Available online at www.sciencedirect.com

SCIENCE DIRECT

ELSEVIER

Journal of Functional Analysis 218 (2005) 475–494

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

The Rokhlin property and the tracial topological rank ☆

Huaxin Lin[a,*] and Hiroyuki Osaka[b]

[a] Department of Mathematics, East China Normal University, Shanghai, China and Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA

[b] Department of Mathematical Sciences, Ritsumeikan University, Kusatsu, Shiga, 525-8577, Japan

Received 17 February 2004; accepted 5 May 2004

Available online 18 August 2004

Communicated by D. Sarason

Abstract

Let A be a unital separable simple C^*-algebra with $\mathrm{TR}(A)\leqslant 1$ and α be an automorphism. We show that if α satisfies the tracially cyclic Rokhlin property then $\mathrm{TR}(A\rtimes_\alpha\mathbb{Z})\leqslant 1$. We also show that whenever A has a unique tracial state and α^m is uniformly outer for each $m(\neq 0)$ and α^r is approximately inner for some $r>0$, α satisfies the tracial cyclic Rokhlin property. By applying the classification theory of nuclear C^*-algebras, we use the above result to prove a conjecture of Kishimoto: if A is a unital simple $A\mathbb{T}$-algebra of real rank zero and $\alpha\in\mathrm{Aut}(A)$ which is approximately inner and if α satisfies some Rokhlin property, then the crossed product $A\rtimes_\alpha\mathbb{Z}$ is again an $A\mathbb{T}$-algebra of real rank zero. As a by-product, we find that one can construct a large class of simple C^*-algebras with tracial rank one (and zero) from crossed products.

MSC: primary 46L55; 46L35

Keywords: The Rokhlin property; Tracial topological rank; Simple C^*-algebras

☆ Research partially supported by NSF grants DMS 0097903 and JSPS Grant for Scientific Research of Japan No.14540217.

* Corresponding author.

E-mail address: hlin@darkwing.uoregon.edu (H. Lin).

0022-1236/$ - see front matter © 2004 Elsevier Inc. All rights reserved.

doi:10.1016/j.jfa.2004.05.005

1. Introduction

The Rokhlin property in ergodic theory was adopted to the context of von Neumann algebras by Connes [2]. It was adopted by Herman and Ocneanu [17] for UHF-algebras. Rørdam [30] and Kishimoto [13] introduced the Rokhlin property to a much more general context of C^*-algebras (see also [9]). Kishimoto had been studying automorphisms on UHF-algebras and more generally, on simple $A\mathbb{T}$-algebras that satisfy a Rokhlin property [12,14]. More recently, Phillips studied finite group actions which satisfy certain type of Rokhlin property on some simple C^*-algebras [26].

A conjecture of Kishimoto can be formulated as follows: Let A be a unital simple $A\mathbb{T}$-algebra of real rank zero and α be an approximately inner automorphism. Suppose that α is "sufficiently outer", then the crossed product of the $A\mathbb{T}$-algebra by α, $A \rtimes_\alpha \mathbb{Z}$ is again a unital simple $A\mathbb{T}$-algebra. In particular, he studied the case that A has a unique tracial state.

Kishimoto proposed that the appropriate notion of the outerness is the Rokhlin property [14]. He also introduced the notion of uniformly outer [13]. In [14], he showed that if A is a unital simple $A\mathbb{T}$-algebra of real rank zero with a unique tracial state and $\alpha \in \mathrm{Aut}(A)$ is approximately inner, then α has the Rokhlin property if and only if α^m is uniformly outer for all $m \neq 0$. He also showed that the Rokhlin property, in this situation, is equivalent to say that $A \rtimes_\alpha \mathbb{Z}$ has real rank zero, and it is equivalent to say that $A \rtimes_\alpha \mathbb{Z}$ has a unique tracial state. Kishimoto showed [12] that, if in addition, A is a UHF-algebra then $A \rtimes_\alpha \mathbb{Z}$ is in fact a unital simple $A\mathbb{T}$-algebra. He also showed in [15] that the conjecture is true for the case that A is assumed to have a unique tracial state, both $K_i(A)$ are finitely generated and $K_1(A) \ncong \mathbb{Z}$, and, in addition, that $\alpha \in \mathrm{Hinn}(A)$, where Hinn(A) is the subgroup of automorphisms which are homotopy to inner automorphisms. Among other things, we prove in this paper the Kishimoto conjecture for all cases that A has a unique tracial state. If the term of "sufficiently outer" is interpreted as "tracial Rokhlin" property, then Kishimoto's conjecture holds: Let A be a unital simple $A\mathbb{T}$-algebra and α be an approximate inner automorphism. Suppose that α has the tracial Rokhlin property, then $A \rtimes_\alpha \mathbb{T}$ is again a unital simple $A\mathbb{T}$-algebra.

We take the advantage of the development in Elliott's program of the classification of nuclear C^*-algebras (see [4,6], for example). In particular, we use the classification result in [23], where unital separable simple C^*-algebras satisfying the Universal Coefficient Theorem and with tracial topological rank zero are classified by their K-theory. Adopting a Phillips's observation, we note that if A is a unital simple C^*-algebra with $\mathrm{TR}(A) \leqslant 1$ and $\alpha \in \mathrm{Aut}(A)$ satisfies a so-called tracial cyclic Rokhlin property, then $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) \leqslant 1$ so that the classification result in [23,24] can be applied. Using Kishimoto's techniques, we show that if α^r is approximately inner (for some integer $r > 0$), the tracial Rokhlin property introduced in [28] implies the tracial cyclic Rokhlin property. Using a result in [28], we actually show a more general result (see Theorem 3.5).

The assumption that α is approximately inner is to insure that the crossed products remain finite (see the introduction of Kishimoto [14]). We relax this restriction slightly by only requiring that α^r is approximately inner for some integer $r>0$. It turns out that in a number of cases, while there are automorphisms which are not approximate inner, all (outer) automorphisms α have this property, i.e., for some integer $r>0$, α^r are approximately inner (see Theorem 4.2). We show that our results also cover many cases in which A may have arbitrary tracial space (see Corollary 4.4).

It is shown by Gong [8] that a unital simple AH-algebra with very slow dimension growth has tracial topological rank one or zero. Moreover, Elliott and Gong [7] show that the class of unital simple AH-algebras with very slow dimension growth can be classified by their K-theoretical data. An improvement of this classification has been made so that unital simple nuclear C^*-algebras with tracial topological rank no more than one which satisfy the Universal Coefficient Theorem can also be classified by their K-theoretical data [24]. However, until now, all interesting examples of unital simple nuclear C^*-algebras that have tracial topological rank one are those AH-algebras with very slow dimension growth (and those of similar inductive limit construction). Theorem 2.7 also provides ways to construct unital simple C^*-algebras with tracial topological rank one by crossed products (see Corollary 4.4 and Example 4.5). It also creates the opportunity to apply the classification results in [24].

2. The Rokhlin properties

The following conventions will be used in this paper. Let A be a unital C^*-algebra.

(i) We denote by $\mathrm{Aut}(A)$ the set of all automorphisms on A and by $\mathrm{T}(A)$ the tracial state space of A.
(ii) Two projections $p, q \in A$ are said to be equivalent if they are Murray–von Neumann equivalent. That is, there exists a partial isometry $w \in A$ such that $w^*w = p$ and $ww^* = q$. Then we write $p \sim q$.
(iii) Let $\mathcal{F}$ and $\mathcal{S}$ be subsets of A and $\varepsilon > 0$. We write $x \in_\varepsilon \mathcal{S}$ if there exists $y \in \mathcal{S}$ such that $||x-y|| < \varepsilon$, and write $\mathcal{F} \subset_\varepsilon \mathcal{S}$ if $x \in_\varepsilon \mathcal{S}$ for all $x \in \mathcal{F}$.
(iv) Let a and b be two positive elements in A. We write $[a] \leqslant [b]$ if there exists an element $x \in A$ such that $a = x^*x$ and $xx^* \in \overline{bAb}$. If $ab = ba = 0$, then we write $[a+b] = [a] + [b]$. Let p be a projection and b non-zero positive element in A. Note that $[p] \leqslant [b]$ implies that p is Murray–von Neumann equivalent to a projection in the hereditary C^*-algebra $\overline{bAb}$.
(v) We denote by $\mathcal{I}^{(0)}$ the class of all finite-dimensional C^*-algebras, and by $\mathcal{I}^{(k)}$ the class of all C^*-algebras with the form $pM_n(C(X))p$, where X is a finite CW complex with dimension k and $p \in M_n(C(X))$ is a projection.

We recall the definition of tracial topological rank of C^*-algebras.

Definition 2.1 (Lin [20, Theorem 6.13]). Let A be a unital simple C^*-algebra and $k \in \mathbb{N}$. Then A is said to have *tracial topological rank no more than k* if and only if for any finite set $\mathcal{F} \subset A$, and $\varepsilon > 0$ and any non-zero positive element $a \in A$, there exists a C^*-subalgebra $B \subset A$ with $B \in \mathcal{I}^{(k)}$ and $\mathrm{id}_B = p$ such that

(1) $\|[x, p]\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) $pxp \in_\varepsilon B$ for all $x \in \mathcal{F}$,
(3) $[1 - p] \leqslant [a]$.

We write $\mathrm{TR}(A) \leqslant k$.

Remark 2.2 (Lin [20, Corollary 6.15]). Let A be a simple unital C^*-algebra with stable rank one which satisfies the Fundamental Comparison Property. Then $\mathrm{TR}(A) \leqslant k$ if and only if for any finite set $\mathcal{F} \subset A$, $\varepsilon > 0$, and any non-zero positive element $a \in A$, there exists a C^*-subalgebra $B \subset A$ with $B \in \mathcal{I}^{(k)}$ and $\mathrm{id}_B = p$ such that

(1) $\|[x, p]\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) $pxp \in_\varepsilon B$ for all $x \in \mathcal{F}$,
(3) $\tau(1 - p) < \varepsilon$ for all $\tau \in \mathrm{T}(A)$.

Recall that A is said to have *the Fundamental Comparison Property* if $p, q \in A$ are two projections with $\tau(p) < \tau(q)$ for all $\tau \in \mathrm{T}(A)$, then p is equivalent to a subprojection of q.

The following is defined in [28, Definition 2.1].

Definition 2.3. Let A be a simple unital C^*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every non-zero positive element $x \in A$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_n \in A$ such that:

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leqslant j \leqslant n - 1$.
(2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leqslant j \leqslant n$ and all $a \in F$.
(3) With $e = \sum_{j=0}^{n} e_j$, $[1 - e] \leqslant [x]$.

We define a slightly stronger version of the tracial Rokhlin property similar to the approximately Rokhlin property in [12, Definition 4.2].

Definition 2.4. Let A be a simple unital C^*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say α has the *tracial cyclic Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every non-zero positive element $x \in A$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_n \in A$ such that

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leqslant j \leqslant n$, where $e_{n+1} = e_0$.
(2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leqslant j \leqslant n$ and all $a \in F$.
(3) With $e = \sum_{j=0}^{n} e_j$, $[1 - e] \leqslant [x]$.

Remark 2.5. (i) The only difference between the tracial Rokhlin property and the tracial cyclic Rokhlin property is that in condition (1) we require that $||\alpha(e_n) - e_0|| < \varepsilon$.

(ii) If A has real rank zero, stable rank one and has weakly unperforated $K_0(A)$ (or if A has SP-property, stable rank one, and the Fundamental Comparison Property), then condition (3) in both Rokhlin property can be replaced by the following condition (3)′ using the standard argument: (3)′ With $e = \sum_{j=0}^{n} e_j$, we have $\tau(1 - e) < \varepsilon$ for all $\tau \in \mathrm{T}(A)$.

(iii) If A is a simple unital C^*-algebra with real rank zero, stable rank one, and has weakly unperforated $K_0(A)$, the Rokhlin property in the sense of Kishimoto [12] implies the tracial Rokhlin property [28].

Recall that a C^*-algebra A is said to have *SP-property* if any non-zero hereditary C^*-subalgebra of A has a non-zero projection.

Obviously, the tracial cyclic Rokhlin property implies the tracial Rokhlin property. The converse is also true in many cases. We will discuss it in the next section.

Before stating the characterization of the tracial Rokhlin property we cite the following notion introduced by Kishimoto [13].

Definition 2.6. Let A be a unital C^*-algebra and $\alpha \in \mathrm{Aut}(A)$. We say α is uniformly outer if for any $a \in A$, any projection $p \in A$, and any $\varepsilon > 0$, there are finite number of projections $p_1, \ldots, p_n$ in A such that $\sum_i p_i = p$ and $||p_i a \alpha(p_i)|| < \varepsilon$ for $i = 1, \ldots, n$.

The following result is the tracial Rokhlin version of Kishimoto's result in the case of simple unital $A\mathbb{T}$-algebras with a unique trace [12, Theorem 2.1].

Theorem 2.7 (Osaka and Phillips [28]). *Let A be a simple unital C^*-algebra with* $\mathrm{TR}(A) = 0$, *and suppose that A has a unique tracial state. Then the following conditions are equivalent*:

(1) *α has the tracial Rokhlin property.*
(2) *α^m is not weakly inner in the GNS representation π_τ for any $m \neq 0$.*
(3) *$A \rtimes_\alpha \mathbb{Z}$ has real rank zero.*
(4) *$A \rtimes_\alpha \mathbb{Z}$ has a unique trace.*

Note that the uniformly outerness implies that α is not weakly inner in the GNS representation π_τ by an α-invariant tracial state τ on A by Kishimoto [13, Lemma 4.4].

Remark 2.8. When A is a simple unital C^*-algebra with tracial topological rank zero, if $\alpha \in \mathrm{Aut}(A)$ has the tracial Rokhlin property, it is proved in [28] that the crossed product $A \rtimes_\alpha \mathbb{Z}$ has real rank zero, stable rank one, and the order on projections over $A \rtimes_\alpha \mathbb{Z}$ is determined by traces. But it is not known that the crossed product $A \rtimes_\alpha \mathbb{Z}$ has tracial topological rank zero. However, if α has the tracial cyclic Rokhlin property, then we have the following result based on an observation of Phillips [26].

Theorem 2.9. *Let A be a simple unital C^*-algebra with $\mathrm{TR}(A) \leqslant 1$. Suppose that $\alpha \in \mathrm{Aut}(A)$ has the tracial cyclic Rokhlin property. Then $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) \leqslant 1$.*

In particular, if A has $\mathrm{TR}(A) = 0$, then $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$.

Proof. We first note that, by [11], $A \rtimes_\alpha \mathbb{Z}$ is a simple C^*-algebra.

Let $\varepsilon > 0$, $n \in \mathbb{N}$, and $F \subset A \rtimes_\alpha \mathbb{Z}$ be a finite set. To simplify notation, without loss of generality, we may assume that

$$F = \{a_i\}_{i=1}^m \cup \{u\},$$

where $a_i \in A$ and $\|a_i\| \leqslant 1$ $(i = 1, 2, \ldots, m)$ and u is a unitary which implements α. Fix $b \in (A \rtimes_\alpha \mathbb{Z})_+ \setminus \{0\}$.

Since A has SP-property and α is outer, $A \rtimes_\alpha \mathbb{Z}$ also has SP-property [10, Theorem 4]. In particular, there is a non-zero projection $r \in \overline{b(A \rtimes_\alpha \mathbb{Z})b}$. Let $r_0 \in A$ be a non-zero projection. Since $A \rtimes_\alpha \mathbb{Z}$ is simple, by 1.8 of [3], it is easy to find a non-zero projection $r' \in r_0 A r_0$ such that r' is equivalent to a subprojection of r (see, for example, [10, Theorem 4]). Hence there are projections $r_1, r_2 \in A$ such that $r_1 r_2 = 0$ and $r_1 + r_2$ is equivalent to a subprojection of r (see, for example, [22, 3.5.7]).

Since α has the tracial cyclic Rokhlin property, for any $\delta > 0$ with $\delta < \frac{\varepsilon}{5}$ there exist projections e_1, e_2 such that

(1) $\|\alpha(e_i) - e_{i+1}\| < \delta$ for $1 \leqslant i \leqslant 2$ $(e_3 = e_1)$.
(2) $\|[e_i, a_k]\| < \delta$ for $1 \leqslant k \leqslant m$.
(3) $[1 - e_1 - e_2] \leqslant [r_1]$.

Set $p = e_1 + e_2$. From (1) above, one estimates that

$$\|up - pu\| = \left\| \sum_{i=1}^{2} ue_i - \sum_{i=1}^{2} e_{i+1}u \right\| = \sum_{i=1}^{2} \|ue_i - e_{i+1}u\| < 2\delta.$$

Hence, together with (2) above, we obtain

(4) $\|[p, a]\| < 2\delta$ for all $a \in F$.

There is a unitary $v \in A \rtimes_\alpha \mathbb{Z}$ such that $\|v - 1\| < \delta$ and $vu^* e_i uv^* = e_{i+1}$ for $1 \leqslant i \leqslant 2$. Set $w = vu^*$, and consider the C^*-algebra D generated by $e_1 A e_1$ and $e_2 w e_1$. Then D is isomorphic to $e_1 A e_1 \otimes M_2(\mathbb{C})$. Note that $pw = e_1 w + e_2 w = we_2 + we_1 = wp$. Moreover, $pwp \in D$. Since $\|pup - pwp\| < \delta$, one has that $pup \in_\delta D$. By (2) again, we have

$$\|pa_j p - (e_1 a_j e_1 + e_2 a_j e_2)\| < 2\delta, \quad j = 1, 2, \ldots, m.$$

It follows that $pFp \subset_{2\delta} D$.

Since A is a simple C^*-algebra with SP-property, there exists a non-zero projection $r_3 \in e_1 A e_1$ such that r_3 is equivalent to a subprojection of r_2. Since $\mathrm{TR}(e_1 A e_1) \leqslant 1$ ($\mathrm{TR}(e_1 A e_1) = 0$ if $\mathrm{TR}(A) = 0$), $\mathrm{TR}(D) \leqslant 1$ ($\mathrm{TR}(D) = 0$ if $\mathrm{TR}(A) = 0$) by Lin [20, Theorem 5.3]. So there exists a C^*-subalgebra $B \in \mathcal{I}^{(k)}$ ($k = 1$ or $k = 0$) and projection $e = 1_B$ such that

(5) $\|[pap, e]\| < \delta < \varepsilon$ for all $a \in F$,
(6) $pFp \subset_{\delta} B$ and
(7) $[p - e] \leqslant [r_3]$.

From (3), (4), (5), and (7) above we estimate that

(8) For any $f \in F$

$$\begin{aligned} \|ef - fe\| &= \|e(pf - fp) + efp - pfe + (pf - fp)e\| \\ &\leqslant \|pf - fp\| + \|epfp - pfpe\| + \|pf - fp\| \\ &< 5\delta < \varepsilon \end{aligned}$$

and

(9)

$$\begin{aligned} [1 - e] &= [1 - p + p - e], \\ &= [1 - p] + [p - e] \\ &\leqslant [r_1] + [r_3] \leqslant [r_1] + [r_2] \leqslant [r] \leqslant [b]. \end{aligned}$$

From (6) and $pFp \subset_{2\delta} D$, we have

(10) $pFp \subset_{4\delta} B$.

Hence from estimates (8)–(10) we conclude that $A \rtimes_{\alpha} \mathbb{Z}$ has tracial topological less than or equal to 1.

In the case of $\mathrm{TR}(A) = 0$ B can be chosen to be finite dimensional. Hence, in that case, $\mathrm{TR}(A \rtimes_{\alpha} \mathbb{Z}) = 0$. $\square$

3. Approximately inner automorphisms

Lemma 3.1. *Let A be a unital separable C^*-algebra and $\alpha : A \to A$ be an approximate inner automorphism. Suppose that $\{p_j\}$ is a central sequence of projections. Then there exists a central sequence of partial isometries $\{w_j\}$ such that $w_j^* w_j = p_j$ and $w_j w_j^* = \alpha(p_j)$, $j = 1, 2, \ldots$.*

Proof. Fix a finite subset $\mathcal{F} \subset A$ which is in the unit ball of A. Let $\varepsilon > 0$. Choose a unitary $v \in U(A)$ such that

$$\|\alpha(a) - v^*av\| < \varepsilon/8 \quad \text{for all } a \in \mathcal{F}.$$

Since α is an automorphism, $\alpha(p_j)$ is also a central sequence of projections. Choose a sufficiently large j so that

$$\|p_ja - ap_j\| < \varepsilon/8 \quad \text{for all } a \in \mathcal{F} \quad \text{and} \quad \|\alpha(p_j)v - v\alpha(p_j)\| < \varepsilon/8.$$

Since α is approximately inner, we obtain another unitary $z \in U(A)$ such that

$$\|z^*p_jz - \alpha(p_j)\| < \varepsilon/8 \quad \text{and} \quad \|z^*az - \alpha(a)\| < \varepsilon/8 \quad \text{for all } a \in \mathcal{F}.$$

It follows that

$$\|(vz^*)p_j(zv^*) - \alpha(p_j)\| \leqslant \|vz^*p_jzv^* - v\alpha(p_j)v^*\| + \|v\alpha(p_j)v^* - \alpha(p_j)\| < \varepsilon/4.$$

Let $x_j = vz^*p_j$. Then $x_j^*x_j = p_j$ and

$$\|x_jx_j^* - \alpha(p_j)\| < \varepsilon/4.$$

From the above we also have

$$\|vz^*azv^* - a\| < \varepsilon/4 \quad \text{and} \quad \|vz^*a - avz^*\| < \varepsilon/4 \quad \text{for all } a \in \mathcal{F}.$$

On the other hand, for any $a \in \mathcal{F}$,

$$\|x_ja - ax_j\| \leqslant \|vz^*p_ja - vz^*ap_j\| + \|vz^*ap_j - avz^*p_j\| < \varepsilon/8 + \varepsilon/4 = 3\varepsilon/8.$$

There is a unitary $u \in U(A)$ such that $\|u - 1\| < \varepsilon/4$ such that

$$u(x_jx_j^*)u^* = \alpha(p_j).$$

Define $w_j = ux_j$. Then $w_jw_j^* = p_j$ and $w_jw_j^* = \alpha(p_j)$. Moreover we have that

$$\|w_ja - aw_j\| < \varepsilon \quad \text{for all } a \in \mathcal{F}.$$

Since $\mathcal{F}$ is arbitrary, the lemma follows. □

Lemma 3.2. *Let A be a unital separable C^*-algebra and $\alpha \in \mathrm{Aut}(A)$ for which α^r is approximately inner for some integer $r \geqslant 1$. Let $m \in \mathbb{N}$, $m_0 \geqslant m$ be the smallest integer such that $m_0 = 0 \bmod r$ and $l = m + (r-1)(m_0+1)$.*

Suppose that $\{e_i^{(n)}\}$, $i = 0, 1, \ldots, l$, $n = 1, 2, \ldots$, are $l+1$ sequences of projections in A satisfying the following:

$$\|\alpha(e_i^{(n)}) - e_{i+1}^{(n)}\| < \delta_n, \quad \lim_{n\to\infty} \delta_n = 0,$$

$$e_i^{(n)}e_j^{(n)} = 0 \quad \text{if } i \neq j, \quad e_i^{(n)} \sim e_j^{(n)} \quad \text{in } A$$

and for each i, $\{e_i^{(n)}\}$ *is a central sequence.*

Then for each $i = 0, 1, 2, \ldots, m$, *there is a central sequence of partial isometries* $\{w_i^{(n)}\}$ *such that*

$$(w_i^{(n)})^* w_i^{(n)} = p_i^{(n)} \quad \text{and} \quad w_i^{(n)}(w_i^{(n)})^* = p_{i+1}^{(n)}, \quad i = 0, 1, \ldots, m-1,$$

where $p_i^{(n)} = \sum_{j=0}^{r-1} e_{i+j(m_0+1)}^{(n)}$. *Moreover, for each* i,

$$\lim_{n\to\infty} ||\alpha(p_i^{(n)}) - p_{i+1}^{(n)}|| = 0.$$

Proof. Since α^r is approximately inner, by applying Lemma 3.1, for each $i, j = 0, 1, \ldots, l$, one obtains central sequences of partial isometries $\{z(i,j,n)\}$ such that

$$z(i,j,n)^* z(i,j,n) = e_i^{(n)} \quad \text{and} \quad z(i,j,n)z(i,j,n)^* = \alpha^{rj}(e_i^{(n)}).$$

Note that

$$||\alpha^{rj}(e_i^{(n)}) - e_{i+rj}^{(n)}|| < rj\delta_n.$$

There is a unitary $u(i,j,n) \in U(A)$, for each i and j, such that

$$||u(i,j,n) - 1|| < 2(rl)\delta_n \quad \text{and} \quad u(i,j,n)^* \alpha^{rj}(e_i^{(n)}) u(i,j,n) = e_{i+rj}^{(n)}.$$

Since $\lim_{n\to\infty} \delta_n = 0$, for each i and j, $\{u(i,j,n)\}$ is central. Therefore, to simplify notation, we may assume that

$$z(i,j,n)^* z(i,j,n) = e_i^{(n)} \quad \text{and} \quad z(i,j,n)z(i,j,n)^* = e_{i+rj}^{(n)}.$$

Define

$$p_i^{(n)} = \sum_{j=0}^{r-1} e_{i+j(m_0+1)}^{(n)}.$$

Then for each i, one checks easily that there are central sequences of partial isometries $\{w(i,n)\}$ such that

$$w(i,n)^* w(i,n) = p_i^{(n)} \quad \text{and} \quad w(i,n)w(i,n)^* = p_{i+1}^{(n)}, \quad i = 0, 1, \ldots, m-1.$$

For example, (with $m_0 = kr$), one defines

$$w(0,n) = z(1,k,n)^* + z(1+(m_0+1),k,n)^* + \cdots + z(1+(r-2)(m_0+1),k,n)^* + z(0,(r-1)k+1,n).$$

Then (with $m_0 = kr$)

$$\begin{aligned} & w(0,n)w(0,n)^* \\ &= z(1,k,n)^* z(1,k,n) + z(1+(m_0+1),k,n)^* z(1+(m_0+1),k,n) + \cdots \\ &\quad + z(1+(r-2)(m_0+1),k,n)^* z(1+(r-2)(m_0+1),k,n) \\ &\quad + z(0,(r-1)k+1,n) z(0,(r-1)k+1,n)^* \\ &= e_1^{(n)} + e_{1+(m_0+1)}^{(n)} + \cdots + e_{1+(r-2)(m_0+1)}^{(n)} + e_{1+(r-1)(m_0+1)}^{(n)} \\ &= p_1^{(n)} \end{aligned}$$

(note that $e^{(n)}_{((r-1)k+1)r} = e^{(n)}_{(r-1)kr+r} = e^{(n)}_{(r-1)m_0+r} = e^{(n)}_{1+(r-1)(m_0+1)}$) and

$$\begin{aligned} & w(0,n)^* w(0,n) \\ &= z(1,k,n) z(1,k,n)^* + z(1+(m_0+1),k,n) z(1+(m_0+1),k,n)^* + \cdots \\ &\quad + z(1+(r-2)(m_0+1),k,n) z(1+(r-2)(m_0+1),k,n)^* \\ &\quad + z(0,(r-1)k+1,n)^* z(0,(r-1)k+1,n) \\ &= e_{1+kr}^{(n)} + e_{1+(m_0+1)+kr}^{(n)} + \cdots + e_{1+(r-2)(m_0+1)+kr}^{(n)} + e_0^{(n)} \\ &= e_{m_0+1}^{(n)} + e_{2(m_0+1)}^{(n)} + \cdots + e_{(r-1)(m_0+1)}^{(n)} + e_0^{(n)} \\ &= p_0^{(n)}. \end{aligned}$$

Since, for each i and j, $\{z(i,j,n)\}$ is central, so is $\{w(i,n)\}$.
From the construction we know that for each i

$$\begin{aligned} \|\alpha(p_i^{(n)}) - p_{i+1}^{(n)}\| &\leqslant \sum_{j=0}^{r-1} \|\alpha(e^{(n)}_{i+j(m_0+1)}) - e^{(n)}_{i+1+j(m_0+1)}\| \\ &\leqslant r\delta_n \to 0 \ (n \to \infty). \qquad \square \end{aligned}$$

Let $\{E_{i,j}\}$ be a system of matrix units and $\mathcal{K}$ be the compact operators on $\ell^2(\mathbb{Z})$ where we identify $E_{i,i}$ with the one-dimensional projection onto the functions supported by $\{i\} \subset \mathbb{Z}$. Let S be the canonical shift operator on $\ell^2(\mathbb{Z})$. Define an automorphism σ of $\mathcal{K}$ by $\sigma(x) = SxS^*$ for all $x \in \mathcal{K}$. Then $\sigma(E_{i,j}) = E_{i+1,j+1}$. For any $N \in \mathbb{N}$ let $P_N = \sum_{i=0}^{N-1} E_{i,i}$.

Lemma 3.3 (Kishimoto [12, 2.1]). *For any* $\eta > 0$ *and* $n \in \mathbb{N}$ *there exist* $N \in \mathbb{N}$ *and projections* $e_0, e_1, \ldots, e_{n-1}$ *in* $\mathcal{K}$ *such that*

$$\sum_{i=0}^{n-1} e_i \leqslant P_N,$$

$$||\sigma(e_i) - e_{i+1}|| < \eta, \quad i = 0, \ldots, n-1, \quad e_n = e_0,$$

$$\frac{n \dim e_0}{N} > 1 - \eta.$$

Theorem 3.4. *Let* A *be a unital separable simple* C^**-algebra with* $\mathrm{TR}(A) \leqslant 1$ *and* $\alpha^r \in \mathrm{Aut}(A)$ *be an approximately inner automorphism for some integer* $r \geqslant 1$*. Suppose that* α *has the tracial Rokhlin property then* α *has the tracial cyclic Rokhlin property.*

Proof. Let $\varepsilon > 0$. Let $\varepsilon/2 > \eta > 0$ and $m \in \mathbb{N}$ be given. Choose N which satisfies the conclusion of Lemma 3.3 (with this η and $n = m$). Identify $P_N \mathcal{K} P_N$ with M_N. Let $\mathcal{G} = \{E_{i+1,i} : i = 0, 1, \ldots, N-1\}$ be a set of generators of M_N. Let $e_0, e_1, \ldots, e_{m-1}$ be as in the conclusion of Lemma 3.3.

For any $\varepsilon > 0$, there is $\delta > 0$ depends only on N such that, if

$$||ag - ga|| < \delta$$

for $g \in \mathcal{G}$, then

$$||ae_i - e_i a|| < \varepsilon/2, \quad i = 0, 1, \ldots, n.$$

We assume that $\delta < \eta$. Fix a finite subset $\mathcal{F}_0 \subset A$.

Choose $m_0 \in \mathbb{N}$ such that $m_0 \geqslant m$ is the smallest integer with $m_0 = 0 \bmod r$. Let $L = N + (r-1)(m_0 + 1)$.

Since α has the tracial Rokhlin property, there exists a sequence of projections $\{e_i^{(k)} : i = 0, 1, \ldots, L\}$ satisfying the following:

$$|\alpha(e_i^{(k)}) - e_{i+1}^{(k)}|| < \delta/4rN, \quad 0 \leqslant i \leqslant L-1, \quad e_i^{(k)} e_j^{(k)} = 0, \quad \text{if } i \neq j,$$

$$\lim_{k \to \infty} ||e_i^{(k)} a - a e_i^{(k)}|| = 0 \quad \text{for all } a \in A, \ i = 0, 1, \ldots, L$$

and

$$\tau\left(1-\sum_{i=0}^{L-1} e_i^{(k)}\right)<\eta \quad \text{for all } \tau \in \mathrm{T}(A),\ k=1,2,\ldots .$$

By applying Lemma 3.2, we obtain a central sequence $\{w_i^{(k)}\}$ in A such that

$$(w_i^{(k)})^* w_i^{(k)} = P_0^{(k)}$$

and

$$w_i^{(k)}(w_i^{(k)})^* = P_i^{(k)}, \quad k=0,1,\ldots,\ i=0,1,\ldots,N,$$

$$P_i^{(k)} P_j^{(k)} = 0, \quad i \neq j,$$

$$\|\alpha(P_i^{(k)}) - P_{i+1}^{(k)}\| < \frac{\delta}{4L}, \quad k=0,1,\ldots,\ i=0,1,\ldots,N-1,$$

$$\tau\left(1-\sum_{i=0}^{N-1} P_i^{(k)}\right)<\eta \quad \text{for all } \tau \in \mathrm{T}(A)$$

where $P_i^{(k)} = \sum_{j=0}^{r-1} e_{i+(m_0+1)j}^{(k)}$ for $i=0,1,\ldots,N$.

It follows that for each i, $\{\alpha^l(w_i^{(k)})\}$, $l=0,1,\ldots,N$ are all central sequences. As the same argument in Lemma 3.2 there is a unitary $u_k \in U(A)$ with $\|u_k - 1\| < \delta/2N$ such that $\mathrm{ad}\, u_k \circ \alpha(P_i^{(k)}) = P_{i+1}^{(k)}$, $i=0,1,\ldots,N-1$. Put $\beta_k = \mathrm{ad}\, u_k \circ \alpha$, and $w^{(k)} = w_0^{(k)}$. Choose a large k, such that

$$\|\beta_k^l(w^{(k)})a - a\beta_k^l(w^{(k)})\| < \delta \quad \text{for all } a \in \mathcal{F}_0,$$

$l=0,1,\ldots,N$.

Now let C_1 and C_2 be the C^*-algebras generated by $w^{(k)}, \beta_k^1(w^{(k)}), \ldots, \beta_k^{N-1}(w^{(k)})$ and by $w^{(k)}, \beta_k^1(w^{(k)}), \ldots, \beta_k^N(w^{(k)})$, respectively. Note that $C_1 \cong M_N$, $C_2 \cong M_{N+1}$. Define a homomorphism $\Phi : C_1 \to \mathcal{K}$ by

$$\Phi(\beta_k^i(w^{(k)})) = E_{i+1,i}, \quad i=0,1,\ldots,N-1$$

(see Lemma 3.3). Then one has $\sigma \circ \Phi|_{C_1} = \Phi \circ \beta_k|_{C_1}$ and $\Phi(C_1) = P_N \mathcal{K} P_N$. Now we apply Lemma 3.3 to obtain mutually orthogonal projections $e_0, e_1, \ldots, e_{m-1}$ in M_N such that

$$\|\sigma(e_i) - e_{i+1}\| < \eta, \quad 0 \leqslant i \leqslant m-1, \quad e_m = e_0,$$

Let $p_i = \Phi^{-1}(e_i)$, $i = 0, 1, \ldots, m-1$. One estimates that

$$\tau\left(\sum_{i=0}^{N-1} P_i^{(k)} - \sum_{i=0}^{m-1} p_i\right) < 1 - \sum_{i=0}^{m-1} \frac{\dim(e_0)}{N} = 1 - \frac{m \dim(e_0)}{N} < \eta < \frac{\varepsilon}{2}$$

for all $\tau \in \mathrm{T}(A)$. So one has mutually orthogonal projections $p_0, p_1, p_2, \ldots, p_{m-1}$ such that

$$||\beta_k(p_i) - p_{i+1}|| < \frac{\varepsilon}{2}, \quad i = 0, 1, 2, \ldots, m-1, \quad p_m = p_0.$$

By the choice of δ, one also has

$$||ap_i - p_i a|| < \varepsilon, \quad i = 0, 1, \ldots, m-1 \quad \text{for all } a \in \mathcal{F}_0$$

and

$$\tau\left(1 - \sum_{i=0}^{m-1} p_i\right) < \tau\left(1 - \sum_{i=0}^{N-1} P_i^{(k)}\right) + \frac{\varepsilon}{2} < \eta + \frac{\varepsilon}{2} < \varepsilon$$

for all $\tau \in \mathrm{T}(A)$. Since

$$||\beta_k - \alpha|| < \delta/2 < \varepsilon/2$$

one finally has

$$||\alpha(p_i) - p_{i+1}|| < \varepsilon, \; i = 0, 1, \ldots, m-1, \quad p_m = p_0.$$

In other words, α has the tracial cyclic Rokhlin property. $\square$

Theorem 3.5. *Let A be a unital separable simple C^*-algebra with* $\mathrm{TR}(A) = 0$ *which has a unique tracial state and satisfies the Universal Coefficient Theorem. Suppose that $\alpha^r \in \mathrm{Aut}(A)$ is approximately inner for some integer $r \geqslant 1$ and that α^m is uniformly outer for any integer $m \neq 0$. Then $A \rtimes_\alpha \mathbb{Z}$ is a simple AH-algebras with no dimension growth with real rank zero.*

Proof. Note that since α is outer, $A \rtimes_\alpha \mathbb{Z}$ is simple by Kishimoto [11]. From Theorem 2.7 α has the tracial Rokhlin property. Since $\alpha^r \in \mathrm{Aut}(A)$ is approximately inner for some integer $r \geqslant 1$, this implies that α has the tracial cyclic Rokhlin property by Theorem 3.4. So from Theorem 2.9 $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$. Using the classification theorem of Lin [23] we conclude that $A \rtimes_\alpha \mathbb{Z}$ is a simple AH-algebra with no dimension growth with real rank zero. $\square$

The following shows that the Kishimoto's conjecture that we mentioned in the introduction is true at least for the case that the simple $A\mathbb{T}$-algebra has a unique tracial state. In Corollary 3.7, we show that if one agrees that the "sufficiently outer" means the automorphism has the tracially Rokhlin property then we do not need to assume that A has a unique tracial state.

Corollary 3.6. *Let A be a unital simple $A\mathbb{T}$-algebra with a unique trace and real rank zero, and let $\alpha \in \mathrm{Aut}(A)$ such that α is approximately inner. If α^m is uniformly outer for any integer $m \neq 0$, or α has the tracial Rokhlin property, then $A \rtimes_\alpha \mathbb{Z}$ is a unital simple $A\mathbb{T}$-algebra of real rank zero.*

Proof. From the following Pimsner–Voiculescu exact sequence [27],

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\mathrm{id}-\alpha_*^{-1}} & K_0(A) & \xrightarrow{\imath_*} & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \xleftarrow[\imath_*]{} & K_1(A) & \xleftarrow[\mathrm{id}-\alpha_*^{-1}]{} & K_1(A). \end{array}$$

We see that $K_0(A \rtimes_\alpha \mathbb{Z})$ and $K_1(A \times_\alpha \mathbb{Z})$ are torsion free. From Theorems 2.7, 3.4, and 2.9 we know that $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$ and $A \rtimes_\alpha \mathbb{Z}$ satisfies the UCT. Therefore $K_0(A \rtimes_\alpha \mathbb{Z})$ is a weakly unperforated Riesz group. It follows from [5] that there is a unital simple $A\mathbb{T}$-algebra B with real rank zero which has the same ordered scaled K-theory of $A \rtimes_\alpha \mathbb{Z}$. It follows from Theorem 5.1 of Lin [23] that $A \cong B$. □

Corollary 3.7. *Let A be a unital simple $A\mathbb{T}$-algebra (with real rank zero) and $\alpha \in \mathrm{Aut}(A)$. Suppose that α is approximately inner and α has the tracial Rokhlin property. Then $A \rtimes_\alpha \mathbb{Z}$ is a unital simple $A\mathbb{T}$-algebra (with real rank zero.)*

Proof. It follows from Theorem 3.4 that α actually has the tracial cyclic Rokhlin property. Then, by Theorem 2.9, $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) \leqslant 1$. As in the proof of Corollary 3.6, $A \rtimes_\alpha \mathbb{Z}$ has torsion free K-theory. We then apply the classification theorem in [23] (for real rank zero case) or apply [24] (for real rank one case) to conclude that $A \rtimes_\alpha \mathbb{Z}$ is a unital simple AT-algebra (and with real rank zero). □

Remark 3.8. Kishimoto in [12,14,15] proved that if A is a simple unital $A\mathbb{T}$-algebra of real rank zero with a unique trace, and $\alpha \in \mathrm{Aut}(A)$ is an approximately inner with the Rokhlin property, then $A \rtimes_\alpha \mathbb{Z}$ is also a simple unital $A\mathbb{T}$-algebra under the assumption that both $K_0(A)$ and $K_1(A)$ are finitely generated with $K_1(A) \neq \mathbb{Z}$ and $\alpha \in \mathrm{HInn}(A)$. Corollary 3.6 shows that the extra conditions of $K_*(A)$ and $\alpha \in \mathrm{HInn}(A)$ are not necessary. Corollary 3.7 shows that Kishimoto's conjecture holds in general (without assuming that A has the unique tracial state) if the "sufficiently outer" is replaced by the tracial Rokhlin property. One should note that the tracial Rokhlin property is weaker than the Rokhlin property used in Kishimoto's work. (See Remark 2.5(iii).) Moreover, tracially cyclic Rokhlin property is related to "approximate Rokhlin" property in 4.2 of Kishimoto [12] which is also weaker than the Rokhlin property used in Kishimoto's work. If one allows the "sufficiently outer" replaced by tracially cyclic Rokhlin property, then $A \rtimes_\alpha \mathbb{Z}$ is always a unital simple $A\mathbb{T}$-algebra without even assuming that α^r is approximately inner but assuming $A \rtimes_\alpha \mathbb{Z}$ has torsion free K-theory.

Theorem 3.9. *Let A be a unital separable simple C^*-algebra with $\mathrm{TR}(A) = 0$ or $\mathrm{TR}(A) = 1$ and $\alpha \in \mathrm{Aut}(A)$ such that α^r is approximately inner for some integer $r > 0$. Suppose that α has tracial Rokhlin property. Then $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$, or $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 1$. Furthermore, if, in addition, A satisfies the Universal Coefficient Theorem, then $A \rtimes_\alpha \mathbb{Z}$ is a simple AH-algebra with no dimension growth.*

Proof. The first part follows from Theorem 3.4 and Theorem 2.9. For the last part, by Lin [24], A is a simple AH-algebra with no dimension growth. By the first part, $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) \leqslant 1$, it follows from [24] again that $A \rtimes_\alpha \mathbb{Z}$ is also a simple AH-algebra with no dimension growth. $\square$

Remark 3.10. In Theorem 3.4, we assume that $\mathrm{TR}(A) \leqslant 1$. In fact, we only need to assume that A has the property (SP) and has the Fundamental Comparison Property. Suppose that A is a unital separable simple C^*-algebra with $\mathrm{TR}(A) = 0$ and with a unique tracial state. Suppose also that $A \rtimes_\alpha \mathbb{Z}$ has a unique tracial state (unique ergodic). Then by applying Theorems 3.4, 2.9 and 2.7, $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$. On the other hand, in Corollaries 3.6 and 3.7, if we assume only that α^r is approximate inner (for $r > 1$) and α has the tracial Rokhlin property, then $A \rtimes_\alpha \mathbb{Z}$ may not be an $A\mathbb{T}$-algebra. This is because $A \rtimes_\alpha \mathbb{Z}$ may have torsion. However, it is a unital AH-algebra with no dimension growth by Theorem 3.5. But, in Corollaries 3.6 and 3.7, if we assume that α^r is approximate inner for some integer r and $A \rtimes_\alpha \mathbb{Z}$ has torsion free K-theory, then conclusion of both Corollaries 3.6 and 3.7 hold. To allow torsion, related to the Kishimoto's conjecture, we proved (in Theorem 3.9) the following: If A is a unital simple AH-algebra with no dimension growth (with real rank zero) and $\alpha \in \mathrm{Aut}(A)$ has the tracial Rokhlin property and α^r is approximate inner for some integer $r > 0$, then $A \rtimes_\alpha \mathbb{Z}$ is again a unital simple AH-algebra with no dimension growth (and with real rank zero).

4. Examples

Let G and F be abelian groups. Recall that $P\mathrm{ext}(G, F)$ is the subgroup of those extensions

$$0 \to F \to E \to G \to 0$$

so that each finitely generated subgroup of G lifts. If A is a separable C^*-algebra which satisfies the Universal Coefficient Theorem, then, for any σ-unital C^*-algebra B, $KL(A, B) = KK(A, B)/P\mathrm{ext}(K_*(A), K_{*-1}(B))$.

Lemma 4.1. *Let A be a separable amenable C^*-algebra satisfying the UCT. Suppose that $\alpha \in \mathrm{Aut}(A)$ such that $(\alpha)_{*i} = \mathrm{id}_{K_i(A)}$, $i = 0, 1$. Suppose that* $\mathrm{ext}_{\mathbb{Z}}(K_{i-1}(A), K_i(A))/$

$P\,\mathrm{ext}(K_{i-1}(A), K_i(A))$ is finite. Then there are integers $r>0$ and $k>0$ such that

$$[\alpha^{r+k}] = [\alpha^k] \quad \textit{in } KL(A, A).$$

Proof. Consider $[\alpha^m] - [\alpha]$, for $m = 1, 2, \ldots$. Since $(\alpha)_{*i} = \mathrm{id}_{K_i(A)}$, $i = 0, 1$, by the Universal Coefficient Theorem [31], one computes that

$$[\alpha^m] - [\alpha] \in \mathrm{ext}_{\mathbb{Z}}(K_{*+1}(A), K_*(A)).$$

Since $\mathrm{ext}_{\mathbb{Z}}(K_{i-1}(A), K_i(A))/P\,\mathrm{ext}(K_{i-1}(A), K_i(A))$ is finite, there are positive integers r and k such that

$$([\alpha^{r+k}] - [\alpha]) = ([\alpha^k] - [\alpha]) \quad \text{in } KL(A, A).$$

It follows that

$$[\alpha^{r+k}] = [\alpha^k] \quad \text{in } KL(A, A). \qquad \square$$

Theorem 4.2. *Let A be a unital separable simple C^*-algebra with $\mathrm{TR}(A) = 0$ satisfying the UCT. Suppose that $\alpha \in \mathrm{Aut}(A)$. In any of the following cases, α^r is approximately inner for some integer $r>0$. Consequently, if α^m is uniformly outer for all $m \in \mathbb{Z}\backslash\{0\}$ (or α has tracial Rokhlin property), α has tracial cyclic Rokhlin property and $\mathrm{TR}(A \rtimes_\alpha \mathbb{Z}) = 0$. In particular, $A \rtimes_\alpha \mathbb{Z}$ is a simple AH-algebra with no dimension growth and real rank zero.*

(1) *$K_0(A) = D$, where D is a countable dense subgroup of $\mathbb{R}$ and $K_1(A) = \mathbb{Z}$, or $K_1(A) = \{0\}$;*
(2) *$K_0(A) = D$, where D is a finitely generated countable dense subgroup of $\mathbb{R}$ and $K_1(A) = \mathbb{Z}$ or $K_1(A)$ is finite;*
(3) *$K_0(A) = D \oplus G$, with*

$$K_0(A)_+ = \{(r, x) \mid r \in D, r>0, x \in G\} \cup \{(0, 0)\}$$

and D is a dense subgroup of $\mathbb{R}$ such that for any non-zero element $d \in D$ and any integer $n \geqslant 1$, there is $e \in D$ such that $me = d$ for some $m \geqslant n$, where $G = \mathbb{Z}$ or G is finite and $K_1(A) = \mathbb{Z}$, or $K_1(A) = \{0\}$;
(4) *$K_0(A) = \mathbb{Q} \oplus G$, where $G = \mathbb{Z}$ or G is finite and $K_1(A) = \mathbb{Z}$, or $K_1(A)$ is a finite group.*

Proof. In all cases, it suffices to show that α^r is approximately inner for some integer $r \geqslant 1$.

For (1), it is clear that $\alpha_{*0} = \mathrm{id}_{K_0(A)}$. If $K_1(A) = \mathbb{Z}$, since α_{*1} is an isomorphism, $\alpha_{*1}(1) = \pm 1$. Therefore $\alpha_{*i}^2 = \mathrm{id}_{K_i(A)}$. Since $K_i(A)$ are torsion free, $[\alpha^2] = [\mathrm{id}_A]$ in $KL(A, A)$. It follows from Theorem 2.4 of Lin [21] that α^2 is approximately inner.

For (2), as in (1), $\alpha_{*0} = \mathrm{id}_{K_0(A)}$. Also if $K_1(A) = \mathbb{Z}$, then $\alpha_{*1}^2 = \mathrm{id}_{K_1(A)}$. If $K_1(A)$ is finite, since α_{*1} is an isomorphism, there exists $r_1 \geqslant 1$ such that $\alpha_{*1}^{r_1} = \mathrm{id}_{K_1(A)}$. Let $\beta = \alpha^{2r_1}$. Then $\beta_{*i} = \mathrm{id}_{K_i(A)}$, $i = 0, 1$. However, in this case,

$$\mathrm{ext}_{\mathbb{Z}}(D, K_1(A)) = \{0\} \quad \text{and} \quad \mathrm{ext}_{\mathbb{Z}}(K_1(A), K_0(A)) \text{ is finite.}$$

It follows from Lemma 4.1 that $[\beta^{m+k}] = [\beta^k]$ in $KL(A, A)$ for some integer $m, k \geqslant 1$. By Theorem 2.3 of Lin [21], there exists a sequence of unitaries such that

$$\lim_{n \to \infty} \mathrm{ad}\, u_n \circ \beta^k(a) = \beta^{m+k}(a) \quad \text{for all } a \in A.$$

Since β^k is an automorphism, it follows that β^m is approximately inner, or $\alpha^{m(2r_1)}$ is approximately inner.

For (3), as above, one has that $\alpha_{*1}^2 = \mathrm{id}_{K_1(A)}$. The assumption on D implies that there is no non-zero homomorphism from D to $\mathbb{Z}$ or a finite group. One then checks that there is an integer $r_1 \geqslant 1$ such that $\alpha_{*0}^{r_1} = \mathrm{id}_{K_0(A)}$. Put $\beta = \alpha^{2r_1}$. Then $\beta_{*i} = \mathrm{id}_{K_i(A)}$, $i = 0, 1$. To see that β^m is approximately inner, we note that $\mathrm{ext}_{\mathbb{Z}}(D, K_1(A)) = P\,\mathrm{ext}(D, K_1(A))$ since D is torsion free. One then computes that

$$\mathrm{ext}_{\mathbb{Z}}(K_0(A), K_1(A))/P\,\mathrm{ext}(K_0(A), K_1(A)) = \mathrm{ext}_{\mathbb{Z}}(G, K_1(A))/P\,\mathrm{ext}(K_0(A), K_1(A))$$

which is finite, and $\mathrm{ext}_{\mathbb{Z}}(K_1(A), K_0(A)) = \{0\}$. Thus one can apply the same argument as in case (2) by applying Lemma 4.1.

For (4), as in cases (2) and (3), there is $r_1 \geqslant 1$ such that $\alpha_{*i}^{r_1} = \mathrm{id}_{K_i(A)}$, $i = 0, 1$. Moreover, since $\mathbb{Q}$ is divisible,

$$\mathrm{ext}_{\mathbb{Z}}(K_1(A), \mathbb{Q}) = \{0\}.$$

Because $K_1(A) = \mathbb{Z}$, or $K_1(A)$ is finite,

$$\mathrm{ext}_{\mathbb{Z}}(K_1(A), K_0(A)) = \mathrm{ext}_{\mathbb{Z}}(K_1(A), G) \text{ is also finite.}$$

Since $\mathbb{Q}$ is torsion free, $\mathrm{ext}_{\mathbb{Z}}(\mathbb{Q}, K_1(A)) = P\,\mathrm{ext}(\mathbb{Q}, K_1(A))$. One then computes that

$$\mathrm{ext}_{\mathbb{Z}}(K_0(A), K_1(A))/P\,\mathrm{ext}(K_0(A), K_1(A)) = \mathrm{ext}_{\mathbb{Z}}(G, K_1(A))/P\,\mathrm{ext}(K_0(A), K_1(A))$$

which is finite. Thus the argument in case (3) applies. □

Proposition 4.3. *Let A be a simple unital C^*-algebra with* $\mathrm{TR}(A) = 0$, *and B be a simple unital C^*-algebra with* $\mathrm{TR}(B) \leqslant 1$. *Suppose that $\alpha \in \mathrm{Aut}(A)$ has the tracial cyclic Rokhlin property. Then for any $\beta \in \mathrm{Aut}(B)$ $\alpha \otimes \beta \in \mathrm{Aut}(A \otimes_{\min} B)$ has the tracial cyclic Rokhlin property.*

Proof. It follows from [18] $\mathrm{TR}(A \otimes_{\min} B) \leqslant 1$. Hence $A \otimes_{\min} B$ has SP-property, stable rank one and the Fundamental Comparison Property [20, Proposition 6.2, Theorems 6.9, 6.11].

Let $F \subset A \otimes_{\min} B$ be a finite set, $n \in \mathbb{N}$, and $\varepsilon > 0$. Without loss of generality, we may assume that there exist a finite set $F_A \subset A$ and $F_B \subset B$ such that $F = F_A \otimes F_B$.

Since α has the tracial cyclic Rokhlin property, there exist mutually orthogonal projections $e_0, e_1, \ldots, e_n \in A$ such that

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leqslant j \leqslant n$, where $e_{n+1} = e_0$.
(2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leqslant j \leqslant n-1$ and all $a \in F_A$.
(3) $\tau\left(1 - \sum_{j=0}^{n} e_j\right) < \varepsilon$ for all tracial states τ on A.

(See Remark 2.5(ii).)

Set $f_i = e_i \otimes 1_B$ for $0 \leqslant i \leqslant n$. Then f_i are mutually orthogonal projections in $A \otimes_{\min} B$ such that

(1) $\|(\alpha \otimes \beta)(f_j) - f_{j+1}\| < \varepsilon$ for $0 \leqslant j \leqslant n$, where $f_{n+1} = f_0$.
(2) $\|f_j a - a f_j\| < \varepsilon$ for $0 \leqslant j \leqslant n-1$ and all $a \in F$.
(3) $\tau\left(1 - \sum_{j=0}^{n} f_j\right) < \varepsilon$ for all tracial states τ on $A \otimes_{\min} B$.

This means that $\alpha \otimes \beta$ has the tracial cyclic Rokhlin property by Remark 2.5(ii). □

Corollary 4.4. *Let A be a separable simple amenable unital C^*-algebra with* $\mathrm{TR}(A) = 0$ *which satisfies the UCT, and B be a simple amenable unital C^*-algebra with* $\mathrm{TR}(B) \leqslant 1$. *Suppose also that A has a unique tracial state and $\alpha \in \mathrm{Aut}(A)$ such that α^m is uniformly outer for all $m \neq 0$ and α^r is approximately inner for some integer $r \geqslant 1$. Then for any $\beta \in \mathrm{Aut}(B)$, $\alpha \otimes \beta$ has the tracial cyclic Rokhlin property and* $\mathrm{TR}(D) \leqslant 1$, *where $D = (A \otimes B) \rtimes_{\alpha \otimes \beta} \mathbb{Z}$.*

An unexpected consequence of the above corollary is that it provides a new way to construct unital simple C^*-algebras with tracial topological rank one. All previous examples are inductive limit construction (see [8]). Since there is basically no restriction on B and β, a great number of those simple C^*-algebras D with $\mathrm{TR}(D) = 1$ can be obtained from Corollary 4.4. Since $\mathrm{TR}(B) = 1$, one certainly expects that most such D has $\mathrm{TR}(D) = 1$ but not $\mathrm{TR}(D) = 0$. To convince the reader that it is likely the case, we compute the tracial rank in a very special case below. From its construction, it should be clear how other example can be constructed.

Denote $\mathrm{Aff}(A)$ the space of all affine continuous functions on $\mathrm{T}(A)$. Given a projection $p \in M_n(A)$ for some integer $n \geqslant 1$ we define $\rho_A(p)(\tau) = (\tau \otimes \mathrm{Tr})(p)$ for all $\tau \in \mathrm{T}(A)$, where Tr is the standard trace on $M_n(\mathbb{C})$. Then $\rho_A(p) \in \mathrm{Aff}(\mathrm{T}(A))$.

Example 4.5. Let A be a unital UHF-algebra with $K_0(A) = \mathbb{Q}$ and α be in $\mathrm{Aut}(A)$ so that α^m is uniformly outer for all $m \neq 0$ (or α is uniquely ergodic). Then α has the tracial cyclic Rokhlin property by Kishimoto [12, Lemma 4.3] and Theorem 3.5. Let B be a unital simple $A\mathbb{T}$-algebra for which $K_0(B) = \mathbb{Q}$ and $K_1(B) = \mathbb{Z} \oplus \mathbb{Z}$ and $\mathrm{Aff}(\mathrm{T}(B)) = C_{\mathbb{R}}([0,1])$. Existence of such simple $A\mathbb{T}$-algebra was given by

Thomsen [32]. It follows from Section 9 of Thomsen [32] that there is $\beta \in \mathrm{Aut}(B)$ such that $\beta_{*1}(x,y) = (-x,y)$ for $(x,y) \in \mathbb{Z} \oplus \mathbb{Z}$, $\beta_{*0} = \mathrm{id}_{K_0(B)}$ and $\tau \circ \beta(b) = \tau(b)$ for all $b \in B$ and $\tau \in \mathrm{T}(B)$. It should be noted that $\tau \circ \alpha(a) = \tau(a)$ for all $a \in A$ and $\tau \in \mathrm{T}(A)$. Put $\gamma = \alpha \otimes \beta$ and $C = A \otimes B$ and $D = C \rtimes_\gamma \mathbb{Z}$.

By the Kunneth formula one computes that C is a unital simple ($A\mathbb{T}$-algebra) with $K_0(C) = \mathbb{Q}$ and $K_1(C) = \mathbb{Q} \oplus \mathbb{Q}$. One computes that $\gamma_{*0} = \mathrm{id}_{K_0(C)}$ and $\gamma_{*1}((x,y)) = (-x,y)$ for $(x,y) \in \mathbb{Q} \oplus \mathbb{Q}$.

It follows from Proposition 4.3 that γ has the tracial cyclic Rokhlin property. Moreover $\mathrm{TR}(D) \leqslant 1$. To check that $\mathrm{TR}(D) = 1$, we first compute that, by Pimsner–Voiculescu's exact sequence and by the divisibility of $\mathbb{Q}$, we have $K_0(D) = \mathbb{Q} \oplus \mathbb{Q}$ and $K_1(D) = \mathbb{Q} \oplus \mathbb{Q}$. Consider tracial states with the form $t \otimes \tau$, where $t \in \mathrm{T}(A)$ and $\tau \in \mathrm{T}(B)$. Note all these tracial states are γ invariant. Thus they give tracial states on D. Note that $\mathrm{T}(A)$ is a single point. Thus we may identify $\mathrm{T}(B)$ with $\mathrm{T}(A) \otimes \mathrm{T}(B)$. Hence $\mathrm{Aff}(\mathrm{T}(A) \otimes \mathrm{T}(B)) = C_{\mathbb{R}}([0,1])$. Let $e_1 = \rho_D((1,0))$ and $e_2 = \rho_D((0,1))$. Then

$$\rho_D(K_0(D)) = \{xe_1 + ye_2 : , x, y \in \mathbb{Q}\}.$$

We view $\mathrm{T}(B) \subset \mathrm{T}(D)$. Thus one has a surjective affine homomorphism $\Lambda : \mathrm{Aff}(\mathrm{T}(D)) \to \mathrm{Aff}(\mathrm{T}(B))$. It is easy to see that $\Lambda \circ \rho_D(K_0(D))$ being rank two cannot be dense in $C_{\mathbb{R}}([0,1])$. It follows from [1, Theorem 6.9] that D has real rank other than zero (actually one). It follows from Theorem 7.1(c) of Lin [20] that $\mathrm{TR}(D) = 1$.

If one insists to get non-zero torsion in K-theory, one may start, for example, with $K_0(A) = \mathbb{Q}$ and $K_1(A) = \mathbb{Z}/p\mathbb{Z}$.

Acknowledgments

Most of this work was done when the first author was visiting Ritsmeikan University. He acknowledges the support of JSPS Grant for Scientific Research of Japan. During the work he was also supported by National Science Foundation of USA. The second author also acknowledges the support of JSPS Grant for Scientific Research of Japan. Both authors would like to thank Masaru Nagisa for a fruitful discussion.

References

[1] B. Blackadar, K-theory for operator algebras, Mathematical Sciences Research Institute Publications, Vol. 5, Springer, New York, 1986.
[2] A. Connes, Outer conjugacy class of automorphisms of factors, Ann. Sci. Ecole Norm. Sup. 8 (1975) 383–420.
[3] J. Cuntz, The structure of multiplication and addition in simple C^*-algebras, Math. Scand. 40 (1977) 215–233.
[4] G.A. Elliott, On the classification of C^*-algebras of real rank zero, J. Reine Angew. Math. 443 (1993) 179–219.
[5] G.A. Elliott, Dimension groups with torsion, Internat. J. Math. 1 (1990) 361–380.

[6] G.A. Elliott, G. Gong, On the classification of C^*-algebras of real rank zero, II, Ann. of Math. 144 (1996) 497–610.
[7] G.A. Elliott, G. Gong, L. Li, On the classification of simple inductive limit C^*-algebras, II: The isomorphism theorem, preprint.
[8] G. Gong, On the classification of simple inductive limit C^*-algebras, I: The reduction theorem, preprint.
[9] M. Izumi, The Rokhlin property for automorphisms of C^*-algebras, Mathematical Physics in Mathematics and Physics (Siena, 2000), Fields Inst. Commun., Vol. 30, American Mathematical Society, Providence, RI, 2001, pp. 191–206.
[10] J.A. Jeong, H. Osaka, Extremal rich C^*-crossed products and the cancellation property, J. Austral. Math. Soc. (Series A) 64 (1998) 285–301.
[11] A. Kishimoto, Outer automorphisms and reduced crossed products of simple C^*-algebra, Comm. Math. Phys. 81 (1981) 429–435.
[12] A. Kishimoto, The Rokhlin property for automorphisms of UHF algebras, J. Reine Angew. Math. 465 (1995) 183–196.
[13] A. Kishimoto, The Rokhlin property for shifts on UHF algebras and automorphisms of Cuntz algebras, J. Funct. Anal. 140 (1996) 100–123.
[14] A. Kishimoto, Automorphisms of $A\mathbb{T}$ algebras with the Rokhlin property, J. Operator Theory 40 (1998) 277–294.
[15] A. Kishimoto, Unbounded derivation in $A\mathbb{T}$-algebras, J. Funct. Anal. 160 (1998) 270–311.
[17] R. Herman, A. Ocneanu, Stability for integer actions on UHF C^*-algebras, J. Funct. Anal. 59 (1984) 132–144.
[18] S. Hu, H. Lin, Y. Xu, The tracial topological rank of C^*-algebras (II), Indiana Univ. Math. J., to appear.
[20] H. Lin, Tracial topological ranks of C^*-algebras, Proc. London Math. Soc. 83 (2001) 199–234.
[21] H. Lin, Classification of simple TAF C^*-algebras, Canad. J. Math. 53 (2001) 161–194.
[22] H. Lin, An Introduction to the Classification of Amenable C^*-algebras, World Scientific, New Jersey, London, Singapore, Hong Kong, Bangalore, 2001.
[23] H. Lin, Classification of simple C^*-algebras with tracial topological rank zero, Duke Math. J., to appear.
[24] H. Lin, Simple C^*-algebras with tracial topological rank one, arXiv.org math.OA/0401240.
[26] N.C. Phillips, Crossed products by finite cyclic group actions with the tracial Rokhlin property, arXiv.org math.OA/0306410.
[27] M. Pimsner, D. Voiculescu, Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras, J. Operator Theory 4 (1) (1980) 93–118.
[28] H. Osaka, N.C. Phillips, Furstenberg transformations on irrational rotation algebras, in preprint.
[30] M. Rørdam, Classification of certain infinite simple C^*-algebras, J. Funct. Anal. 131 (1995) 415–458.
[31] J. Rosenberg, C. Schochet, The Ku''nneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor, Duke Math. J. 55 (1987) 431–474.
[32] K. Thomsen, Limits of certain subhomogeneous C^*-algebras, Me'm. Soc. Math. Fr. (N.S.) 71 (1997) (1998)

Further reading

A. Kishimoto, Non-commutative shifts and crossed products, J. Funct. Anal. 200 (2003) 281–300.
H. Lin, Tracially AF C^*-algebras, Trans. Amer. Math. Soc. 353 (2001) 693–722.
Q. Lin, N.C. Phillips, C^*-algebras of minimal diffeomorphisms, preprint 2000.