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Asymptotic behaviour of a Bingham fluid in thin layers

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Abstract

A nonlinear stationary model describing the behaviour of a Bingham fluid is considered in a thin layer in $\mathbb{R}^3$. The limit problem obtained after transforming the original problem into one posed over a fixed reference domain and then letting ε (the parameter representing the thickness of the layer) tend to zero is studied. Existence and uniqueness results and a lower-dimensional 'Bingham-like' constitutive law are obtained. An identical study of a two-dimensional problem yields a one-dimensional model prevalent in engineering literature.

1. Introduction

A Bingham fluid, which is a visco-plastic medium, obeys the general laws of continuum mechanics and has a special nonlinear constitutive law. It is used to model the behaviour of a variety of fluids such as paint, lava and fluid mud (a clay–water mixture with a high concentration of cohesive mineral particles).

It is a non-Newtonian fluid which moves like a rigid body when a certain function of the stress tensor is below a certain threshold (sometimes called the yield stress). Beyond this yield stress, it obeys a nonlinear constitutive law.

In this paper, we are interested in the asymptotic behaviour of a Bingham fluid in a thin layer represented by a 'thin' domain in $\mathbb{R}^3$. Starting from the three-dimensional variational

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inequality giving the velocity and pressure, as formulated by Duvaut and Lions [2], the problem is transformed into one over a fixed reference domain, thus explicitly bringing out the dependence on ε (the parameter representing the thickness of the domain) in the variational formulation. The limit problem, as ε tends to zero, is then obtained. An identical study of a two-dimensional problem yields a one-dimensional constitutive law, prevalent in engineering literature (cf., for instance, Liu and Mei [6]).

The paper is organized as follows. Section 2 describes the three-dimensional problem and transforms it to one over a fixed reference domain by a standard change of variable and a priori estimates are obtained. Section 3 is devoted to the study of a class of function spaces of Sobolev type which will be needed in the sequel. Just as the theorem of de Rham characterizes the annihilators of divergence free vector fields as gradients of scalar functions, the annihilators of a certain space studied here are characterized as the gradients of functions in the horizontal variable alone. This helps in the recovery of the pressure later. In Section 4, the limit problem and its well-posedness are studied. In Section 5, the lower-dimensional constitutive law and the differential equation satisfied by the limit variables in the nonrigid zone are obtained. The corresponding results for the two-dimensional problem are stated.

2. Problem statement and basic estimates

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with sufficiently smooth boundary. Let $h : \omega \to \mathbb{R}$ be a sufficiently smooth function such that

$$0 < h_0 \leqslant h(x, y) \leqslant h_1 \tag{2.1}$$

for all $(x, y) \in \omega$, where h_0 and h_1 are constants. Let $\varepsilon > 0$. Set

$$\begin{aligned} \Omega &= \big\{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \omega,\ 0 < z < h(x, y)\big\}, \\ \Omega_\varepsilon &= \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in \omega,\ 0 < x_3 < \varepsilon h(x_1, x_2)\big\}. \end{aligned} \tag{2.2}$$

We will repeatedly use the bijection between the points of Ω_ε and those of Ω given by

$$(x_1, x_2, x_3) \in \Omega_\varepsilon \quad \leftrightarrow \quad (x, y, z) \in \Omega, \quad x = x_1,\ y = x_2,\ z = x_3/\varepsilon. \tag{2.3}$$

This automatically produces a bijection between functions $\varphi : \Omega_\varepsilon \to \mathbb{R}$ and $\hat{\varphi} : \Omega \to \mathbb{R}$ given by

$$\hat{\varphi}(x, y, z) = \varphi(x_1, x_2, x_3). \tag{2.4}$$

Notation. We will denote vector fields in three dimensions using bold face (e.g., $\mathbf{f} = (f_1, f_2, f_3)$) and vector fields in two dimensions using an underscore (e.g., $\underline{v} = (v_1, v_2)$). We will denote the Euclidean norm in $\mathbb{R}^2$ or $\mathbb{R}^3$ of these vector fields using the modulus (i.e., $|\mathbf{f}|$ or $|\underline{v}|$). We will denote integration with respect to the (Lebesgue) measure in $\mathbb{R}^3$ by $d\mathbf{x}$.

Let $\mathbf{f} \in (L^2(\Omega))^3$ be given. Let $\mathbf{f}_\varepsilon \in (L^2(\Omega_\varepsilon))^3$ be defined by

$$\mathbf{f}_\varepsilon(x_1, x_2, x_3) = \mathbf{f}(x_1, x_2, x_3/\varepsilon) \ \big(= \mathbf{f}(x, y, z)\big). \tag{2.5}$$

Consider an incompressible Bingham fluid occupying the region Ω_ε with viscosity and yield stress given (after nondimensionalization) by $\mu\varepsilon^2$ and $g\varepsilon$, respectively (where $\mu > 0$ and $g > 0$ are constants independent of ε), and acted upon by a body force of density given by $\mathbf{f}_\varepsilon$ defined by (2.5) (cf. Bourgeat and Mikelić [1] or Lions and Sanchez-Palencia [5]). A typical situation would be when the forces depend only on x_1 and x_2.

If $\mathbf{u}_\varepsilon$ and p_ε are the velocity and pressure, respectively, then the stress tensor can be written as $\sigma^\varepsilon = -p_\varepsilon I + \sigma^{D,\varepsilon}$. We set

$$D_{ij}(\mathbf{u}_\varepsilon) = \frac{1}{2}\left(\frac{\partial u_{\varepsilon,i}}{\partial x_j} + \frac{\partial u_{\varepsilon,j}}{\partial x_i}\right), \quad 1 \leqslant i, j \leqslant 3,$$

$$D_{II}(\mathbf{u}_\varepsilon) = \frac{1}{2}\sum_{i,j=1}^{3} D_{ij}(\mathbf{u}_\varepsilon) D_{ij}(\mathbf{u}_\varepsilon),$$

$$\sigma_{II}^\varepsilon = \frac{1}{2}\sum_{i,j=1}^{3} \sigma_{ij}^{D,\varepsilon}\sigma_{ij}^{D,\varepsilon}.$$

Then the constitutive relation is given by

$$\begin{cases} (\sigma_{II}^\varepsilon)^{1/2} \leqslant g\varepsilon & \Leftrightarrow \quad D_{II}(\mathbf{u}_\varepsilon) = 0, \\ (\sigma_{II}^\varepsilon)^{1/2} > g\varepsilon & \Leftrightarrow \quad D_{ij}(\mathbf{u}_\varepsilon) = \frac{1}{2\mu}\left(1 - \frac{g\varepsilon}{(\sigma_{II}^\varepsilon)^{1/2}}\right)\sigma_{ij}^{D,\varepsilon}. \end{cases} \tag{2.6}$$

Let

$$V_\varepsilon = \left\{\mathbf{v} \in \left(H_0^1(\Omega_\varepsilon)\right)^3 \mid \operatorname{div}(\mathbf{v}) = 0\right\}.$$

Then, the velocity $\mathbf{u}_\varepsilon$ is the unique solution of the following variational inequality (cf. Duvaut and Lions [2]).

(P_ε) Find $\mathbf{u}_\varepsilon \in V_\varepsilon$ such that

$$\mu\varepsilon^2 \int_{\Omega_\varepsilon} \nabla \mathbf{u}_\varepsilon . \nabla(\mathbf{v} - \mathbf{u}_\varepsilon)\, d\mathbf{x} + g\varepsilon \int_{\Omega_\varepsilon} |\nabla \mathbf{v}|\, d\mathbf{x} - g\varepsilon \int_{\Omega_\varepsilon} |\nabla \mathbf{u}_\varepsilon|\, d\mathbf{x} \geqslant \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon . (\mathbf{v} - \mathbf{u}_\varepsilon)\, d\mathbf{x} \tag{2.7}$$

for every $\mathbf{v} \in V_\varepsilon$.

Equivalently (cf. Bourgeat and Mikelić [1]), there exists $p_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbb{R}$ such that the couple $(\mathbf{u}_\varepsilon, p_\varepsilon)$ satisfies the following:

$$\mu\varepsilon^2 \int_{\Omega_\varepsilon} \nabla \mathbf{u}_\varepsilon . \nabla(\mathbf{v} - \mathbf{u}_\varepsilon)\, d\mathbf{x} + g\varepsilon \int_{\Omega_\varepsilon} |\nabla \mathbf{v}|\, d\mathbf{x} - g\varepsilon \int_{\Omega_\varepsilon} |\nabla \mathbf{u}_\varepsilon|\, d\mathbf{x} \geqslant \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon . (\mathbf{v} - \mathbf{u}_\varepsilon)\, d\mathbf{x} + \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div}(\mathbf{v} - \mathbf{u}_\varepsilon)\, d\mathbf{x} \tag{2.8}$$

for every $\mathbf{v} \in (H_0^1(\Omega_\varepsilon))^3$.

Notation. We will denote the norm in $L^2(U)$ (or $(L^2(U))^N$, $N = 2, 3$) of a domain U by $|\cdot|_{0,U}$ and the norm in $H^s(U)$ by $\|\cdot\|_{s,U}$.

Let $\hat{\mathbf{u}}_\varepsilon \in (H_0^1(\Omega))^3$ and $\hat{p}_\varepsilon \in L^2(\Omega)$ denote the transformed functions defined over Ω as per the rule (2.4). We now proceed to obtain a priori estimates for these functions.

Lemma 2.1. *There exists a constant $C > 0$, independent of ε, such that*

$$\left|\frac{\partial \hat{\mathbf{u}}_\varepsilon}{\partial x}\right|_{0,\Omega}, \left|\frac{\partial \hat{\mathbf{u}}_\varepsilon}{\partial y}\right|_{0,\Omega} \leqslant C\varepsilon^{-1}, \quad \left|\frac{\partial \hat{\mathbf{u}}_\varepsilon}{\partial z}\right|_{0,\Omega} \leqslant C, \quad |\hat{\mathbf{u}}_\varepsilon|_{0,\Omega} \leqslant C. \tag{2.9}$$

Proof. The proof follows by setting $\mathbf{v} = 2\mathbf{u}_\varepsilon$ and $\mathbf{v} = \mathbf{0}$ successively in (2.7), using the transformations suggested by (2.3) and (2.4) and by applying the classical Poincaré's inequality which, for the domain Ω_ε, reads as

$$|\varphi|_{0,\Omega_\varepsilon} \leqslant C\varepsilon|\nabla\varphi|_{0,\Omega_\varepsilon}$$

for any $\varphi \in H_0^1(\Omega_\varepsilon)$, where $C > 0$ is independent of ε. $\square$

Lemma 2.2. *There exists a constant $C > 0$, independent of ε, such that*

$$|\hat{p}_\varepsilon|_{0,\Omega} \leqslant C, \quad \left\|\frac{\partial \hat{p}_\varepsilon}{\partial x}\right\|_{-1,\Omega}, \left\|\frac{\partial \hat{p}_\varepsilon}{\partial y}\right\|_{-1,\Omega} \leqslant C, \quad \left\|\frac{\partial \hat{p}_\varepsilon}{\partial z}\right\|_{-1,\Omega} \leqslant C\varepsilon. \tag{2.10}$$

Proof. Let $\mathbf{w} \in (H_0^1(\Omega))^3$. Defining $\mathbf{w}_\varepsilon(x_1, x_2, x_3) = \mathbf{w}(x_1, x_2, x_3/\varepsilon) \in (H_0^1(\Omega_\varepsilon))^3$, and setting $\mathbf{v} = \mathbf{w}_\varepsilon + \mathbf{u}_\varepsilon$ in (2.8), we can deduce the last two estimates in (2.10). Consequently it follows that (cf. Girault and Raviart [3, Chapter I, Corollary 2.1]) there exists a representative of $\hat{p}_\varepsilon \in L^2(\Omega)/\mathbb{R}$ such that

$$|\hat{p}_\varepsilon|_{0,\Omega} \leqslant C\|\nabla\hat{p}_\varepsilon\|_{-1,\Omega} \leqslant C,$$

since Ω is a Lipschitz domain. This completes the proof. $\square$

3. Some function spaces

It follows from Lemma 2.1 that, for a subsequence, $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}$ and that $\partial\mathbf{u}_\varepsilon/\partial z \rightharpoonup \partial\mathbf{u}/\partial z$ weakly in $(L^2(\Omega))^3$. We lose information on the derivatives in the x and y directions. Hence we are led to consider the space $\tilde{W}$ of functions $v \in L^2(\Omega)$ such that $\partial v/\partial z \in L^2(\Omega)$.

We now introduce the linear mapping $T : L^2(\Omega) \to L^2(\omega)$ given by

$$T(v)(x, y) = \int_0^{h(x,y)} v(x, y, z)\, dz.$$

Lemma 3.1. *We have*

$$T \in \mathcal{L}\big(L^2(\Omega), L^2(\omega)\big) \cap \mathcal{L}\big(H_0^1(\Omega), H_0^1(\omega)\big)$$

and, for every $v \in H_0^1(\Omega)$,

$$\frac{\partial}{\partial x}\big(T(v)\big) = T\left(\frac{\partial v}{\partial x}\right), \qquad \frac{\partial}{\partial y}\big(T(v)\big) = T\left(\frac{\partial v}{\partial y}\right). \tag{3.1}$$

Lemma 3.2. *Let $w \in L^2(\Omega)$ such that $\partial w/\partial z = 0$. Then there exists $\tilde{w} \in L^2(\omega)$ such that $w(x, y, z) = \tilde{w}(x, y)$, i.e., for every $v \in L^2(\Omega)$,*

$$\int_\Omega wv\,d\mathbf{x} = \int_\omega \tilde{w}(x, y)T(v)(x, y)\,dx\,dy. \tag{3.2}$$

Corollary 3.1. *Let $w \in L^2(\Omega)$ such that, for all $v \in H^1(\Omega)$,*

$$\int_\Omega w\frac{\partial v}{\partial z}\,d\mathbf{x} = 0. \tag{3.3}$$

Then $w = 0$.

Proof. By the preceding lemma, since (3.3) implies that $\partial w/\partial z = 0$, we have that $w(x, y, z) = \tilde{w}(x, y)$. If $\varphi \in \mathcal{D}(\omega)$, setting $v(x, y, z) = \varphi(x, y)z$, we deduce from (3.3) that

$$\int_\omega \tilde{w}(x, y)\varphi(x, y)h(x, y)\,dx\,dy = 0.$$

Since φ was arbitrary, it follows that $\tilde{w}h = 0$, i.e., $w = 0$ (cf. (2.1)). □

Let us now set

$$\Gamma_0 = \big\{(x, y, 0) \mid (x, y) \in \bar{\omega}\big\}, \qquad \Gamma_1 = \big\{\big(x, y, h(x, y)\big) \mid (x, y) \in \bar{\omega}\big\}.$$

Definition 3.1. We say that $u \in \tilde{W}$ vanishes on $\Gamma = \Gamma_0 \cup \Gamma_1$ if, for any $w \in \tilde{W}$, we have

$$\int_\Omega u\frac{\partial w}{\partial z}\,d\mathbf{x} = -\int_\Omega \frac{\partial u}{\partial z}w\,d\mathbf{x}.$$

Let

$$H(\mathrm{div}; \omega) = \big\{\underline{\Phi} \in \big(L^2(\omega)\big)^2 \mid \mathrm{div}(\underline{\Phi}) \in L^2(\omega)\big\}.$$

If $\underline{\Phi} \in H(\mathrm{div}; \omega)$, then we can define the trace $\underline{\Phi}.\nu$ on $\partial\omega$, where ν is the unit outer normal on the boundary of ω. If this trace is zero, then, for any $\psi \in H^1(\omega)$, we have

$$\int_\omega \psi\,\mathrm{div}(\underline{\Phi})\,dx\,dy = -\int_\omega \nabla\psi.\underline{\Phi}\,dx\,dy.$$

We denote the space of such vector fields with vanishing trace by $H_0(\mathrm{div}; \omega)$.

We now introduce the space

$$W = \big\{\underline{v} \in (\tilde{W})^2 \mid \underline{v} \text{ vanishes on } \Gamma \text{ and } T(\underline{v}) \in H_0(\mathrm{div}; \omega)\big\}, \tag{3.4}$$

where $T(\underline{v}) = (T(v_1), T(v_2))$ if $\underline{v} = (v_1, v_2)$. It is easy to see that this is a Hilbert space for the inner-product defined by

$$(\underline{v}, \underline{w})_W = \int_\Omega \left(\underline{v}.\underline{w} + \frac{\partial \underline{v}}{\partial z}.\frac{\partial \underline{w}}{\partial z} \right) dx\,dy + \int_\omega \operatorname{div}\big(T(\underline{v})\big) \operatorname{div}\big(T(\underline{w})\big)\,dx\,dy. \tag{3.5}$$

Proposition 3.1. *$(H_0^1(\Omega))^2$ is dense in W.*

Proof. By Lemma 3.1, we know that $(H_0^1(\Omega))^2$ is contained in W. Let $\underline{v} \in W$ such that $(\underline{v}, \underline{\varphi})_W = 0$ for all $\underline{\varphi} \in (H_0^1(\Omega))^2$. Our aim is to establish that $\underline{v} = 0$, which will complete the proof. We do this in several steps.

Step 1. If $\xi \in \mathcal{D}(0, h_0)$ such that $\int_0^{h_0} \xi(z)\,dz = 1$, and if $w \in \mathcal{D}(\omega)$ (respectively, in $H_0^1(\omega)$), then setting $\varphi(x, y, z) = w(x, y)\xi(z)$, we have $\varphi \in \mathcal{D}(\Omega)$ (respectively, in $H_0^1(\Omega)$) and, further, $T(\varphi) = w$. Similarly, if $\underline{\phi} \in H_0(\operatorname{div}; \omega)$, we have that $\underline{\psi} = \underline{\phi}\xi \in W$ and $T(\underline{\psi}) = \underline{\phi}$.

Step 2. If $\underline{\varphi} \in (\mathcal{D}(\omega))^2$ and $\underline{\psi} = \underline{\phi}\xi$, we have $(\underline{v}, \underline{\psi})_W = 0$. Thus

$$\int_\Omega (\underline{v}.\underline{\phi})\xi(z)\,d\mathbf{x} + \int_\Omega \frac{\partial \underline{v}}{\partial z}.\underline{\phi}\xi'(z)\,d\mathbf{x} + \int_\omega \operatorname{div}\big(T(\underline{v})\big) \operatorname{div}(\underline{\phi})\,dx\,dy = 0.$$

Hence,

$$\left| \int_\omega \operatorname{div}\big(T(\underline{v})\big) \operatorname{div}(\underline{\phi})\,dx\,dy \right| \leqslant C|\underline{\phi}|_{0,\omega}.$$

It follows that $\operatorname{div}(T(\underline{v})) \in H^1(\omega)$.

Step 3. Let $\underline{\varphi} \in (\mathcal{D}(\Omega))^2$. Using the result of Step 2, we deduce from the relation $(\underline{v}, \underline{\varphi})_W = 0$, that

$$\left| \int_\Omega \frac{\partial \underline{v}}{\partial z} \frac{\partial \underline{\varphi}}{\partial z}\,d\mathbf{x} \right| \leqslant |\underline{v}|_{0,\Omega}|\underline{\varphi}|_{0,\Omega} + \left| \int_\omega \nabla\big(\operatorname{div}\big(T(\underline{v})\big)\big).T(\underline{\varphi})\,dx\,dy \right| \leqslant C|\underline{\varphi}|_{0,\Omega}.$$

Since $\underline{\varphi}$ was arbitrary, it follows that $\partial^2 \underline{v}/\partial z^2 \in (L^2(\Omega))^2$.

Step 4. If $\underline{\psi} \in (\mathcal{D}(\Omega))^2$ and if we set $\underline{\varphi} = \partial \underline{\psi}/\partial z$, then $\underline{\varphi} \in (\mathcal{D}(\Omega))^2$ and $T(\underline{\varphi}) = 0$. Thus, for all $\underline{\psi} \in (\mathcal{D}(\Omega))^2$, the relation $(\underline{v}, \partial \underline{\psi}/\partial z)_W = 0$ yields

$$\frac{\partial \underline{v}}{\partial z} - \frac{\partial^3 \underline{v}}{\partial z^3} = 0.$$

It then follows from Lemma 3.4 that $\underline{v} - \partial^2 \underline{v}/\partial z^2 = \underline{c}(x, y)$, and $\underline{c} \in (L^2(\omega))^2$.

Step 5. Thus, for all $\underline{\varphi} \in (\mathcal{D}(\Omega))^2$, we can rewrite $(\underline{v}, \underline{\varphi})_W = 0$ as

$$\int_\omega \big[\underline{c} - \nabla\big(\operatorname{div}\big(T(\underline{v})\big)\big)\big].T(\underline{\varphi})(x, y)\,dx\,dy = 0.$$

But by Step 1, the map $T:(\mathcal{D}(\Omega))^2 \to (\mathcal{D}(\omega))^2$ is surjective and thus it follows that

$$\underline{c}(x,y) = \nabla\big(\operatorname{div}\big(T(\underline{v})\big)\big)(x,y)$$

as elements in $(L^2(0,1))^2$.

Step 6. Finally, let $\underline{\varphi} \in W$. Then, by the preceding steps and Green's formula (cf. the definition of W),

$$(\underline{v},\underline{\varphi})_W = \int_\omega \big[\underline{c} - \nabla\big(\operatorname{div}\big(T(\underline{v})\big)\big)\big].T(\underline{\varphi})\,dx\,dy = 0.$$

Thus $\underline{v} = 0$ and the proof is complete. □

We now introduce a subspace of W which will be needed in the sequel. Let

$$W_0 = \big\{\underline{v} \in W \mid \operatorname{div}\big(T(\underline{v})\big) = 0\big\}. \tag{3.6}$$

This space will play the role similar to that of vector fields with vanishing divergence in the original problem. Just as the annihilator of such vector fields are gradients of scalar functions, we have a characterization of the annihilator of W_0.

Proposition 3.2. *Let $F \in W'$, the dual of W, such that $F(\underline{v}) = 0$ for all $\underline{v} \in W_0$. Then, there exists $p \in L^2(\omega)$ such that for every $\underline{v} \in W$,*

$$F(\underline{v}) = \int_\omega p(x,y)\operatorname{div}\big(T(\underline{v})\big)(x,y)\,dx\,dy. \tag{3.7}$$

Proof. *Step* 1. Let $\xi \in \mathcal{D}(0,h_0)$ such that $\int_0^{h_0}\xi(z)\,dz = 1$. If $\underline{v} \in W$, then $\underline{\chi} \in W_0$, where $\chi_i(x,y,z) = v_i(x,y,z) - T(v_i)(x,y)\xi(z),\ i = 1,2$. Hence, $F(\underline{\chi}) = 0$. By the Riesz representation theorem, there exists $\underline{w} \in W$ such that, for all $\underline{v} \in W$, $F(\underline{v}) = (\underline{v},\underline{w})_W$. Thus,

$$\begin{aligned} F(\underline{v}) &= F\big(T(\underline{v})\xi\big) = \big(\underline{w}, T(\underline{v})\xi\big)_W \\ &= \sum_{i=1}^2 \int_\omega r_i(x,y)T(v_i)(x,y)\,dx\,dy + \int_\omega \operatorname{div}\big(T(\underline{w})\big)\operatorname{div}\big(T(\underline{v})\big)\,dx\,dy, \end{aligned}$$

where $r_i \in L^2(\omega)$ is given by

$$r_i(x,y) = \int_0^{h(x,y)} w_i(x,y,z)\xi(z)\,dz + \int_0^{h(x,y)} \frac{\partial w_i}{\partial z}(x,y,z)\xi'(z)\,dz.$$

Step 2. On $(H_0^1(\omega))^2$, define the linear functional

$$\Phi(\underline{\varphi}) = \sum_{i=1}^2 \int_\omega r_i\varphi_i\,dx\,dy + \int_\omega \operatorname{div}\big(T(\underline{w})\big)\operatorname{div}(\underline{\varphi})\,dx\,dy$$

for any $\underline{\varphi} \in (H_0^1(\omega))^2$, $\underline{\varphi}\xi \in (H_0^1(\Omega))^2 \subset W$ and $T(\underline{\varphi}\xi) = \underline{\varphi}$. Thus, if $\operatorname{div}(\underline{\varphi}) = 0$, it follows from Step 1 that $\Phi(\underline{\varphi}) = 0$ and so, by de Rham's theorem, there exists $p \in L^2(\omega)$ such that

$$\Phi(\underline{\varphi}) = \int_\omega p(x, y) \operatorname{div}(\underline{\varphi})(x, y)\, dx\, dy.$$

For $\underline{v} \in (H_0^1(\Omega))^2$, we have that $T(\underline{v}) \in (H_0^1(\omega))^2$ and $F(\underline{v}) = \Phi(T(\underline{v}))$. The result now follows from the density of $(H_0^1(\Omega))^2$ in W. □

Remark 3.1. Proceeding exactly as in the proof of Poincaré's inequality (cf., for instance, Kesavan [4]), we can show that there exists a constant $C > 0$ such that for $\underline{v} \in W$,

$$|\underline{v}|_{0,\Omega} \leqslant C \left| \frac{\partial \underline{v}}{\partial z} \right|_{0,\Omega}.$$

Thus, in W_0, since $\operatorname{div}(T(\underline{v})) = 0$, the function $v \mapsto |\partial \underline{v}/\partial z|_{0,\Omega}$ defines a norm equivalent to the norm in W.

4. The limit problem

From the a priori estimates (2.9), we deduce that, for a subsequence, $\hat{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u}$, $\partial\hat{\mathbf{u}}_\varepsilon/\partial z \rightharpoonup \partial\mathbf{u}/\partial z$ and $\varepsilon(\partial\hat{\mathbf{u}}_\varepsilon/\partial x) \rightharpoonup \mathbf{z}_1$, $\varepsilon(\partial\hat{\mathbf{u}}_\varepsilon/\partial y) \rightharpoonup \mathbf{z}_2$ weakly in $(L^2(\Omega))^3$. But since $\{\partial\hat{\mathbf{u}}_\varepsilon/\partial x\}$ is bounded in $(H^{-1}(\Omega))^3$, it follows that $\mathbf{z}_1 = \mathbf{0}$. In the same way, $\mathbf{z}_2 = \mathbf{0}$.

Similarly, from the estimates (2.10), there exists a subsequence for which $\hat{p}_\varepsilon \rightharpoonup p$ weakly in $L^2(\Omega)$ and since $\partial\hat{p}_\varepsilon/\partial z \to 0$ in $H^{-1}(\Omega)$, it follows that $\partial p/\partial z = 0$ and, by Lemma 3.2, that $p(x, y, z) = p(x, y)$.

We will, henceforth, consider a subsequence (indexed yet again by ε, for convenience) for which all the above convergences are valid.

We first deduce some properties of $\mathbf{u}$ coming out of the incompressibility condition $\operatorname{div}(\mathbf{u}_\varepsilon) = 0$.

Lemma 4.1. *Let $\hat{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u}$, $\partial\hat{\mathbf{u}}_\varepsilon/\partial z \rightharpoonup \partial\mathbf{u}/\partial z$ and $\varepsilon(\partial\hat{\mathbf{u}}_\varepsilon/\partial x), \varepsilon(\partial\hat{\mathbf{u}}_\varepsilon/\partial y) \rightharpoonup \mathbf{0}$ weakly in $(L^2(\Omega))^3$, where $\mathbf{u} = (u_1, u_2, u_3)$. Then, $u_3 = 0$ and $\operatorname{div}(T(\underline{u})) = 0$, where $\underline{u} = (u_1, u_2)$.*

Proof. Since $\operatorname{div}(\mathbf{u}_\varepsilon) = 0$, we have

$$\frac{\partial\hat{u}_{\varepsilon,1}}{\partial x} + \frac{\partial\hat{u}_{\varepsilon,2}}{\partial y} + \frac{1}{\varepsilon}\frac{\partial\hat{u}_{\varepsilon,3}}{\partial z} = 0. \tag{4.1}$$

If $v \in H^1(\Omega)$, then, multiplying (4.1) by v and integrating by parts, and then passing to the limit, using the convergences stated in the hypotheses, we deduce that

$$\int_\Omega u_3 \frac{\partial v}{\partial z}\, d\mathbf{x} = 0$$

for all $v \in H^1(\Omega)$ and so $u_3 = 0$ by Corollary 3.1.

Now, let $\varphi \in \mathcal{D}(\omega)$. Once again, from (4.1), we deduce that

$$\int_\Omega \frac{\partial \hat{u}_{\varepsilon,1}}{\partial x}\varphi\,d\mathbf{x} + \int_\Omega \frac{\partial \hat{u}_{\varepsilon,2}}{\partial y}\varphi\,d\mathbf{x} + \frac{1}{\varepsilon}\int_\Omega \frac{\partial \hat{u}_{\varepsilon,3}}{\partial z}\varphi\,d\mathbf{x} = 0.$$

But, since $\hat{u}_{\varepsilon,3} \in H_0^1(\Omega)$ and φ is independent of z, the third integral vanishes and so, thanks to (3.1) and the fact that $\underline{\hat{u}}_\varepsilon = (\hat{u}_{\varepsilon,1}, \hat{u}_{\varepsilon,2}) \in (H_0^1(\Omega))^2$, it follows that $\operatorname{div}(T(\underline{\hat{u}}_\varepsilon)) = 0$. Since T is continuous and linear, it is weakly continuous and so $\operatorname{div}(T(\underline{u})) = 0$. □

Henceforth, we will set $\mathbf{u} = (\underline{u}, 0)$. Our limit problem will, therefore, be one satisfied by $\underline{u}$.

Proposition 4.1. *Let $(\mathbf{u}_\varepsilon, p_\varepsilon)$ be solution of (2.8) such that $\hat{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u} = (\underline{u}, 0)$ in $(L^2(\Omega))^3$ weakly and let $\hat{p}_\varepsilon \rightharpoonup p$ in $L^2(\Omega)$ weakly, so that $\varepsilon(\partial\hat{\mathbf{u}}_\varepsilon/\partial x), \varepsilon(\partial\hat{\mathbf{u}}_\varepsilon/\partial y) \rightharpoonup \mathbf{0}$ and $\partial\hat{\mathbf{u}}_\varepsilon/\partial z \rightharpoonup \partial\mathbf{u}/\partial z$ weakly in $(L^2(\Omega))^3$. Then $(\underline{u}, p) \in W_0 \times L^2(\Omega)$ and satisfies the following variational inequality*:

$$\mu\int_\Omega \frac{\partial \underline{u}}{\partial z}\frac{\partial}{\partial z}(\underline{v} - \underline{u})\,d\mathbf{x} + g\int_\Omega \left|\frac{\partial \underline{v}}{\partial z}\right| d\mathbf{x} - g\int_\Omega \left|\frac{\partial \underline{u}}{\partial z}\right| d\mathbf{x}$$
$$\geqslant \int_\Omega \underline{f}.(\underline{v} - \underline{u})\,d\mathbf{x} + \int_\omega p\operatorname{div}\big(T(\underline{v} - \underline{u})\big)\,dx\,dy \tag{4.2}$$

for every $\underline{v} \in W$, where $\mathbf{f} = (f_1, f_2, f_3) = (\underline{f}, f_3)$. Further, $p = p(x, y)$.

Proof. We have already observed that $p = p(x, y)$ and that (cf. Lemma 4.1) $u \in W_0$.

Let $\mathbf{v} = (v_1, v_2, v_3) \in (H_0^1(\Omega))^3$ and set $\mathbf{v}_\varepsilon(x_1, x_2, x_3) = \mathbf{v}(x_1, x_2, x_3/\varepsilon) \in (H_0^1(\Omega_\varepsilon))^3$. It then follows from (2.8) that

$$\mu\varepsilon^2 \int_\Omega \sum_{i=1}^3 \left(\frac{\partial \hat{u}_{\varepsilon,i}}{\partial x}\frac{\partial v_i}{\partial x} + \frac{\partial \hat{u}_{\varepsilon,i}}{\partial y}\frac{\partial v_i}{\partial y} + \frac{1}{\varepsilon^2}\frac{\partial \hat{u}_{\varepsilon,i}}{\partial z}\frac{\partial v_i}{\partial z}\right) d\mathbf{x}$$
$$+ g\varepsilon\int_\Omega \left[\sum_{i=1}^3 \left(\left(\frac{\partial v_i}{\partial x}\right)^2 + \left(\frac{\partial v_i}{\partial y}\right)^2 + \frac{1}{\varepsilon^2}\left(\frac{\partial v_i}{\partial z}\right)^2\right)\right]^{1/2} d\mathbf{x}$$
$$\geqslant \mu\varepsilon^2 \int_\Omega \sum_{i=1}^3 \left(\left(\frac{\partial \hat{u}_{\varepsilon,i}}{\partial x}\right)^2 + \left(\frac{\partial \hat{u}_{\varepsilon,i}}{\partial y}\right)^2 + \frac{1}{\varepsilon^2}\left(\frac{\partial \hat{u}_{\varepsilon,i}}{\partial z}\right)^2\right) d\mathbf{x}$$
$$+ g\varepsilon\int_\Omega \left[\sum_{i=1}^3 \left(\left(\frac{\partial \hat{u}_{\varepsilon,i}}{\partial x}\right)^2 + \left(\frac{\partial \hat{u}_{\varepsilon,i}}{\partial y}\right)^2 + \frac{1}{\varepsilon^2}\left(\frac{\partial \hat{u}_{\varepsilon,i}}{\partial z}\right)^2\right)\right]^{1/2} d\mathbf{x}$$
$$+ \int_\Omega \sum_{i=1}^3 f_i(v_i - \hat{u}_{\varepsilon,i})\,d\mathbf{x} + \int_\Omega \hat{p}_\varepsilon\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{1}{\varepsilon}\frac{\partial v_3}{\partial z}\right) d\mathbf{x}$$

since $\operatorname{div}(\mathbf{u}_\varepsilon) = 0$. We now choose $\mathbf{v} = (\underline{v}, 0)$, with $\underline{v} \in (H_0^1(\Omega))^2$. Then, ignoring some positive terms on the right-hand side and passing to the limit as $\varepsilon \to 0$, we get, using the various convergences announced earlier,

$$\mu \int_\Omega \frac{\partial \underline{u}}{\partial z} \frac{\partial \underline{v}}{\partial z}\, d\mathbf{x} + g \int_\Omega \left| \frac{\partial \underline{v}}{\partial z} \right| d\mathbf{x}$$
$$\geqslant \mu \int_\Omega \left| \frac{\partial \underline{u}}{\partial z} \right|^2 d\mathbf{x} + g \int_\Omega \left| \frac{\partial \underline{u}}{\partial z} \right| d\mathbf{x} + \int_\Omega \underline{f}.(\underline{v} - \underline{u})\, d\mathbf{x} + \int_\Omega p \operatorname{div}(\underline{v})\, d\mathbf{x}.$$

Finally, since $p = p(x, y)$ and $\operatorname{div}(T(\underline{u})) = 0$, we can replace the last integral on the right by $\int_\omega p \operatorname{div}(T(\underline{v} - \underline{u}))\, dx\, dy$. Thus we get (4.2) for all $\underline{v} \in (H_0^1(\Omega))^2$ and the result follows from the density of this space in W (cf. Proposition 3.1). □

If $\underline{v} \in W_0$, then (4.2) reads as

$$\mu \int_\Omega \frac{\partial u}{\partial z} \frac{\partial}{\partial z}(\underline{v} - \underline{u})\, d\mathbf{x} + g \int_\Omega \left| \frac{\partial \underline{v}}{\partial z} \right| d\mathbf{x} - g \int_\Omega \left| \frac{\partial \underline{u}}{\partial z} \right| d\mathbf{x} \geqslant \int_\Omega \underline{f}.(\underline{v} - \underline{u})\, d\mathbf{x}. \tag{4.3}$$

Thus, we get a variational inequality in the space W_0. The 'pressure' p can be recovered from (4.3) by proceeding in a manner similar to that outlined by Duvaut and Lions [2], which we now detail below.

As usual, setting $\underline{v} = 2\underline{u}$ and $\underline{v} = \underline{0}$ successively in (4.3), we deduce that it is equivalent to the system

$$\begin{cases} \mu \int_\Omega \left|\frac{\partial \underline{u}}{\partial z}\right|^2 d\mathbf{x} + g \int_\Omega \left|\frac{\partial \underline{u}}{\partial z}\right| d\mathbf{x} - \int_\Omega \underline{f}.\underline{u}\, d\mathbf{x} = 0, \\ \mu \int_\Omega \frac{\partial \underline{u}}{\partial z} \frac{\partial \underline{v}}{\partial z}\, d\mathbf{x} + g \int_\Omega \left|\frac{\partial \underline{v}}{\partial z}\right| d\mathbf{x} - \int_\Omega \underline{f}.\underline{v}\, d\mathbf{x} \geqslant 0, \end{cases} \tag{4.4}$$

for every $\underline{v} \in W_0$. Changing $\underline{v}$ to $-\underline{v}$, we get that, for all $\underline{v} \in W_0$,

$$\left| \mu \int_\Omega \frac{\partial \underline{u}}{\partial z} \frac{\partial \underline{v}}{\partial z}\, d\mathbf{x} - \int_\Omega \underline{f}.\underline{v}\, d\mathbf{x} \right| \leqslant g \int_\Omega \left| \frac{\partial \underline{v}}{\partial z} \right| d\mathbf{x}. \tag{4.5}$$

Thus, setting

$$F(\underline{v}) = \mu \int_\Omega \frac{\partial \underline{u}}{\partial z} \frac{\partial \underline{v}}{\partial z}\, d\mathbf{x} - \int_\Omega \underline{f}.\underline{v}\, d\mathbf{x},$$

(4.5) tells us that F is a continuous linear functional on the subspace of $(L^1(\Omega))^2$ which is the image of W_0 under the mapping $\underline{v} \mapsto \pi(\underline{v}) = \partial \underline{v}/\partial z$. Hence, by the Hahn–Banach theorem, there exists $\underline{m} \in (L^\infty(\Omega))^2$, with $\| |\underline{m}| \|_\infty \leqslant 1$, such that for all $\underline{v} \in W_0$,

$$F(\underline{v}) = -g \int_\Omega \underline{m}.\frac{\partial \underline{v}}{\partial z}\, d\mathbf{x}. \tag{4.6}$$

In particular, it follows from (4.4) that

$$\int_\Omega \underline{m}.\frac{\partial \underline{u}}{\partial z}\, d\mathbf{x} = \int_\Omega \left| \frac{\partial \underline{u}}{\partial z} \right| d\mathbf{x}. \tag{4.7}$$

Rewriting (4.6) as

$$\mu \int_{\Omega} \frac{\partial \underline{u}}{\partial z} \frac{\partial \underline{v}}{\partial z} \, d\mathbf{x} + g \int_{\Omega} \underline{m} . \frac{\partial \underline{v}}{\partial z} \, d\mathbf{x} - \int_{\Omega} \underline{f} . \underline{v} \, d\mathbf{x} = 0$$

for every $\underline{v} \in W_0$, we deduce, from Proposition 3.2, the existence of $p \in L^2(\omega)$ such that

$$\mu \int_{\Omega} \frac{\partial \underline{u}}{\partial z} \frac{\partial \underline{v}}{\partial z} \, d\mathbf{x} + g \int_{\Omega} \underline{m} . \frac{\partial \underline{v}}{\partial z} \, dx\, dy - \int_{\Omega} \underline{f} . \underline{v} \, d\mathbf{x} = \int_{\omega} p \operatorname{div}\bigl(T(\underline{v})\bigr) \, dx\, dy \tag{4.8}$$

for all $\underline{v} \in W$. Thus, if for $\underline{v} \in W$, we set

$$X = \mu \int_{\Omega} \frac{\partial \underline{u}}{\partial z} \frac{\partial}{\partial z} (\underline{v} - \underline{u}) \, d\mathbf{x} + g \int_{\Omega} \left| \frac{\partial \underline{v}}{\partial z} \right| d\mathbf{x} - g \int_{\Omega} \left| \frac{\partial \underline{u}}{\partial z} \right| d\mathbf{x}$$
$$- \int_{\Omega} \underline{f} . (\underline{v} - \underline{u}) \, d\mathbf{x} - \int_{\omega} p \operatorname{div}\bigl(T(\underline{v} - \underline{u})\bigr) \, dx\, dy,$$

it follows, from the preceding considerations, that

$$X = g \int_{\Omega} \left| \frac{\partial \underline{v}}{\partial z} \right| d\mathbf{x} - g \int_{\Omega} \underline{m} . \frac{\partial \underline{v}}{\partial z} \, d\mathbf{x} \geqslant 0$$

since $\| |\underline{m}| \|_\infty \leqslant 1$. Thus, $(\underline{u}, p)$ satisfies (4.2).

Consequently, it is now enough to consider (4.3) over the space W_0 as the limit problem (for the unknown u).

If $\underline{u}_1$ and $\underline{u}_2$ are two solutions, then using $\underline{u}_2$ as a test function in the inequality for $\underline{u}_1$ and vice versa, we get, in addition,

$$-\mu \int_{\Omega} \left| \frac{\partial}{\partial z} (\underline{u}_1 - \underline{u}_2) \right|^2 d\mathbf{x} \geqslant 0.$$

Since $\underline{u}_1 - \underline{u}_2 \in W_0$, it follows that (cf. Remark 3.2) $\underline{u}_1 - \underline{u}_2 = 0$. We know that the limit problem possesses a solution, viz. the limit $\underline{u}$ of $(\hat{u}_{\varepsilon,1}, \hat{u}_{\varepsilon,2})$. We can also prove this independently, using the Galerkin method. Thus, the problem (4.3) admits a unique solution in W_0.

Remark 4.1. In view of the uniqueness of the solution of the limit problem, we deduce that the entire sequence $(\hat{u}_{\varepsilon,1}, \hat{u}_{\varepsilon,2})$ converges to $\underline{u}$. We have no result on the uniqueness of p, even up to an additive constant.

5. Discussion

We now examine the implications of the limit problem (4.2) (or, equivalently, (4.3)) obtained in the previous section.

Let us set $\underline{\sigma} = -\nabla p + \underline{\tilde{\sigma}}$, where

$$\underline{\tilde{\sigma}} = \mu \frac{\partial \underline{u}}{\partial z} + g\underline{m},$$

$\underline{m}$ being as in (4.6). Thus, if $\partial \underline{u}/\partial z = 0$, it follows that $|\underline{\tilde{\sigma}}| \leqslant g$ since $\| \, |\underline{m}| \, \|_\infty \leqslant 1$. Now, rewriting (4.7) as

$$\int\limits_{|\partial \underline{u}/\partial z| \neq 0} \left(\left| \frac{\partial \underline{u}}{\partial z} \right| - \underline{m}.\frac{\partial \underline{u}}{\partial z} \right) d\mathbf{x} = 0,$$

and taking into account the fact that $|\underline{m}| \leqslant 1$, we deduce that

$$\underline{m}.\frac{\partial \underline{u}}{\partial z} = \left| \frac{\partial \underline{u}}{\partial z} \right|$$

on the set where $|\partial \underline{u}/\partial z| \neq 0$. Hence, if $|\partial \underline{u}/\partial z| \neq 0$, we get

$$\underline{\tilde{\sigma}} = \mu \frac{\partial \underline{u}}{\partial z} + g \frac{\partial \underline{u}/\partial z}{|\partial \underline{u}/\partial z|}. \tag{5.1}$$

In this case, clearly, $|\underline{\tilde{\sigma}}| > g$. We can thus write

$$\mu \frac{\partial \underline{u}}{\partial z} = \begin{cases} 0, & \text{if } |\underline{\tilde{\sigma}}| \leqslant g, \\ \underline{\tilde{\sigma}} - g \frac{\partial \underline{u}/\partial z}{|\partial \underline{u}/\partial z|}, & \text{if } |\tilde{\sigma}| > g, \end{cases} \tag{5.2}$$

which is a lower-dimensional 'Bingham-like' law.

If we now take into account (4.8), we get

$$\int\limits_{\Omega} \underline{\tilde{\sigma}} \frac{\partial \underline{v}}{\partial z} \, d\mathbf{x} = \int\limits_{\Omega} \underline{f}.\underline{v} \, d\mathbf{x} - \int\limits_{\Omega} \nabla p(x, y).\underline{v} \, d\mathbf{x}$$

for all $\underline{v} \in W$. Thus,

$$-\frac{\partial \underline{\tilde{\sigma}}}{\partial z} = \underline{f} - \nabla p(x, y) \quad \text{in } \Omega$$

and on the set where $|\partial \underline{u}/\partial z| \neq 0$, we get the system of differential equations (using (5.1))

$$-\frac{\partial}{\partial z}\left[\mu \frac{\partial \underline{u}}{\partial z} + g \frac{\partial \underline{u}/\partial z}{|\partial \underline{u}/\partial z|} \right] = \underline{f} - \nabla p(x, y). \tag{5.3}$$

We can perform an identical analysis on a two-dimensional model with reference domain

$$\Omega = \{(x, y) \mid 0 < y < h(x)\},$$

where h is a sufficiently smooth function and the thin layer given by

$$\Omega_\varepsilon = \{(x, y) \mid 0 < y < \varepsilon h(x)\}.$$

The limit problem will be one similar to (4.2) or (4.3) involving only derivatives in y. In this case the spaces W and W_0 will be as follows:

$$W = \left\{ v \in L^2(\Omega) \;\middle|\; \frac{\partial v}{\partial y} \in L^2(\Omega),\ v = 0 \text{ on } \Gamma,\ T(v) \in H_0^1(0,1) \right\},$$
$$W_0 = \{ v \in W \mid T(v) = 0 \},$$

where Γ denotes, as before, the upper and lower boundaries of Ω and T is defined by

$$T(v)(x) = \int_0^{h(x)} v(x,y)\, dy.$$

We can again derive, mathematically, the following one-dimensional 'Bingham-like' law:

$$\mu \frac{\partial u}{\partial y} = \begin{cases} 0, & \text{if } |\tilde{\sigma}| \leqslant g, \\ \tilde{\sigma} - g \operatorname{sgn}\left(\frac{\partial u}{\partial y}\right), & \text{if } |\tilde{\sigma}| \geqslant g. \end{cases}$$

This has been used by engineers to model a Bingham fluid in thin layers (cf., for instance, Liu and Mei [6]). The differential equation satisfied in the nonrigid zone will then turn out to be

$$-\frac{\partial}{\partial y}\left[\mu \frac{\partial u}{\partial y} + g \operatorname{sgn}\left(\frac{\partial u}{\partial y}\right)\right] = f_1 - p'(x),$$

where $p = p(x)$ is the pressure in the limit. We can integrate this equation in the nonrigid zone and obtain the following result, which we state without proof.

Proposition 5.1. *Assume that f_1 is a function of x alone and that $\partial u/\partial y$ is continuous in $\bar{\Omega}$.*

(i) *If*

$$\frac{\partial u}{\partial y}(x,0)\frac{\partial u}{\partial y}(x,1) > 0,$$

then $(\partial u/\partial y)(x,y) \neq 0$ for all $y \in [0, h(x)]$ and thus the vertical line through $(x,0)$ does not traverse the rigid zone.

(ii) *If*

$$\frac{\partial u}{\partial y}(x,0)\frac{\partial u}{\partial y}(x,1) < 0,$$

then we can find $0 < v_0(x) < v_1(x) < h(x)$ such that $(\partial u/\partial y)(x,y)$ vanishes only in a subset of $[v_0(x), v_1(x)]$. In this case, necessarily,

$$g \leqslant h(x)\left|f_1(x) - p'(x)\right|$$

and

$$\left|\frac{\partial u}{\partial y}(x,0)\right|\left|\frac{\partial u}{\partial y}(x,1)\right| \leqslant \mu^{-2}\left(h(x)\left|f_1(x) - p'(x)\right| - g\right)^2.$$

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