Properties of oblique dual frames in shift-invariant systems

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ABSTRACT

Gramian analysis is used to study properties of a shift-invariant system $X = \{\phi(\cdot - Bk): \phi \in \Phi, k \in \mathbb{Z}^n\}$, where $B$ is an invertible $n \times n$ matrix and $\Phi$ a finite or countable subset of $L^2(\mathbb{R}^n)$ under the assumption that the system forms a frame for the closed subspace $M$ of $L^2(\mathbb{R}^n)$. In particular, the relationship between various features of such system, such as being a frame for the whole space $L^2(\mathbb{R}^n)$, being a Riesz sequence and having a unique shift-generated dual of type I or II is discussed in details. Several interesting examples are presented.

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1. Introduction

Shift-invariant spaces lie at the very heart of several areas of modern Fourier analysis such as the theory of wavelets, spline systems, Gabor systems and approximation theory. Let $B$ be an invertible matrix in $M_n$, the space of real matrices of size $n \times n$. In this work, we consider a finite or countable family $\Phi$ of functions in $L^2(\mathbb{R}^n)$ and study certain properties of the dual of the associated shift-invariant system

$$X = \{\phi(\cdot - Bk): \phi \in \Phi, k \in \mathbb{Z}^n\},$$

which we will assume to form a frame for the closure of its linear span. Before going any further into the theory of shift-invariant spaces, we need to recall a few basic definitions and terminology related to the theory of frames (see [3] for more details). Given $\mathcal{H}$ an infinite-dimensional separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and a countable index set $\mathcal{N}$, the collection $X = \{x_n\}_{n \in \mathcal{N}}$ in $\mathcal{H}$ is a frame for its closed linear span $\mathcal{M}$ (also called a frame sequence) if there exist constants $C_1, C_2 > 0$, called the frame bounds, such that

$$C_1 \|x\|^2 \leq \sum_{n \in \mathcal{N}} |\langle x, x_n \rangle|^2 \leq C_2 \|x\|^2, \quad x \in \mathcal{M}.$$  

(1.2)

$X$ is called a right frame for $\mathcal{M}$ if $C_1 = C_2$ in (1.2) and it is a Parseval frame for $\mathcal{M}$ if $C_1 = C_2 = 1$. $X$ is a Bessel collection, with (Bessel) constant $C_2$, if the right-hand inequality in (1.2) holds for all $x \in \mathcal{M}$. A collection $X$ is fundamental if its finite linear span, span$(X)$, is dense in $\mathcal{H}$. The collection $X = \{x_n\}_{n \in \mathcal{N}}$ is called a Riesz family or Riesz sequence with constants $C_1, C_2$ (with $C_1 > 0$), if the inequalities

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C_1 \sum_{n \in \mathcal{N}} |a_n|^2 \leq \left\| \sum_{n \in \mathcal{N}} a_n x_n \right\|^2 \leq C_2 \sum_{n \in \mathcal{N}} |a_n|^2,

hold for all (finitely supported) sequences \{a_n\} of complex numbers. If a Riesz family \(X\) is fundamental, we say that \(X\) forms a Riesz basis. When \(\mathcal{M} = \mathcal{H}\), a frame for \(\mathcal{M}\) is often simply called a frame. Let \(\ell^2(\mathcal{N})\) denote the Hilbert space of complex-valued square-summable sequences indexed by \(\mathcal{N}\). If \(X\) is a Bessel collection, we can define the analysis operator or frame transform associated with \(X\), \(T_X : \mathcal{M} \to \ell^2(\mathcal{N})\), by

\[
T_X(x) = \{ (x, x_n) \}_{n \in \mathcal{N}}, \quad x \in \mathcal{M},
\]

while its adjoint, the synthesis operator, \(T_X^* : \ell^2(\mathcal{N}) \to \mathcal{M}\), is defined by

\[
T_X^*([c_n]_{n \in \mathcal{N}}) = \sum_{n \in \mathcal{N}} c_n x_n, \quad \{c_n\}_{n \in \mathcal{N}} \in \ell^2(\mathcal{N}).
\]

The operator \(S = T_X^* T_X : \mathcal{M} \to \mathcal{M}\) is called the frame operator and is computed via the formula

\[
Sx = \sum_{n \in \mathcal{N}} (x, x_n) x_n, \quad x \in \mathcal{M}.
\]

If \(X\) is a frame for \(\mathcal{M}\), then \(S\) is a bounded and invertible operator from \(\mathcal{M}\) onto \(\mathcal{M}\). The collection \(\{S^{-1} x_n\}_{n \in \mathcal{N}}\) is called the standard dual frame of the frame \(X\). It provides a reconstruction formula for the elements of \(\mathcal{M}\) in terms of their inner products with the frame elements:

\[
x = \sum_{n \in \mathcal{N}} (x, x_n) S^{-1} x_n = \sum_{n \in \mathcal{N}} (x, S^{-1} x_n) x_n, \quad x \in \mathcal{M}.
\]

More generally, a Bessel collection \(Y = \{y_n\}_{n \in \mathcal{N}}\) in \(\mathcal{H}\) is called an oblique dual for the frame \(X\) if

\[
x = \sum_{n \in \mathcal{N}} (x, y_n) x_n, \quad x \in \mathcal{M}.
\]

The term dual frame was used instead of “oblique dual” in [1] but we prefer to use here the latter as it emphasizes the fact that we do not require that \(y_n \in \mathcal{M}\) in this definition. However, if this last property holds for all \(n \in \mathcal{N}\), then \(Y\) is called a dual frame of type I. If the inclusion \(\text{Range} T_Y \subset \text{Range} T_X\) holds, then \(Y\) is called a dual frame of type II (see [1] for the motivation behind these two last definitions). The standard dual is always a dual of both type I and type II. It is well known that the standard dual is the unique dual of type I for a frame \(X\) if and only if \(X\) is a Riesz sequence and that it is the unique dual of type II for \(X\) if and only if \(X\) is fundamental (i.e. \(\text{span}(X) = \mathcal{H}\)) (see [1] for more details).

When considering frames built from shift-invariant systems such as in (1.1), it is natural to look for dual systems which are of the same form, i.e. generated by the same family of shifts, as made more precise in the following definition. These duals will be called shift-generated duals or SG-duals for short. To simplify the notation, we associate with the system (1.1) the index set \(\mathcal{I} = \Phi \times \mathbb{Z}^n\).

**Definition 1.1.** Let \(X = \{\phi(\cdot - Bk)\}_{(\phi,k) \in \mathcal{I}}\) be a frame for the closed subspace \(\mathcal{M}\) of \(L^2(\mathbb{R}^n)\). Let \(R : \Phi \to L^2(\mathbb{R}^n)\) be a mapping and let \(Y\) be the collection \(Y = \{(R\phi)(\cdot - Bk)\}_{(\phi,k) \in \mathcal{I}}\).

(i) If \(Y\) is an oblique dual for the frame \(X\), we call \(Y\) an oblique SG-dual frame for the frame \(X\).

(ii) We say that \(Y\) is a SG-dual of type I for the frame \(X\) if \(Y\) is an oblique SG-dual for \(X\) and all the elements of \(Y\) belong to \(\mathcal{M}\).

(iii) We say that \(Y\) is a SG-dual of type II for \(X\) if \(Y\) is an oblique SG-dual for \(X\) and we have the inclusion \(\text{Range}(T_Y) \subset \text{Range}(T_X)\).

The standard dual is always a SG-dual of both type I and type II. Since the definition of an oblique SG-dual is more restrictive than that of a general oblique dual, the uniqueness of the SG-dual of type I (resp. type II) does not necessarily imply that of a general oblique dual of type I (resp. type II). The main purpose of this article is to clarify the relationship between the uniqueness of these types of oblique duals in relation with the number of generators in the corresponding shift-invariant system and to point out conditions on the generators under which the uniqueness of the SG-dual of type I (resp. type II) is equivalent to that of the general dual of type I (resp. type II).

This paper is organized as follows. In Section 2, we recall some basic definitions, and facts from the theory of shift-invariant spaces and Gramian analysis, mainly due to Ron and Shen [5] (see also [2]), as well as the main results of our paper [1]. In Section 3, we consider separately the cases where the system is generated by one, finitely many, or countably many functions. Several examples are worked out in details in all cases. Finally, in Section 4, we point out that, under some mild conditions, the property that a system has a unique SG-dual of type I (resp. type II) is actually equivalent to this system being a Riesz basis (resp. fundamental).
2. The uniqueness of shift-generated duals

In this section we will identify $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ with its fundamental domain $[0,1)^n$ and recall some notation and basic results, most of whom are taken from [1,2,5]. The Fourier transform of a function $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is defined by the formula

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx, \quad \xi \in \mathbb{R}^n,$$

where $\cdot$ denotes the standard inner product in $\mathbb{R}^n$. It can be extended as a unitary operator acting on $L^2(\mathbb{R}^n)$. Let us consider a shift-invariant system $X = \{\phi(-Bk) : (\phi, k) \in \mathcal{I}\}$ forming a frame for its closed linear span $\mathcal{M}$. We define, for a.e. $\xi \in [0,1)^n$, the pre-Gramian operator $K(\xi) : \ell^2(\Phi) \to \ell^2(\mathbb{Z}^n)$ by its matrix representation

$$K(\xi)_{i,\phi} = \left[\sqrt{\hat{\phi}^2(D(\xi + j))}\right]_{i,\phi}, \quad (\phi, j) \in \mathcal{I},$$

where $D = (B^t)^{-1}$ and $q = |\text{det}(D)|$. Its adjoint $K^*(\xi) : \ell^2(\mathbb{Z}^n) \to \ell^2(\Phi)$ has the corresponding matrix representation given by

$$K^*(\xi)_{\phi,j} = \left[\sqrt{\hat{\phi}^2(D(\xi + j))}\right]_{\phi,j}, \quad (\phi, j) \in \mathcal{I}, \quad \xi \in \mathbb{R}^n.$$

For a.e. $\xi \in [0,1)^n$, the Gramian operator $G(\xi) : \ell^2(\Phi) \to \ell^2(\Phi)$ and dual Gramian operator, $\tilde{G} : \ell^2(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n)$ are the positive operators defined by

$$G(\xi) = K^*(\xi)K(\xi), \quad \tilde{G}(\xi) = K(\xi)K^*(\xi),$$

respectively. Their corresponding matrix representations are given by

$$G_{\phi_1,\phi_2}(\xi) = q \sum_{j \in \mathbb{Z}^n} \phi_1(D(\xi + j))\hat{\phi}_2(D(\xi + j)), \quad \phi_1, \phi_2 \in \Phi, \quad (2.1)$$

and

$$\tilde{G}_{j_1,j_2}(\xi) = q \sum_{\phi \in \Phi} \hat{\phi}(D(\xi + j_1))\hat{\phi}(D(\xi + j_2)), \quad j_1, j_2 \in \mathbb{Z}^n. \quad (2.2)$$

The importance of the Gramian and dual Gramian lies in fact that various properties of the original shift-invariant system $X$ corresponds to analogous properties holding uniformly on $[0,1)^n$ for the Gramian or dual Gramian operators. For example, the following characterization of the frame property was first obtained by Ron and Shen [5] (see also [2]), although the version that we state here is phrased in a slightly different way (see [1]).

**Theorem 2.1.** The following are equivalent:

(a) The system $X$ is a frame with frame bounds $C_1$ and $C_2$ for $\mathcal{M}$.
(b) For a.e. $\xi \in \mathbb{R}^n$, $C_1G(\xi) \leq \tilde{G}^2(\xi) \leq C_2G(\xi)$.
(c) For a.e. $\xi \in \mathbb{R}^n$, $C_1\tilde{G}(\xi) \leq G^2(\xi) \leq C_2\tilde{G}(\xi)$.

Similarly, the properties of forming a Riesz sequence or of being fundamental have a natural interpretation in terms of the Gramian and dual Gramian, respectively. Letting $I_1$ be the identity operator acting on $\ell^2(\Phi)$ and $I_2$ be the identity operator acting on $\ell^2(\mathbb{Z}^n)$, the following characterizations are well known [5] (see also [1,2]).

**Theorem 2.2.** Suppose that the shift-invariant system $X$ is a frame with frame bounds $C_1$ and $C_2$. Then, the following are equivalent:

(a) $X$ is a Riesz basis for $\mathcal{M}$.
(b) The inequalities $C_1I_1 \leq G(\xi) \leq C_2I_1$ hold for a.e. $\xi \in [0,1)^n$.

**Theorem 2.3.** Under the same conditions as in the previous theorem, the following are equivalent:

(a) $X$ is fundamental, i.e. $\mathcal{M} = L^2(\mathbb{R}^n)$.
(b) The inequalities $C_1I_2 \leq \tilde{G}(\xi) \leq C_2I_2$ hold for a.e. $\xi \in [0,1)^n$.

The uniqueness of the $SG$-dual of type I (resp. type II) has also a simple interpretation in terms of the Gramian (resp. dual Gramian) as proved in [1].
Theorem 2.4. Suppose that the shift-invariant system $X$ is a frame with frame bounds $C_1$ and $C_2$. Then, the following are equivalent:

(a) The standard dual is the unique SG-dual of type I for $X$.
(b) For a.e. $\xi \in [0, 1]^n$, we have either

\[ G(\xi) = 0 \quad \text{or} \quad C_1 I_1 \leq G(\xi) \leq C_2 I_1. \]

Theorem 2.5. Under the same conditions as in the previous theorem, the following are equivalent:

(a) The standard dual is the unique SG-dual of type II for $X$.
(b) For a.e. $\xi \in [0, 1]^n$, we have either

\[ \tilde{G}(\xi) = 0 \quad \text{or} \quad C_1 I_2 \leq \tilde{G}(\xi) \leq C_2 I_2. \]

It is clear, by comparing the statements of Theorems 2.2 and 2.4, that the uniqueness of the SG-dual of type I is implied by the fact that $X$ is a Riesz sequence and, by comparing the statements of Theorems 2.3 and 2.5, that the uniqueness of the SG-dual of type II is implied by the fact that $X$ is fundamental.

To conclude this section, we mention one last result from [1] dealing with the uniqueness of SG-duals.

Theorem 2.6. Suppose that the shift-invariant system $X$ is a frame for $\mathcal{M}$.

(a) If $X$ is fundamental, then either $X$ is a Riesz basis or $X$ admits more than one SG-dual of type I.
(b) If $X$ is Riesz sequence, then either $X$ is fundamental or $X$ admits more than one SG-dual of type II.

As we will show in the next section, the existence of duals having some of the properties we just described will depend in an essential way on the number of generators for the shift-invariant system $X$, i.e. the cardinality of $\Phi$.

3. SG-duals in principal, finitely many and infinitely many generated shift-invariant spaces

We will apply the results of [1] to the situation where $\Phi$ contains only one function, i.e. $\Phi = \{\phi\}$, for some $\phi \in L^2(\mathbb{R}^n)$, in which case the associated space $\mathcal{M}$ generated by $X = \{\phi(\cdot - Bk)\}_{k \in \mathbb{Z}^n}$ is called a Principal Shift Invariant subspace (PSI) according to terminology used in [5]. The operator-valued function $G(\cdot) = K^*(\cdot)K(\cdot)$ can then be identified with the scalar-valued function

\[ G(\xi) = q \sum_{k \in \mathbb{Z}^n} |\hat{\phi}(D(\xi + k))|^2, \quad \xi \in [0, 1]^n. \]

The system $X$ constitutes a frame for $\mathcal{M}$ with frame bounds $C_1$ and $C_2$ if and only if

\[ C_1 \leq G(\xi) \leq C_2, \quad \text{for a.e. } \xi \in \{\xi \in [0, 1]^n, \ G(\xi) \neq 0\}. \quad (3.1) \]

$X$ forms a Riesz basis for $\mathcal{M}$ if and only if the inequalities in (3.1) holds for a.e. $\xi \in [0, 1]^n$. The following result was first obtained by O. Christensen and Y.C. Eldar [4, Corollary 4.4]. We provide here a short proof based on Theorem 2.4.

Proposition 3.1. A frame $\{\phi(\cdot - Bk)\}_{k \in \mathbb{Z}^n}$ generating a PSI space $\mathcal{M}$ always admits a unique SG-dual of type I.

Proof. Using the equivalence of (a) and (b) in Theorem 2.4, we easily see that the conditions for obtaining a SG-dual of type I are equivalent to those for obtaining a frame in the case of a PSI space. \qed

It is thus very easy to construct example of frames for PSI spaces that are not Riesz sequences but admit a unique SG-dual of type I, since this is equivalent to the condition that $G$ vanishes on a subset of $[0, 1]^n$ having positive measure. One such example is given by the PSI spaces generated by $\phi$ defined by

\[ \hat{\phi}(\xi) = \chi_{[0, 1]^n}(\xi), \quad \xi \in \mathbb{R}^n, \]

where $B = I$, the identity matrix on $\mathbb{R}^n$. The operator $\tilde{G}(\xi) : \ell^2(\mathbb{Z}^n) \to \ell^2(\mathbb{Z}^n)$ has a matrix representation with entries

\[ (\tilde{G}(\xi))_{j_1, j_2} = q \hat{\phi}(D(\xi + j_1)) \hat{\phi}(D(\xi + j_2)), \quad j_1, j_2 \in \mathbb{Z}^n. \]

It follows that for a.e. $\xi \in [0, 1]^n$, $\tilde{G}(\xi)$ is either 0 or a rank one operator and $\tilde{G}(\xi)$ is never invertible. Hence, $X$ can never be fundamental. Moreover, we have the following.

\[ \tilde{G}(\xi) = \sum_{k \in \mathbb{Z}^n} |\hat{\phi}(D(\xi + k))|^2, \quad \xi \in [0, 1]^n. \]
Proposition 3.2. A frame \( X = \{ \phi(\cdot - Bk) \}_{k \in \mathbb{Z}^n} \) with \( \phi \neq 0 \) generating a PSI space \( \mathcal{M} \) always admits more than one SG-dual of type II.

Proof. The conditions for having a unique SG-dual of type II given in Theorem 2.5 imply that \( \tilde{G} \) must be either invertible or 0 a.e. on \([0,1)^n\) in that case. Since \( \tilde{G} \) is never invertible, it must be identically 0 which would then imply that \( \phi = 0 \), contrary to our assumption. \( \square \)

We now turn our attention to the case where \( \Phi = \{ \phi_1, \ldots, \phi_m \} \) is a finite set. The corresponding space \( \mathcal{M} \) is then called a Finitely generated Shift-Invariant space (FSI) in the terminology used in [5]. The operator-valued function \( G(\cdot) = K^*(\cdot)K(\cdot) \) can then be identified with a function taking values in the space of \( m \times m \) matrices, with the entries of \( G(\cdot) \) given by

\[
[G(\xi)]_{ijk} = q \sum_{j \in \mathbb{Z}^n} \tilde{\phi}_k(D(\xi + j)) \tilde{\phi}_l(D(\xi + j)), \quad k, l = 1, 2, \ldots, m.
\]

As will be illustrated in the examples below, if \( X \) is generated by functions \( \phi_1, \ldots, \phi_m \), where \( m \geq 2 \), there are 3 possibilities:

(i) \( X \) has more than one SG-dual of type I.
(ii) \( X \) admits a unique SG-dual of type I, but is not a Riesz sequence.
(iii) \( X \) is a Riesz sequence.

As for the SG-duals of type II, note that the operator \( \tilde{G}(\xi) : l^2(\mathbb{Z}^n) \rightarrow l^2(\mathbb{Z}^n) \) has the matrix representation

\[
[\tilde{G}(\xi)]_{j_1j_2} = q \sum_{k=1}^m \tilde{\phi}_k(D(\xi + j_1)) \tilde{\phi}_l(D(\xi + j_2)), \quad j_1, j_2 \in \mathbb{Z}^n,
\]

showing that it is a finite rank operator from \( l^2(\mathbb{Z}^n) \) to itself and that it is therefore never invertible. We have thus the following result whose proof is similar to that of Proposition 3.2.

Proposition 3.3. A frame \( X = \{ \phi(\cdot - Bk) \}_{(\phi,k) \in \mathcal{I}} \), with \( \Phi \) a finite collection in \( L^2(\mathbb{R}^n) \), generating a FSI space \( \mathcal{M} \neq \{0\} \) always admits more than one SG-dual of type II.

Example 3.4. Let \( \phi_1, \phi_2 \in L^2(\mathbb{R}) \) be defined by

\[
\hat{\phi}_1 = \chi_{[0,1/2)} \quad \text{and} \quad \hat{\phi}_2 = \chi_{[1/2,1)}
\]

and consider the system \( X = \{ \phi_1(\cdot - k), \phi_2(\cdot - k) : k \in \mathbb{Z} \} \). Since

\[
\sum_{k \in \mathbb{Z}} |\hat{\phi}_1(\xi - k)|^2 = \begin{cases} 1, & \xi \in [0,1/2), \\ 0, & \xi \in [1/2,1), \end{cases}
\]

and

\[
\sum_{k \in \mathbb{Z}} |\hat{\phi}_2(\xi - k)|^2 = \begin{cases} 0, & \xi \in [0,1/2), \\ 1, & \xi \in [1/2,1), \end{cases}
\]

we have

\[
G(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \xi \in [0,1/2), \quad G(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \xi \in [1/2,1).
\]

Thus \( G \) is never identically zero or invertible. The frame \( X \) admits thus more than one SG-dual of type I. It is easy in this case to compute all the SG-duals of type I. Note that the frame operator is the identity and thus the standard dual coincides with the original frame. Furthermore, the subspace \( \mathcal{M} \) in this example is the Paley-Wiener space of all functions in \( L^2(\mathbb{R}) \) having their Fourier transform supported on \([0,1]\). If \( \psi_i \in \mathcal{M} \) and \( \widehat{\psi_i} \in L^\infty(\mathbb{R}) \), for \( i = 1, 2 \), and \( f \) belong to \( \mathcal{M} \), the identity

\[
0 = \sum_{n \in \mathbb{Z}} \left[ \langle f, \psi_1(\cdot - n) \rangle \phi_1(\cdot - n) + \langle f, \psi_2(\cdot - n) \rangle \phi_2(\cdot - n) \right]
\]

is equivalent to

\[
0 = \sum_{k \in \mathbb{Z}} \hat{f}(\xi - k) \hat{\psi}_1(\xi - k) \hat{\phi}_1(\xi) + \sum_{k \in \mathbb{Z}} \hat{f}(\xi - k) \hat{\psi}_2(\xi - k) \hat{\phi}_2(\xi)
\]

or

\[
0 = \hat{f}(\xi) \left[ \hat{\psi}_1(\xi) \hat{\phi}_1(\xi) + \hat{\psi}_2(\xi) \hat{\phi}_2(\xi) \right].
\]
This last identity holds for all \( f \in \mathcal{M} \) if and only if
\[
\overline{\psi_1(\xi)} \chi_{(0,1/2)} + \overline{\psi_2(\xi)} \chi_{(1/2,1)} = 0.
\]
This is equivalent to \( \hat{\psi}_1 = 0 \) a.e. on \([0,1/2)\) and \( \hat{\psi}_2 = 0 \) a.e. on \([1/2,1)\). Thus a collection \( \{\rho_1(\cdot - n), \rho_2(\cdot - n)\}_{n \in \mathbb{Z}} \) forms a SG-dual of type I for \( X \) if and only if both \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) are supported on \([0,1)\), belong to \( L^\infty(\mathbb{R}) \) (to ensure that the Bessel condition is satisfied) and if, furthermore,
\[
\hat{\rho}_1 = 1 \quad \text{a.e. on } [0,1/2) \quad \text{and} \quad \hat{\rho}_2 = 1 \quad \text{a.e. on } [1/2,1).
\]

Next example corresponds to case (ii) above.

**Example 3.5.** Let \( \phi_1, \phi_2 \in L^2(\mathbb{R}) \) be defined by
\[
\hat{\phi}_1 = \chi_{(0,1/2)} \quad \text{and} \quad \hat{\phi}_2 = \chi_{(1,3/2)},
\]
and consider the system \( X = \{\phi_1(\cdot - k), \phi_2(\cdot - k) : k \in \mathbb{Z}\} \). Then,
\[
G(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \xi \in [0,1/2), \quad G(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \xi \in [1/2,1),
\]
and the corresponding system admits a unique SG-dual of type I, which coincides with the original frame since, as can be easily checked, \( X \) is a Parseval frame. \( X \) does not constitute a Riesz sequence since condition (b) in Theorem 2.2 clearly fails.

The next example covers case (iii).

**Example 3.6.** Let \( \phi_1, \phi_2 \in L^2(\mathbb{R}) \) be defined by
\[
\hat{\phi}_1 = \chi_{(0,1)} \quad \text{and} \quad \hat{\phi}_2 = \chi_{(1,2)}
\]
and consider the system \( X = \{\phi_1(\cdot - k), \phi_2(\cdot - k) : k \in \mathbb{Z}\} \). Clearly \( X \) forms an orthonormal basis for \( \mathcal{M} \), the Paley–Wiener space of functions in \( L^2(\mathbb{R}) \) whose Fourier transforms are supported on \([0,2)\). In particular, \( X \) form a Riesz basis for \( \mathcal{M} \) and the range of the frame operator \( T_X \) is the whole space \( \ell^2(\mathbb{Z}) \).

Thus any SG-dual will be a SG-dual of type II. We now describe the SG-duals of type II. Note that, if \( \{\psi_1(\cdot - k), \psi_2(\cdot - k) : k \in \mathbb{Z}\} \) is a Bessel collection and \( f \in \mathcal{M} \), the condition
\[
0 = \sum_{n \in \mathbb{Z}} [\langle f, \psi_1(\cdot - n) \rangle \phi_1(\cdot - n) + \langle f, \psi_2(\cdot - n) \rangle \phi_2(\cdot - n)]
\]
is equivalent to
\[
0 = \left\{ \sum_{k \in \mathbb{Z}} \hat{f}(\xi - k) \overline{\psi_1(\xi - k)} \right\} \hat{\phi}_1(\xi) + \left\{ \sum_{k \in \mathbb{Z}} \hat{f}(\xi - k) \overline{\psi_2(\xi - k)} \right\} \hat{\phi}_2(\xi)
\]
or, using the support condition on the Fourier transform of a function in \( \mathcal{M} \), to
\[
0 = \left[ \hat{f}(\xi) \overline{\psi_1(\xi)} + \hat{f}(\xi - 1) \overline{\psi_1(\xi - 1)} \right] \chi_{(0,1)}(\xi) + \left[ \hat{f}(\xi) \overline{\psi_2(\xi)} + \hat{f}(\xi + 1) \overline{\psi_2(\xi + 1)} \right] \chi_{(1,2)}(\xi).
\]
This holds for all \( f \in \mathcal{M} \) if and only if both \( \hat{\psi}_1 \) and \( \hat{\psi}_2 \) are 0 a.e. on \([0,2)\). Thus a collection
\[
\{\rho_1(\cdot - k), \rho_2(\cdot - k) : k \in \mathbb{Z}\}, \quad \rho_1, \rho_2 \in L^2(\mathbb{R}),
\]
forms a SG-dual of type II for \( X \) in this case if there exist constants \( C_1, C_2 > 0 \) such that, for a.e. \( \xi \in \mathbb{R} \),
\[
\sum_k |\hat{\rho}_1(\xi - k)|^2 \leq C_1, \quad \sum_k |\hat{\rho}_2(\xi - k)|^2 \leq C_2
\]
and if
\[
\hat{\rho}_1 = \begin{cases} 1, & \text{on } [0,1), \\ 0, & \text{on } [1,2), \end{cases} \quad \hat{\rho}_2 = \begin{cases} 0, & \text{on } [0,1), \\ 1, & \text{on } [1,2). \end{cases}
\]
Now we apply the results of [1] to a given shift-invariant system $X$ with countably many generators. Table 1 shows the various possibilities for the different types of $SG$-duals for $X$.

Note that the two cases excluded from Table 1 above correspond to the two items in Theorem 2.6. In what follows, we will provide simple concrete examples showing that the other 7 cases can actually occur. The matrix $B$ will be the identity matrix in all these cases and the system $X$ will be of the form $X = \{\phi_i(x - k): i, k \in \mathbb{Z}\}$. To simplify, we will use $G(\xi)_{i,j}$ instead of $G(\xi)\phi_i \phi_j$ to denote the elements of the Gramian matrix and we introduce the operators $A$, $B$, $C$, $D$ acting on $l^2(\mathbb{Z}^n)$ and defined by their matrix representations:

$$A_{i,j} = \begin{cases} 1, & i = j \in 2\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases} \quad B_{i,j} = \begin{cases} 1, & i = j \in 2\mathbb{Z} + 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$C_{i,j} = \begin{cases} 1, & i \neq j \neq 1, \\ 0, & \text{otherwise}, \end{cases} \quad D_{i,j} = \begin{cases} 1, & i \neq j \neq 0, \\ 0, & \text{otherwise}. \end{cases}$$

Note that these four operators are non-trivial projections and thus all non-invertible. In the following examples $A$, $B$, $C$, and $D$ are the above operators.

**Example 3.7.** Let $\phi_i \in L^2(\mathbb{R})$ be defined by $\hat{\phi}_i = \chi_{(i,i+1)}$ for each $i \in \mathbb{Z}$. Then, the corresponding system $X$ is clearly an orthonormal basis for $L^2(\mathbb{R})$. This corresponds to case 1.

**Example 3.8.** Let $\phi_i \in L^2(\mathbb{R})$ be defined by $\hat{\phi}_i = \chi_{(i/2,i/2+1/2)}$ for each $i \in \mathbb{Z}$. Then, $G(\xi) = A \chi_{(0,1/2)}(\xi) + B \chi_{(1/2,1)}(\xi)$ for $\xi \in [0, 1)$. The Gramian matrix is thus never identically zero or invertible. Hence, the $SG$-dual of type I is not unique by Theorem 2.4. We have also $G = I$, the identity operator, and $X$ is fundamental by Theorem 2.3. This covers case 2.

**Example 3.9.** Let $\phi_i \in L^2(\mathbb{R})$ be defined by $\hat{\phi}_i = \chi_{(i,i+1/2)}$ for each $i \in \mathbb{Z}$. Then, $G(\xi) = I \chi_{(0,1/2)}(\xi)$ for $\xi \in [0, 1)$. So the Gramian is either identically zero or the identity operator and thus $X$ has a unique $SG$-dual of type I by Theorem 2.4, but is not a Riesz sequence by Theorem 2.2. We have $\hat{G}(\xi) = I \chi_{(0,1/2)}(\xi)$ for $\xi \in [0, 1)$. Therefore, the dual Gramian is either identically zero or the identity operator. Hence, $X$ is not fundamental by Theorem 2.3, but $X$ admits a unique $SG$-dual of type II by Theorem 2.5. This corresponds to case 3.

**Example 3.10.** Let $\phi_i \in L^2(\mathbb{R})$ be defined by $\hat{\phi}_i = \chi_{(i/4,1/4)}$, $\hat{\phi}_1 = \chi_{(1/4,1/2)}$, and $\hat{\phi}_2 = \chi_{(1/2,1/4)}$, for $i \leq -1$. Then, $G(\xi) = C \chi_{(0,1/4)}(\xi) + D \chi_{(1/4,1/2)}(\xi)$ for $\xi \in [0, 1)$. The Gramian matrix is thus both non-identically zero and non-invertible on a set of positive measure. The $SG$-dual of type I for $X$ is thus non-unique by Theorem 2.4 and, of course, $X$ is not a Riesz sequence. Since $G(\xi) = I \chi_{(0,1/2)}(\xi)$ for $\xi \in [0, 1)$, $X$ is not fundamental by Theorem 2.3 but has a unique $SG$-dual of type II by Theorem 2.5. This corresponds to case 4.

**Example 3.11.** Let $\phi_i \in L^2(\mathbb{R})$ be defined by $\hat{\phi}_i = \chi_{(2i,2i+1)}$ for each $i \in \mathbb{Z}$. Then, $G(\xi) = I$ for all $\xi \in [0, 1)$ and $X$ is Riesz sequence by Theorem 2.2. We have also $\hat{G}(\xi) = A$ for all $\xi \in [0, 1)$. Therefore the dual Gramian is always both non-identically zero and non-invertible. Thus, the $SG$-dual of type II for $X$ is non-unique by Theorem 2.5. This corresponds to case 5.

**Example 3.12.** Let $\phi_i \in L^2(\mathbb{R})$ be defined by $\hat{\phi}_i = \chi_{(2i,2i+1/2)}$ for each $i \in \mathbb{Z}$. Then, $G(\xi) = I \chi_{(0,1/2)}(\xi)$ for all $\xi \in [0, 1)$. So the Gramian matrix is either identically zero or the identity matrix, and thus $X$ is not a Riesz sequence by Theorem 2.2 but it does admit a unique $SG$-dual of type I by Theorem 2.4. We have $\hat{G}(\xi) = A \chi_{(0,1/2)}(\xi)$ for $\xi \in [0, 1)$. Therefore the dual Gramian matrix is both non-identically zero and non-invertible on a set of positive measure, which implies the non-uniqueness of the $SG$-dual of type II by Theorem 2.5. This corresponds to case 6.

<table>
<thead>
<tr>
<th>Table 1</th>
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<tr>
<td>$X$ is a Riesz sequence</td>
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<tr>
<td>is fundamental</td>
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<tr>
<td>is not fundamental but has a unique $SG$-dual of type II</td>
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<tr>
<td>is not fundamental and has non-unique $SG$-dual of type II</td>
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</table>
Example 3.13. Let $\phi_1 \in L^2(\mathbb{R})$ be defined by $\hat{\phi}_1 = X_{(1,1/2)}$, if $i$ is even, and by $\hat{\phi}_1 = X_{(i+1/2,i+1)}$ if $i$ is odd. Then, $G(\xi) = A_X(\xi) + B_X(\xi)$ for $\xi \in [0, 1)$, and the Gramian is always both non-identically zero and non-invertible. Thus $X$ is a Riesz sequence by Theorem 2.2 and the SG-dual of type I for $X$ is not unique by Theorem 2.4. Similarly, since $\hat{G}(\xi) = A_X(\xi) + B_X(\xi)$ for $\xi \in [0, 1)$, it follows that $X$ is not fundamental by Theorem 2.3 and that the SG-dual of type II for $X$ is non-unique by Theorem 2.5. This corresponds to case 7.

4. The unique dual properties and certain continuity requirements

In this last section, our goal is to show that, under some mild requirements on the system $X$, the notion of Riesz basis and that of having a unique SG-dual of type I are equivalent. Similarly, under some other mild conditions on $X$, the notion of being fundamental and that of having a unique SG-dual of type II are equivalent.

Proposition 4.1. Let $X = \{\phi(\cdot - Bk)\}_{(\phi, k) \in \mathbb{Z}}$ be a shift-invariant system generating a subspace $M \neq \{0\}$ and suppose that there exists a function $\phi_0 \in \Phi$ with the property that

$$\psi(\xi) := \sum_{k \in \mathbb{Z}^n} |\phi_0(D(\xi + k))|^2$$

(4.1)

is continuous on $(0, 1)^n$, where $D = (B^{-1})^T$. Then, the following are equivalent:

(a) $X$ is a Riesz sequence.
(b) The standard dual is the unique SG-dual of type I for $X$.

Proof. Note that the function defined in (4.1) is, up to a non-zero constant multiple, one of the diagonal entries of the Gramian matrix $G(\cdot)$. Clearly (a) always implies (b). On the other hand if (b) holds, by Theorem 2.4 shows that $G(\cdot)$ is either identically zero or satisfies the inequalities

$$C_1 I \leq G(\cdot) \leq C_2 I$$

(4.2)

for two constants $C_1, C_2 > 0$, independent of $G(\cdot)$. Clearly, on the set where $G = 0$ we must have $\psi = 0$, while on the set where $G$ satisfies (4.2), we must have $C_1 \leq \psi \leq C_2$. Since $M \neq \{0\}$, the second case must occur on a set $E \subset (0, 1)^n$ having positive measure. The continuity of $\psi$ then implies that $C_1 \leq \psi \leq C_2$ on $(0, 1)^n$. Thus (4.2) must hold on $(0, 1)^n$ and thus a.e. on $\mathbb{R}^n$, using the $\mathbb{Z}^n$-periodicity of $G$. Theorem 2.5 in [1] shows that $X$ must then be a Riesz sequence.

Remark 4.2. The condition that the function $\psi$ defined in (4.1) be continuous is actually a way of expressing the fact that the function $\phi_0$ is (spatially) well-localized. Indeed, note that $\psi$ can also be written as the Fourier series

$$\psi(\xi) = q \sum_{j \in \mathbb{Z}^n} \left| \int S_{\xi, j} (\xi) \phi_0(x) \phi_0(x + Bj) \, dx \right|^2 e^{2\pi i \xi j}.$$  

(4.3)

In particular if $\phi_0$ has compact support or, more generally, if

$$\sum_{j \in \mathbb{Z}^n} \left| \int S_{\xi, j} (\xi) \phi_0(x + Bj) \, dx \right| < \infty,$$

the continuity of $\psi$ will follow, since the series in (4.3) is an absolutely convergent Fourier series in that case. Note that in Examples 3.9 and 3.12, if $\phi_0$ is any function in $\Phi$, the corresponding function $\psi$ fails to be continuous on $(0, 1)^n$.

Proposition 4.3. Let $X = \{\phi(\cdot - Bk)\}_{(\phi, k) \in \mathbb{Z}}$ be a shift-invariant system generating a subspace $M \neq \{0\}$ and suppose that there exists $k_0 \in \mathbb{Z}^n$ such that

$$\rho(\xi) := \sum_{\phi \in \Phi} |\hat{\phi}(D(\xi + k_0))|^2$$

(4.4)

is continuous on $(0, 1)^n$, where $D = (B^{-1})^T$. Then, the following are equivalent:

(a) $X$ is fundamental.
(b) The standard dual is the unique SG-dual of type II for $X$.

Proof. The proof is similar to that of Proposition 4.1 and uses the fact that the function $\rho$ defined in (4.4) is a non-zero constant multiple of one of the diagonal elements of the dual Gramian $\tilde{G}$. 

Remark 4.4. Note that the condition that the function $\rho$ in (4.4) be continuous is a condition involving all the functions in $\Phi$ and depending on the behavior of their Fourier transform on some $n$-dimensional cube $k_0 + [0, 1]^n$. In particular, if all functions in $\Phi$ have a Fourier transform that vanishes on $k_0 + [0, 1]^n$, then the condition on the function $\rho$ in (4.4) is automatically verified. Since, clearly $X$ cannot be fundamental in that case, it follows from the equivalence above that the $SG$-dual of type II for $X$ is non-unique. On the other hand, a slight modification of Example 3.9 shows that the same conclusion is no longer true if we replace $[0, 1]^n$ by a proper measurable subset $A$ of it with $|(0, 1)^n \setminus A| > 0$. Note also that, for any $k_0 \in \mathbb{Z}$, the function $\rho$ in (4.4) corresponding to the system in Examples 3.9 and 3.10 fails to be continuous on the interval $(0, 1)$.

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References