DIFFERENTIAL OPERATORS ON COMPLEX MANIFOLDS WITH A FLAT PROJECTIVE STRUCTURE

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Abstract. - Some natural differential operators on a complex manifold equipped with a flat projective structure have been constructed. As an application, a higher dimensional analog of the Schwarzian derivative has been defined. This higher dimensional analog shares the characteristic properties of the usual one dimensional Schwarzian derivative with respect to the projective transformations.

A canonical decomposition of the space of all differential operators between certain line bundles over a Riemann surface equipped with a projective structure has been described. © Elsevier, Paris

Key words and phrases: Projective structure, differential operator, Schwarzian derivative.

1. Introduction

A flat projective structure on a complex manifold, of complex dimension \(d\), is defined by giving a covering by holomorphic coordinate charts such that all the transition functions are given by the natural action of \(SL(d+1,\mathbb{C})\) on \(\mathbb{C}P^d\). With an abuse of notation, in this sequel we will call a flat projective structure simply a projective structure. For a Riemann surface, there is no distinction between these two notions.

Let \(X\) be a complex manifold of dimension \(d\) equipped with a projective structure. Let \(\mathcal{L}\) be a holomorphic line bundle over \(X\) together with an isomorphism of \(\mathcal{L}^{\oplus(d+1)}\) with the canonical line bundle \(K_X\). The existence of such a line bundle and isomorphism is ensured by the projective structure. Let \(\mathcal{V}\) be the flat vector bundle over \(X\) associated to the projective structure for the standard action of \(SL(d+1,\mathbb{C})\) on \(\mathbb{C}^{d+1}\). In Theorem 3.7 we prove the following theorem:

Theorem A. - For any \(n \geq 0\), the jet bundle \(J^n(\mathcal{L}^{-n})\) over \(X\) has a natural flat connection. Moreover, the flat vector bundle \(J^n(\mathcal{L}^{-n})\) is identified with \(S^n(\mathcal{V}^*)\). For \(m \geq n\), the restriction homomorphism,

\[
J^m(\mathcal{L}^{-n}) \longrightarrow J^n(\mathcal{L}^{-n})
\]

admits a canonical splitting. In particular, setting \(m = 1\), a differential operator

\[
\mathcal{D}_X(n + 1) \in H^0(X, \text{Diff}_X^{n+1}(\mathcal{L}^{-n}, S^{n+1}(\Omega^1_X) \otimes \mathcal{L}^{-n}))
\]

is obtained, whose symbol is the identity homomorphism of \(S^{n+1}(\Omega^1_X)\).
Theorem A is obtained by studying the jet bundles of the line bundle $\mathcal{U}_{CP^d}(k)$ over $\mathcal{L}^{md}$.

It is well-known that there are rich interconnections between projective structures on a Riemann surface and differential equations. A key factor in these interplay is a (nonlinear) differential equation known as the Schwarzian derivative.

The main point in Section 4 is the construction of a higher dimensional analog of the Schwarzian derivative. In (4.6) we define a general Schwarzian derivative which generalizes the usual Schwarzian derivative in one variable.

The usual Schwarzian derivative of a nonzero function $f$ defined on an open set of $\mathbb{C}$ coincides with that of $G \circ f$, where $G$ is a projective transformation. Moreover the Schwarzian derivative of a nonzero function vanishes identically if and only if it is a projective transformation. In fact the Schwarzian derivative can be characterized using these two properties. These two key properties continue to hold for the higher dimensional analog of the Schwarzian derivative.

From Section 5 onwards we restrict ourselves to the case of Riemann surfaces.

Let $X$ be a Riemann surface equipped with a projective structure. The projective structure on $X$ gives a pair consisting of a holomorphic line bundle $\mathcal{L}$ over $X$ along with an isomorphism of $\mathcal{L}^\otimes 2$ with $K_X$. This pair is determined up to a tensor with a flat line bundle of order two.

We will denote by $\text{Diff}^n(M, M')$ the sheaf of differential operators of order $n$ from sections of a line bundle $M$ over $X$ to sections of another line bundle $M'$ over $X$.

In Section 6 we give a construction of lift of a symbol of differential operator, between certain line bundles, to an actual differential operator. Using this construction of lift of a symbol, the following decomposition of global differential operators is established in Theorem 6.3:

**THEOREM B.** -- Let $X$ be a Riemann surface equipped with a projective structure. Let $k, l \in \mathbb{Z}$, and $n \in \mathbb{N}$ be such that $k \neq [-n + 1, 0]$, and $l - k - j \neq \{0, 1\}$ for any integer $j \in [1, n]$. Then the space of global differential operators of order $n$ from $\mathcal{L}^k$ to $\mathcal{L}^l$, namely $H^0(X, \text{Diff}^n_X(\mathcal{L}^k, \mathcal{L}^l))$, is canonically isomorphic to the direct sum

$$\bigoplus_{i=0}^{n} H^0(X, \mathcal{L}^{-k-2n+2i}) = \bigoplus_{i=0}^{n} H^0(X, \mathcal{L}^{-k} \otimes K_X^{-i})$$

with the property that the image of $H^0(X, \mathcal{L}^{-k-2j})$ by this isomorphism is contained in the subspace $H^0(X, \text{Diff}^j_X(\mathcal{L}^{k}, \mathcal{L}^l))$.

Theorem B implies that $\bigoplus_{m=0}^{j} H^0(X, \mathcal{L}^{-k-2m})$ is canonically isomorphic to the subspace $H^0(X, \text{Diff}^j_X(\mathcal{L}^{k}, \mathcal{L}^l))$ of $H^0(X, \text{Diff}^n_X(\mathcal{L}^{k}, \mathcal{L}^l))$. In other words, the point of Theorem B is to produce a canonical splitting (semisimplification) of the natural filtration of $H^0(X, \text{Diff}^n_X(\mathcal{L}^{k}, \mathcal{L}^l))$, given by its subspaces $H^0(X, \text{Diff}^j_X(\mathcal{L}^{k}, \mathcal{L}^l))$, where $0 \leq j \leq n$.

Theorem B and a construction in Section 3 together have the following corollary (Corollary 6.6 and equation (6.7)):

**COROLLARY C.** -- Let $X$ be a compact Riemann surface equipped with a projective structure. The space of global differential operators of order $n$ from $\mathcal{L}^{-n}$ to $\mathcal{L}^{n+2}$ admits the following natural decomposition:

$$H^0(X, \text{Diff}_{X}^{n+1}(\mathcal{L}^{-n}, \mathcal{L}^{n+2})) = \bigoplus_{i=0}^{n+1} H^0(X, K_X^i).$$
Let X be a compact Riemann surface of genus g equipped with a projective structure. In Theorem 6.9 we prove the following:

**THEOREM D.** Let k, l and n be as in Theorem B, i.e., $k \notin \{-n+1, 0\}$ and $l \in \{-n+k, 0\}$ for any integer $j \in \{1, n\}$. Let $M$ and $M'$ be line bundles over $X$ of degree $k(g-1)$ and $l(g-1)$ respectively. Then $H^0(X, \text{Diff}^\infty_X(M, M'))$ admits the following natural decomposition:

$$H^0(X, \text{Diff}^\infty_X(M, M')) = \bigoplus_{i=0}^{n} H^0(X, \text{Hom}(M, M') \otimes K_X^{-i}).$$

satisfying the property that the image of $H^0(X, \text{Hom}(M, M') \otimes K_X^{-i})$ by this isomorphism is contained in the subspace $H^0(X, \text{Diff}^\infty_X(M, M'))$ of $H^0(X, \text{Diff}^\infty_X(M, M'))$.

Theorem D implies that the direct sum $\bigoplus_{m=0}^{i} H^0(X, \text{Hom}(M, M') \otimes K_X^{-m})$ is canonically isomorphic to the subspace $H^0(X, \text{Diff}^\infty_X(M, M'))$ of $H^0(X, \text{Diff}^\infty_X(M, M'))$.

For a compact Riemann surface $X$, consider the diagonal divisor $D$ on $S = X \times X$. The line bundle $K_S \otimes \mathcal{O}_S(2D)$ is trivializable over any infinitesimal neighborhood of $D$, and it has a canonical trivialization over $2D$. In [BR1] the following result was proved:

There is a natural one-to-one correspondence between the space of all projective structures on $X$ and the space of all trivializations of $K_S \otimes \mathcal{O}_S(2D)$ over $3D$ which coincide with the canonical trivialization over $2D$.

It will be interesting to be able to directly construct the decomposition in Theorem D from such a trivialization over $3D$.

In Section 7 we compute the infinitesimal deformations of a projective structure on a Riemann surface in terms of a third order differential operator constructed in Section 3 [Lemma 7.4]. More precisely, the infinitesimal deformations are parametrized by the first hypercohomology of the two term complex defined by the differential operator.

There is a natural local biholomorphism $H$ from the space of all equivalence classes of projective structures $\mathcal{P}(S)$ on a compact oriented smooth real manifold $S$ to the space of all isomorphism classes of $SL(2, \mathbb{C})$ connections over $S$. The differential of $H$ is described in Proposition 7.10.

The natural symplectic structure on the space of flat connections induces a symplectic structure on $\mathcal{P}(S)$ using the map $H$. This symplectic structure on $\mathcal{P}(S)$ has been expressed in terms of the above mentioned description of the space of infinitesimal deformations of a projective structure as the hypercohomology of a certain complex [Proposition 7.13].

## 2. Projective and affine structures

Let $V$ be a vector space over $\mathbb{C}$ of dimension $d+1$. Let $SL(V)$ denote the group of automorphisms of $V$ whose induced action on the top exterior product, namely $\wedge^{d+1} V$, is the trivial action.

Let $P(V)$ denote the projective space comprising of all lines in $V$. The natural homomorphism

$$\rho : SL(V) \longrightarrow \text{Aut}(P(V))$$

satisfies the property that the image of $SL(V)$ by this isomorphism is contained in the subspace $\text{Aut}(P(V))$ of $\text{Aut}(P(V))$. This isomorphism is described in Proposition 7.10.
is surjective with the finite group \text{center}(SL(V)) as its kernel.

Let \( X \) be a complex manifold of complex dimension \( d \). By a holomorphic coordinate chart on \( X \) we will mean a pair of the form \( (U, \phi) \), where \( U \subseteq X \) is an open subset and \( \phi \) is a biholomorphism from \( U \) onto an open set in \( P(V) \).

**Definition 2.2.** A projective structure on \( X \) is a covering of \( X \) by holomorphic coordinate charts \( \{U_\alpha, \phi_\alpha\} \alpha \in I \), and for each pair \( \alpha, \beta \in I \), is given an element \( G(\alpha, \beta) \in SL(V) \) satisfying the following two conditions:
1. \( \phi_\alpha \circ \phi_\beta^{-1} \) is the restriction of the automorphism \( \rho(G(\alpha, \beta)) \) to \( \phi_\beta(U_\beta) \);
2. \( G(\alpha, \beta) = G(\beta, \alpha)^{-1} \) and \( G(\alpha, \beta) \circ G(\beta, \gamma) \circ G(\gamma, \alpha) = \text{Id} \) (in other words, \( \{G(\alpha, \beta)\} \) is a \( SL(V) \)-valued one cocycle).

Two projective structures \( \{U_\alpha, \phi_\alpha, G(\alpha, \beta)\} \) and \( \{U'_\alpha, \phi'_\alpha, G'(\alpha, \beta)\} \) are called equivalent if their union, namely \( \{U_\alpha \cup U'_\alpha, \phi_\alpha \cup \phi'_\alpha, G(\alpha, \beta) \cup G'(\alpha, \beta)\} \), is again a part of the data for a projective structure.

The above definition of a projective structure is slightly different from the definition of a projective structure according to [Gu1], [Gu2]; in the definition given in [Gu1] and [Gu2], \( SL(V) \) is replaced by \( PGL(V) \). When \( X \) is a Riemann surface, a projective structure according to above Definition 2.2 is equivalent to a projective structure in the sense of [Gu1] and [Gu2], along with a choice of a theta characteristic, i.e., a square-root of the canonical bundle of \( X \). More generally, the obstruction to lifting a \( PGL(V) \) structure on a \( d \) dimensional complex manifold \( X \) to a \( SL(V) \) structure on \( X \) is the image in \( H^2(X, \mathbb{Z}/(d+1)) \) of the first Chern class of the canonical line bundle \( K_X \) over \( X \) [S].

Henceforth, by a projective structure we will always mean an equivalence class of projective structures.

Any Riemann surface admits a projective structure, but in the case of higher dimensions there are nontrivial obstructions for the existence of a projective structure [Gu1], [Gu2].

The tautological line bundle bundle \( \mathcal{O}_{P(V)}(-1) \) over \( P(V) \) will be denoted by \( L \). The action of \( SL(V) \) on \( P(V) \) lifts to an action on \( L \). This implies that for a projective structure on \( X \), the action of \( SL(V) \) on \( L \) induces a line bundle over \( X \). To describe this line bundle over \( X \) in more details, fix a covering of \( X \) by coordinate charts \( \{U_\alpha, \phi_\alpha\} \), compatible with the projective structure. Now consider the pullback of \( L \) over \( U_\alpha \) using \( \phi_\alpha \). Using the lift of the action of \( SL(V) \) to \( L \), these pullback bundles patch together to define a line bundle on \( X \). The patching condition is ensured by the cocycle condition in Definition 2.2.

We will denote by \( \mathcal{L} \) this line bundle over \( X \) obtained above.

Choose and fix, once and for all, a nonzero element, \( \theta \), in the line \( \wedge^{d+1}V \).

The top exterior product \( \wedge^{d+1}T_{P(V)} \) is canonically isomorphic to

\[
L^{-d-1} \otimes (P(V) \times \wedge^{d+1}V),
\]

where \( P(V) \times \wedge^{d+1}V \) denotes the trivial line bundle over \( P(V) \) with \( \wedge^{d+1}V \) as the typical fiber. So trivializing \( \wedge^{d+1}V \) using \( \theta \), we have:

\[
(2.3) \quad K_{P(V)} = L^{d+1},
\]

where \( K_{P(V)} \) denotes the canonical bundle of \( P(V) \).
The above isomorphism (2.3) commutes with the actions of $SL(V)$ on $L$ and $K_{P(V)}$. This implies that

\[(2.4) \quad \mathcal{L}^{d+1} = K_X.\]

Indeed, local identifications between $\mathcal{L}^{d+1}$ and $K_X$, using coordinate charts, compatible with the projective structure, patch compatibly. That they patch compatibly clearly follows from the fact that the isomorphism in (2.3) is equivariant for the actions of $SL(V)$. In other words, local isomorphisms, obtained using (2.3), patch together to give a global isomorphism in (2.4).

This line of argument identifying $SL(V)$ equivariant objects on $Z'(V)$ with objects on a manifold equipped with a projective structure will be a recurrent theme in this paper.

In order to deduce consequences of a projective structure we need to define a special class of projective structure, known as affine structures, where the transition functions are restrictions of affine transformations.

Let $\text{Aff}(V)$ denote the group of affine transformations of $V$, which is the space of all diffeomorphisms of $V$ of the form

\[(2.5) \quad v \mapsto T(v) = Av + w,\]

where $A \in GL(V)$ and $w \in V$.

Let $Y$ be a complex manifold of dimension $d + 1$. An affine structure on $Y$ is an equivalence class of coverings of $Y$ by coordinate charts of the form $\{U_\alpha, \phi_\alpha\}_{\alpha \in \mathcal{I}}$, where $\phi_\alpha$ is a biholomorphism from the open subset $U_\alpha$ (of $Y$) to an open subset of $V$, such that any transition function $\phi_\alpha \circ \phi_\beta^{-1}$ is the restriction (to $\phi_\alpha(U_\beta)$) of an affine transformation.

A compact Riemann surface admits an affine structure if and only if its genus is one [Gu1].

Given an affine structure on $Y$, using the multiplication by scalars on $V$, it is possible to choose a covering of $Y$ by compatible affine coordinate charts such that the transition functions are of the form

\[v \mapsto Av + w,\]

(as in (2.5)) with $A \in SL(V)$. Henceforth, whenever we will consider a covering by affine coordinate charts, the above condition will be assumed to be valid.

Consider the vector space $\overline{V} := \mathbb{C} \oplus V$. The subgroup of $\text{Aff}(V)$ consisting of automorphisms as in (2.5) with $A \in SL(V)$ admits a natural embedding in $SL(\overline{V})$ using the homomorphism defined by

\[T := (A, w) \mapsto \begin{pmatrix} 1 & 0 \\ w & A \end{pmatrix} \in SL(V),\]

where $T$ as in (2.5).

Thus, an affine structure modeled on $V$ gives a projective structure modeled on $\overline{V}$ (on a different manifold). Conversely, starting from a manifold with a projective structures, we will next describe the construction of a complex manifold equipped with an affine structure.
Let $\mathcal{L}$ denote the zero section of $L$ consisting of all zero vectors in $L$. The space $L - Z$ consisting of all nonzero vectors in $L$ is:

$$L - Z = V' := V - \{0\}.$$ 

Now $V'$ has a natural affine structure (simply using its inclusion in $V$ as the coordinate chart). The action of $SL(V)$ on $V'$ obviously preserves this affine structure.

Let $X$ be a complex manifold equipped with a projective structure. Let $Z$ denote the zero section of the line bundle $L$ over $X$ defined earlier.

Using its $SL(V)$ equivariance property, the affine structure on $L - Z$ induces an affine structure on the complement $L - Z$ consisting of all nonzero vectors of $L$. Since the multiplication by a constant scalar preserves the affine structure of $V'$, the affine structure on $L - Z$ induces an affine structure on the quotient

$$\mathcal{L} - Z \{\exp(2\pi \sqrt{-1}/(d + 1))\},$$

where $\{\exp(2\pi \sqrt{-1}/(d + 1))\}$ is the group of $(d + 1)$-th roots of 1.

Let $K'_X := K_X - \{0\}$ denote the space of nonzero vectors in $K_X$. The space $K'_X$ gets identified with the quotient space in (2.6) using the isomorphism (2.4) and the étale covering map

$$p : \mathcal{L} - Z \rightarrow K'_X$$

defined by $p(v) = v^{\otimes (d + 1)}$.

Using the above identification, the affine structure of the quotient space in (2.6) induces an affine structure on $K'_X$.

We put down the above observations in the form of the following lemma:

**Lemma 2.7.** For a complex manifold $X$ equipped with a projective structure, the space of nonzero vectors in its canonical bundle, namely $K'_X$, has a natural affine structure.

We will now derive some consequences of an affine structure, which will, finally, be used to derive consequences of a projective structure.

The tangent bundle over $V$, which is the trivial bundle with $V$ itself as fiber, has a natural flat connection given by this trivialization. This flat connection, which we will denote by $\nabla^V$, clearly commutes with the action of $\text{Aff}(V)$ on the base manifold $V$. The action of $\text{Aff}(V)$ on the fiber is the trivial action.

Let $Y$ be a complex manifold equipped with an affine structure.

The $\text{Aff}(V)$ equivariance property of $\nabla^V$ mentioned above implies that it induces a flat connection on the tangent bundle of $Y$; this connection on $T_Y$ will be denoted by $\nabla^Y$.

Since the connection $\nabla^V$ is torsion-free, $\nabla^Y$ is also torsion-free.

The vector $\theta$ gives a section of the canonical bundle $K_Y$, which is flat with respect to the connection on $K_Y$, induced by $\nabla^V$.

Thus, using Lemma 2.7 the above observations give the following theorem:

**Theorem 2.8.** For a complex manifold $Y$ equipped with an affine structure, the tangent bundle $T_Y$ has a natural flat torsion-free connection. The canonical bundle $K_Y$ admits a
natural flat (nonzero) section. In particular, for any \( X \) equipped with a projective structure, the tangent bundle of \( K'_X \) (nonzero vectors in \( K'_X \)) has a natural flat torsion-free connection; and furthermore, the canonical bundle of \( K'_X \) admits a natural flat (nonzero) section.

It is easy to see that the canonical bundle of \( K'_X \) has a tautological section (\( X \) is not required to have a projective structure). Indeed, the pullback of \( K_X \) over \( X \), to \( K'_X \), has a nowhere zero tautological section; now, this, tensored with the natural section of the relative canonical bundle (for the projection of \( K'_X \) to \( X \)), gives a section of the canonical bundle of \( K'_X \). Evidently, this section coincides with the section of the canonical bundle of \( K'_X \) obtained in Theorem 2.8.

Theorem 2.8 immediately implies that if a compact Riemann surface admits an affine structure then its genus must be one. Also, if \( X \) is a complex manifold with \( K_X \) trivial, and not all Chern classes \( c_i(T_X) \), \( i \geq 1 \), vanish (for example, a \( K3 \) surface or a smooth quintic threefold), then \( X \) cannot admit a projective structure.

Let \( \tilde{V} \) denote the trivial vector bundle over \( P(V) \) with \( V \) as the typical fiber. The diagonal action of \( SL(V) \) on \( P(V) \times V \) preserves the natural flat connection on \( \tilde{V} \) given by its trivialization. This property of the action of \( SL(V) \) implies that the flat connection on \( \tilde{V} \) induces a vector bundle equipped with a natural flat connection on any \( X \) equipped with a projective structure.

We will denote by \( \mathcal{V} \) this flat vector bundle over \( X \) obtained above.

The quotient vector bundle \( \tilde{V}/L \) over \( P(V) \) is denoted by \( Q \). On \( P(V) \) we have the following exact sequence of vector bundles:

\[
0 \longrightarrow \mathcal{O}_{P(V)} \longrightarrow L^{\ast} \bigotimes \tilde{V} \longrightarrow T_{P(V)} = \text{Hom}(L, Q) \longrightarrow 0.
\]

This exact sequence is equivariant for the actions of \( SL(V) \) on all the factors in the sequence. The action on \( \mathcal{O}_{P(V)} \) is the trivial action on the fiber factor. Therefore, (2.9) induces an exact sequence

\[
0 \longrightarrow \mathcal{O}_X \longrightarrow L^{\ast} \bigotimes \mathcal{V} \longrightarrow T_X \longrightarrow 0
\]

of vector bundles over \( X \).

Since \( \mathcal{V} \) has a flat connection, its Chern character has the following simple form:

\[
\text{Ch}(\mathcal{V}) = d + 1.
\]

Therefore, the exact sequence (2.10) puts the following restriction on the Chern classes of \( T_X \):

**Proposition 2.11.** - For a complex manifold \( X \) equipped with a projective structure, the Chern character of the tangent bundle \( T_X \) is

\[
\text{Ch}(T_X) = (d + 1) \exp(-c_1(L)) - 1 = (d + 1) \exp(-c_1(K_X)/(d + 1)) - 1.
\]

(The last equality follows from (2.3).)

The above proposition is valid for any manifold admitting a holomorphic projective connection [Gu2], Theorem 5, page 94. The Proposition 2.11 provides a very simple proof for the special case of flat projective connections.

In the next section we will construct some natural differential operators on a manifold equipped with a projective structure.
3. Jet bundles over a manifold with a projective structure

In this section we will construct some natural differential operators on a manifold equipped with a projective structure. This will be used in the next section to construct a Schwarzian derivative for higher dimensional manifolds with a projective structure.

Let $E$ be a holomorphic vector bundle over a complex manifold $X$. The $n$-th jet bundle, namely $J^n(E)$, is defined to be the following direct image on $X$:

$$p_1^*((p_2^*E) \otimes (O_{X \times X}/\mathcal{I}^{n+1})),$$

where $p_i$ is the projection of $X \times X$ onto its $i$-th factor, and $\mathcal{I} \subset O_{X \times X}$ is the ideal sheaf for the diagonal in $X \times X$.

The jet bundles fit into the following exact sequence of vector bundles over $X$:

$$0 \rightarrow S^n(\Omega^1_X) \otimes E \rightarrow J^n(E) \rightarrow J^{n-1}(E) \rightarrow 0,$$

where $S^n(\Omega^1_X)$ is the $n$-th symmetric power of the (holomorphic) cotangent bundle $\Omega^1_X$.

The above surjective homomorphism $J^n(E) \rightarrow J^{n-1}(E)$ is simply the restriction of a section of $E$ over the $(n+1)$-th order infinitesimal neighborhood of a point in $X$ to the $n$-th order infinitesimal neighborhood.

The inclusion of the vector bundle $S^n(\Omega^1_X) \otimes E$ in $J^n(E)$ is defined as follows: given any point $x \in X$, for a local holomorphic function $f$ with $f(x) = 0$ and a local section $s$ of $E$ around $x$, the image of $(df)^{\otimes n} \otimes s$ in $J^n(E)_x$ coincides with the image of $f^ns$.

Let $\hat{V}^*(n)$ denote the trivial vector bundle over $P(V)$ with $S^n(V^*)$ as the typical fiber; $\hat{V}^*(0)$ will denote the trivial line bundle with $\mathbb{C}$ as the typical fiber.

**Lemma 3.2.** - For any $n \geq 0$, the vector bundle $J^n(L^{-n})$ over $P(V)$ is canonically isomorphic to $\hat{V}^*(n)$. Moreover, for any $m \geq n$, the surjective (restriction) homomorphism, $J^m(L^{-m}) \rightarrow J^n(L^{-n})$

given by (3.1), admits a canonical splitting.

**Proof.** - For $n \geq 0$, we have

$$H^0(P(V), L^{-n}) = S^n(V^*).$$

$S^0(V^*)$ is defined to be $\mathbb{C}$.

For any $x \in P(V)$ consider the homomorphism

$$(3.3) S^n(V^*) = H^0(P(V), L^{-n}) \rightarrow J^n(L^{-n})_x$$

obtained by restricting a section of $L^{-n}$ to the $n$-th order infinitesimal neighborhood of $x$. ($J^n(L^{-n})_x$ is the fiber of $J^n(L^{-n})$ at $x$.)

We want to prove that the above homomorphism in (3.3) is actually an isomorphism. Towards this first note that:

$$\dim H^0(P(V), L^{-n}) = \left(\begin{array}{c} d+n \\ d \end{array}\right) = \dim J^n(L^{-n})_x.$$
So in order to prove that the homomorphism (3.3) is an isomorphism it is enough to show that it is injective. Assume that a nonzero section \( s \in H^0(P(V), L^{-n}) \) vanishes at \( x \) of order \( n + 1 \). If \( d \geq 2 \) then take a line \( P^1 \) in \( P(V) \) passing through \( x \) and not contained in the divisor for \( s \). The section \( s \) gives a nonzero section
\[
\tilde{s} \in H^0(P^1, \mathcal{O}_{P^1}(n))
\]
which vanishes of order \( n + 1 \) at \( x \in P^1; \mathcal{O}_{P^1}(n) \) is the line bundle of degree \( n \) over \( P^1 \) (note that the restriction of \( L^{-n} \) to \( P^1 \) is \( \mathcal{O}_{P^1}(n) \)). However, this is impossible since a section of \( \mathcal{O}_{P^1}(n) \) vanishing of order \( n + 1 \) at \( x \) is actually a section of \( \mathcal{O}_{P^1}(-1) \), and \( \mathcal{O}_{P^1}(-1) \) does not have any nonzero section. If \( d = 1 \), then repeat the above argument with \( P^1 = P(V) \). This completes the proof of the first part of Lemma 3.2.

To prove the second part of the lemma, simply consider the homomorphism:
\[
H^0(P(V), L^{-n}) \longrightarrow J^m(L^{-n})_x
\]
defined by restricting a section of \( L^{-n} \) to the \( m \)-th order infinitesimal neighborhood of \( x \). Now the isomorphism (3.3) gives the required splitting homomorphism, namely
\[
\rho(n, m) : J^n(L^{-n}) \longrightarrow J^m(L^{-n}),
\]
whose composition with the (restriction) homomorphism, given by (3.1), is the identity homomorphism of \( J^n(L^{-n}) \). This completes the proof of the lemma. \( \square \)

The vector bundle \( J^1(L^{-n}) \) admits a natural lift of the action of \( SL(V) \) on \( P(V) \), induced by the action of \( SL(V) \) on \( L \). Evidently, the above homomorphism \( \rho(n, m) \) commutes with this action of \( SL(V) \).

The sheaf of differential operators of order \( n \), from sections of a vector bundle \( E \) over \( X \) to the sections of another vector bundle \( F \), denoted by \( \text{Diff}^n_X(E, F) \), is defined as follows:
\[
(3.4) \quad \text{Diff}^n_X(E, F) := \text{Hom}(J^n(E), F).
\]
Given a homomorphism \( J^n(E) \to F \), we may restrict it to \( S^n(\Omega^1_X) \otimes E \) (using (3.1)). The homomorphism,
\[
(3.5) \quad \text{Diff}^n_X(E, F) \longrightarrow \text{Hom}(S^n(\Omega^1_X) \otimes E, F),
\]
thus obtained, is known as the symbol map.

Taking \( m = n + 1 \) in Lemma 3.2, the splitting \( \rho(n, n + 1) \) of the exact sequence (3.1) gives a homomorphism,
\[
J^{n+1}(L^{-n}) \longrightarrow S^{n+1}(\Omega^1_{P(V)}) \otimes L^{-n},
\]
which, by the definition (3.4), is a differential operator of order \( n + 1 \). Let
\[
(3.6) \quad \mathcal{D}(n+1) \in H^0(P(V), \text{Diff}_{P(V)}^{n+1}(L^{-n}, S^{n+1}(\Omega^1_{P(V)}) \otimes L^{-n}))
\]
denote the differential operator obtained this way. The symbol of \( \mathcal{D}(n) \) (defined in (3.5)) is a section of \( \text{Hom}(S^n(\Omega^1_{P(V)}), S^n(\Omega^1_{P(V)}) \otimes L^{-n}) \). Since \( \mathcal{D}(n) \) is given by a splitting of the jet.
sequence (3.1), it is easy enough to check that its symbol is the identity homomorphism of $S^n(\Omega^1_P(V))$.

From the canonical nature of the construction of $\mathcal{D}(n + 1)$ it is evident that the section $\mathcal{D}(n + 1)$ is an invariant for the action of $SL(V)$ on

$$\text{Diff}^{n+1}_P(L^{-n}, S^{n+1}(\Omega^1_P(V)) \otimes L^{-n})$$

induced by the actions of $SL(V)$ on $P(V)$ and $L^{-n}$.

Let $\mathcal{D}(0)$ is the exterior differential operator, i.e., it maps a locally defined holomorphic function $f$ to its differential $df$.

Let $X$ be a complex manifold equipped with a projective structure. Recall the line bundle $\mathcal{L}$ defined in Section 2. Lemma 3.2 and the above constructions give the following theorem on jets of $\mathcal{L}$:

**Theorem 3.7.** For any $n \geq 0$, the jet bundle $J^n(\mathcal{L}^{-n})$ over $X$ has a natural flat connection. Moreover, the flat vector bundle $J^n(\mathcal{L}^{-n})$ is identified with $S^n(\mathcal{V}^*)$. (The flat vector bundle $\mathcal{V}$ was defined in Section 2.) For $m \geq n$, the restriction homomorphism,

$$J^m(\mathcal{L}^{-m}) \rightarrow J^n(\mathcal{L}^{-n})$$

admits a canonical splitting. In particular, setting $m = 1$, a differential operator

$$\mathcal{D}_X(n + 1) \in H^0(X, \text{Diff}^{n+1}_X(\mathcal{L}^{-n}, S^{n+1}(\Omega^1_X) \otimes \mathcal{L}^{-n}))$$

is obtained, whose symbol is the identity homomorphism of $S^{n+1}(\Omega^1_X)$.

Since all our constructions are equivariant for the actions of $SL(V)$, Theorem 3.7 follows immediately. The flat connection on $J^n(\mathcal{L}^{-n})$ is induced by the natural flat connection on $\mathcal{V}^*(n)$ (= $J^n(\mathcal{L}^{-n})$) over $P(V)$ given by its trivialization.

In the special case where $X$ is a Riemann surface, the above construction of differential operators $\mathcal{D}_X(n)$ was done in [Bi]. A different construction of $\mathcal{D}_X(n)$, where $X$ is Riemann surface, can also be found in [BR2].

### 4. The Schwarzian derivative

In this section we will construct a higher dimensional analog of the Schwarzian derivative.

For a holomorphic function, $f(z)$, in one variable, the **Schwarzian derivative** is defined as follows:

$$SD(f)(z) := \frac{2f''(z)f'''(z) - 3(f''(z))^2}{2(f'(z))^2} \tag{4.1}$$

which has the property that if

$$G(z) = \frac{az + b}{cz + d}$$

is a Möbius transformation with $ad - bc = 1$, then we have:

$$SD(G \circ f) = SD(f) \tag{4.2}$$
Furthermore, for a nonconstant function $f$, the equality $\mathcal{S}D(f) = 0$ holds if and only if it is a Möbius transformation.

We will first relate the Schwarzian derivative with the operator $\mathcal{D}_X(2)$ constructed in Theorem 3.7.

Let $f$ be a holomorphic function defined over some neighborhood $U$ of $x \in \mathbb{C}$. Assume that $f'(x) = 0$. Therefore, $f$ is locally a biholomorphism. Shrinking the domain, if necessary, we will assume $f$ to be a biholomorphism from $U$ onto its image.

Take $X$ to be the open set $U$. Let $\mathcal{D} := \mathcal{D}_X(2)$ be the operator given by Theorem 3.7 for the obvious projective structure on $U$ given by its inclusion in $\mathbb{C}$. Since $f$ is a biholomorphism, we may pullback the obvious projective structure on $f(U)$ to get a possibly new projective structure on $U$. Let $\mathcal{D}'$ be the second order operator given by Theorem 3.7 for this (possibly new) projective structure on $U$. Since the symbols of $\mathcal{D}$ and $\mathcal{D}'$ coincide, their difference, namely $\mathcal{D}' - \mathcal{D}$, is a differential operator, of order at most one, mapping (local) sections of $L^{-1}$ to that of $K^2 \otimes L^{-1}$. The following lemma describes the difference $\mathcal{D}' - \mathcal{D}$.

**Lemma 4.3.** The order of the differential operator $\mathcal{D}' - \mathcal{D}$ is zero. More precisely, for a local section, $s$, of $L^{-1}$ over $U$,

$$\mathcal{D}'(s) - \mathcal{D}(s) = \frac{\mathcal{S}D(f)(dz)^{0,2}s}{4},$$

where $z$ is the natural holomorphic coordinate on $U$ given by its inclusion in $\mathbb{C}$.

**Proof.** Let $X$ be a Riemann surface equipped with a projective structure, and let $(W, z)$ be a coordinate chart compatible with the projective structure. By $(dz)^{1/2}$ we will mean a local section of $\mathcal{L}$ over $W$ such that the section $(dz)^{1/2} \otimes (dz)^{1/2}$ (of $\mathcal{L}^{0,2}$) gets identified with the section $dz$ (of $K_W$) using the isomorphism (2.4). The $i$ th tensor power of the section $(dz)^{1/2}$ will be denoted by $(dz)^{i/2}$.

It is easy enough to check the following property of the operator $\mathcal{D}_X(n)$ constructed in Theorem 3.7: for a section $(dz)^{-(n-1)/2}h(z)$ of $L^{-(n-1)}$, where $h$ is a holomorphic function on $W$, the equality

$$(4.4)\quad \mathcal{D}_X(n)((dz)^{-(n-1)/2}h(z)) = \frac{1}{n!}(dz)^{(n+1)/2}\frac{d^n h(z)}{dz^n}$$

holds.

Now using the property (4.2), and the fact that the operator $\mathcal{D}(2)$ in (3.6) is an invariant for the action of $SL(V)$, we may assume that $x = 0$, and that $f$ is of the following type (recall $f'(x) \neq 0)$:

$$f(z) = z + \sum_{i=2}^{\infty} a_i z^i,$$

by composing any $f$ with a suitable Möbius transformation to bring it to the above form.

Let $h$ be the holomorphic function on $U$ such that $s = (dz)^{-1/2}h(z)$, where $z$ is the obvious coordinate function on $U$ given by its inclusion in $\mathbb{C}$. 


Using (4.4) it is an easy computation to check that

\[(D' - D)((dz)^{-1/2} h(z))(0) = \frac{3}{2} \frac{(dz)^{3/2}}{2} (a_3 - a_2^2) h(0).\]

On the other hand, we have

\[SD(f)(0) = 6(a_3 - a_2^2).\]

Comparing this with (4.5) we get that \((D' - D)((dz)^{-1/2} h(z))(0) = SD(f)(0)/4.\) In other words, the equality in Lemma 4.3 is valid at \(z = 0\). Now, for any other point \(y \in U\), using the equality (4.2), and the \(SL(V)\) invariance of \(D(2)\), the equality in Lemma 4.3 is automatically valid, since it is valid at \(z = 0\). Indeed, using some Möbius transformation we may map \(y\) to 0, and then invoke the above argument for \(z = 0\). This completes the proof of the lemma.

In the light of Lemma 4.3, we might view the Schwarzian derivative as a (nonlinear) differential operator mapping sections of \(L^*\) to sections of \(K_X^2 \otimes L^*\); namely, it sends \((dz)^{-1/2} f\) to \((dz)^{3/2} SD(f)\).

Let \(P\) and \(P'\) be two projective structures on a Riemann surface \(X\). The space of projective structures on \(X\) is an affine space for \(H^0(X, K_X^2)\) [Gu1], Theorem 19, page 170]. Let \(\omega \in H^0(X, K_X^2)\) be such that

\[P' - P = \omega.\]

Let us quickly recall the affine space structure (for \(H^0(X, K_X^2)\)) of the space of projective structures on a Riemann surface. Take a coordinate chart \((U,z)\) compatible with the projective structure \(P\). Let \(g\) be the holomorphic function on \(U\) such that

\[\omega(z) = g(z)(dz)^{\otimes 2}.\]

Let \(\psi\) be a function on \(U\) such that \(SD(\psi) = g\). Now \((U, \psi \circ z)\) is a coordinate chart compatible with the projective structure \(P'\).

Let \(D_X(2)\) and \(D_X'(2)\) be the two operators of order two given by Theorem 3.7 corresponding to the projective structures \(P\) and \(P'\) respectively. By Lemma 4.3

\[D_X'(2) - D_X(2)\]

is of order zero, and hence it is a section of \(K_X^2\). This is because an operator of order zero from \(\mathcal{L}^*\) to \(K_X^2 \otimes \mathcal{L}^*\) is simply a section of \(\text{Hom}(\mathcal{L}^*, K_X^2 \otimes \mathcal{L}^*) = K_X^2\).

Take the function \(f\) in Lemma 4.3 to be the function \(\psi\) satisfying the equation \(SD(\psi) = g\), and take \(s = (dz)^{-1/2}\). Combining the equality in Lemma 4.3 with the fact that \((U, \psi \circ z)\) is a coordinate chart compatible with \(P'\), we conclude the following equality:

\[D_X'(2) - D_X(2) = \frac{\omega}{4}.\]

In other words, the operator \(D_X(2)\) in Theorem 3.7 determines the projective structure (on a Riemann surface).
Lemma 4.3 indicates how one might possibly try to generalize the Schwarzian derivative for the case of higher dimensions.

We take $V = \mathbb{C}^d \oplus \mathbb{C}$. So $P(V)$ is the compactification of $\mathbb{C}^d$ obtained by adding the hyperplane at infinity. Let $F: U \rightarrow P(V)$ be a holomorphic map defined on an open subset $U \subseteq \mathbb{C}^d$. Assume that $F$ is a biholomorphism of $U$ onto its image. Let $\mathcal{D}$ and $\mathcal{D}'$ be the two differential operators, of order two, given by Theorem 3.7, for the two projective structures on $U$, namely one given by the inclusion $U \subseteq \mathbb{C}^d$ and the other given by the pullback of the projective structure on $F(U)$ using $F$.

As before, we consider the difference $\mathcal{D}' - \mathcal{D}$, which is a differential operator of order at most one (since the symbols of $\mathcal{D}'$ and $\mathcal{D}$ coincide). Though in the case of $d = 1$, $\mathcal{D}' - \mathcal{D}$ is of order zero, if $d \geq 2$ then there are functions $F$ such that $\mathcal{D}' - \mathcal{D}$ is actually of order one.

Define the following differential operator (of order at most one)

$$(4.6) \quad SD(F) := 2(d + 1)(\mathcal{D}' - \mathcal{D}) \in H^0(U, \text{Diff}_U^2(L^*, S^2(\Omega_U^2) \otimes L^*))$$

over $U$, which maps sections of $L^*$ to sections of $S^2(\Omega_U^2) \otimes L^*$. Define the Schwarzian derivative of $F$ to be the differential operator $SD(F)$, of order at most one, obtained above.

When $d = 1$, the operator $SD(F)$ is of order zero, and it is simply the tensor-product operation with the usual Schwarzian derivative of $F$.

Let $(dz)^{1/(d+1)}$ denote a section of $L$ over $U$ such that $((dz)^{1/(d+1)} \otimes (d+1))$ is the section $dz = dz_1 \wedge \cdots \wedge dz_{d-1} \wedge dz_d$ (of $K_U$) using the isomorphism (2.3) (for the natural projective structure of $U$ given by its inclusion in $\mathbb{C}^d$). Let $SD(F)_0$ be the section of $S^2(\Omega_U^2)$ which satisfies the condition

$$(4.7) \quad SD(F)((dz)^{-1/(d+1)}) = SD(F)_0 \otimes (dz)^{-1/(d+1)}$$

over $U$. Note that since any two choices of $(dz)^{1/(d+1)}$ differ by multiplication with a $(d+1)$-th root of 1, the section $SD(F)_0$ does not depend upon the choice of $(dz)^{1/(d+1)}$.

Let

$$\sigma(SD(F)) := \text{symbol}(SD(F)) \in H^0(U, T_U \otimes S^2(\Omega_U^2))$$

be the symbol of the Schwarzian derivative $SD(F)$ defined in (4.6).

The Schwarzian derivative $SD(F)$ evidently determines, and is determined by, the section

$$SD(F)_0 \oplus \sigma(SD(F)) \in H^0(U, S^2(\Omega_U^2) \oplus (T_U \otimes S^2(\Omega_U^2)))$$

over $U$. If $d = 1$, then $SD(F)_0$ is usual Schwarzian derivative of $F$.

For any $G \in SL(V)$, the two projective structures on $U$ induced by $F$ and $G \circ F$ coincide. This immediately implies that

$$SD(G \circ F) = SD(F),$$

which is a generalization of the property (4.2). For any $G \in SL(V)$ we have $SD(G) = 0$. A projective transformation is clearly determined by its action on the second order infinitesimal neighborhood of a point. This implies that if $SD(F) = 0$, then $F$ must be a projective transformation. Thus, $F$ is a projective transformation if and only if its Schwarzian derivative $SD(F)$ vanishes identically. So we see that the higher dimensional
analog of the Schwarzian derivative shares all the properties of the usual Schwarzian derivative with respect to the projective transformations.

Let \( X \) be a complex manifold equipped with two projective structures, say \( \mathcal{P} \) and \( \mathcal{P} \) respectively. Let \( \phi \) and \( \tilde{\phi} \) be two coordinate charts on an open subset \( U' \subseteq X \), compatible with \( \mathcal{P} \) and \( \mathcal{P} \) respectively. Setting \( U = \phi(U') \) and \( F = \phi \circ \tilde{\phi}^{-1} \), construct the Schwarzian derivative \( SD(F) \). Using \( \phi \) identify \( SD(F) \) as a differential operator

\[
SD(F) \in H^0(U', \text{Diff}^1_{U'}(L^*, S^2(\Omega^1_{U'})) \otimes L^*)
\]

over \( U' \), where \( L \) is the line bundle over \( X \) corresponding to the projective structure \( \mathcal{P} \) (defined in Section 2).

The compatibility properties of the Schwarzian derivative with projective transformations imply that such operators \( SD(F) \) over open sets of \( X \) patch compatibly to give a global differential operator

\[
(4.8) \quad SD(\mathcal{P}, \mathcal{P}) \in H^0(X, \text{Diff}^1_X(L^*, S^2(\Omega^1_X) \otimes L^*))
\]

on \( X \). If \( SD(\mathcal{P}, \mathcal{P}) = 0 \) then \( \mathcal{P} \) and \( \mathcal{P} \) are the two lifts to \( SL(V) \) of the same \( PGL(V) \) structure on \( X \). Indeed, this is a consequence of the earlier observation that \( SD(F) = 0 \) implies that \( F \) must be a projective transformation.

Taking the symbol of \( SD(\mathcal{P}, \mathcal{P}) \) in (4.8), the section

\[
\text{symbol}(SD(\mathcal{P}, \mathcal{P})) \in H^0(X, T_X \otimes S^2(\Omega^1_X))
\]

is obtained on \( X \). However, the section \( SD(F)_0 \), which was constructed locally in (4.7), cannot be made global — the local sections do not patch compatibly.

The higher dimensional generalization of the Schwarzian derivative constructed above was inspired by the work [BR1], which is a mathematical study of problems related to Conformal Field Theory. Different generalizations of the Schwarzian derivative to functions from \( \mathbb{H}^n \) to \( \mathbb{H}^n \) have been proposed in [A] and [OS].

5. A pairing of jet bundles over the projective line

Henceforth, we will always restrict ourselves to the case of a Riemann surface. In other words, it is assumed that \( \dim P(V) = d = 1 \). A projective structure on a Riemann surface is usually defined by giving a covering using holomorphic coordinate charts such that all the transition functions are projective transformations [Gu1]. However any such projective structure has a subcover such that the transition functions lift as an one cocycle with values in \( SL(V) \). Any two such lifts differ by a homomorphism of the fundamental group of the Riemann surface into \( \mathbb{Z}/2\mathbb{Z} \).

As in Section 2, we fix once and for all a nonzero vector \( \theta \in \mathbb{A}^2V \). So \( \theta \) gives a symplectic structure on \( V \).

Take any \( x \in P(V) \); for any \( k \in \mathbb{Z} \) consider following short exact sequence of sheaves over \( P(V) \):

\[
0 \longrightarrow L^\otimes k \otimes \mathcal{O}_{P(V)}(-(n+1)x) \longrightarrow L^\otimes k \longrightarrow J^n(L^\otimes k)_x \longrightarrow 0,
\]
where $J^n(L^k)_{x}$ is the fiber of $J^n(L^k)$ over $x$. The long exact sequence of cohomologies for the above short exact sequence yields the following two observations:

1: If $k > 0$, then the equality

$$J^n(L^k)_{x} = \text{kernel}(H^1(P(V), L^k \otimes \mathcal{O}_{P(V)}(-(n+1)x)) \to H^1(P(V), L^k))$$

holds. (Since $H^0(P(V), L^k) = 0$.)

2: If $k + n \leq 0$, then the equality

$$J^n(L^k)_{x} = \frac{H^0(P(V), L^k)}{H^0(P(V), L^k \otimes \mathcal{O}_{P(V)}(-(n+1)x))}$$

holds. (Since $H^1(P(V), L^k \otimes \mathcal{O}_{P(V)}(-(n+1)x)) = 0$ if and only if $k + n \leq 0$.)

Choose and fix an isomorphism between the two line bundles $L$ and $\mathcal{O}_{P(V)}(-x)$ over $P(V)$. A choice of such an isomorphism corresponds to a choice of a nonzero vector $v \in V$ which is represented by $x \in P(V)$. Indeed, using the Poincaré adjunction formula, the fiber $\mathcal{O}_{P(V)}(-x)_{x}$ is the fiber of the canonical bundle, namely $(K_{P(V)})_{x}$. Now the equality (2.3) implies that any isomorphism from $L_{x}$ to $\mathcal{O}_{P(V)}(-x)_{x}$ is given by tensoring with a nonzero vector $v \in V$ represented by $x$.

Let $v \in V$ denote the vector that gives the isomorphism between $L$ and $\mathcal{O}_{P(V)}(-x)$ that has been fixed. The symplectic structure $\theta$ on $V$ induces an isomorphism between $V$ and $V^*$. Let $v^* \in V^*$ denote the image of $v$ by this isomorphism.

Assume that $k \leq -n$. The right-hand side of (5.3) can be written as follows:

$$\frac{H^0(P(V), L^k)}{H^0(P(V), L^k \otimes \mathcal{O}_{P(V)}(-(n+1)x))} = \frac{H^0(P(V), L^k)}{H^0(P(V), L^{k+n+1})} = S_{-k-n-1}(V^*) = S^n(V^*).$$

(We use the convention that $S^0(V^*) = \mathbb{C}$ and $S^{-1}(V^*) = 0$.) The vector space $S_{-k-n-1}(V^*)$ is realized as a subspace of $S^{-k}(V^*)$, in (5.4), by mapping any $w \in S_{-k-n-1}(V^*)$ to the symmetrization of $w \otimes (v^*)^{\otimes (n+1)}$. The quotient vector space $S_{-k-n-1}(V^*)$ is identified with $S^n(V^*)$ in the following way: consider the surjective homomorphism

$$\psi_w : S^{-k}(V^*) \to S^n(V^*)$$

defined by contracting with $v^\otimes (-k-n)$. It is easy enough to check that the kernel of $\psi_w$ is precisely the image of $S_{-k-n-1}(V^*)$ in $S^{-k}(V^*)$.

Now (5.3) and (5.4) combine together to give an isomorphism from $J^n(L^k)_{x}$ to $S^n(V^*)$. We will denote this isomorphism by $F(x, v)$. If $v$ is replaced by $\lambda v$, where $\lambda \in \mathbb{C}^*$ is any nonzero scalar, then this isomorphism changes by multiplication with the scalar $\lambda^{-k-n}$, i.e.,

$$F(x, \lambda v) = \lambda^{-k-n} F(x, v).$$

Consider the isomorphism

$$f(k,n)_{x} : J^n(L^k)_{x} \to S^n(V^*) \otimes L^k_{x}$$
defined by \( w \mapsto F(x, v)(w) \otimes v^{(k+n)} \). The above equality implies that \( f(k, n) \) does not change if \( v \) is replaced by \( \lambda v \). In other word, \( f(k, n) \) is independent of the choice of the vector \( v \). However \( f(k, n) \) actually depends upon the choice of the symplectic structure \( \theta \).

We will now assume that \( k > 0 \), and try to simplify the right-hand side of (5.2).

Using Serre duality and the equality (2.3), we have the following equality:

\[
(5.6) \quad H^1(P(V), L^k) = H^0(P(V), L^0 \otimes (2-k)^*) = S^{k-2}(V^*) = S^{k-2}(V^*) = S^{k-2}(V^*).
\]

The last equality in (5.6) is given by the symplectic form \( \theta \), which identifies \( V \) with \( V^* \).

As before, choose a nonzero vector, \( v \), in the line in \( V \) represented by \( x \), to get an isomorphism between \( L \) and \( O_{P(V)}(-x) \). So (5.6) gives the following isomorphism:

\[
H^1(P(V), L^k \otimes O_{P(V)}(-(n+1)x)) = S^{k+n-1}(V^*).
\]

The projection, namely

\[
\psi : S^{k+n-1}(V^*) = H^1(P(V), L^k \otimes O((-n+1)x)) \to H^1(P(V), L^k) = S^{k-2}(V^*),
\]

in the right-hand side of (5.2), is the contraction by \( v^{(n+1)} \). The kernel of \( \psi \) is \( S^n(V^*) \), where \( S^n(V^*) \) is realized as a subspace of \( S^{k+n-1}(V^*) \) by mapping any \( w \in S^n(V^*) \) to the symmetrization of \( w \otimes (v^*)^\otimes (k-1) \); the vector \( v^* \in V^* \), as before, is the image of \( v \) by the isomorphism between \( V \) and \( V^* \) given by \( \theta \).

The above observations and the equality (5.2) combine together to give an isomorphism between \( J^n(L^k) \) and \( S^n(V^*) \). It is easy enough to check that if \( v \) is replaced by \( \lambda v \), where \( \lambda \in \mathbb{C} - 0 \), then this isomorphism changes by multiplication with the scalar \( \lambda^{-k-n} \). Thus, the natural isomorphism obtained in (5.5) extends to the case where \( k > 0 \).

As in Section 3, let \( V^*(n) \) denote the trivial vector bundle over \( P(V) \) with \( S^n(V^*) \) as the typical fiber.

From the construction of the isomorphism (5.5) it is evident that the vector bundle isomorphism

\[
f(k, n) : J^n(L^k) \to V^*(n) \otimes L^{k+n},
\]

given by (5.5), is equivariant for the actions of \( SL(V) \). Note that the vector \( \theta \) is left invariant by the action \( SL(V) \) on \( \wedge^2 V \).

We put down the above observations in the form of the following lemma.

**Lemma 5.7.** If \( k > 0 \) or \( k \leq -n \), then we have a natural isomorphism

\[
f(k, n) : J^n(L^k) \to V^*(n) \otimes L^{k+n}
\]

of vector bundles over \( P(V) \). Moreover, this isomorphism is equivariant for the natural actions of \( SL(V) \) on the vector bundles \( J^n(L^k) \) and \( V^*(n) \otimes L^{k+n} \).

Consider the diagonal action of \( SL(V) \) on the tensor product \( S^n(V) \otimes S^n(V) \). It admits the following direct sum decomposition into irreducible \( SL(V) \) representations ([FH], page 151, Exercise 11.11):

\[
S^n(V) \otimes S^n(V) = S^{2n}(V) \oplus S^{2n-2}(V) \oplus S^{2n-4}(V) \oplus \cdots \oplus S^2(V) \oplus S^0(V).
\]

We will first describe the trivial subrepresentation \( S^0(V) \) in the above decomposition.
The symplectic form $\theta$ induces a nondegenerate bilinear form on any symmetric tensor power $S^n(V)$, which is symmetric if $n$ is even and skew-symmetric if $n$ is odd, and it has the further property that for any $u, v \in V$, the following equality is valid:
\[
\langle u \otimes^n v \otimes^n, u \otimes^n v \otimes^n \rangle = \langle u, v \rangle^n.
\]

Since completely decomposable vectors generate the vector space $S^n(V)$, the above condition actually determines the bilinear form. We will denote by $\omega$ this bilinear form on $S^n(V)$. Clearly $\omega$ is an invariant for the natural action of $SL(V)$ on $S^n(V)$. The form $\omega$, being nondegenerate, can be thought of as an element in $S^n(V) \otimes S^n(V)$. We are again using that $\theta$ identifies $V$ with $V^\ast$.

The above element in $S^n(V) \otimes S^n(V)$ given by $\omega$, which we will also denote by $\omega$, generates the trivial subrepresentation $S^0(V)$ in the decomposition (5.8).

We will now give an explicit description of the projection of the tensor product $S^n(V) \otimes S^n(V)$ onto the direct summand $S^0(V)$. The form $\omega$ induces a nondegenerate symmetric bilinear form on $S^n(V) \otimes S^n(V)$ using the following condition: for any $u, v \in S^n(V)$,
\[
\langle u \otimes v, u \otimes v \rangle = \langle u, u \rangle \langle v, v \rangle.
\]

Now consider the surjective homomorphism
\[
S^n(V) \otimes S^n(V) \longrightarrow S^0(V) \quad \text{defined by} \quad v \longmapsto \frac{\langle v, \omega \rangle}{\langle \omega, \omega \rangle} \omega,
\]
which is clearly $SL(V)$ equivariant, and it maps $\omega$ to $\omega$. Hence it must be the projection of $S^n(V) \otimes S^n(V)$ onto the factor $S^0(V)$ in (5.8).

Identifying $S^n(V) \otimes S^n(V)$ with $\text{End}(S^n(V))$ using $\theta$, the direct summand $S^0(V)$ in (5.8) is precisely the space of endomorphisms of the form $\lambda Id_{S^n(V)}$, where $\lambda \in \mathbb{C}$. In this identification $\omega$ corresponds to the identity map, and the projection to $S^0(V)$ is the homomorphism defined by
\[
A \longmapsto \frac{\text{trace}(A)}{\dim S^n(V)}.
\]

Since $S^n(V)$ is an irreducible $SL(V)$-module, by Schur's lemma $Id_{S^n(V)}$ spans the space of $SL(V)$-invariants in $\text{End}(S^n(V))$.

Take pairs of integers $(k_1, n)$ and $(k_2, n)$ satisfying the assumption in Lemma 5.7, i.e., $k_1 \notin [-n + 1, 0]$ and $k_2 \notin [-n + 1, 0]$. Consider the tensor product of homomorphisms given by Lemma 5.7, namely
\[
\text{(5.9)} \quad f(k_1, n) \otimes f(k_2, n) : J^n(L^{\otimes k_1}) \otimes J^n(L^{\otimes k_2}) \longrightarrow \tilde{V}^*(n) \otimes \tilde{V}^*(n) \otimes L^{k_1+k_2+2n}.
\]

Note that $S^0(V) = \mathbb{C}$ using the isomorphism which maps any $\lambda \in \mathbb{C}$ to $\lambda \omega$.

Using the symplectic structure $\theta$, we have $V = V^\ast$. Now using the projection of $S^n(V^\ast) \otimes S^n(V^\ast)$, which is the typical fiber of the trivial vector bundle $\tilde{V}^*(n) \otimes \tilde{V}^*(n)$, onto the line $S^0(V)$ in the decomposition (5.8), the homomorphism in (5.9) induces a homomorphism
\[
\text{(5.10)} \quad \Gamma(k_1, k_2, n) : J^n(L^{\otimes k_1}) \otimes J^n(L^{\otimes k_2}) \longrightarrow L^{k_1+k_2+2n}.
\]
From the construction of the homomorphism $\Gamma(k_1, k_2, n)$ it is evident that this homomorphism is equivariant for natural actions of $SL(V)$ on all the vector bundles in (5.10).

For any $s \in H^0(P(V), L^{k_2})$, with $\tilde{s}$ being the corresponding section of $J^n(L^{k_2})$, the homomorphism

$$D(s) : J^n(L^{k_1}) \rightarrow L^{k_1+k_2+2n},$$

defined by $t \mapsto \Gamma(k_1, k_2, n)(t, \tilde{s})$, gives a global section of the sheaf of differential operators $\text{Diff}^n_P(L^{k_1}, L^{k_1+k_2+2n})$. It is a straightforward calculation to check that the symbol (which is a section of $L^{k_2}$) of the differential operator $D(s)$ is the section $s$ itself. In fact, it is obvious that the symbol must be a constant scalar multiple of $s$. Indeed, the endomorphism of $H^0(P(V), L^{k_2}) = S^{-k_2}(V^*)$, obtained by mapping any $s$ to the symbol of $D(s)$, commutes with the action of $SL(V)$. On the other hand, $S^{-k_2}(V^*)$ is an irreducible $SL(V)$ module. Hence by Schur's lemma, this endomorphism must be a constant scalar multiplication.

For any $s' \in H^0(P(V), L^{k_1})$, using $\Gamma(k_1, k_2, n)$ we may similarly construct an order $n$ differential operator $\tilde{D}(s') \in H^0(P(V), \text{Diff}^n_P(L^{k_2}, L^{k_1+k_2+2n}))$. The symbol of $\tilde{D}(s')$ is $(-1)^n s'$.

6. Jet bundles over a Riemann surface with a projective structure

Let $X$ be a Riemann surface, not necessarily compact, equipped with a projective structure.

As in Section 2 (see above 2.9), let $\mathcal{V}$ denote the flat vector bundle over $X$ that corresponds to the trivial vector bundle over $P(V)$ with $V$ as the typical fiber. As in Section 2, the line bundle over $X$ corresponding to $L$ will be denoted by $L$.

Since the isomorphism $f(k, n)$ in Lemma 5.7 is equivariant for the actions of $SL(V)$, it implies the following:

**Lemma 6.1.** If $k \notin [-n + 1, 0]$, then there is a natural isomorphism

$$f_X(k, n) : J^n(\mathcal{L}^{\otimes k}) \rightarrow S^n(\mathcal{V}^*) \otimes \mathcal{L}^{k+n}$$

of vector bundles over $X$.

Similarly, the homomorphism $\Gamma(k_1, k_2, n)$ in (5.10) induces a homomorphism

$$\Gamma_X(k_1, k_2, n) : J^n(\mathcal{L}^{\otimes k_1}) \otimes J^n(\mathcal{L}^{\otimes k_2}) \rightarrow \mathcal{L}^{k_1+k_2+2n}$$

whenever the pairs of integers $(k_1, n)$ and $(k_2, n)$ satisfy the condition $k_1, k_2 \notin [-n + 1, 0]$.

As described in section 5, for a section $s$ of $\mathcal{L}^{k_2}$ we may construct a differential operator of order $n$ from $\mathcal{L}^{k_1}$ to $\mathcal{L}^{k_1+k_2+2n}$ simply by mapping a local section $t$ of $\mathcal{L}^{k_1}$ to $\Gamma_X(k_1, k_2, n)(t, s)$. The symbol of this differential operator is the section $s$.

Thus, the homomorphism $\Gamma_X(k_1, k_2, n)$ provides us with a way to lift a symbol of differential operator to an actual differential operator. In view of this prescription for lifting symbols of differential operators, it is natural to expect that the space of all differential operators of order $n$, between certain line bundles, may be decomposable into a direct sum.
of symbols of order less than or equal to \( n \). In other words, the filtration (given by order) of the space of all differential operators between certain line bundles may have a natural splitting (semisimplification). The following theorem is in this direction.

**Theorem 6.3.** Let \( X \) be a Riemann surface equipped with a projective structure. Let \( k, l \in \mathbb{Z} \), and \( n \in \mathbb{N} \) be such that \( k \notin [-n + 1, 0] \), and \( l - k - j \notin \{0, 1\} \) for any integer \( j \in [1, n] \). Then the space of global differential operators of order \( n \) from \( \mathcal{L}^k \) to \( \mathcal{L}^l \), namely \( H^0(X, \text{Diff}_X(x, \mathcal{L}^k, \mathcal{L}^l)) \), is canonically isomorphic to the direct sum

\[
\bigoplus_{i=0}^{n} H^0(X, \mathcal{L}^{l-k-2n+2i}) \quad \bigoplus_{i=0}^{n} H^0(X, \mathcal{L}^{l-k} \otimes K_X^{-i})
\]

with the property that the image of \( H^0(X, \mathcal{L}^{l-k-2j}) \) by this isomorphism is contained in the subspace \( H^0(X, \text{Diff}_X(x, \mathcal{L}^k, \mathcal{L}^l)) \) of \( H^0(X, \text{Diff}_X(x, \mathcal{L}^k, \mathcal{L}^l)) \).

**Proof.** For any section \( s \in H^0(X, \mathcal{L}^{l-k-2j}) \), where \( 0 \leq j \leq n \), consider the differential operator of order \( j \) defined by

\[
t \mapsto \Gamma_X(k, l - k - 2j, j)(t, s),
\]

which we will denote by \( F_j(s) \). Using the obvious inclusion, namely \( \text{Diff}_X(x, \mathcal{L}^k, \mathcal{L}^l) \subseteq \text{Diff}_X(x, \mathcal{L}^k, \mathcal{L}^l) \), we will consider \( F_j(s) \) as a section of \( \text{Diff}_X(x, \mathcal{L}^k, \mathcal{L}^l) \). Thus, we have a homomorphism from the direct sum in Theorem 6.3 to \( H^0(X, \text{Diff}_X(x, \mathcal{L}^k, \mathcal{L}^l)) \), defined by

\[
(s_0, s_1, \ldots, s_n) \mapsto \sum_{j=0}^{n} F_j(s_j),
\]

where \( s_j \in H^0(X, \mathcal{L}^{l-k-2j}) \).

In order to construct the inverse homomorphism of (6.4), take any differential operator \( D \in H^0(X, \text{Diff}_X(x, \mathcal{L}^k, \mathcal{L}^l)) \); let \( \sigma(D) \in H^0(X, \mathcal{L}^{l-k-2n}) \) be the symbol of \( D \). Consider the differential operator, \( \Delta_n \) defined by

\[
t \mapsto \Gamma_X(k, l - k - 2n, n)(t, \sigma(D)).
\]

Since the symbol of \( \Delta_n \) is also \( \sigma(D) \), the differential operator given by the difference, namely

\[
D_{n-1} := D - \Delta_n
\]

is of order at most \( n - 1 \). Using the natural inclusion of the space of lower order differential operator into the space of higher order operators, we will consider \( D_{n-1} \) as a section of \( \text{Diff}_X^{n-1}(\mathcal{L}^k, \mathcal{L}^l) \).

Repeat the above construction of \( \Delta_n \) by replacing \( D \) with \( D_{n-1} \) and call the differential operators of order \( n - 1 \), obtained this way, as \( \Delta_{n-1} \). Then define \( D_{n-2} := D_{n-1} - \Delta_{n-1} \). Now we may replace \( D \) with \( D_{n-2} \). Iterating this process we get operators \( D_i \) and \( \Delta_i \) of order \( i \), where \( 0 \leq i \leq n - 1 \), with

\[
D_i = D_{i+1} - \Delta_{i+1}
\]

for \( i \geq 0 \), and \( D_0 = \Delta_0 \).
At the \( j \)-th inductive step the homomorphism \( \Gamma_X(k, l - k - 2(n - j), n - j) \) is used in order to construct \( \Delta_{n-j} \). The numerical conditions imposed on \( k \) and \( l \) ensure that at each step of the above inductive argument, \( k \) and \( l - k - 2(n - j) \) are in the allowed range for \( \Gamma_X(k, l - k - 2(n - j), n - j) \), namely \( k, l - k - 2(n - j) \notin [j - n + 1, 0] \).

Mapping \( D \) to

\[
\sigma(D) + \sum_{i=0}^{n-1} \sigma(D_i),
\]

where \( \sigma \) is the symbol map, we get the required inverse homomorphism of (6.4). That it is actually the inverse homomorphism follows from the equality, namely \( D = \sum_{i=0}^{n} \Delta_i \), and the fact that the symbols of \( D_i \) and \( \Delta_i \) coincide. This completes the proof of the theorem. \( \square \)

Evidently, Theorem 6.3 implies that for a Riemann surface equipped with a projective structure, the direct sum \( \bigoplus_{m=0}^{j} H^0(X, \mathcal{L}^{l-k-2m}) \) is canonically isomorphic to the subspace \( H^0(X, \mathcal{D}^0_X(\mathcal{L}^k, \mathcal{L}^l)) \) of \( H^0(X, \mathcal{D}^n_X(\mathcal{L}^k, \mathcal{L}^l)) \).

We will make two remarks about the decomposition of \( H^0(X, \mathcal{D}^n_X(\mathcal{L}^k, \mathcal{L}^l)) \) in Theorem 6.3.

**Remark 6.5.**

1. The isomorphism of the spaces of sections in Theorem 6.3, namely

\[
H^0(X, \mathcal{D}^n_X(\mathcal{L}^k, \mathcal{L}^l)) = \bigoplus_{i=0}^{n} H^0(X, \mathcal{L}^{l-k} \otimes K_X^{-i}),
\]

is not induced by some vector bundle homomorphism between \( \mathcal{D}^n_X(\mathcal{L}^k, \mathcal{L}^l) \) and \( \bigoplus_{i=0}^{n} \mathcal{L}^{l-k} \otimes K_X^{-i} \) — it does not commute with the multiplication by meromorphic functions (whenever applicable). Also, the vector bundles \( J^n(\mathcal{L}^k) \) are not, in general, decomposable into a direct sum of line bundles.

2. Theorem 6.3 perhaps suggests the possible validity of the following general decomposition:

\[
H^0(X, \mathcal{D}^n_X(\mathcal{L}^k, \mathcal{L}^l)) = \bigoplus_{i=0}^{n} H^0(X, \mathcal{L}^{l-k} \otimes K_X^{-i})
\]

for any \( k, l, n \). However, this general statement is not valid. Indeed, for example we have

\[
H^0(X, K_X) \oplus H^0(X, \mathcal{O}_X) \neq H^0(X, \mathcal{D}^1_X(\mathcal{L}, \mathcal{L}^3)) = H^0(X, K_X).
\]

Thus, some numerical conditions on \( k \), \( l \) and \( n \) are essential.

Setting \( k = -n \) and \( l = n + 2 \) in Theorem 6.3, and using the equality \( L^{\otimes 2} = K_X \), we obtain the following corollary:

**Corollary 6.6.** Let \( X \) be a compact Riemann surface equipped with a projective structure. The space of global differential operators of order \( n \) from \( \mathcal{L}^{-n} \) to \( \mathcal{L}^{n+2} \) admits the following natural decomposition:

\[
H^0(X, \mathcal{D}^n_X(\mathcal{L}^{-n}, \mathcal{L}^{n+2})) = \bigoplus_{i=0}^{n+1} H^0(X, K_X^i).
\]
In Theorem 3.7 we constructed an operator
\[ D_X(n + 1) \in H^0(X, \text{Diff}_{X}^{n+1}(\mathcal{L}^{-n}, \mathcal{L}^{n+2})) \]
whose symbol is the constant function 1. Combining this construction with Corollary 6.6, we conclude that for a compact Riemann surface \( X \), equipped with a projective structure, the following natural decomposition of global differential operators holds:
\[
H^0(X, \text{Diff}^{n+1}_{X}(\mathcal{L}^{-n}, \mathcal{L}^{n+2})) = \bigoplus_{i=0}^{n+1} H^0(X, K^i_X).
\]

Let \( X \) be a compact Riemann surface of genus \( g \) equipped with a projective structure. Let \( M \) and \( M' \) be two line bundles over it of degree \( k_1(g - 1) \) and \( k_2(g - 1) \) respectively. So
\[
M = \mathcal{L}^{k_1} \otimes \xi \quad \text{and} \quad M' = \mathcal{L}^{k_2} \otimes \xi',
\]
where \( \xi \) and \( \xi' \) are degree zero line bundles. This implies that \( \xi \) and \( \xi' \) have natural unitary flat connections, which we will denote by \( \nabla \) and \( \nabla' \) respectively. Assume that the pairs \( (k_1, n) \) and \( (k_2, n) \) satisfy the condition in Lemma 6.1, i.e., \( k_1, k_2 \notin [-n + 1, 0] \). It is straight-forwards to check that using \( \nabla \) and \( \nabla' \), the homomorphism \( \Gamma_X(k_1, k_2, n) \), defined in (6.2), induces a homomorphism
\[
(6.8) \quad \Gamma_X^\prime : J^n(M) \otimes J^n(M') \rightarrow M \otimes M' \otimes K^n_X.
\]
Indeed, \( \Gamma_X^\prime \) is determined by the following condition: for (local) sections \( s \) and \( t \) of \( \mathcal{L}^{k_1} \) and \( \mathcal{L}^{k_2} \) respectively, and (local) flat sections \( u \) and \( v \) of \( \xi \) and \( \xi' \) respectively,
\[
\Gamma_X^\prime(s \otimes u, t \otimes v) = \Gamma_X(k_1, k_2, n)(s, t) \otimes u \otimes v.
\]

Now we have the following generalization of Theorem 6.3 for a compact Riemann surface with a projective structure:

**Theorem 6.9.** Let \( X \) be a compact Riemann surface of genus \( g \) equipped with a projective structure. Let \( k, l \in \mathbb{Z} \) and \( n \in \mathbb{N} \) be as in Theorem 6.3, i.e., \( k \notin [-n + 1, 0] \) and \( l \neq j \notin \{0, 1\} \) for any integer \( j \in [1, n] \). Let \( M \) and \( M' \) be line bundles over \( X \) of degree \( k(g - 1) \) and \( l(g - 1) \) respectively. Then \( H^0(X, \text{Diff}_{X}^{n}(M, M')) \) admits the following natural following decomposition:
\[
H^0(X, \text{Diff}_{X}^{n}(M, M')) = \bigoplus_{i=0}^{n} H^0(X, \text{Hom}(M, M') \otimes K^{-i}_X),
\]
satisfying the property that the image of \( H^0(X, \text{Hom}(M, M') \otimes K^{-1}_X) \) by this isomorphism is contained in the subspace \( H^0(X, \text{Diff}_{X}^{n}(M, M')) \) of \( H^0(X, \text{Diff}_{X}^{n}(M, M')) \).

The proof of Theorem 6.9 is exactly the same as that of Theorem 6.3. As in the proof of Theorem 6.3, the key point in the argument is the following: for any section \( s \) of \( M' \), the symbol of the order \( n \) differential operator \( (\in H^0(X, \text{Diff}_{X}^{n}(M, M' \otimes K^{-n}_X))) \), defined by \( t \mapsto \Gamma_X(t, s) \), is \( s \). The validity of this property is actually an immediate consequence of the fact that the differential operators constructed using the homomorphism \( \Gamma_X(k_1, k_2, n) \), defined in (6.2), have this property.

Theorem 6.9 implies that for a compact Riemann surface equipped with a projective structure, the direct sum \( \bigoplus_{n=0}^{\infty} H^0(X, \text{Hom}(M, M') \otimes K^{-n}_X) \) is canonically isomorphic to the subspace \( H^0(X, \text{Diff}_{X}^{n}(M, M')) \) of \( H^0(X, \text{Diff}_{X}^{n}(M, M')) \).

It is easy enough to see that the decompositions of global differential operators given in Theorems 6.3 and 6.9 actually depend upon the projective structure on \( X \).
7. Infinitesimal deformations of a projective structure

Let $S$ be a connected oriented $C^\infty$ real manifold of dimension two. Assume the following condition on its genus:

$$g = \text{genus}(S) \geq 2.$$ 

Let $\mathcal{P}(S)$ denote the space of all equivalence classes $\mathcal{P}$ of projective structure on $S$ such that the orientation of $S$ is compatible with the orientation induced by the projective structure. (Recall that a projective structure on $S$ induces a complex structure on $S$.) We will call two projective structures, say $P$ and $Q$, on $S$ to be equivalent if there is a diffeomorphism of $S$, say $f$, which is homotopic to the identity map of $S$, and $f^*P = Q$.

The space $\mathcal{P}(S)$ has a natural structure of a complex manifold of dimension $6g - 6$. Given a projective structure, we consider the monodromy representation of the flat vector bundle $V$, defined in Section 2 (above 2.9). The map

$$H : \mathcal{P}(S) \rightarrow \frac{\text{Hom}(\pi_1(S), SL(V))}{SL(V)}$$

obtained this way is known to be a local biholomorphism. The natural complex structure on $SL(V)$ induces a complex structure on $\text{Hom}(\pi_1(S), SL(V))/SL(V)$.

Let $\mathcal{T}(S)$ denote the Teichmüller space for $S$. It is the space of all complex structures on $S$, compatible with its orientation, quotiented by the group of diffeomorphisms of $S$ homotopic to the identity map.

Since a projective structure on $S$ induces a complex structure on $S$, there is a natural projection

$$\rho : \mathcal{P}(S) \rightarrow \mathcal{T}(S),$$

which is actually a $\Omega^1_{\mathcal{T}(S)}$-torsor. This means that $\rho$ is a holomorphic submersion, and for any $c \in \mathcal{T}(S)$, the fiber $\rho^{-1}(c)$ is an affine space for the fiber $(\Omega^1_{\mathcal{T}(S)})_c$ of the holomorphic cotangent space; moreover, for any (local) holomorphic sections $s$ and $t$ of $\rho$ and $\Omega^1_{\mathcal{T}(S)}$ respectively, $s + t$ is also a holomorphic section of $\rho$. That $\rho$ is a $\Omega^1_{\mathcal{T}(S)}$-torsor follows from the fact that the space of all projective structures on a Riemann surface $X$ is an affine space for $H^0(X, K_X^2)$ in a natural way [D].

Take any $P \in \mathcal{P}(S)$. So $P$ gives a Riemann surface structure on $S$, which we will denote by $X$, equipped with a projective structure. For $P$, consider the differential operator $D_X(3)$ constructed in Theorem 3.7. Let $C^\cdot$ denote the following complex of sheaves on $X$:

$$C^\cdot : C^0 = T_X \xrightarrow{D_X(3)} C^1 \rightarrow K_X^2,$$

where $C^0$ is in the 0-th position.

The following lemma was inspired by [W].

**Lemma 7.4.** - The tangent space $T_P \mathcal{P}(S)$ is naturally parametrized by $H^1(X, C^\cdot)$, the first hypercohomology of the complex $C^\cdot$.

**Proof.** - Let $\pi : X_U \rightarrow U$ be a holomorphic family of Riemann surfaces with projective structure parametrized by the unit disk $U \subset \mathbb{C}$, such that $X_0 = \pi^{-1}(0)$ is the
given Riemann surface $X$ with projective structure $P$. Let $\{U_\alpha\}$ be an open cover of $X_U$ such that each $U_\alpha$ is Stein. Fix a coordinate function $(z_\alpha, t)$ on $U_\alpha$, where $t$ is the obvious coordinate function on $U$.

For $t \in U$, on the Riemann surface $X_t = \pi^{-1}(t)$, equipped with a projective structure, let $\mathcal{D}_t$ denote the operator $\mathcal{D}_X(3)$ constructed in Theorem 3.7.

On the open set $U_\alpha \cap X_t$, the action of $\mathcal{D}_t$ on $\frac{\partial}{\partial z_\alpha}$, namely $\mathcal{D}_t(\frac{\partial}{\partial z_\alpha})$, is a section of $K_X^2$, $(\frac{\partial}{\partial z_\alpha}$ is a local section of $T_{X_t}$). Let $\mathcal{D}_t(\frac{\partial}{\partial z_\alpha}) = g(z_\alpha, t)(dz_\alpha)^{\otimes 2}$. Consider

$$\phi_\alpha := \frac{\partial}{\partial t} \left( \mathcal{D}_t \left( \frac{\partial}{\partial z_\alpha} \right) \right) \bigg|_{t=0} = \frac{\partial}{\partial t} (g(z_\alpha, t))(dz_\alpha)^{\otimes 2},$$

which is a section of $K_X^2$ over $U_\alpha \cap X$.

The difference

$$\zeta(\alpha, \beta) := \frac{\partial}{\partial z_\alpha} - \frac{\partial}{\partial z_\beta}$$

is a vertical vector field (for the projection $\pi$) over $U_\alpha \cap U_\beta$.

It is easy enough to see that the pair $\{(\zeta(\alpha, \beta)), \{\phi_\alpha\}\}$ is an one cocycle for $C$. For two different choices of coverings (and coordinates) on $X_U$, the two corresponding cocycles for $C$ differ by a coboundary.

Thus, we have a homomorphism

$$(7.5) \quad f(P) : T_P \mathcal{P}(S) \longrightarrow \mathcal{H}^1(X, C),$$

which is easily seen to be injective.

Consider the following short exact sequence of complexes:

$$(7.6) \quad 0 \longrightarrow K_X^2[1] \longrightarrow C : \longrightarrow T_X \longrightarrow 0,$$

where $K_X^2[1]$ denotes the shifted complex $0 \longrightarrow K_X^2$, with $K_X^2$ at the first position; $T_X$ in (7.6) is the one term complex with $T_X$ at the 0-th position. Since

$$H^1(X, K_X^2) = 0 = H^0(X, T_X),$$

(recall that $g \geq 2$) the long exact sequence of hypercohomologies for (7.6) gives that

$$\dim \mathcal{H}^1(X, C) - \dim H^0(X, K_X^2) + \dim H^1(X, T_X) - 6g - 6.$$

Since $\dim \mathcal{P}(S) = 6g - 6$, the homomorphism $f(P)$ in (7.5) must be an isomorphism. This completes the proof of the lemma. $\square$

From the construction of the homomorphism $f(P)$ in (7.5) it is clear that the projection $H^1(X, C) \longrightarrow H^1(X, T_X)$, induced by (7.6), is the differential, at $P$, of the map $\rho$ defined in (7.2).

In (4.4) we saw that the operator $\mathcal{D}(3)$, constructed in (3.6), is $\frac{1}{3!} \frac{\partial^3}{\partial z^3}$, where $z$ is the natural coordinate on $P(V)$ (with values in $C \cup \{\infty\}$ obtained after choosing a basis of $V$). This implies that the kernel of $\mathcal{D}(3)$ is the constant subsheaf of $T_P(V)$ given by the image of $H^0(P(V), T_P(V))$. This subsheaf is precisely $\tilde{V}^*(2)$ in Lemma 3.2.
Now from the proof of Theorem 3.7 we see that the kernel of the differential operator $D_X(3)$ is the locally constant sheaf corresponding to the flat vector bundle

$$S^2(V^*) = Ad(V),$$

where $Ad(V)$ is the Lie algebra bundle consisting of trace zero endomorphisms of $V$. The natural isomorphism between $S^2(V^*)$ and $Ad(V)$ can be described as follows: using contraction we have a homomorphism

$$(7.7) \quad S^2(V^*) \otimes V \longrightarrow V^*.$$ 

On the other hand, the symplectic structure on the fibers of $V$ (induced by $\theta$) identifies $V$ with $V^*$. Combining this isomorphism with the homomorphism in (7.7) we get an isomorphism between $S^2(V^*)$ and $Ad(V)$.

We will denote by $Ad(V)$ the locally constant sheaf on $X$ corresponding to the flat vector bundle $Ad(V)$.

Since $Ad(V)$ is the kernel of $D_X(3)$, the complex $C^.$ is quasi-isomorphic to $Ad(V)$. Therefore, we have the following isomorphism:

$$(7.8) \quad H^1(X, C^.) = H^1(X, Ad(V))).$$

This identification obtained in (7.8) is evidently the differential, at $P$, of the map $H$ defined in (7.1). The differential of $\rho \circ H^{-1}$, where $H^{-1}$ is a local inverse of $H$, can be seen as follows: Consider the isomorphisms of vector bundles

$$Ad(V) = S^2(V^*) = J^2(T_X),$$

where the last isomorphism is given by Theorem 3.7. Now using the natural inclusion of $Ad(V)$ in $Ad(V)$, and the obvious projection $J^2(T_X) \longrightarrow T_X$, we get the following homomorphisms:

$$(7.9) \quad H^1(X, Ad(V))) \longrightarrow H^1(X, J^2(T_X)) \longrightarrow H^1(X, T_X).$$

The composition of the above two homomorphisms is the differential, at $H(P)$, of the map $\rho \circ H^{-1}$.

Thus, we have proved the following proposition:

**Proposition 7.10.** – The differential at $P \in \mathcal{P}(S)$ of the local biholomorphism $H$, defined in (7.1), coincides with the isomorphism obtained in (7.8). Furthermore, the differential at $H(P)$ of the locally defined holomorphic map $\rho \circ H^{-1}$ coincides with the projection obtained in (7.9).

The vector space $H^1(X, Ad(V))$ has a skew-symmetric pairing which defines a symplectic form on $\Hom(\pi_1(S), SL(V))/SL(V)$, [AB], [Go1]. This pairing is defined by

$$(7.11) \quad \alpha \otimes \beta \longmapsto \text{trace}(\alpha \cup \beta) \cap [X] \in \mathbb{C},$$

where $\cap [X]$ is the cap product with the oriented generator of $H_2(X, \mathbb{Z})$. 

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The corresponding symplectic pairing on the left-hand side of (7.8) also has a simple
description.

Consider the homomorphism from the tensor product $C^* \otimes_C C^*$, namely

$$C^* \otimes_C C^* : T_X \otimes_C T_X \rightarrow (T_X \otimes_C K_X^2) \oplus (K_X^2 \otimes_C T_X) \rightarrow \cdots,$$

to the complex $K_X[1]$, namely

$$K_X[1] : 0 \rightarrow K_X,$$

defined using the natural contraction of $T_X$ with $K_X^2$. This homomorphism of complexes
induces the following pairing:

$$(7.12)\quad H^1(X, C^*) \otimes_{C} H^1(X, C^*) \rightarrow H^2(X, C^* \otimes_C C^*) \rightarrow H^2(X, K_X[1]) = H^1(X, K_X) = \mathbb{C}.$$  

This pairing corresponds to the symplectic pairing (7.11) using the isomorphism (7.8).
Thus, we have,

**PROPOSITION 7.13.** – The natural symplectic structure on the space of all projective
structures $P(S)$ on $S$ coincides with the natural symplectic pairing on the hypercohomology
$H^1(X, C^*)$ defined in (7.12).

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(Manuscript received June 10, 1998; revised July 15, 1998.)

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