one of $S_4^2, S_3^1 \ast S_2^0, S_2^0 \ast S_2^0, S_2^0 \ast S_1^1, S_2^0 \ast S_0^0, S_2^0 \ast P_6^2, \Sigma_1, \ldots, \Sigma_5$ or $R$ (of Example 1 and Example 2(a)).

**Proof.** Let $\mathcal{S}$ be the list of simplicial complexes in the statement of this lemma. We find by inspection that $\mathcal{S}$ is closed under generalized bistellar 1-moves.

If $f_0(U) \leq 5$ then $U$ is a weak pseudomanifold and hence, by Proposition 2.1, $U \in \mathcal{S}$. So assume $f_0(U) = 6$ or 7. The proof is by induction on the number $n(U)$ of degree 4 edges in $U$. If $n(U) = 0$ then $U$ is a weak pseudomanifold and hence, by Propositions 2.1 and 2.2, $U \in \mathcal{S}$. So let $n(U) > 0$ and suppose that we have the result for all smaller values of $n(U)$.

By the assumption, all the edges of $U$ are of degree 2 or 4. Therefore, a two-way counting yields $4n(U) + 2(f_1(U) - n(U)) = 3 f_2(U) \leq 30$. Thus, $n(U) + f_1(U) \leq 15$. Therefore,$$f_1(U) < 15, \quad (2)$$showing that $U$ has at least one non-edge. Fix an edge $ab$ of degree 4 in $U$. Let $W$ be the link of $ab$. If each pair of vertices in $W$ formed an edge in $U$ then $f_1(U)$ would be $\geq 15$, contradicting (2). So, there exist $c, d \in W$ such that $cd$ is a non-edge in $U$. 


Let $A = \{a, b, c, d\}$. Then $\kappa_A$ is a generalized bistellar 1-move and hence $\kappa_A(U)$ also satisfies the hypothesis of the lemma, and $n(\kappa_A(U)) = n(U) - 1$. Therefore, by the induction hypothesis, $\kappa_A(U) \in \mathcal{S}$. Since $\mathcal{S}$ is closed under generalized bistellar 1-moves, $U = \kappa_A(\kappa_A(U)) \in \mathcal{S}$. □

**Lemma 3.4.** Let $X$ be a 7-vertex three-dimensional simplicial complex. Suppose (a) $X$ is $\mathbb{Z}_2$-acyclic, (b) $X$ is not collapsible, and (c) $X$ is minimal subject to (a) and (b). Then the $f$-vector of $X$ is $(7, 20, 30, 16, 7, 21, 32, 17, 7, 21, 33, 18, 7, 21, 34, 19)$ or $(7, 21, 35, 20)$.

**Proof.** For $0 \leq i \leq 3$, let $f_i$ be the number of $i$-faces of $X$. For $i > 0$, let $t_i$ be the number of triangles of degree $i$ in $X$. By Lemma 3.1, we have $t_i = 0$ for $i \leq 1$. Two way counting yields

$$\sum_{i=2}^{4} t_i = f_2, \quad \sum_{i=2}^{4} it_i = 4f_3$$

and hence

$$t_3 \leq t_3 + 2t_4 = 4f_3 - 2f_2. \quad (3)$$

Say that a triangle of $X$ is odd (respectively even) if it is in an odd (respectively even) number of tetrahedra of $X$. By the same argument as in Lemma 3.2, each edge is in an even number of odd triangles, so that the number $t_3$ of odd triangles is $0$ or $\geq 4$.

If there is no odd triangle then the sum of all the tetrahedra gives a non-zero element of $H_3(X, \mathbb{Z}_2)$, a contradiction. So, $t_3 \geq 4$. Combining this with (3) we get

$$2f_3 - f_2 \geq 2. \quad (4)$$

Since $X$ is $\mathbb{Z}_2$-acyclic, by a result of Stanley [13], $X$ has a two-dimensional subcomplex $Y$ such that the $f$-vector of $X$ equals the $f$-vector of a cone over $Y$. (In [13], the author uses the vanishing of the reduced cohomology groups as his definition of acyclicity, while we have used the homology definition. However, since the coefficient ring used is a field, these two definitions coincide.) Let $(g_0, g_1, g_2)$ be the $f$-vector of $Y$. Thus, $g_0 = 6$ and

$$f_1 = g_1 + 6, \quad f_2 = g_1 + g_2, \quad f_3 = g_2. \quad (5)$$

Hence (4) yields

$$g_2 \geq g_1 + 2. \quad (6)$$

Let $m = \binom{6}{2} - g_1$, $n = \binom{6}{3} - g_2$ be the number of non-edges and non-triangles of $Y$, respectively. Since each non-edge is in exactly four non-triangles and any two non-edges are shared by at most one non-triangle, we have $n \geq 4m - \binom{m}{2}$. Also, from (6) we get $n \leq m + 3$. Hence $m + 3 \geq 4m - \binom{m}{2}$ or $(m - 1)(m - 6) \geq 0$. So, either $m \leq 1$ or $m \geq 6$.

First suppose $m \geq 6$, i.e., $g_1 \leq 9$. If each edge of $Y$ was in $\leq 3$ triangles then we would have $g_2 \leq g_1$, contradicting (6). So, there is an edge of $Y$ contained in four triangles, together covering all the nine edges of $Y$. But, apart from the four triangles already seen, no three of these nine edges form a triangle of $Y$. Thus $g_2 = 4, g_1 = 9$—contradicting (6). So, $m \leq 1$, i.e., $g_1 = 14$ or 15.
If \( g_1 = 14 \) then the four triangles through the missing edge are missing from \( Y \), so that \( g_2 \leq 16 \). Thus, by (6), \( (g_1, g_2) = (14, 16), (15, 17), (15, 18), (15, 19) \) or \( (15, 20) \). The lemma now follows from (5). \( \square \)

**Lemma 3.5.** Let \( X \) be a 7-vertex three-dimensional \( \mathbb{Z}_2 \)-acyclic simplicial complex. Then \( X \) is collapsible.

**Proof.** Let \( X \) be a minimal counter example. As before, each edge is in an even number of odd triangles. Let \( f_i 's \) and \( t_j 's \) be as in the proof of Lemma 3.4. Then, by Lemma 3.4, \( t_3 + 2t_4 = 4f_3 - 2f_2 \leq 10 \) and hence the number \( t_3 \) of odd triangles is \( \leq 10 \).

Let \( U \) denote the pure two-dimensional simplicial complex whose facets are the odd triangles of \( X \). Then each edge of \( U \) is in an even number of triangles of \( U \). Therefore, by Lemma 3.3, we get the following cases:

**Case 1:** \( U \) is the union of two combinatorial spheres with no common triangle (on 4 or 5 vertices), say on vertex sets \( A \) and \( B \).

First suppose \( \#(A) = \#(B) = 4 \). If both \( A \) and \( B \) are 3-faces in \( X \) then the pure simplicial complex \( \tilde{X} \) whose facets are those of \( X \) other than \( A \) and \( B \) is a three-dimensional weak pseudomanifold. This implies that the sum of all the tetrahedra, excepting \( A \) and \( B \), gives a non-zero element of \( H_3(X, \mathbb{Z}_2) \), a contradiction. So, without loss of generality \( A \not\subseteq X \).

Since each of the four triangles inside \( A \) is of degree 3 in \( X \), the three vertices (say \( x, y, z \)) outside \( A \) occur in the link of all the four triangles. Then the 3-sphere \( S_4^2(A) \ast S_2^0(x, y) \) occurs as a subcomplex of \( X \), forcing \( H_3(X, \mathbb{Z}_2) \neq 0 \), a contradiction.

In the remaining case \( \#(A) = 4, \#(B) = 5 \) (since \( U \) has at most 10 triangles, the case \( \#(A) = \#(B) = 5 \) does not arise). Write \( B = \{ b_1, b_2, b_3, x, y \} \) and \( U = S_4^2(A) \cup (S_3^1(b_1, b_2, b_3) \ast S_2^0(x, y)) \). As above, we must have \( A \in X \).

If both \( b_1b_2b_3x \) and \( b_1b_2b_3y \) are in \( X \), then the sum of the 3-faces other than \( A \), \( b_1b_2b_3x \) and \( b_1b_2b_3y \) gives a non-zero element of \( H_3(X, \mathbb{Z}_2) \), a contradiction. So, without loss of generality, \( b_1b_2b_3x \not\subseteq X \). Since the triangles of \( (S_3^1(b_1, b_2, b_3) \ast S_2^0(x, y)) \) are degree 3 triangles in \( X \), it follows that \( b_1b_2xy \), \( b_1b_3xy \), \( b_2b_3xy \) \( \in X \). Then the sum of the tetrahedra other than \( A \) and these three tetrahedra gives a non-zero element of \( H_3(X, \mathbb{Z}_2) \), a contradiction.

**Case 2:** \( U = S_4^2 \). We get a contradiction as in Case 1.

**Case 3:** \( U = S_3^3 \ast S_2^0 \). We get a contradiction as in Case 1.

**Observation 1.** As \( t_3 \geq 8 \) in the remaining cases, we have \( 2f_3 - f_2 \geq 4 \) and hence only the following two possibilities survive for the \( f \)-vector of \( X \): \( (7, 21, 34, 19) \) and \( (7, 21, 35, 20) \). Therefore, \( X \) has at most one missing triangle and at most one triangle of degree 4, and these two cases are exclusive. It follows that, if \( x \) is a vertex not covered by the odd triangles, then \( \text{Lk}_X(x) \) is a 6-vertex two-dimensional neighbourly weak pseudomanifold. But, from Proposition 2.1, we see that \( \mathbb{R}P_6^2 \) is the only possibility. Thus, \( \text{Lk}_X(x) = \mathbb{R}P_6^2 \). This implies that if \( V_1 \subseteq V(U) \) is a 3-set then exactly one of \( V_1 \) and \( V(U) \setminus V_1 \) is a simplex in \( \text{Lk}_X(x) \). In particular, any two triangles in \( \text{Lk}_X(x) \) intersect.

**Case 4:** \( U = S_2^0(a_1, a_2) \ast S_2^0(b_1, b_2) \ast S_2^0(c_1, c_2) \). Then the odd triangles of \( X \) are \( a_ib_jck, 1 \leq i, j, k \leq 2 \). If \( \{a_1a_2b_jck : 1 \leq j, k \leq 2 \} \subseteq X \), then the sum of the remaining
tetrahedra gives a non-zero element of \( H_3(X, \mathbb{Z}_2) \), a contradiction. So, without loss of generality, \( a_1a_2b_1c_1 \notin X \). As \( a_1b_1c_1, a_2b_1c_1 \) are degree 3 triangles, it follows that \( a_1b_1b_2c_1b \notin X \). If both \( a_1b_1b_2c_2 \) and \( a_2b_1b_2c_2 \) are in \( X \) then \( X \supseteq \{ a_1b_1b_2c_k : 1 \leq i, k \leq 2 \} \), hence we get a contradiction as before. So, without loss of generality, \( a_2b_1b_2c_2 \notin X \).

Since \( a_1a_2b_1c_1, a_2b_1b_2c_2 \notin X \) and \( a_1b_1c_1, a_2b_2c_2 \) are degree 3 triangles, it follows that these two disjoint triangles occur in the link of \( x \). But this contradicts Observation 1.

Case 5: \( U = \Sigma_1 \) of Example 1. Thus, the odd triangles are 125, 126, 156, 235, 236, 345, 346 and 456. If 1256, 3456 \( \notin X \) then, since 125 and 346 are degree 3 triangles, they are disjoint triangles in \( \text{Lk}_X(x) \), contradicting Observation 1. So, without loss of generality, 1256 \( \in X \).

If 3456 \( \notin X \) then, since 345, 346, 456 are degree 3 triangles, 2345, 2346, 2456, 2465 \( \in X \). Then the sum of all the tetrahedra, excepting 1256, 2345, 2346, 2456, gives a non-zero element of \( H_3(X, \mathbb{Z}_2) \). Thus, 3456 \( \in X \).

If 2356 \( \in X \), then the sum of all the tetrahedra, excepting 1256, 2356, 3456, gives a non-zero element of \( H_3(X, \mathbb{Z}_2) \). Therefore 2356 \( \notin X \).

Since 235 and 236 are degree 3 triangles, 2345, 2346 \( \in X \). First, suppose that at least one of 1356, 2456 is in \( X \). Without loss, say 2456 \( \in X \). Then the sum of all the tetrahedra, excepting 1256, 2456, 2345, 2346, gives a non-zero element of \( H_3(X, \mathbb{Z}_2) \). Thus 1356, 2456 \( \notin X \). Then, since 156, 456 are degree 3 triangles, 156x, 456x \( \in X \).

Since 2356, 2456 \( \notin X \), \( x \in \text{Lk}_X(256) \), i.e., 256x \( \in X \). Similarly, looking at 356, we conclude that 356x \( \in X \). Thus, 56x is a degree 4 triangle in \( X \). But this is not possible since, by Observation 1, \( \text{Lk}_X(x) \) is \( \mathbb{R}P^2 \).

Observation 2. In the remaining cases, \( t_3 = 10 \) and hence the \( f \)-vector of \( X \) is \( (7, 21, 35, 20) \). In consequence, \( t_4 = 0 \). Thus, all triangles are of degree 2 or 3. Since \( f_3 = \binom{3}{3} \), each edge in \( X \) has degree 5. Thus, if \( e \) is an edge outside \( U \) then the link of \( e \) is a pentagon \( (S^5) \).

Case 6: \( U = \mathbb{R}P^2 \). In this case, all the 4-sets of vertices not containing \( x \) contain exactly two odd triangles each. In particular, all the tetrahedra of \( X \) not containing \( x \) contain exactly two odd triangles each. Trivially, each tetrahedron through \( x \) contains at most one odd triangle. Thus, letting \( z_i, i \geq 0 \), denote the number of tetrahedra of \( X \) containing exactly \( i \) odd triangles, we have \( x_2 = 20 - 10 = 10 \) and \( z_0 + z_1 = 10 \). But two way counting yields \( z_1 + 2x_2 = 10 \times 3 = 30 \). Hence \( z_1 = 10 \), \( z_0 = 0 \). Thus, \( x \) occurs in the link of each odd triangle and hence \( \text{Lk}_X(x) = U \). Therefore, the 10 tetrahedra of \( X \) not passing through \( x \) add up to a non-zero element of \( H_3(X, \mathbb{Z}_2) \), a contradiction.

Case 7: \( U = R \) of Example 2(a). Thus, the odd triangles are 123, 124, 125, 126, 135, 146, 236, 245, 345 and 346. We claim that \( \text{Lk}_X(12) \supseteq \begin{array}{c}
5 \bullet \\
4 \bullet \\
3 \bullet \\
2 \bullet \\
1 \bullet
\end{array} \). If, for instance, 1236 \( \notin X \) then, as 123, 126, 236 are degree 3 triangles, \( x \) belongs to the link of each of these triangles. Then \( \text{Lk}_X(2x) \supseteq \begin{array}{c}
6 \bullet \\
5 \bullet \\
4 \bullet \\
3 \bullet
\end{array} \) contradicting Observation 2. This proves the claim.

Since 3, 4, 5, 6 are of degree 3 and \( x \) is of degree 2 in \( \text{Lk}_X(12) \), it follows that \( \text{Lk}_X(12) = \begin{array}{c}
5 \bullet \\
4 \bullet \\
3 \bullet \\
6 \bullet \\
x
\end{array} \) or \( = \begin{array}{c}
6 \bullet \\
5 \bullet \\
4 \bullet \\
x
\end{array} \).
In the first case, 125, 126 ∈ Lk_X(x). Hence, by Observation 1, 345, 346 ∉ Lk_X(x). Since these two are degree 3 triangles, it follows that Lk_X(345) = {1, 2, 6} and Lk_X(346) = {1, 2, 5}. Since 1, 2 are of degree 2 in Lk_X(34), this forces Lk_X(34) = 3 and hence x ∉ Lk_X(34). This is a contradiction since X is 3-neighbourly.

In the second case, 125, 126 ∉ Lk_X(x) and hence, by Observation 1, 345, 346 ∈ Lk_X(x). That is, 5x, 6x ∈ Lk_X(34). Also, as 34 ∉ Lk_X(12), we have 12 ∉ Lk_X(34). Since 5, 6 are of degree 3 and 1, 2, x are of degree 2 in Lk_X(34), it follows that Lk_X(34) = 356.

Hence 1345, 2347, 456, 457 and 567. By the above claim, 1247, 1457, 1567, 2347, 4567.

Claim. In the remaining cases, if F is a set of four vertices of U containing at least two odd triangles, then either F ∈ X or F ⊆ V(Lk_U(x)) for some vertex x.

In these cases, V(U) = V(X). If F ∉ X contains two odd triangles, then on the average, a vertex outside F occurs in the links (in X) of ≥ 3 × 2 + 2 × 2/3 = 3 of the four triangles inside F. Thus, there is a vertex x in the link of all these triangles. If F ∉ V(Lk_U(x)) for this x, then choose a vertex y ∈ F such that xy ∉ U. Then Lk_X(xy) ⊆ S^1(F\{y}), contradicting Observation 2. This proves the claim.

Case 8: U = S^1(Z_5) ∗ S^0_2(u, v). In this case, the above claim implies that X contains the five tetrahedra {u, v, i, i + 1}, i ∈ Z_5. Then the sum of the remaining 15 tetrahedra gives a non-zero element of H_3(X, Z_2), a contradiction.

Case 9: U = Σ_2 of Example 1. Thus, the odd triangles are 126, 127, 167, 236, 237, 346, 345, 457, 456 and 567. By the above claim, 1267, 2367, 3467, 4567 ∈ X. Then the sum of the remaining 16 tetrahedra gives a non-zero element of H_3(X, Z_2), a contradiction.

Case 10: U = Σ_3 of Example 1. Thus, the odd triangles are 126, 127, 167, 234, 237, 246, 347, 456, 457 and 567. By the claim, 1267, 2347, 4567 ∈ X.

If 2467 ∈ X then the sum of all the tetrahedra, excepting 1267, 2347, 4567, 2467, gives a non-zero element of H_3(X, Z_2), a contradiction. So, 2467 ∉ X. Then Lk_X(246) = {1, 3, 5}.

Since deg(247) = 2 and 2347 ∈ X, assume without loss of generality, that 2457 ∈ X and 1247 ∉ X. Then Lk_X(127) = {3, 5, 6}.

So, 2456, 2457 ∈ X and deg(245) = 2. Hence 2345 ∉ X. Then Lk_X(234) = {1, 6, 7}.

Now, 1234, 1237 ∈ X and deg(123) = 2. Therefore, 1236 ∉ X. Then Lk_X(126) = {4, 5, 7}.

This implies that 1 ∧ 2 ∧ 3 ∧ 6 ⊆ Lk_X(25), a contradiction to Observation 2.

Case 11: U = Σ_4 of Example 1. Thus, the odd triangles are 124, 127, 145, 156, 167, 234, 237, 347, 457 and 567. By the claim, 1247, 1457, 1567, 2347 ∈ X. Then the sum of the remaining 16 tetrahedra gives a non-zero element of H_3(X, Z_2), a contradiction.

Case 12: U = Σ_5 of Example 1. Thus, the odd triangles are 123, 126, 135, 156, 234, 246, 345, 457, 467, 567. By the claim, 1234, 1235, 1246, 1256, 1345, 2345, 3457, 4567 ∈ X.

Thus Lk_X(14) ⊇ 6 and Lk_X(25) ⊇ 4. Since 14 and 25 are not in U, Observation
2 implies that \( \text{Lk}_X(14) = \{3\} \cup \{2\} \) and \( \text{Lk}_X(25) = \{3\} \cup \{1\} \). Thus 1457, 2457 \( \subseteq X \). Then the triangle 457 is of degree 4 in \( X \), a contradiction. This completes the proof. \( \Box \)

**Proof of Theorem 1.** Let \( Y \) be a minimal counter example. So, \( Y \) is an \( n \)-vertex (for some \( n \leq 7 \)) \( \mathbb{Z}_2 \)-acyclic simplicial complex which is not collapsible to any proper subcomplex.

If \( n < 7 \) then choose a facet \( x \) of \( Y \) and an element \( v \notin V(Y) \). Let \( \tilde{Y} \) be obtained from \( Y \) by the bistellar 0-move \( \kappa_{2\{v\}[x]} \). Then \( \tilde{Y} \) is an \((n+1)\)-vertex \( \mathbb{Z}_2 \)-acyclic simplicial complex. Since \( Y \) has no free face, \( \tilde{Y} \) has no free face and hence \( \tilde{Y} \) is not collapsible to any proper subcomplex. Repeating this construction (if necessary) we get a 7-vertex \( \mathbb{Z}_2 \)-acyclic simplicial complex \( X \) which is not collapsible to any proper subcomplex. Then, by Lemma 3.1, \( X \) is of dimension 2 or 3. But, this is not possible by Lemmas 3.2 and 3.5. This completes the proof. \( \Box \)

4. Homology spheres

**Lemma 4.1.** Let \( Y \) be a pseudomanifold of dimension \( d \). Let \( Y_1 \) be a proper induced subcomplex of \( Y \) which is pure of dimension \( d \). Put \( L = C(Y_1, Y) \) and \( Y_2 = N(L, Y) \). Then (a) \( Y_1, Y_2 \) are weak pseudomanifolds with boundary, (b) \( \partial Y_2 \) is an induced subcomplex of \( Y_2 \) and (c) \( \partial Y_2 = \partial Y_1 = Y_1 \cap Y_2 \).

**Proof.** Since \( Y \) is a pseudomanifold and \( Y_1 \subset Y \) is pure of maximum dimension, \( Y_1 \) is a weak pseudomanifold with boundary. Since the maximal simplices of \( Y_2 \) are those maximal simplices of \( Y \) which intersect \( V(L) \), \( Y_2 \) is pure of dimension \( d \) and each \( d \)-simplex of \( Y \) is either in \( Y_1 \) or in \( Y_2 \) but not both. This implies that \( Y_2 \) is a weak pseudomanifold with boundary. This proves (a).

Let \( V_1 = V(Y_1) \), \( V_2 = V(L) \). Then \( V(Y) = V_1 \cup V_2 \). Now, \( \tau \) is a facet of \( \partial Y_2 \) \( \iff \) there exists a unique \( d \)-face \( \sigma_2 \in Y_2 \) containing \( \tau \) \( \iff \) there exists a unique \( d \)-face \( \sigma_1 \in Y_1 \) containing \( \tau \) \( \iff \) \( \tau \) is a facet of \( \partial Y_1 \). Therefore, \( \partial Y_2 = \partial Y_1 \subseteq Y_1 \cap Y_2 \).

Since \( \partial Y_2 = \partial Y_1 \), \( \partial Y_2 \subseteq Y_2[V_1] = Y_2[V_1 \cap V(Y_2)] \). Conversely, let \( \tau \) be a maximal face in \( Y_2[V_1] \). Since \( Y_2 \) is pure, there exists a \( d \)-simplex \( \sigma_2 \in Y_2 \) such that \( \tau \subseteq \sigma_2 \). Since \( Y_1 = Y[V_1] \), \( \tau \in Y_1 \) and hence there exists a \( d \)-simplex \( \sigma_1 \in Y_1 \) such that \( \tau \subseteq \sigma_1 \). This implies that \( \tau \in \partial Y_1 \). Thus, \( Y_2[V_1] \subseteq \partial Y_1 = \partial Y_2 \). So, \( Y_2[V_1] = \partial Y_2 \). This proves (b).

Since \( \tau \in Y_1 \cap Y_2 \) implies \( \tau \in Y_2[V_1] = \partial Y_2 \), \( Y_1 \cap Y_2 \subseteq \partial Y_2 \). Therefore, \( Y_1 \cap Y_2 = \partial Y_2 \). This completes the proof. \( \Box \)

**Lemma 4.2.** Let \( X \) be a connected combinatorial \( d \)-manifold. Let \( X_1 \) be an induced subcomplex of \( X \) which is a combinatorial \( d \)-ball. Put \( L = C(X_1, X) \) and \( X_2 = N(L, X) \). Then

(a) \( X_2 \) is a connected combinatorial \( d \)-manifold with boundary.
(b) \( |X_2| \subseteq |L| \).
(c) If, further, \( L \) is collapsible then \( X \) is a combinatorial sphere.
Proof. Let $V_1 = V(X_1)$, $V_2 = V(L)$. Then $V(X) = V_1 \cup V_2$. As in the proof of Lemma 4.1, $X_2$ is pure of dimension $d$ and each $d$-simplex of $X$ is either in $X_1$ or in $X_2$ but not in both.

Let $v$ be a vertex of $X_2$. Notice that $v \in X_1 \setminus \partial X_1 \Rightarrow \text{Lk}_{X_1}(v) \subseteq \text{Lk}_X(v)$ are $(d-1)$-spheres $\Rightarrow \text{Lk}_{X_1}(v) = \text{Lk}_X(v) \Rightarrow v \notin X_2$, a contradiction. So, either $v \in V_2$ or $v \in \partial X_1$.

If $v \in V_2$ then each $d$-simplex of $X$ containing $v$ is in $X_2$ and hence $\text{Lk}_{X_2}(v) = \text{Lk}_X(v)$ is a combinatorial $(d-1)$-sphere.

If $v \in \partial X_1$ then $(Y, Y_1, Y_2) : = (\text{Lk}_X(v), \text{Lk}_{X_1}(v), \text{Lk}_{X_2}(v))$ satisfies the hypothesis of Lemma 4.1. Therefore, by Lemma 4.1, $\text{Lk}_{X_1}(v) \cap \text{Lk}_{X_2}(v) = \partial(\text{Lk}_{X_1}(v))$. This implies that the closure of $|\text{Lk}_X(v)| \setminus |\text{Lk}_{X_1}(v)|$ in $|\text{Lk}_X(v)|$ is $|\text{Lk}_{X_2}(v)|$. Since $|\text{Lk}_X(v)|$ is a pl $(d-1)$-sphere and $|\text{Lk}_{X_1}(v)|$ is a pl $(d-1)$-ball, by Proposition 2.5, $|\text{Lk}_{X_2}(v)|$ is a pl $(d-1)$-ball. Thus, $\text{Lk}_{X_2}(v)$ is a combinatorial $(d-1)$-ball.

Thus $X_2$ is a combinatorial $d$-manifold with boundary such that $\partial X_2 = \partial X_1$, by Lemma 4.1 is connected. Therefore, if $X_2$ were disconnected, it would have a $d$-dimensional weak pseudomanifold as a component. This is not possible since $X$ is a $d$-dimensional pseudo-manifold. Therefore, $X_2$ is connected. This proves (a).

As $L = X[V_2]$, we have $L \subseteq X_2$ and hence $L = X_2[V_2]$. Since, by Lemma 4.1, $\partial X_2$ is the induced subcomplex of $X_2$ on $V_1 \cap V(X_2)$, this implies that $L$ is the simplicial complement of $\partial X_2$ in $X_2$. Then, by Proposition 2.3, $|X_2| \setminus |L|$. This proves (b).

Now, if $L \setminus \partial X_2 \neq \emptyset$ then $|L| \setminus |X_2|$ and hence $|L| \setminus |X_2|$ does not exist. So, by Proposition 2.6, $|X_2| \setminus |L|$ is a pl ball. Let $\sigma$ be a $d$-simplex in $S_{d+2}^d$. Let $B_1 = |\sigma|$ and $B_2 = |S_{d+2}^d \setminus |\sigma||$. Then $B_1$ and $B_2$ are pl $d$-balls. Let $f_2 : B_2 \rightarrow |X_2|$ be a pl homeomorphism. Let $f = f_2|\partial B_2$. Since $\partial B_1 = \partial B_2$ and $\partial(|X_1|) = |\partial X_1| = |\partial X_2|$, $f : \partial B_1 \rightarrow \partial(|X_1|)$ is a pl homeomorphism. By Proposition 2.4, there exists a pl homeomorphism $f_1 : B_1 \rightarrow |X_1|$ such that $f_1|\partial B_1 = f = f_2|\partial B_2$. Then $f_1 \cup f_2$ is a pl homeomorphism from $|S_{d+2}^d|$ to $|X|$. This proves (c). □

Lemma 4.3. Let $X$ be a combinatorial triangulation of a $\mathbb{Z}_2$-homology $d$-sphere. Let $X_1$ be an induced subcomplex of $X$ which is a combinatorial $d$-ball. Let $L = C(X_1, X)$ and $X_2 = N(L, X)$. Then $X_2$ is $\mathbb{Z}_2$-acyclic.

Proof. Let $J = X_1 \cap X_2$. Then, by Lemma 4.1, $J = \partial X_1$. So, $J$ is a combinatorial $(d-1)$-sphere. Therefore, $H_{q-1}(J, \mathbb{Z}_2) = \mathbb{Z}_2$ and $\tilde{H}_q(J, \mathbb{Z}_2) = 0$ for all $q \neq d-1$. Also $\tilde{H}_q(X_1, \mathbb{Z}_2) = 0$ for all $q \geq 0$. For $q \geq 1$, we have the following exact Mayer–Vietoris sequence of homology groups with coefficients in $\mathbb{Z}_2$ (see [9,12]):

$$\cdots \rightarrow H_{q+1}(X) \rightarrow H_q(J) \rightarrow H_q(X_1) \oplus H_q(X_2) \rightarrow H_q(X) \rightarrow \tilde{H}_{q-1}(X) \rightarrow \cdots$$  

(7)

Now, $H_d(X, \mathbb{Z}_2) = \mathbb{Z}_2$ and $\tilde{H}_q(X, \mathbb{Z}_2) = 0$ for $q \neq d$. By Lemma 4.2, $|X_2|$ is a connected $d$-manifold with non-trivial boundary. Therefore, $H_d(X_2, \mathbb{Z}_2) = 0$ and $H_0(X_2, \mathbb{Z}_2) = \mathbb{Z}_2$. Then, by (7), $H_q(X_2, \mathbb{Z}_2) = 0$ for $0 < q < d-1$ and for $q = d-1$ we get the following short exact sequence of abelian groups:

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow H_{d-1}(X_2, \mathbb{Z}_2) \rightarrow 0.$$

Clearly, this implies $H_{d-1}(X_2, \mathbb{Z}_2) = 0$. Thus, $\tilde{H}_q(X_2, \mathbb{Z}_2) = 0$ for all $q \geq 0$. □
Proof of Theorem 2. Let $X_1$ be an $m$-vertex induced subcomplex of $M$ which is a combinatorial $d$-ball. Let $L = C(X_1, M)$ and $X_2 = N(L, M)$. Then, by Part (b) of Lemma 4.2, $|X_2| \leq |L|$.

Again, by Lemma 4.3, $X_2$ is $\mathbb{Z}_2$-acyclic and hence $L$ is $\mathbb{Z}_2$-acyclic. Since $n \leq m + 7$, the number of vertices in $L$ is $\leq 7$. Therefore, by Theorem 1, $L$ is collapsible. Then, by Part (c) of Lemma 4.2, $M$ is a combinatorial sphere. □

Proof of Corollary 3. If $\sigma$ is a $d$-simplex of $M$ then the induced subcomplex $A_d^{d+1}(\sigma)$ is a $(d + 1)$-vertex combinatorial $d$-ball. Therefore, by Theorem 2, $M$ is a combinatorial sphere. □

Proof of Corollary 4. Assume, if possible, that $M$ admits a bistellar $i$-move $\kappa_A$ for some $i > 0$. Let $\beta$ be the core of $A$ and $\alpha = A \setminus \beta$. Then $M[A] = A_i^{i+1}(\alpha) \ast S_{d-i+1}^{d-i-1}(\beta)$ is a $(d + 2)$-vertex combinatorial $d$-ball. Therefore, by Theorem 2, $M$ is a combinatorial sphere, a contradiction. This proves the corollary. □

References