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Characterization of rough set approximations in Atanassov intuitionistic fuzzy set theory

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ABSTRACT

The primitive notions in rough set theory are lower and upper approximation operators defined by a fixed binary relation and satisfying many interesting properties. Many types of generalized rough set models have been proposed in the literature. This paper discusses the rough approximations of Atanassov intuitionistic fuzzy sets in crisp and fuzzy approximation spaces in which both constructive and axiomatic approaches are used. In the constructive approach, concepts of rough intuitionistic fuzzy sets and intuitionistic fuzzy rough sets are defined, properties of rough intuitionistic fuzzy approximation operators and intuitionistic fuzzy rough approximation operators are examined. Different classes of rough intuitionistic fuzzy set algebras and intuitionistic fuzzy rough set algebras are obtained from different types of fuzzy relations. In the axiomatic approach, an operator-oriented characterization of rough sets is proposed, that is, rough intuitionistic fuzzy approximation operators and intuitionistic fuzzy rough approximation operators are defined by axioms. Different axiom sets of upper and lower intuitionistic fuzzy set-theoretic operators guarantee the existence of different types of crisp/fuzzy relations which produce the same operators.

1. Introduction

The theory of rough sets was originally proposed by Pawlak [1] as a formal tool for modeling and processing intelligent systems characterized by insufficient and incomplete information. The basic structure of rough set theory is an approximation space consisting of a universe of discourse and a binary relation imposed on it. By introducing the concepts of lower and upper approximations of all decision classes with respect to an approximation space induced from the conditional attribute set, knowledge hidden in information tables may be unraveled and expressed in the form of decision rules. We have witnessed a rapid development of and a fast growing interest in rough set theory recently and many models and methods have been proposed and studied (see e.g. the literature cited in [2,3]).

In the classical Pawlak rough set model [1], an equivalence relation is a key and primitive notion in the construction of an approximation space. This equivalence relation, however, seems to be a very restrictive condition that may limit the application domain of the rough set model. Thus one of the main directions in the research of rough set theory is naturally the generalization of concepts of Pawlak rough set approximation operators. There are two ways to define rough set approximation operators: the constructive and the axiomatic approach. In the constructive approach, binary relations on a universe of discourse, partitions or coverings of the universe of discourse, neighborhood systems, and Boolean algebras are all primitive notions. The lower and upper approximation operators are constructed by means of these notions

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[4,1,5–8]. Constructive definitions of rough sets can also be generalized to the fuzzy environment [9–17]. For example, by using an equivalence relation on a universe of discourse, Dubois and Prade defined lower and upper approximations of fuzzy sets in the Pawlak approximation space to obtain an extended notion called rough fuzzy set [10]. A similar fuzzy relation can be used to replace an equivalence relation, the result is a deviation of rough set theory called fuzzy rough set [10,13]. More generalizations of fuzzy rough sets were defined by employing an implicator and a triangular norm on [0, 1] [11,14,17]. Based on arbitrary fuzzy relations, fuzzy partitions on the universe of discourse, and Boolean subalgebras of the power set of the universe of discourse, extended notions called rough fuzzy sets and fuzzy rough sets were also obtained [11,12,4,14–16]. Alternatively, a rough fuzzy set is the approximation of a fuzzy set in a crisp approximation space. The rough fuzzy set model may be used to handle knowledge acquisition in information systems with fuzzy decisions. And a fuzzy rough set is the approximation of a crisp set or a fuzzy set in a fuzzy approximation space. The fuzzy rough set model may be used to unravel knowledge hidden in fuzzy decision systems.

On the other hand, the axiomatic approach, which is appropriate for studying the structures of rough set algebras, considers the reverse problem, that is, the lower and upper approximation operators are taken as primitive notions. A set of axioms is used to characterize approximation operators that are the same as those derived by using the constructive approach. Various classes of approximation operators are characterized by different axiom sets, and the axiom sets of approximation operators guarantee the existence of certain types of binary relations producing the same operators. Under this point of view, rough set theory may be interpreted as an extension of the classical set theory with two additional unary operators. Many authors explored and developed the axiomatic approach in the study of rough set theory [18–22]. The most important axiomatic studies on crisp rough sets were contributed by Yao [8,23], in which various classes of crisp rough set approximation operators were characterized by different sets of axioms. Furthermore, the research of the axiomatic approach has also been extended to approximation operators in the fuzzy environment [9,11,24,25,22,26,14–16]. For instance, Thiele [22,26] investigated axiomatic characterizations of fuzzy rough approximation operators and rough fuzzy approximation operators within modal logic. Radzikowska [13] defined a broad family called $(\mathcal{I}, \mathcal{T})$-fuzzy rough sets determined by an implicator $\mathcal{I}$ and a triangular norm $\mathcal{T}$ on [0, 1] and suggested axiomatic sets for characterizing the operators. The studies of axiomatic research on various generalized approximation operators in fuzzy environment were made by Wu et al. [14–16], in which various classes of fuzzy approximation operators were characterized by different sets of axioms.

In 1986, Atanassov [27] introduced the concept of an intuitionistic fuzzy (IF) set. An AIF set[1] is considered as a generalization of fuzzy set [33] and has been found to be useful to deal with vagueness. In the sense of Atanassov an IF set is characterized by a pair of functions valued in [0, 1]: the membership function and the non-membership function. The evaluation degrees of membership and non-membership are independent. Thus an AIF set is more material and concise to describe the essence of fuzziness, and AIF set theory may be more suitable than fuzzy set theory for dealing with imperfect knowledge in many problems. In recent years, several authors attempted to investigate in combining AIF set theory and rough set theory, various tentative definitions of the concept of an ''IF rough set'' were explored [34–36,13,37]. For example, based on fuzzy rough sets in the sense of Nanda and Majumda [4], Chakrabarty et al. [34] proposed the concept of an IF rough set. In Chakrabarty's opinion, an IF rough set is a generalization of fuzzy rough set and the upper and lower approximations are both AIF sets. Such a notion was reintroduced by Jena and Ghosh in [36]. Samanta and Mondal [37] also introduced this notion, they called it a rough IF set in which the membership and non-membership functions are no longer fuzzy sets but fuzzy rough sets in the sense of Nanda and Majumda. It is well known that fuzzy rough sets in the sense of Nanda and Majumda [4] are not constructed from an approximation space, that is, fuzzy rough sets in the sense of Nanda and Majumda are not defined by binary relations, thus the above mentioned IF rough sets and rough IF sets are not defined by an approximation space. In comparison with the above approaches and along the lines of the Pawlak rough sets, Rizvi et al. [38] introduced a concept of a rough IF set by employing a Pawlak approximation space (U, R), however, in such a case, the lower and upper approximations are not AIF sets in the universe U but AIF sets in the class of equivalence classes of the equivalence relation R. To overcome this drawback, by employing an approximation space constituted from an IF triangular norm $\mathcal{T}$, an IF implicator $\mathcal{I}$, and a $\mathcal{T}$-equivalence AIF relation on the universe of discourse, Cornelis et al. [35] defined a concept of an IF rough set in which the lower and upper approximations are AIF sets in the universe of discourse. Such an IF rough set is indeed a natural generalization of Pawlak's original concept of rough sets. We observe that, on the one hand, a $\mathcal{T}$-equivalence AIF relation will become an equivalence crisp relation in the degenerated case, thus IF rough sets induced from the $\mathcal{T}$-equivalence AIF relation will limit the application of rough set theory in complex systems, therefore, just as in the crisp and fuzzy cases, the requirement of equivalence relation in an approximation space should be relaxed. As a complement for Cornelis' studies, Zhou et al. [39] explore the rough approximations of AIF sets based on IF implicators. On the other hand, the algebraic structure of a class of IF rough sets has not been discussed in detail, meanwhile, the approximations of AIF sets with respect to a crisp/fuzzy approximation space have not been discussed which may be useful to make IF decisions.

The present paper studies rough IF approximation operators and IF rough approximation operators in which both the constructive and axiomatic approaches are used. In the constructive approach, based on an arbitrary crisp relation

[1] Though the term ''intuitionistic fuzzy set'' has been the argument of a large debate [28–32], we will use AIF set instead of intuitionistic fuzzy set in the sense of Atanassov due to its underlying mathematical structure and being a popular topic of investigation with increasing literature in the fuzzy set community.

(and an arbitrary fuzzy relation, respectively), a pair of upper and lower rough IF approximation operators (and IF rough approximation operators, respectively) are defined and their properties are also investigated. In an axiomatic approach, rough IF approximation operators and IF rough approximation operators are axiomatized by abstract operators. Various classes of intuitionistic fuzzy approximation operators are characterized by different sets of axioms, and certain axiom sets of approximation operators guarantee the existence of the corresponding crisp/fuzzy relations producing the same operators.

2. Preliminaries

In this section, we introduce some basic definitions and properties which will be used in this paper. The family of all subsets (respectively, fuzzy subsets) of a set X is denoted by $\mathscr{P}(X)$ (respectively, $\mathscr{F}(X)$). The complement of a set A (whatever A may be) is denoted by $\sim A$. For $\alpha \in [0, 1]$, $\widehat{\alpha}$ will be denoted by the constant fuzzy set, i.e., $\widehat{\alpha}(x) = a$ for all $x \in U$.

Definition 2.1 (*[40,41]*). Let a set U be fixed. An *Atanassov intuitionistic fuzzy set* (we will use "AIF" instead of "Atanassov intuitionistic fuzzy" hereinafter) A in U is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in U\},$$

where the functions $\mu_A : U \to [0, 1]$ and $\gamma_A : U \to [0, 1]$ satisfy $0 \le \mu_A(x) + \nu_A(x) \le 1$ for all $x \in U$, and $\mu_A(x)$ and $\nu_A(x)$ define the degree of membership and the degree of non-membership of the element $x \in U$ to A, respectively. The family of all AIF subsets in U is denoted by $\mathcal{IF}(U)$. The complement of an AIF set A is denoted by $\sim A = \{\langle x, \gamma_A(x), \mu_A(x)\rangle \mid x \in U\}$.

Formally, an AIF set A associates two fuzzy sets $\mu_A : U \to [0, 1]$ and $\gamma_A : U \to [0, 1]$ and can be represented as $A = (\mu_A, \gamma_A)$. Obviously, any fuzzy set $A = \mu_A = \{\langle x, \mu_A(x)\rangle \mid x \in U\}$ may be identified with the AIF set in the form $A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in U\}$. Thus an AIF set is an extension of a fuzzy set.

We introduce operations on $\mathcal{IF}(U)$ as follows [27,41]: $\forall A, B \in \mathcal{IF}(U)$,

- $A \subseteq B$ if and only if (iff) $\mu_A(x) \le \mu_B(x)$ and $\gamma_A(x) \ge \gamma_B(x)$ for all $x \in U$,
- $A \supseteq B$ iff $B \subseteq A$,
- $A = B$ iff $A \subseteq B$ and $B \subseteq A$,
- $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\gamma_A(x), \gamma_B(x))\rangle \mid x \in U\}$,
- $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\gamma_A(x), \gamma_B(x))\rangle \mid x \in U\}$.

Here we define some special AIF sets: a constant AIF set $\widehat{(\alpha, \beta)} = \{\langle x, \alpha, \beta\rangle \mid x \in U\}$, where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \le 1$; the AIF universe set is $U = 1_U = \widehat{(1, 0)} = \{\langle x, 1, 0\rangle \mid x \in U\}$ and the AIF empty set is $\emptyset = 0_U = \widehat{(0, 1)} = \{\langle x, 0, 1\rangle \mid x \in U\}$.

For any $y \in U$, AIF sets 1_y and $1_{U-\{y\}}$ are, respectively, defined by: $\forall x \in U$,

$$\mu_{1_y}(x) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases} \qquad \gamma_{1_y}(x) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

$$\mu_{1_{U-\{y\}}}(x) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases} \qquad \gamma_{1_{U-\{y\}}}(x) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$$

According to the above definitions, the following basic properties about AIF sets can be easily derived:

(1) $A \subseteq B$ and $B \subseteq C \Longrightarrow A \subseteq C$;
(2) $A \cap B \subseteq A \cup B$;
(3) $A \subseteq B$ and $C \subseteq D \Longrightarrow A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D$;
(4) $\sim (\sim A) = A$;
(5) $\sim (A \cup B) = (\sim A) \cap (\sim B)$, $\sim (A \cap B) = (\sim A) \cup (\sim B)$;
(6) $\sim U = \emptyset$, $\sim \emptyset = U$, $1_y = \sim 1_{U-\{y\}}$, $1_{U-\{y\}} = \sim 1_y$.

Definition 2.2. Let $R \in \mathscr{P}(U \times U)$ be a crisp binary relation on U. R is referred to as serial if for any $x \in U$ there exists a $y \in U$ such that $(x, y) \in R$; R is referred to as reflexive if $(x, x) \in R$ for all x; R is referred to as symmetric if for any $x, y \in U$, $(x, y) \in R$ implies $(y, x) \in R$; R is referred to as transitive if for any $x, y, z \in U$, $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$; R is referred to as equivalent if R is reflexive, symmetric, and transitive.

Definition 2.3. Let $R \in \mathscr{F}(U \times U)$ be a fuzzy binary relation on U. $R(x, y)$ is the degree of the relation between x and y, where $(x, y) \in U \times U$. R is referred to as a serial fuzzy relation if $\vee_{y\in U} R(x, y) = 1$ for all $x \in U$; R is referred to as a reflexive fuzzy relation if $R(x, x) = 1$ for all $x \in U$; R is referred to as a symmetric fuzzy relation if $R(x, y) = R(y, x)$ for all $x, y \in U$; R is referred to as a transitive fuzzy relation if $R(x, z) \ge \vee_{y\in U}(R(x, y) \wedge R(y, z))$ for all $x, z \in U$; R is referred to as an equivalence fuzzy relation if R is a reflexive, symmetric, and transitive fuzzy relation.

Definition 2.4. Let U be a nonempty and finite universe of discourse. For an arbitrary crisp relation R on U, we can define a set-valued function $R_s : U \to \mathscr{P}(U)$ by

$$R_s(x) = \{y \in U | (x, y) \in R\}, \quad x \in U.$$

$R_s(x)$ is referred to as the successor neighborhood of x with respect to (w.r.t.) R. The pair (U, R) is called a crisp approximation space. For any $A \subseteq U$, the lower and upper approximations of A w.r.t. (U, R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are, respectively, defined as follows:

$$\underline{R}(A) = \{x \in U | R_s(x) \subseteq A\},$$
$$\overline{R}(A) = \{x \in U | R_s(x) \cap A \neq \emptyset\}.$$

The pair $(\underline{R}(A), \overline{R}(A))$ is referred to as a crisp rough set, and $\underline{R}, \overline{R} : \mathscr{P}(U) \to \mathscr{P}(U)$ are referred to as lower and upper crisp approximation operators, respectively. If R is an equivalence relation on U, then the pair (U, R) is called a Pawlak approximation space and $(\underline{R}(A), \overline{R}(A))$ is called a Pawlak rough set [1].

From the definition, the following theorem can be easily derived [42].

Theorem 2.1. *Let (U, R) be a crisp approximation space, and $\underline{R}$ and $\overline{R}$ the lower and upper crisp approximation operators defined by Definition 2.4. Then*

(1)

$$\begin{aligned} R \text{ is serial} &\iff \text{(L0) } \underline{R}(\emptyset) = \emptyset, \\ &\iff \text{(U0) } \overline{R}(U) = U, \\ &\iff \text{(LU0)} \underline{R}(A) \subseteq \overline{R}(A), \quad \forall A \in \mathscr{P}(U). \end{aligned}$$

(2)

$$\begin{aligned} R \text{ is reflexive} &\iff \text{(LR) } \underline{R}(A) \subseteq A, \quad \forall A \in \mathscr{P}(U), \\ &\iff \text{(UR) } A \subseteq \overline{R}(A), \quad \forall A \in \mathscr{P}(U). \end{aligned}$$

(3)

$$\begin{aligned} R \text{ is symmetric} &\iff \text{(LS) } \overline{R}(\underline{R}(A)) \subseteq A, \quad \forall A \in \mathscr{P}(U), \\ &\iff \text{(US) } A \subseteq \underline{R}(\overline{R}(A)), \quad \forall A \in \mathscr{P}(U). \end{aligned}$$

(4)

$$\begin{aligned} R \text{ is transitive} &\iff \text{(LT) } \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)), \quad \forall A \in \mathscr{P}(U), \\ &\iff \text{(UT) } \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A), \quad \forall A \in \mathscr{P}(U). \end{aligned}$$

Definition 2.5. Let U be a nonempty and finite universe of discourse and R a fuzzy relation on U, the pair (U, R) is called a generalized fuzzy approximation space. For any set $A \in \mathcal{F}(U)$, the lower and upper approximations of A w.r.t. (U, R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are fuzzy sets of U whose membership functions are, respectively, defined as follows:

$$\overline{R}(A)(x) = \bigvee_{y \in U} [R(x, y) \wedge A(y)], \quad x \in U,$$
$$\underline{R}(A)(x) = \bigwedge_{y \in U} [(1 - R(x, y)) \vee A(y)], \quad x \in U.$$

The pair $(\underline{R}(A), \overline{R}(A))$ is referred to as a generalized *fuzzy rough set*, and $\underline{R}$ and $\overline{R} : \mathcal{F}(U) \to \mathcal{F}(U)$ are, respectively, called lower and upper generalized *fuzzy rough approximation operators*.

Especially, if R is a crisp relation on U, that is, (U, R) is a crisp approximation space, then the fuzzy rough approximation operators degenerate to *rough fuzzy approximation operators*. It can be easily checked that

$$\overline{R}(A)(x) = \bigvee_{y \in R_s(x)} A(y), \quad x \in U,$$
$$\underline{R}(A)(x) = \bigwedge_{y \in R_s(x)} A(y), \quad x \in U.$$

Under such a circumstance, the pair $(\underline{R}(A), \overline{R}(A))$ is referred to as a generalized *rough fuzzy set*.

Theorem 2.2 ([16]). *The lower and upper fuzzy rough (respectively, rough fuzzy) approximation operators, $\underline{R}$ and $\overline{R}$, defined in Definition 2.5) satisfy the properties:*
$\forall A, B \in \mathcal{F}(U), \forall \alpha \in [0, 1]$,

(FL1)	$\underline{R}(A) = \sim \overline{R}(\sim A)$,	(FU1)	$\overline{R}(A) = \sim \underline{R}(\sim A)$,
(FL2)	$\underline{R}(A \cup \widehat{\alpha}) = \underline{R}(A) \cup \widehat{\alpha}$,	(FU2)	$\overline{R}(A \cap \widehat{\alpha}) = \overline{R}(A) \cap \widehat{\alpha}$;
(FL3)	$\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$,	(FU3)	$\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$.

The relationships between special fuzzy/crisp relations and fuzzy approximation operators are summarized as follows:

Theorem 2.3 (*[16]*). *Let (U, R) be a fuzzy/crisp approximation space, and $\underline{R}$ and $\overline{R}$ the lower and upper approximation operators defined in Definition 2.5. Then*

(1)

$$\begin{aligned} R \text{ is serial} &\iff (\text{FL0})\ \underline{R}(\emptyset) = \emptyset, \\ &\iff (\text{FU0})\ \overline{R}(U) = U, \\ &\iff (\text{FL0})'\ \underline{R}(\widehat{\alpha}) = \widehat{\alpha}, \quad \forall \alpha \in [0, 1], \\ &\iff (\text{FU0})'\ \overline{R}(\widehat{\alpha}) = \widehat{\alpha}, \quad \forall \alpha \in [0, 1], \\ &\iff (\text{FLU0})\ \underline{R}(A) \subseteq \overline{R}(A), \quad \forall A \in \mathscr{F}(U). \end{aligned}$$

(2)

$$\begin{aligned} R \text{ is reflexive} &\iff (\text{FLR})\ \underline{R}(A) \subseteq A, \quad \forall A \in \mathscr{F}(U), \\ &\iff (\text{FUR})\ A \subseteq \overline{R}(A), \quad \forall A \in \mathscr{F}(U). \end{aligned}$$

(3)

$$\begin{aligned} R \text{ is symmetric} &\iff (\text{FLS})\ \underline{R}(1_{U-\{x\}})(y) = \underline{R}(1_{U-\{y\}})(x), \quad \forall (x, y) \in U \times U, \\ &\iff (\text{FUS})\ \overline{R}(1_x)(y) = \overline{R}(1_y)(x), \quad \forall (x, y) \in U \times U. \end{aligned}$$

(4)

$$\begin{aligned} R \text{ is transitive} &\iff (\text{FLT})\ \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)), \quad \forall A \in \mathscr{F}(U), \\ &\iff (\text{FUT})\ \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A), \quad \forall A \in \mathscr{F}(U). \end{aligned}$$

Remark 2.1. If (U, R) is a crisp approximation space, then [16]

$$\begin{aligned} R \text{ is symmetric} &\iff (\text{FLS})'\ \overline{R}(\underline{R}(A)) \subseteq A, \quad \forall A \in \mathscr{F}(U), \\ &\iff (\text{FUS})'\ A \subseteq \underline{R}(\overline{R}(A)), \quad \forall A \in \mathscr{F}(U). \end{aligned}$$

3. Construction of rough intuitionistic fuzzy sets

Just as a rough fuzzy set is the result of approximation of a fuzzy set in a crisp approximation space, a rough IF set is the result of approximation of an AIF set in a crisp approximation space. In this section, we introduce the concept of rough IF sets and investigate the properties of rough IF approximation operators.

Definition 3.1. Let (U, R) be a crisp approximation space, for $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in U\} \in \mathcal{IF}(U)$, the upper and lower approximations of A w.r.t. (U, R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$, are, respectively, defined as follows:

$$\overline{R}(A) = \{\langle x, \mu_{\overline{R}(A)}(x), \gamma_{\overline{R}(A)}(x)\rangle \mid x \in U\},$$
$$\underline{R}(A) = \{\langle x, \mu_{\underline{R}(A)}(x), \gamma_{\underline{R}(A)}(x)\rangle \mid x \in U\},$$

where

$$\mu_{\overline{R}(A)}(x) = \bigvee_{y \in R_s(x)} \mu_A(y), \qquad \gamma_{\overline{R}(A)}(x) = \bigwedge_{y \in R_s(x)} \gamma_A(y);$$
$$\mu_{\underline{R}(A)}(x) = \bigwedge_{y \in R_s(x)} \mu_A(y), \qquad \gamma_{\underline{R}(A)}(x) = \bigvee_{y \in R_s(x)} \gamma_A(y).$$

It is easy to observe that $\overline{R}(A)$ and $\underline{R}(A)$ are two AIF sets in U, thus AIF mappings $\overline{R}, \underline{R} : \mathcal{IF}(U) \to \mathcal{IF}(U)$ are, respectively, referred to as the upper and lower rough IF approximation operators, and the pair $(\underline{R}(A), \overline{R}(A))$ is called the *rough IF set* of A w.r.t. (U, R).

For any $A \in \mathcal{IF}(U)$, according to Definition 2.5, we can observe that

$$\mu_{\overline{R}(A)} = \overline{R}(\mu_A), \qquad \gamma_{\overline{R}(A)} = \underline{R}(\gamma_A).$$

Then

$$\overline{R}(A) = (\mu_{\overline{R}(A)}, \gamma_{\overline{R}(A)}) = (\overline{R}(\mu_A), \underline{R}(\gamma_A)). \tag{1}$$

Similarly,

$$\underline{R}(A) = (\mu_{\underline{R}(A)}, \gamma_{\underline{R}(A)}) = (\underline{R}(\mu_A), \overline{R}(\gamma_A)). \tag{2}$$

Remark 3.1. When $A \in \mathscr{F}(U)$, $\mu_A(x) + \gamma_A(x) = 1$ for all $x \in U$, then it is easy to observe that $(\underline{R}(A), \overline{R}(A))$ is a rough fuzzy set [16].

Theorem 3.1. *Let (U, R) be a crisp approximation space, then the upper and lower rough IF approximation operators defined in Definition 3.1 satisfy the following properties:*
$\forall A, B \in \mathcal{IF}(U), \forall \alpha, \beta \in [0, 1]$ *with* $\alpha + \beta \le 1$,

(ILc)	$\underline{R}(1_{U-\{y\}}) \in \mathscr{P}(U), \forall y \in U$,	(IUc)	$\overline{R}(1_y) \in \mathscr{P}(U), \forall y \in U$;
(IL1)	$\underline{R}(\sim A) = \sim \overline{R}(A)$,	(IU1)	$\overline{R}(\sim A) = \sim \underline{R}(A)$;
(IL2)	$\underline{R}(U) = U$,	(IU2)	$\overline{R}(\emptyset) = \emptyset$;
(IL3)	$\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$,	(IU3)	$\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$;
(IL4)	$\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$,	(IU4)	$\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$;
(IL5)	$A \subseteq B \Longrightarrow \underline{R}(A) \subseteq \underline{R}(B)$,	(IU5)	$A \subseteq B \Longrightarrow \overline{R}(A) \subseteq \overline{R}(B)$;
(IL6)	$\underline{R}(A \cup \widehat{(\alpha, \beta)}) = \underline{R}(A) \cup \widehat{(\alpha, \beta)}$,	(IU6)	$\overline{R}(A \cap \widehat{(\alpha, \beta)}) = \overline{R}(A) \cap \widehat{(\alpha, \beta)}$.

Proof. We only prove property (IL3), the others can be proved similarly.

According to Eqs. (1) and (2), by Theorem 2.2, we have

$$\begin{aligned}
\underline{R}(A \cap B) &= (\mu_{\underline{R}(A\cap B)}, \gamma_{\underline{R}(A\cap B)}) \\
&= (\underline{R}(\mu_{A\cap B}), \overline{R}(\gamma_{A\cap B})) \\
&= (\underline{R}(\mu_A \cap \mu_B), \overline{R}(\gamma_A \cup \gamma_B)) \\
&= (\underline{R}(\mu_A) \cap \underline{R}(\mu_B), \overline{R}(\gamma_A) \cup \overline{R}(\gamma_B)) \\
&= (\mu_{\underline{R}(A)} \cap \mu_{\underline{R}(B)}, \gamma_{\underline{R}(A)} \cup \gamma_{\underline{R}(B)}) \\
&= (\mu_{\underline{R}(A)\cap\underline{R}(B)}, \gamma_{\underline{R}(A)\cap\underline{R}(B)}).
\end{aligned}$$

Thus we conclude (IL3). □

Properties (IL1) and (IU1) in Theorem 3.1 show that $\underline{R}$ and $\overline{R}$ are dual with each other. With respect to certain special types, say, serial, reflexive, symmetric, and transitive crisp binary relations on the universe of discourse U, the rough IF approximation operators have additional properties.

Theorem 3.2. *Let (U, R) be a crisp approximation space, and $\underline{R}, \overline{R} : \mathcal{IF}(U) \to \mathcal{IF}(U)$ the lower and upper rough IF approximation operators. Then*

(1)

$$\begin{aligned}
R \text{ is serial} &\Longleftrightarrow \text{(IL0) } \underline{R}(\emptyset) = \emptyset, \\
&\Longleftrightarrow \text{(IU0) } \overline{R}(U) = U, \\
&\Longleftrightarrow \text{(IL0)}' \ \underline{R}(\widehat{(\alpha, \beta)}) = \widehat{(\alpha, \beta)}, \quad \forall \alpha, \beta \in [0, 1], \alpha + \beta \le 1, \\
&\Longleftrightarrow \text{(IU0)}' \ \overline{R}(\widehat{(\alpha, \beta)}) = \widehat{(\alpha, \beta)}, \quad \forall \alpha, \beta \in [0, 1], \alpha + \beta \le 1, \\
&\Longleftrightarrow \text{(ILU0) } \underline{R}(A) \subseteq \overline{R}(A), \quad \forall A \in \mathcal{IF}(U).
\end{aligned}$$

(2)

$$\begin{aligned}
R \text{ is reflexive} &\Longleftrightarrow \text{(ILR) } \underline{R}(A) \subseteq A, \quad \forall A \in \mathcal{IF}(U), \\
&\Longleftrightarrow \text{(IUR) } A \subseteq \overline{R}(A), \quad \forall A \in \mathcal{IF}(U).
\end{aligned}$$

(3)

$$\begin{aligned}
R \text{ is symmetric} &\Longleftrightarrow \text{(ILS) } \overline{R}(\underline{R}(A)) \subseteq A, \quad \forall A \in \mathcal{IF}(U), \\
&\Longleftrightarrow \text{(IUS) } A \subseteq \underline{R}(\overline{R}(A)), \quad \forall A \in \mathcal{IF}(U), \\
&\Longleftrightarrow \text{(ILS)}' \ \mu_{\underline{R}(1_{U-\{x\}})}(y) = \mu_{\underline{R}(1_{U-\{y\}})}(x), \quad \forall (x, y) \in U \times U, \\
&\Longleftrightarrow \text{(IUS)}' \ \mu_{\overline{R}(1_x)}(y) = \mu_{\overline{R}(1_y)}(x), \quad \forall (x, y) \in U \times U, \\
&\Longleftrightarrow \text{(ILS)}'' \ \gamma_{\underline{R}(1_{U-\{x\}})}(y) = \gamma_{\underline{R}(1_{U-\{y\}})}(x), \quad \forall (x, y) \in U \times U, \\
&\Longleftrightarrow \text{(IUS)}'' \ \gamma_{\overline{R}(1_x)}(y) = \gamma_{\overline{R}(1_y)}(x), \quad \forall (x, y) \in U \times U.
\end{aligned}$$

(4)

$$\begin{aligned}
R \text{ is transitive} &\Longleftrightarrow \text{(ILT) } \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)), \quad \forall A \in \mathcal{IF}(U), \\
&\Longleftrightarrow \text{(IUT) } \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A), \quad \forall A \in \mathcal{IF}(U).
\end{aligned}$$

Proof. We only prove (1), the others can be proved similarly.

(1) Firstly, according to Eqs. (1) and (2), by Theorem 2.3, it is easy to verify that

$$R \text{ is serial} \iff \text{(IL0)} \iff \text{(IU0)}.$$

Secondly, by Theorem 2.3, we have

$$\begin{aligned} R \text{ is serial} &\iff \underline{R}(\widehat{\alpha}) = \widehat{\alpha}, \overline{R}(\widehat{\beta}) = \widehat{\beta}, \quad \forall \alpha, \beta \in [0, 1] \\ &\iff \underline{R}(\widehat{(\alpha, \beta)}) = \left(\mu_{\underline{R}(\widehat{(\alpha,\beta)})}, \gamma_{\underline{R}(\widehat{(\alpha,\beta)})}\right) \\ &\quad = \left(\underline{R}\big(\mu_{\widehat{(\alpha,\beta)}}\big), \overline{R}\big(\gamma_{\widehat{(\alpha,\beta)}}\big)\right) \\ &\quad = \big(\underline{R}(\widehat{\alpha}), \overline{R}(\widehat{\beta})\big) \\ &\quad = (\widehat{\alpha}, \widehat{\beta}) \\ &\quad = \widehat{(\alpha, \beta)}, \quad \forall \alpha, \beta \in [0, 1], \alpha + \beta \le 1. \end{aligned}$$

By the duality of $\underline{R}$ and $\overline{R}$, we then conclude that

$$R \text{ is serial} \iff \text{(IU0)}'.$$

Finally, by Theorem 2.3, we have

$$\begin{aligned} R \text{ is serial} &\iff \underline{R}(\mu_A) \subseteq \overline{R}(\mu_A), \underline{R}(\gamma_A) \subseteq \overline{R}(\gamma_A), \quad \forall A \in \mathcal{IF}(U) \\ &\iff \mu_{\underline{R}(A)} \subseteq \mu_{\overline{R}(A)}, \gamma_{\overline{R}(A)} \subseteq \gamma_{\underline{R}(A)}, \quad \forall A \in \mathcal{IF}(U) \\ &\iff \big(\mu_{\underline{R}(A)}, \gamma_{\underline{R}(A)}\big) \subseteq \big(\mu_{\overline{R}(A)}, \gamma_{\overline{R}(A)}\big), \quad \forall A \in \mathcal{IF}(U) \\ &\iff \text{(ILU0) } \underline{R}(A) \subseteq \overline{R}(A), \quad \forall A \in \mathcal{IF}(U). \quad \Box \end{aligned}$$

4. Construction of intuitionistic fuzzy rough sets

In this section, we introduce the constructive definition of IF rough sets and investigate the properties of IF rough approximation operators. We first review a special lattice on $[0, 1]^2$ with its logical operations [43,44].

Definition 4.1 (*[43] (Lattice $(L^*, \le_{L^*})$)*). Let $L^* = \{(x_1, x_2) \in [0, 1]^2 \mid x_1 + x_2 \le 1\}$. Define a relation $\le_{L^*}$ on L^* as follows: $\forall (x_1, x_2), (y_1, y_2) \in L^*$,

$$(x_1, x_2) \le_{L^*} (y_1, y_2) \iff x_1 \le y_1 \quad \text{and} \quad x_2 \ge y_2.$$

Then $\le_{L^*}$ is a partial ordering on L^* and the pair $(L^*, \le_{L^*})$ is a complete lattice with the smallest element $0_{L^*} = (0, 1)$ and the greatest element $1_{L^*} = (1, 0)$ [43]. The meet operator $\wedge$ and the join operator $\vee$ on $(L^*, \le_{L^*})$ which are linked to the ordering $\le_{L^*}$ are, respectively, defined as follows:

$\forall (x_1, x_2), (y_1, y_2) \in L^*$,

$$(x_1, x_2) \wedge (y_1, y_2) = (\min(x_1, y_1), \max(x_2, y_2)),$$
$$(x_1, x_2) \vee (y_1, y_2) = (\max(x_1, y_1), \min(x_2, y_2)).$$

For an AIF set $A \in \mathcal{IF}(U)$, we write $A(x) = (\mu_A(x), \gamma_A(x))$ for $x \in U$, then it is clear that $A \in \mathcal{IF}(U)$ iff $A(x) \in L^*$ for all $x \in U$. For any $A, B \in \mathcal{IF}(U)$, we can represent the corresponding AIF sets by using L^* as follows:

- $A(x) = (\mu_A(x), \gamma_A(x)) \in L^*, x \in U$,
- $U(x) = (1, 0) = 1_{L^*}, \forall x \in U$,
- $\emptyset(x) = (0, 1) = 0_{L^*}, \forall x \in U$,
- $x = y \Longrightarrow 1_y(x) = 1_{L^*}$ and $1_{U-\{y\}}(x) = 0_{L^*}, x, y \in U$,
- $x \neq y \Longrightarrow 1_y(x) = 0_{L^*}$ and $1_{U-\{y\}}(x) = 1_{L^*}, x, y \in U$,
- $A \subseteq B \iff A(x) \le_{L^*} B(x), \forall x \in U \iff B(x) \ge_{L^*} A(x), \forall x \in U$,
- $\big(A \cap B\big)(x) = A(x) \wedge B(x) = \big(\mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)\big) \in L^*, x \in U$,
- $\big(A \cup B\big)(x) = A(x) \vee B(x) = \big(\mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)\big) \in L^*, x \in U$.

Definition 4.2. Let (U, R) be a fuzzy approximation space. For any $A \in \mathcal{IF}(U)$, we define the upper and lower approximations of A w.r.t. (U, R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, as follows:

$$\overline{R}(A) = \left\{\left\langle x, \mu_{\overline{R}(A)}(x), \gamma_{\overline{R}(A)}(x)\right\rangle \mid x \in U\right\},$$
$$\underline{R}(A) = \left\{\left\langle x, \mu_{\underline{R}(A)}(x), \gamma_{\underline{R}(A)}(x)\right\rangle \mid x \in U\right\},$$

where

$$\mu_{\overline{R}(A)}(x) = \bigvee_{y\in U}[R(x,y) \wedge \mu_A(y)], \qquad \gamma_{\overline{R}(A)}(x) = \bigwedge_{y\in U}[(1-R(x,y)) \vee \gamma_A(y)];$$
$$\mu_{\underline{R}(A)}(x) = \bigwedge_{y\in U}[(1-R(x,y)) \vee \mu_A(y)], \qquad \gamma_{\underline{R}(A)}(x) = \bigvee_{y\in U}[R(x,y) \wedge \gamma_A(y)].$$

The pair $(\underline{R}(A), \overline{R}(A))$ is referred to as the *IF rough set* of A w.r.t. (U, R).

It can be seen that $\overline{R}(A)$ and $\underline{R}(A)$ are AIF sets in U. (In fact, $\forall x \in U$, notice that $A \in \mathcal{IF}(U)$, we have $\gamma_A(x) \leq 1 - \mu_A(x)$, then

$$\begin{aligned}\mu_{\overline{R}(A)}(x) &= \bigvee_{y\in U}[R(x,y) \wedge \mu_A(y)] \\ &= 1 - \bigwedge_{y\in U}[(1-R(x,y)) \vee (1-\mu_A(y))] \\ &\leq 1 - \bigwedge_{y\in U}[(1-R(x,y)) \vee \gamma_A(y)].\end{aligned}$$

Consequently,

$$\begin{aligned}\mu_{\overline{R}(A)}(x) + \gamma_{\overline{R}(A)}(x) &= \bigvee_{y\in U}[R(x,y) \wedge \mu_A(y)] + \bigwedge_{y\in U}[(1-R(x,y)) \vee \gamma_A(y)] \\ &\leq 1 - \bigwedge_{y\in U}[(1-R(x,y)) \vee \gamma_A(y)] + \bigwedge_{y\in U}[(1-R(x,y)) \vee \gamma_A(y)] = 1.\end{aligned}$$

Thus we have proved that $\overline{R}(A) \in \mathcal{IF}(U)$. Similarly, we can verify that $(\mu_{\underline{R}(A)}(x), \gamma_{\underline{R}(A)}(x)) \in L^*$ for all $x \in U$, i.e., $\underline{R}(A) \in \mathcal{IF}(U)$.) Based on this observation, we call $\overline{R}, \underline{R} : \mathcal{IF}(U) \to \mathcal{IF}(U)$ the upper and lower IF rough approximation operators, respectively.

For any $A \in \mathcal{IF}(U)$, according to Definition 2.5, we can observe that

$$\mu_{\overline{R}(A)} = \overline{R}(\mu_A), \qquad \gamma_{\overline{R}(A)} = \underline{R}(\gamma_A).$$

Then

$$\overline{R}(A) = (\mu_{\overline{R}(A)}, \gamma_{\overline{R}(A)}) = (\overline{R}(\mu_A), \underline{R}(\gamma_A)). \tag{3}$$

Similarly,

$$\underline{R}(A) = (\mu_{\underline{R}(A)}, \gamma_{\underline{R}(A)}) = (\underline{R}(\mu_A), \overline{R}(\gamma_A)). \tag{4}$$

Remark 4.1. When $A \in \mathscr{F}(U)$, $\mu_A(x) + \gamma_A(x) = 1$ for all $x \in U$, then it is easy to observe that $(\underline{R}(A), \overline{R}(A))$ is no other than a fuzzy rough set [16]. On the other hand, if R in Definition 4.2 is a crisp binary relation on U, then Definition 4.2 degenerates to Definition 3.1, thus an IF rough set can be regarded as a generalization of a rough IF set.

The following Theorem 4.1 presents some basic properties of IF rough approximation operators.

Theorem 4.1. *Let (U, R) be a fuzzy approximation space. Then the IF rough approximation operators defined in Definition 4.2 satisfy the following properties:*

$\forall A, B \in \mathcal{IF}(U)$, $\forall (\alpha, \beta) \in L^*$,

(ILf)	$\underline{R}(1_{U-\{y\}}) \in \mathscr{F}(U), \quad \forall y \in U,$	(IUf)	$\overline{R}(1_y) \in \mathscr{F}(U), \quad \forall y \in U;$
(IL1)	$\underline{R}(\sim A) =\sim \overline{R}(A),$	(IU1)	$\overline{R}(\sim A) =\sim \underline{R}(A);$
(IL2)	$\underline{R}(U) = U,$	(IU2)	$\overline{R}(\emptyset) = \emptyset;$
(IL3)	$\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B),$	(IU3)	$\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B);$
(IL4)	$\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B),$	(IU4)	$\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B);$
(IL5)	$A \subseteq B \Longrightarrow \underline{R}(A) \subseteq \underline{R}(B),$	(IU5)	$A \subseteq B \Longrightarrow \overline{R}(A) \subseteq \overline{R}(B);$
(IL6)	$\underline{R}(A \cup \widehat{(\alpha, \beta)}) = \underline{R}(A) \cup \widehat{(\alpha, \beta)},$	(IU6)	$\overline{R}(A \cap \widehat{(\alpha, \beta)}) = \overline{R}(A) \cap \widehat{(\alpha, \beta)}.$

Proof. It is similar to the proof of Theorem 3.1. □

Definition 4.3. Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in U\} \in \mathcal{IF}(U)$ and $(\alpha, \beta) \in L^*$, we introduce the (α, β)-level cut set of the AIF set A, denoted by A_α^β, as follows:

$$A_\alpha^\beta = \{x \in U \mid \mu_A(x) \geq \alpha, \gamma_A(x) \leq \beta\}.$$

Meanwhile,

(a) we respectively call the sets

$$A_\alpha = \{x \in U \mid \mu_A(x) \geq \alpha\} \quad \text{and} \quad A_{\alpha+} = \{x \in U \mid \mu_A(x) > \alpha\}$$

the α-level cut set and the strong α-level cut set of membership degree generated by A;

(b) we respectively call the sets

$$A^\beta = \{x \in U \mid \gamma_A(x) \leq \beta\} \quad \text{and} \quad A^{\beta+} = \{x \in U \mid \gamma_A(x) < \beta\}$$

the β-level cut set and the strong β-level cut set of non-membership degree generated by A.

Likewise, we define other types of cut sets of the AIF set A as follows:

$A_{\alpha+}^\beta = \{x \in U \mid \mu_A(x) > \alpha, \gamma_A(x) \leq \beta\}$, which is called $(\alpha+, \beta)$-level cut set of A;

$A_\alpha^{\beta+} = \{x \in U \mid \mu_A(x) \geq \alpha, \gamma_A(x) < \beta\}$, which is called $(\alpha, \beta+)$-level cut set of A;

$A_{\alpha+}^{\beta+} = \{x \in U \mid \mu_A(x) > \alpha, \gamma_A(x) < \beta\}$, which is called $(\alpha+, \beta+)$-level cut set of A.

Theorem 4.2. *The cut sets of AIF sets on (α, β)-level satisfy the following properties:*
$\forall A, B \in \mathcal{IF}(U), \forall (\alpha, \beta) \in L^*$,

(1) $A_\alpha^\beta = A_\alpha \cap A^\beta$;
(2) $(\sim A)_\alpha = \sim A^{\alpha+}, \quad (\sim A)^\beta = \sim A_{\beta+}$;
(3) $A \subseteq B \Longrightarrow A_\alpha^\beta \subseteq B_\alpha^\beta$;
(4) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha,\ (A \cap B)^\beta = A^\beta \cap B^\beta,\ (A \cap B)_\alpha^\beta = A_\alpha^\beta \cap B_\alpha^\beta$;
(5) $(A \cup B)_\alpha = A_\alpha \cup B_\alpha,\ (A \cup B)^\beta = A^\beta \cup B^\beta,\ (A \cup B)_\alpha^\beta \supseteq A_\alpha^\beta \cup B_\alpha^\beta$.

Proof. It follows directly from Definition 4.3. □

The following lemma is well known [16].

Lemma 4.1. *Assume that R is a fuzzy relation on U, denote*

$$R_\alpha = \{(x, y) | R(x, y) \geq \alpha\}, \qquad R_\alpha(x) = \{y \in U | R(x, y) \geq \alpha\}, \quad \alpha \in [0, 1],$$
$$R_{\alpha+} = \{(x, y) | R(x, y) > \alpha\}, \qquad R_{\alpha+}(x) = \{y \in U | R(x, y) > \alpha\}, \quad \alpha \in [0, 1).$$

Then R_α and $R_{\alpha+}$ are two crisp relations on U and

(1) *if R is reflexive, then R_α and $R_{\alpha+}$ are also reflexive;*
(2) *if R is symmetric, then R_α and $R_{\alpha+}$ are also symmetric;*
(3) *if R is transitive, then R_α and $R_{\alpha+}$ are also transitive.*

In what follows, for simplicity we call both rough IF approximation operators and IF rough approximation operators the intuitionistic fuzzy approximation operators. The following Theorems 4.3–4.5 show that intuitionistic fuzzy approximation operators can be represented by crisp approximation operators.

Theorem 4.3. *Let (U, R) be a fuzzy approximation space and $\overline{R}$ the upper IF rough approximation operator defined in Definition 4.2. Then:*
$\forall A \in \mathcal{IF}(U), \forall x \in U$,

(1)

$$\begin{aligned}\mu_{\overline{R}(A)}(x) &= \bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R_\alpha}(A_\alpha)(x)] = \bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R_\alpha}(A_{\alpha+})(x)] \\ &= \bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R_{\alpha+}}(A_\alpha)(x)] = \bigvee_{\alpha \in [0,1]} [\alpha \wedge \overline{R_{\alpha+}}(A_{\alpha+})(x)].\end{aligned}$$

(2)

$$\begin{aligned}\gamma_{\overline{R}(A)}(x) &= \bigwedge_{\alpha \in [0,1]} [\alpha \vee (1 - \overline{R_{1-\alpha}}(A^\alpha)(x))] = \bigwedge_{\alpha \in [0,1]} [\alpha \vee (1 - \overline{R_{1-\alpha}}(A^{\alpha+})(x))] \\ &= \bigwedge_{\alpha \in [0,1]} [\alpha \vee (1 - \overline{R_{(1-\alpha)+}}(A^\alpha)(x))] = \bigwedge_{\alpha \in [0,1]} [\alpha \vee (1 - \overline{R_{(1-\alpha)+}}(A^{\alpha+})(x))].\end{aligned}$$

And moreover, for any $\alpha \in [0, 1]$,

(3) $[\overline{R}(A)]_{\alpha+} \subseteq \overline{R_{\alpha+}}(A_{\alpha+}) \subseteq \overline{R_{\alpha+}}(A_{\alpha}) \subseteq \overline{R_{\alpha}}(A_{\alpha}) \subseteq [\overline{R}(A)]_{\alpha}$.
(4) $[\overline{R}(A)]^{\alpha+} \subseteq \overline{R_{(1-\alpha)+}}(A^{\alpha+}) \subseteq \overline{R_{(1-\alpha)+}}(A^{\alpha}) \subseteq \overline{R_{1-\alpha}}(A^{\alpha}) \subseteq [\overline{R}(A)]^{\alpha}$.
(3)′ $[\overline{R}(A)]_{\alpha+} \subseteq \overline{R_{\alpha+}}(A_{\alpha+}) \subseteq \overline{R_{\alpha}}(A_{\alpha+}) \subseteq \overline{R_{\alpha}}(A_{\alpha}) \subseteq [\overline{R}(A)]_{\alpha}$.
(4)′ $[\overline{R}(A)]^{\alpha+} \subseteq \overline{R_{(1-\alpha)+}}(A^{\alpha+}) \subseteq \overline{R_{1-\alpha}}(A^{\alpha+}) \subseteq \overline{R_{1-\alpha}}(A^{\alpha}) \subseteq [\overline{R}(A)]^{\alpha}$.

Proof. It follows directly from Eqs. (3) and (4) and Theorem 3 in [16]. □

Likewise, for the lower IF approximation operator, we have the following

Theorem 4.4. *Let* (U, R) *be a fuzzy approximation space and* $\underline{R}$ *the lower IF rough approximation operator defined in Definition 4.2. Then:*
$\forall A \in \mathcal{IF}(U), \forall x \in U$,

(1)

$$\mu_{\underline{R}(A)}(x) = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \underline{R_{1-\alpha}}(A_{\alpha+})(x)] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \underline{R_{(1-\alpha)+}}(A_{\alpha})(x)]$$
$$= \bigwedge_{\alpha\in[0,1]} [\alpha \vee \underline{R_{(1-\alpha)+}}(A_{\alpha+})(x)] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \underline{R_{1-\alpha}}(A_{\alpha})(x)].$$

(2)

$$\gamma_{\underline{R}(A)}(x) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_{\alpha}}(A^{\alpha})(x))] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_{\alpha+}}(A^{\alpha})(x))]$$
$$= \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_{\alpha+}}(A^{\alpha+})(x))] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R_{\alpha}}(A^{\alpha+})(x))].$$

And moreover for any $\alpha \in [0, 1]$,

(3) $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R_{1-\alpha}}(A_{\alpha+}) \subseteq \underline{R_{(1-\alpha)+}}(A_{\alpha+}) \subseteq \underline{R_{(1-\alpha)+}}(A_{\alpha}) \subseteq [\underline{R}(A)]_{\alpha}$.
(4) $[\underline{R}(A)]^{\alpha+} \subseteq \underline{R_{\alpha}}(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^{\alpha}) \subseteq [\underline{R}(A)]^{\alpha}$.
(3)′ $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R_{1-\alpha}}(A_{\alpha+}) \subseteq \underline{R_{(1-\alpha)}}(A_{\alpha}) \subseteq \underline{R_{(1-\alpha)+}}(A_{\alpha}) \subseteq [\underline{R}(A)]_{\alpha}$.
(4)′ $[\underline{R}(A)]^{\alpha+} \subseteq \underline{R_{\alpha}}(A^{\alpha+}) \subseteq \underline{R_{\alpha}}(A^{\alpha}) \subseteq \underline{R_{\alpha+}}(A^{\alpha}) \subseteq [\underline{R}(A)]^{\alpha}$.

According to Definition 3.1, Theorems 4.3 and 4.4, we can conclude the following

Theorem 4.5. *Let* (U, R) *be a crisp approximation space, and* $\overline{R}$ *and* $\underline{R}$ *the upper and lower rough IF approximation operators defined in Definition 3.1. Then:*
$\forall A \in \mathcal{IF}(U), \forall x \in U, \alpha \in [0, 1]$,

(1)

$$\mu_{\overline{R}(A)}(x) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R}(A_{\alpha})(x)] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge \overline{R}(A_{\alpha+})(x)],$$
$$\gamma_{\overline{R}(A)}(x) = \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \overline{R}(A^{\alpha})(x))] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee (1 - \overline{R}(A^{\alpha+})(x))].$$

(2)

$$\mu_{\underline{R}(A)}(x) = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \underline{R}(A_{\alpha})(x)] = \bigwedge_{\alpha\in[0,1]} [\alpha \vee \underline{R}(A_{\alpha+})(x)],$$
$$\gamma_{\underline{R}(A)}(x) = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R}(A^{\alpha})(x))] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - \underline{R}(A^{\alpha+})(x))].$$

(3)

$$[\overline{R}(A)]_{\alpha+} \subseteq \overline{R}(A_{\alpha+}) \subseteq \overline{R}(A_{\alpha}) \subseteq [\overline{R}(A)]_{\alpha},$$
$$[\overline{R}(A)]^{\alpha+} \subseteq \overline{R}(A^{\alpha+}) \subseteq \overline{R}(A^{\alpha}) \subseteq [\overline{R}(A)]^{\alpha}.$$

(4)

$$[\underline{R}(A)]_{\alpha+} \subseteq \underline{R}(A_{\alpha+}) \subseteq \underline{R}(A_{\alpha}) \subseteq [\underline{R}(A)]_{\alpha},$$
$$[\underline{R}(A)]^{\alpha+} \subseteq \underline{R}(A^{\alpha+}) \subseteq \underline{R}(A^{\alpha}) \subseteq [\underline{R}(A)]^{\alpha}.$$

By Eqs. (3) and (4), in terms of Theorem 2.3, we can conclude the following Theorem 4.6 which shows that a special type of fuzzy relation can be characterized by properties of IF rough approximation operators.

Theorem 4.6. *Let* (U, R) *be a fuzzy approximation space, and* $\underline{R}$ *and* $\overline{R}$ *the lower and upper IF rough approximation operators defined in Definition 4.2. Then*

(1)

$$\begin{aligned} R \text{ is serial} &\iff \text{(IL0) } \underline{R}(\emptyset) = \emptyset, \\ &\iff \text{(IU0) } \overline{R}(U) = U, \\ &\iff \text{(ILU0) } \underline{R}(A) \subseteq \overline{R}(A), \quad \forall A \in \mathcal{IF}(U). \end{aligned}$$

(2)

$$\begin{aligned} R \text{ is reflexive} &\iff \text{(ILR) } \underline{R}(A) \subseteq A, \quad \forall A \in \mathcal{IF}(U), \\ &\iff \text{(IUR) } A \subseteq \overline{R}(A), \quad \forall A \in \mathcal{IF}(U). \end{aligned}$$

(3)

$$\begin{aligned} R \text{ is symmetric} &\iff \text{(ILS)}' \ \mu_{\underline{R}(1_{U-\{x\}})}(y) = \mu_{\underline{R}(1_{U-\{y\}})}(x), \quad \forall (x, y) \in U \times U, \\ &\iff \text{(IUS)}' \ \mu_{\overline{R}(1_x)}(y) = \mu_{\overline{R}(1_y)}(x), \quad \forall (x, y) \in U \times U, \\ &\iff \text{(ILS)}'' \ \gamma_{\underline{R}(1_{U-\{x\}})}(y) = \gamma_{\underline{R}(1_{U-\{y\}})}(x), \quad \forall (x, y) \in U \times U, \\ &\iff \text{(IUS)}'' \ \gamma_{\overline{R}(1_x)}(y) = \gamma_{\overline{R}(1_y)}(x), \quad \forall (x, y) \in U \times U. \end{aligned}$$

(4)

$$\begin{aligned} R \text{ is transitive} &\iff \text{(ILT) } \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)), \quad \forall A \in \mathcal{IF}(U), \\ &\iff \text{(IUT) } \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A), \quad \forall A \in \mathcal{IF}(U). \end{aligned}$$

5. Axiomatic characterization of intuitionistic fuzzy approximation operators

In an axiomatic approach, rough sets are axiomatized by abstract operators. For the case of rough IF sets and IF rough sets, the primitive notion is a system $(\mathcal{IF}(U), \cap, \cup, \sim, L, H)$, where $L, H : \mathcal{IF}(U) \to \mathcal{IF}(U)$ are AIF operators from $\mathcal{IF}(U)$ to $\mathcal{IF}(U)$. In this section, we show that intuitionistic fuzzy approximation operators can be characterized by axioms.

For simplicity, we denote the family of membership functions and the family of non-membership functions on U by $f(U)$ and $\bar{f}(U)$, respectively. For any $A, B \in \mathcal{IF}(U)$, clearly, $\mu_A, \mu_B \in f(U)$ and $\gamma_A, \gamma_B \in \bar{f}(U)$. Moreover, we define: $\mu_A \subseteq \mu_B$ iff $\mu_A(x) \leq \mu_B(x)$ for all $x \in U$, likewise, $\gamma_A \subseteq \gamma_B$ iff $\gamma_A(x) \leq \gamma_B(x)$ for all $x \in U$; $\mu_A = \mu_B$ iff $\mu_A(x) = \mu_B(x)$ for all $x \in U$, likewise, $\gamma_A = \gamma_B$ iff $\gamma_A(x) = \gamma_B(x)$ for all $x \in U$. Consider a system $(\mathcal{IF}(U), \cap, \cup, \sim, L, H)$, where $L, H\colon \mathcal{IF}(U) \to \mathcal{IF}(U)$ are operators from $\mathcal{IF}(U)$ to $\mathcal{IF}(U)$. Furthermore, L and H can be, respectively, represented by a pair of operators $L = (L_\mu, L_\gamma)$, $H = (H_\mu, H_\gamma)$, where $L_\mu, H_\mu : f(U) \to f(U)$ and $L_\gamma, H_\gamma : \bar{f}(U) \to \bar{f}(U)$. For $A \in \mathcal{IF}(U)$, $L(A) = (L_\mu(\mu_A), L_\gamma(\gamma_A))$ such that $\mu_{L(A)} = L_\mu(\mu_A)$ and $\gamma_{L(A)} = L_\gamma(\gamma_A)$, $H(A) = (H_\mu(\mu_A), H_\gamma(\gamma_A))$ such that $\mu_{H(A)} = H_\mu(\mu_A)$ and $\gamma_{H(A)} = H_\gamma(\gamma_A)$.

Definition 5.1. Let $L, H\colon \mathcal{IF}(U) \to \mathcal{IF}(U)$ be two operators with $L = (L_\mu, L_\gamma)$ and $H = (H_\mu, H_\gamma)$. L and H are referred to as dual operators if for each $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in U\} \in \mathcal{IF}(U)$, the following axioms are satisfied:

(IFL1) $L(A) =\sim H(\sim A)$, i.e., $L_\mu(\mu_A) = H_\gamma(\mu_A)$, $L_\gamma(\gamma_A) = H_\mu(\gamma_A)$;
(IFU1) $H(A) =\sim L(\sim A)$, i.e., $H_\mu(\mu_A) = L_\gamma(\mu_A)$, $H_\gamma(\gamma_A) = L_\mu(\gamma_A)$.

Theorem 5.1. *Let* $L, H\colon \mathcal{IF}(U) \to \mathcal{IF}(U)$ *be two dual operators. Then there exists a fuzzy relation* R *on* U *such that* $L(A) = \underline{R}(A)$ *and* $H(A) = \overline{R}(A)$ *for all* $A \in \mathcal{IF}(U)$ *iff* L *satisfies the following axioms* (IFLf), (IFL2) *and* (IFL3), *or equivalently,* H *satisfies axioms* (IFUf) , (IFU2) *and* (IFU3) *:* $\forall A, B \in \mathcal{IF}(U), \forall (\alpha, \beta) \in L^*$,

(IFLf) $L(1_{U-\{y\}}) \in \mathscr{F}(U), \forall y \in U$;
(IFL2) $L(A \cup \widehat{(\alpha, \beta)}) = L(A) \cup \widehat{(\alpha, \beta)}$, *i.e.*, $L_\mu(\mu_A \cup \widehat{\alpha}) = L_\mu(\mu_A) \cup \widehat{\alpha}$, $L_\gamma(\gamma_A \cap \widehat{\beta}) = L_\gamma(\gamma_A) \cap \widehat{\beta}$;
(IFL3) $L(A \cap B) = L(A) \cap L(B)$, *i.e.*, $L_\mu(\mu_{A\cap B}) = L_\mu(\mu_A \cap \mu_B) = L_\mu(\mu_A) \cap L_\mu(\mu_B)$, $L_\gamma(\gamma_{A\cap B}) = L_\gamma(\gamma_A \cup \gamma_B) = L_\gamma(\gamma_A) \cup L_\gamma(\gamma_B)$;
(IFUf) $H(1_y) \in \mathscr{F}(U), \forall y \in U$;
(IFU2) $H(A \cap \widehat{(\alpha, \beta)}) = H(A) \cap \widehat{(\alpha, \beta)}$, *i.e.*, $H_\mu(\mu_A \cap \widehat{\alpha}) = H_\mu(\mu_A) \cap \widehat{\alpha}$, $H_\gamma(\gamma_A \cup \widehat{\beta}) = H_\gamma(\gamma_A) \cup \widehat{\beta}$;
(IFU3) $H(A \cup B) = H(A) \cup H(B)$, *i.e.*, $H_\mu(\mu_{A\cup B}) = H_\mu(\mu_A \cup \mu_B) = H_\mu(\mu_A) \cup H_\mu(\mu_B)$, $H_\gamma(\gamma_{A\cup B}) = H_\gamma(\gamma_A \cap \gamma_B) = H_\gamma(\gamma_A) \cap H_\gamma(\gamma_B)$.

Proof. "⇒" follows immediately from Theorem 4.1.

"⇐" Suppose that the operator H satisfies axioms (IFUf). Then we can define a fuzzy relation R on U by H as follows:

$$R(x, y) = H_\mu(\mu_{1_y})(x) = 1 - H_\gamma(\gamma_{1_y})(x), \quad (x, y) \in U \times U.$$

For any $A \in \mathcal{IF}(U)$, notice that

$$\mu_A = \bigcup_{y\in U}(\mu_{1_y} \cap \widehat{\mu_A(y)}), \quad \gamma_A = \bigcap_{y\in U}(\gamma_{1_y} \cup \widehat{\gamma_A(y)}).$$

Then, for any $x \in U$, according to Definition 4.2, (IFU2), and (IFU3), we have

$$\begin{aligned}
\mu_{\overline{R}(A)}(x) &= \bigvee_{y\in U}[R(x,y) \wedge \mu_A(y)] \\
&= \bigvee_{y\in U}[H_\mu(\mu_{1_y})(x) \wedge \mu_A(y)] \\
&= \bigvee_{y\in U}[H_\mu(\mu_{1_y}) \cap \widehat{\mu_A(y)}](x) \\
&= \bigvee_{y\in U} H_\mu[\mu_{1_y} \cap \widehat{\mu_A(y)}](x) \\
&= H_\mu\left[\bigcup_{y\in U}(\mu_{1_y} \cap \widehat{\mu_A(y)})\right](x) \\
&= H_\mu(\mu_A)(x) = \mu_{H(A)}(x),
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{\overline{R}(A)}(x) &= \bigwedge_{y\in U}[(1 - R(x,y)) \vee \gamma_A(y)] \\
&= \bigwedge_{y\in U}[H_\gamma(\gamma_{1_y})(x) \vee \gamma_A(y)] \\
&= \bigwedge_{y\in U}[H_\gamma(\gamma_{1_y}) \cup \widehat{\gamma_A(y)}](x) \\
&= \bigwedge_{y\in U} H_\gamma[\gamma_{1_y} \cup \widehat{\gamma_A(y)}](x) \\
&= H_\gamma\left[\bigcap_{y\in U}(\gamma_{1_y} \cup \widehat{\gamma_A(y)})\right](x) \\
&= H_\gamma(\gamma_A)(x) = \gamma_{H(A)}(x).
\end{aligned}$$

Thus $H(A) = \overline{R}(A)$.

$L(A) = \underline{R}(A)$ follows directly from $H(A) = \overline{R}(A)$ and Definition 5.1. □

Theorem 5.2. *Let $L, H : \mathcal{IF}(U) \to \mathcal{IF}(U)$ be two dual operators. Then there exists a crisp binary relation R on U such that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in \mathcal{IF}(U)$ iff L satisfies the following axioms* (IFLc), (IFL2) *and* (IFL3), *or equivalently, H satisfies axioms* (IFUc), (IFU2) *and* (IFU3)*: $\forall A, B \in \mathcal{IF}(U)$, $\forall(\alpha, \beta) \in L^*$,*

(IFLc) $L(1_{U-\{y\}}) \in \mathscr{P}(U)$, $\forall y \in U$;

(IFL2) $L(A \cup \widehat{(\alpha,\beta)}) = L(A) \cup \widehat{(\alpha,\beta)}$, *i.e.*, $L_\mu(\mu_A \cup \widehat{\alpha}) = L_\mu(\mu_A) \cup \widehat{\alpha}$, $L_\gamma(\gamma_A \cap \widehat{\beta}) = L_\gamma(\gamma_A) \cap \widehat{\beta}$;

(IFL3) $L(A \cap B) = L(A) \cap L(B)$, *i.e.*, $L_\mu(\mu_{A\cap B}) = L_\mu(\mu_A \cap \mu_B) = L_\mu(\mu_A) \cap L_\mu(\mu_B)$, $L_\gamma(\gamma_{A\cap B}) = L_\gamma(\gamma_A \cup \gamma_B) = L_\gamma(\gamma_A) \cup L_\gamma(\gamma_B)$;

(IFUc) $H(1_y) \in \mathscr{P}(U)$, $\forall y \in U$;

(IFU2) $H(A \cap \widehat{(\alpha,\beta)}) = H(A) \cap \widehat{(\alpha,\beta)}$, *i.e.*, $H_\mu(\mu_A \cap \widehat{\alpha}) = H_\mu(\mu_A) \cap \widehat{\alpha}$, $H_\gamma(\gamma_A \cup \widehat{\beta}) = H_\gamma(\gamma_A) \cup \widehat{\beta}$;

(IFU3) $H(A \cup B) = H(A) \cup H(B)$, *i.e.*, $H_\mu(\mu_{A\cup B}) = H_\mu(\mu_A \cup \mu_B) = H_\mu(\mu_A) \cup H_\mu(\mu_B)$, $H_\gamma(\gamma_{A\cup B}) = H_\gamma(\gamma_A \cap \gamma_B) = H_\gamma(\gamma_A) \cap H_\gamma(\gamma_B)$.

Proof. "⇒" follows immediately from Theorem 3.1.

"⇐" Suppose that the operator H satisfies axioms (IFUc), (IFU2), and (IFU3). Then we can define a crisp relation R on U by

$$(x,y) \in R \Longleftrightarrow R(x,y) = 1 \Longleftrightarrow H_\mu(\mu_{1_y})(x) = 1, H_\gamma(\gamma_{1_y})(x) = 0, \quad \forall (x,y) \in U \times U,$$
$$(x,y) \notin R \Longleftrightarrow R(x,y) = 0 \Longleftrightarrow H_\mu(\mu_{1_y})(x) = 0, H_\gamma(\gamma_{1_y})(x) = 1, \quad \forall (x,y) \in U \times U.$$

Thus, similarly to Theorem 5.1, we can conclude that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$. □

Remark 5.1. As can be seen from Theorem 5.1, axioms (IFLf), (IFL1), (IFU1), (IFL2), and (IFL3), or equivalently, axioms (IFUf), (IFL1), (IFU1), (IFU2), and (IFU3) are considered as basic axioms for characterizing IF rough approximation operators. Similarly, according to Theorem 5.2, axioms (IFLc), (IFL1), (IFU1), (IFL2), and (IFL3), or equivalently, axioms (IFUc), (IFL1), (IFU1), (IFU2), and (IFU3) are considered as basic axioms for characterizing rough IF approximation operators. So we have the following definitions of IF rough set algebras and rough IF set algebras.

Definition 5.2. Let $L, H : \mathcal{IF}(U) \to \mathcal{IF}(U)$ be a pair of dual operators. If L satisfies axioms (IFLf), (IFL2) and (IFL3), or equivalently, H satisfies axioms (IFUf), (IFU2) and (IFU3), then the system $(\mathcal{IF}(U), \cap, \cup, \sim, L, H)$ is referred to as an *IF rough set algebra*, L and H are, respectively, called the lower and upper IF rough approximation operators. Furthermore, if L satisfies axioms (IFLc), (IFL2), and (IFL3), or equivalently, H satisfies axioms (IFUc), (IFU2), and (IFU3), then the system $(\mathcal{IF}(U), \cap, \cup, \sim, L, H)$ is referred to as a *rough IF set algebra*, and L and H are, respectively, called the lower and upper rough IF approximation operators.

The following Theorems 5.3–5.7 show that special types of intuitionistic fuzzy approximation operators can be characterized by different axiom sets.

Theorem 5.3. *Let $L, H : \mathcal{IF}(U) \to \mathcal{IF}(U)$ be a pair of dual IF rough approximation operators, i.e., L satisfies axioms* (IFLf),(IFL1),(IFL2), *and* (IFL3), *and H satisfies axioms* (IFUf),(IFU1),(IFU2), *and* (IFU3). *Then there exists a serial fuzzy relation R on U such that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in \mathcal{IF}(U)$ iff L and H satisfy axioms:*

(IFL0) $L(\emptyset) = \emptyset$,
(IFU0) $H(U) = U$,
(IFL0)′ $L(\widehat{(\alpha, \beta)}) = \widehat{(\alpha, \beta)}, \forall (\alpha, \beta) \in L^*$,
(IFU0)′ $H(\widehat{(\alpha, \beta)}) = \widehat{(\alpha, \beta)}, \forall (\alpha, \beta) \in L^*$,
(IFLU0) $L(A) \subseteq H(A)$, *i.e.*, $L_\mu(\mu_A) \subseteq H_\mu(\mu_A), L_\gamma(\gamma_A) \supseteq H_\gamma(\gamma_A), \forall A \in \mathcal{IF}(U)$.

Similarly, if $L, H : \mathcal{IF}(U) \to \mathcal{IF}(U)$ are a pair of dual rough IF approximation operators, i.e., L satisfies axioms (IFLc),(IFL1),(IFL2), *and* (IFL3), *and H satisfies axioms* (IFUc),(IFU1),(IFU2), *and* (IFU3), *then there exists a serial crisp relation R on U such that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in \mathcal{IF}(U)$ iff L and H satisfy axioms* (IFL0),(IFU0),(IFL0)′,(IFU0)′, *and* (IFLU0).

Proof. "⇒" follows from Theorems 3.2 and 4.6, and "⇐" follows from Theorems 3.2, 4.6, 5.1 and 5.2. □

Theorem 5.4. *Let $L, H : \mathcal{IF}(U) \to \mathcal{IF}(U)$ be a pair of dual IF rough approximation operators. Then there exists a reflexive fuzzy relation R on U such that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in \mathcal{IF}(U)$ iff L satisfies axiom* (IFLR), *or equivalently, H satisfies axiom* (IFUR)*:*

(IFLR) $L(A) \subseteq A$, *i.e.*, $L_\mu(\mu_A) \subseteq \mu_A, L_\gamma(\gamma_A) \supseteq \gamma_A, \forall A \in \mathcal{IF}(U)$;
(IFUR) $A \subseteq H(A)$, *i.e.*, $H_\mu(\mu_A) \supseteq \mu_A, H_\gamma(\gamma_A) \subseteq \gamma_A, \forall A \in \mathcal{IF}(U)$.

Similarly, if $L, H : \mathcal{IF}(U) \to \mathcal{IF}(U)$ are a pair of dual rough IF approximation operators, then there exists a reflexive crisp relation R on U such that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in \mathcal{IF}(U)$ iff L satisfies axiom (IFLR), *or equivalently, H satisfies axiom* (IFUR).

Proof. "⇒" follows from Theorems 3.2 and 4.6, and "⇐" follows from Theorems 3.2, 4.6, 5.1 and 5.2. □

Theorem 5.5. *Let $L, H : \mathcal{IF}(U) \to \mathcal{IF}(U)$ be a pair of dual IF rough approximation operators. Then there exists a symmetric fuzzy relation R on U such that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in \mathcal{IF}(U)$ iff L satisfies axiom (IFLS)′, or equivalently, H satisfies axiom* (IFUS)′*:*

(IFLS)′ $L_\mu(\mu_{1_{U-\{x\}}})(y) = L_\mu(\mu_{1_{U-\{y\}}})(x)$, or, $L_\gamma(\gamma_{1_{U-\{x\}}})(y) = L_\gamma(\gamma_{1_{U-\{y\}}})(x), \forall (x, y) \in U \times U$;
(IFUS)′ $H_\mu(\mu_{1_x})(y) = H_\mu(\mu_{1_y})(x)$, or, $H_\gamma(\gamma_{1_x})(y) = H_\gamma(\gamma_{1_y})(x), \forall (x, y) \in U \times U$.

Similarly, if $L, H : \mathcal{IF}(U) \to \mathcal{IF}(U)$ are a pair of dual rough IF approximation operators, then there exists a symmetric crisp relation R on U such that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in \mathcal{IF}(U)$ iff L satisfies axiom (IFLS), *or equivalently, H satisfies axiom* (IFUS)*:*

(IFLS) $A \subseteq L(H(A))$, *i.e.*, $L_\mu(H_\mu(\mu_A)) \supseteq \mu_A, L_\gamma(H_\gamma(\gamma_A)) \subseteq \gamma_A, \forall A \in \mathcal{IF}(U)$;
(IFUS) $H(L(A)) \subseteq A$, *i.e.*, $H_\mu(L_\mu(\mu_A)) \subseteq \mu_A, H_\gamma(L_\gamma(\gamma_A)) \supseteq \gamma_A, \forall A \in \mathcal{IF}(U)$.

Proof. "⇒" follows from Theorems 3.2 and 4.6, and "⇐" follows from Theorems 3.2, 4.6, 5.1 and 5.2. □

Theorem 5.6. *Let $L, H : \mathcal{IF}(U) \to \mathcal{IF}(U)$ be a pair of dual IF rough approximation operators, then there exists a transitive fuzzy relation R on U such that $L(A) = \underline{R}(A)$ and $H(A) = \overline{R}(A)$ for all $A \in \mathcal{IF}(U)$ iff L satisfies axiom* (IFLT), *or equivalently, H satisfies axiom* (IFUT)*:*

(IFLT) $L(A) \subseteq L(L(A))$, *i.e.*, $L_\mu(\mu_A) \subseteq L_\mu(L_\mu(\mu_A)), L_\gamma(\gamma_A) \supseteq L_\gamma(L_\gamma(\gamma_A)), \forall A \in \mathcal{IF}(U)$;
(IFUT) $H(H(A)) \subseteq H(A)$, *i.e.*, $H_\mu(H_\mu(\mu_A)) \subseteq H_\mu(\mu_A), H_\gamma(H_\gamma(\gamma_A)) \supseteq H_\gamma(\gamma_A), \forall A \in \mathcal{IF}(U)$.

Similarly, if $L, H : \mathcal{IF}(U) \to \mathcal{IF}(U)$ *are a pair of dual rough IF approximation operators, then there exists a transitive crisp relation R on U such that* $L(A) = \underline{R}(A)$ *and* $H(A) = \overline{R}(A)$ *for all* $A \in \mathcal{IF}(U)$ *iff L satisfies axiom* (IFLT)*, or equivalently, H satisfies axiom* (IFUT).

Proof. "⇒" follows from Theorems 3.2 and 4.6, and "⇐" follows from Theorems 3.2, 4.6, 5.1 and 5.2. □

According to Theorems 5.4–5.6, we can immediately deduce the following

Theorem 5.7. *Let* $L, H : \mathcal{IF}(U) \to \mathcal{IF}(U)$ *be a pair of dual IF rough approximation operators, then there exists an equivalence fuzzy relation R on U such that* $L(A) = \underline{R}(A)$ *and* $H(A) = \overline{R}(A)$ *for all* $A \in \mathcal{IF}(U)$ *iff L satisfies axioms* (IFLR),(IFLS)′*, and* (IFLT)*, or equivalently, H satisfies axioms* (IFUR),(IFUS)′*, and* (IFUT).

Similarly, if $L, H : \mathcal{IF}(U) \to \mathcal{IF}(U)$ *are a pair of dual rough IF approximation operators, then there exists an equivalence crisp relation R on U such that* $L(A) = \underline{R}(A)$ *and* $H(A) = \overline{R}(A)$ *for all* $A \in \mathcal{IF}(U)$ *iff L satisfies axioms* (IFLR),(IFLS)*, and* (IFLT)*, or equivalently, H satisfies axioms* (IFUR),(IFUS)*, and* (IFUT).

6. Conclusion

Both rough sets and IF sets capture facets of imprecision, a natural extension is to combine the two set theories into a new hybrid one. In this paper, we have introduced two classes of IF approximation operators and investigated their properties. We have defined rough IF sets and IF rough sets which, respectively, resulted from the approximations of AIF sets w.r.t. a crisp approximation space and a fuzzy approximation space. Properties of rough IF approximation operators and IF rough approximation operators corresponding to special approximation spaces have been discussed. An axiomatic approach has been introduced to characterize the intuitionistic fuzzy approximation operators. By this way, we have solved the problem of finding assumptions permitting given AIF set-theoretic operators to represent upper and lower approximations derived from special crisp or fuzzy relations, that is, we have proved that axiom sets of rough IF approximation operators (and IF rough approximation operators, respectively) guarantee the existence of certain types of crisp relations (and fuzzy relations, respectively) producing the same operators.

This work may be viewed as an extension of the study in [16] when the approximated fuzzy sets are replaced by AIF sets. For further study, by employing rough IF sets and IF rough sets we will explore knowledge acquisition in IF information systems.

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References

[1] Z. Pawlak, Rough sets, International Journal of Computer and Information Sciences 11 (1982) 341–356.
[2] Z. Pawlak, A. Skowron, Rudiments of rough sets, Information Sciences 177 (2007) 3–27.
[3] Z. Pawlak, A. Skowron, Rough sets: some extensions, Information Sciences 177 (2007) 28–40.
[4] S. Nanda, S. Majumda, Fuzzy rough sets, Fuzzy Sets and Systems 45 (1992) 157–160.
[5] Z. Pawlak, Rough Sets: Theoretical Aspects of Reasoning about Data, Kluwer Academic Publishers, Boston, 1991.
[6] Y.H. Qian, J.Y. Liang, C.Y. Dang, Converse approximation and rule extraction from decision tables in rough set theory, Computers and Mathematics with Applications 55 (2008) 1754–1765.
[7] E.C.C. Tsang, D.G. Chen, D.S. Yeung, Approximations and reducts with covering generalized rough sets, Computers and Mathematics with Applications 56 (2008) 279–289.
[8] Y.Y. Yao, Constructive and algebraic methods of the theory of rough sets, Journal of Information Sciences 109 (1998) 21–47.
[9] D. Boixader, J. Jacas, J. Recasens, Upper and lower approximations of fuzzy sets, International Journal of General Systems 29 (2000) 555–568.
[10] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, International Journal of General Systems 17 (1990) 191–209.
[11] J.-S. Mi, W.-X. Zhang, An axiomatic characterization of a fuzzy generalization of rough sets, Information Sciences 160 (2004) 235–249.
[12] J.-S. Mi, Y. Leung, H.-Y. Zhao, T. Feng, Generalized fuzzy rough sets determined by a triangular norm, Information Sciences 178 (2008) 3203–3213.
[13] A.M. Radzikowska, Rough approximation operators based on IF sets, in: L. Rutkowski, R. Tadeusiewicz, L.A. Zadeh, J. Zurada (Eds.), Artificial Intelligence and Soft Computing, ICAISC 2006, in: LNAI, vol. 4029, 2006, pp. 528–537.
[14] W.Z. Wu, Y. Leung, J.S. Mi, On characterizations of $(\mathcal{I}, \mathcal{T})$-fuzzy rough approximation operators, Fuzzy Sets and Systems 154 (2005) 76–102.
[15] W.Z. Wu, J.S. Mi, W.X. Zhang, Generalized fuzzy rough sets, Information Sciences 151 (2003) 263–282.
[16] W.Z. Wu, W.X. Zhang, Constructive and axiomatic approaches of fuzzy approximation operators, Information Sciences 159 (2004) 233–254.
[17] D.S. Yeung, D.G. Chen, E.C.C. Tsang, J.W.T. Lee, X.Z. Wang, On the generalization of fuzzy rough sets, IEEE Transactions on Fuzzy Systems 13 (2005) 343–361.
[18] S. Comer, An algebraic approach to the approximation of information, Fundamenta Informaticae 14 (1991) 492–502.
[19] S. Comer, On connections between information systems, rough sets, and algebraic logic, in: C. Rauszer (Ed.), Algebraic Methods in Logic and Computer Science, in: Banach Center Publisher, vol. 28, Polish Academy of Sciences, 1993, pp. 117–127.
[20] T.Y. Lin, Q. Liu, Rough approximate operators: axiomatic rough set theory, in: W. Ziarko (Ed.), Rough Sets, Fuzzy Sets and Knowledge Discovery, Springer, Berlin, 1994, pp. 256–260.
[21] H. Thiele, On axiomatic characterisation of crisp approximation operators, Information Sciences 129 (2000) 221–226.
[22] H. Thiele, On axiomatic characterisation of fuzzy approximation operators I, the fuzzy rough set based case, in: RSCTC 2000, Banff Park Lodge, Bariff, Canada, 2000, Conference Proceedings, pp. 239–247.
[23] Y.Y. Yao, Two views of the theory of rough sets in finite universes, International Journal of Approximate Reasoning 15 (1996) 291–317.

[24] A.M. Radzikowska, E.E. Kerre, A comparative study of fuzzy rough sets, Fuzzy Sets and Systems 126 (2002) 137–155.
[25] A.M. Radzikowska, E.E. Kerre, Characterization of main classes of fuzzy relations using fuzzy modal operators, Fuzzy Sets and Systems 152 (2005) 223–247.
[26] H. Thiele, On axiomatic characterisation of fuzzy approximation operators II, the rough fuzzy set based case, in: Proceeding of the 31st IEEE International Symposium on Multiple-Valued Logic, 2001, pp. 330–335.
[27] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[28] K. Atanassov, Answer to D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk and H. Prade's paper "Terminological dificulties in fuzzy set theory—the case of Intuitionistic Fuzzy Sets", Fuzzy Sets and Systems 156 (2005) 496–499.
[29] G. Cattaneo, D. Ciucci, Basic intuitionistic principles in fuzzy set theories and its extensions (A terminological debate on Atanassov IFS), Fuzzy Sets and Systems 157 (2006) 3198–3219.
[30] D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk, H. Prade, Terminological difficulties in fuzzy set theory—the case of "Intuitionistic Fuzzy Sets", Fuzzy Sets and Systems 156 (2005) 485–491.
[31] J.G. García, S.E. Rodabaugh, Order–theoretic, topological, categorical redundancees of interval-valued sets, grey sets, vague sets, interval-valued "intuitionistic" sets, "intuitionistic" fuzzy sets and topologies, Fuzzy Sets and Systems 156 (2005) 445–484.
[32] P. Grzegorzewski, E. Mrówka, Some notes on (Atanassov's) intuitionistic fuzzy sets, Fuzzy Sets and Systems 156 (2005) 492–495.
[33] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[34] K. Chakrabarty, T. Gedeon, L. Koczy, Intuitionistic fuzzy rough set, in: Proceedings of 4th Joint Conference on Information Sciences, JCIS, Durham, NC, 1998, pp. 211–214.
[35] C. Cornelis, M. De Cock, E.E. Kerre, Intuitionistic fuzzy rough sets: at the crossroads of imperfect knowledge, Expert Systems 20 (2003) 260–270.
[36] S.P. Jena, S.K. Ghosh, Intuitionistic fuzzy rough sets, Notes on Intuitionistic Fuzzy Sets 8 (2002) 1–18.
[37] S.K. Samanta, T.K. Mondal, Intuitionistic fuzzy rough sets and rough intuitionistic fuzzy sets, Journal of Fuzzy Mathematics 9 (2001) 561–582.
[38] S. Rizvi, H.J. Naqvi, D. Nadeem, Rough intuitionistic fuzzy set, in: Proceedings of the 6th Joint Conference on Information Sciences, JCIS, Durham, NC, 2002, pp. 101–104.
[39] Lei Zhou, Wei-Zhi Wu, Wen-Xiu Zhang, On characterization of intuitionistic fuzzy rough sets based on intuitionistic fuzzy implicators, Information Sciences 179 (2009) 883–898.
[40] K. Atanassov, More on intuitionistic fuzzy sets, Fuzzy Sets and Systems 33 (1989) 37–46.
[41] K. Atanassov, Intuitionistic fuzzy Sets, Physica-Verlag, Heidelberg, 1999.
[42] Y.Y. Yao, Generalized rough set model, in: L. Polkowski, A. Skowron (Eds.), Rough Sets in Knowledge Discovery 1. Methodology and Applications, Physica-Verlag, Heidelberg, 1998, pp. 286–318.
[43] C. Cornelis, G. Deschrijver, E.E. Kerre, Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application, International Journal of Approximate Reasoning 35 (2004) 55–95.
[44] G. Deschrijver, E.E. Kerre, On the relationship between some extensions of fuzzy set theory, Fuzzy Sets and Systems 133 (2003) 227–235.