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Spectra of Products and Numerical Ranges[1]

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1. INTRODUCTION

If A is bounded linear transformation from a complex Hilbert space H into itself, then the *numerical range* of A is by definition the set

$$W(A) = \{\langle Ax, x\rangle : \| x \| = 1\}.$$

It is wellknown and easy to prove that if $\sigma(A)$ denotes the spectrum of A, then

$$\sigma(A) \subset \overline{W(A)},$$

where the bar indicates closure.

The purpose of this paper is two-fold. We first present an extension of the foregoing relation and the proceed to indicate how the extension may be used in two other situations, namely bounded linear operators on a Banach space, and certain nonlinear transformations on a real or complex Hilbert space. The extension is mild, Specifically, we will show that if $0 \notin \overline{W(A)}$, then

$$\sigma(A^{-1}B) \subset \overline{W(B)}/\overline{W(A)}$$

for any operator B on H. Here the set on the right is by definition the set of quotients b/a with $b \in \overline{W(B)}$ and $a \in \overline{W(A)}$.

The extension has interesting consequences. For example it implies that if A is strictly positive and $B \geqslant 0$, then the product AB has a nonnegative spectrum. Also, if A is positive and B is self-adjoint then the product AB has real spectrum.

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2. Linear Operators on a Hilbert Space

We begin with the proof of the extension.

THEOREM 1. *Let A and B operators on the complex Hilbert space H. If $0 \notin \overline{W(A)}$ then*

$$\sigma(A^{-1}B) \subset \overline{W(B)}/\overline{W(A)}.$$

PROOF. Observe first of all that since $\sigma(A) \subset \overline{W(A)}$, the hypothesis guarantees that A^{-1} exists (as a bounded linear operator on H). Secondly, the identity

$$A^{-1}B - \lambda = A^{-1}(B - \lambda A)$$

shows that if $\lambda \in \sigma(A^{-1}B)$, then $0 \in \sigma(B - \lambda A)$. This in turn implies that

$$0 \in \overline{W(B - \lambda A)} \subset \overline{W(B)} - \lambda \overline{W(A)},$$

and this means that

$$\lambda \in \overline{W(A)}/\overline{W(A)}.$$

We indicated two corollaries above. To get another we recall that any operator A on H has a "polar decomposition"

$$A = UP,$$

and that if A is invertible, then U is unitary and P is strictly positive. Following Berberian [1] we call the unitary operator U *cramped* if its spectrum is contained in an arc of the unit circle with central angle $< \pi$.

COROLLARY (Berberian). *If $0 \notin \overline{W(A)}$, then the unitary part of A is cramped.*

PROOF. Use the fact that $\overline{W(A)}$ is convex to see that if $0 \notin \overline{W(A)}$, then $\overline{W(A)}$ is contained in a sector

$$S = \{re^{i\theta} : r > 0 : \theta_1 \leqslant \theta \leqslant \theta_2\}$$

with $\theta_2 - \theta_1 < \pi$. Then write $U = A \cdot P^{-1}$ and apply the theorem to see that $\sigma(U)$ is a subset of the arc

$$\{e^{i\theta} : \theta_1 \leqslant \theta \leqslant \theta_2\}.$$

REMARK. (i) The inclusion $\sigma(A^{-1}B) \subset \overline{W(B)}/\overline{W(A)}$ is not valid with the weaker assumption that A is merely invertible. Indeed if A and B are self-adjoint $\sigma(AB)$ need not even be real. This follows from the computation

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

in two-dimensional Hilbert space

(ii). The more symmetic statement

$$\sigma(AB) \subset \overline{W(A)} \cdot \overline{W(B)} \qquad \text{if} \qquad 0 \notin \overline{W(A)} \cup \overline{W(B)}$$

is also not valid. To see this let A be the operator

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then $W(A) = W(A^*)$ is the disk of radius $1/2$ about 1 and so the set $W(A) \cdot W(A^*)$ lies to the left of $\operatorname{Re} z = 9/4$. On the other hand $9/4 < 1/2(3 + \sqrt{5}) \in \sigma(AA^*)$.

Returning to the theorem, the reader will note that the proof really does not concern operators on a Hilbert space at all. Indeed, the essential ingredients are these: An algebra $\mathscr{A}$ with unit, and two mappings $A \to \sigma(A)$, $A \to W(A)$ from $\mathscr{A}$ to subsets of the complex plane which have the following properties:

(1) $W(A + B) \subset W(A) + W(B)$
(2) $W(\lambda A) \subset \lambda W(A)$
(3) $\sigma(A) \subset \overline{W(A)}$
(4) $\lambda \notin \sigma(A)$ if and only if $(A - \lambda)^{-1} \in \mathscr{A}$.

(We write $B^{-1} \in \mathscr{A}$ to mean that the element B of $\mathscr{A}$ has an inverse and that this inverse in fact belongs to $\mathscr{A}$.) In what follows we will indicate how this observation extends the theorem to two other situations.

3. Linear Operators on a Banach Space

For our first application we need a few facts about Banach spaces. First, if X is a Banach space then the Hahn-Banach theorem guarantees that for each $x \in X$ there is an $x^* \in X^*$ of norm 1 such that $\langle x, x^* \rangle = \| x \|$. The space X (or more properly, the unit ball of X) is called *smooth* [2] if there is exactly one such x^* for each $x \in X$. Thus in a smooth space there is a unique map φ form X to X^* such that

$$\| \varphi(x) \| = \| x \|, \qquad \langle x, \varphi(x) \rangle = \| x \|^2 \qquad (x \in X).$$

As an example the reader can easily verify that L^p is smooth for $1 < p < \infty$. The isometry φ sends $f \in L^p$ to

$$f \frac{|f|^{p-2}}{\|f\|^{p-2}}.$$

If X is smooth and φ is the indicated mapping, then it is easy to see that φ is conjugate homogeneous:

$$\varphi(\alpha x) = \bar{\alpha}\varphi(x), \qquad \alpha \text{ complex}.$$

(However, if φ is additive, then the norm in X satisfies the parallelogram law and hence X is a Hilbert space.) Again, if X is smooth and $f \in X^*$ attains its supremum on the unit ball of X, then f belongs to the range of φ. Now a result of Bishop and Phelps [3] states that for any Banach space X the collection of bounded linear functionals on X which attain their suprema on the unit ball of X is always (norm) dense in X^*. By using this fact and the preceding remark it follows that if X is smooth, then the range of φ is dense in X^*.

Now using the function φ we can define a "semi-inner-product" on X by

$$[x, y] = \langle x, \varphi(y)\rangle \qquad (x, y \in X).$$

It is readily verified that the following hold:

$$\begin{aligned}
[x, x] &= \| x \|^2 \\
[x_1 + x_2, y] &= [x_1, y] + [x_2, y] \\
[\lambda x, y] &= \lambda[x, y], \ [x, \lambda y] = \bar{\lambda}[x, y] \\
|[x, y]| &\leqslant \| x \| \| y \|.
\end{aligned}$$

If now A is a bounded linear operator on X we can define the ***numerical range*** of A by setting

$$W(A) = \{[Ax, x] : \| x \| = 1\}.$$

Clearly we will have

$$W(A + B) \subset W(A) + W(B).$$
$$W(\lambda A) \subset \lambda W(A).$$

Lumer [4] also shows that the boundary of $\sigma(A)$ is a subset of $\overline{W(A)}$. We need the following stronger result:

PROPOSITION. $\sigma(A) \subset \overline{W(A)}$.

PROOF. The argument parallels the linear case: If λ is at a positive distance δ from $\overline{W(A)}$, then for unit vectors x

$$\|(A - \lambda)x\| \geqslant |[(A - \lambda)x, x]| = |[Ax, x] - \lambda| \geqslant \delta = \delta\| x \|$$

and

$$\|(A - \lambda)^*\varphi(x)\| \geqslant |\langle x, (A - \lambda)^*\varphi(x)\rangle| = |[(A - \lambda)x, x]| \geqslant \delta = \delta\|\varphi(x)\|.$$

The first of these implies that $A - \lambda$ is one-to-one with a closed range. The second implies that $(A - \lambda)^*$ is bounded below on the range of φ and since this is dense in X^*, $(A - \lambda)^*$ is bounded below, hence one-to-one, and this means that $A - \lambda$ has a dense range. It now follows from the Open Mapping Theorem that $A - \lambda$ has a bounded inverse. Hence $\lambda \notin \overline{W(A)}$ implies $\lambda \notin \sigma(A)$ as asserted.

We may summarize the preceding discussion as follows:

THEOREM 2. *Let X be a smooth Banach space and define $W(A)$ as above. Then if $0 \notin \overline{W(A)}$ we have*

$$\sigma(A^{-1}B) \subset \overline{W(B)}/\overline{W(A)}$$

for any operator B on X.

If the Banach space X is not smooth then there will be many isometries φ_α from X to x^* satisfying

$$\langle x, \varphi_\alpha(x)\rangle = \| x \|^2 \qquad (x \in X).$$

Each of these maps defines a semi-inner product $[\ ,\ ]_\alpha$ on X and a bounded linear operator T on X has corresponding numerical ranges $W_\alpha(T)$. It is natural to define the *numerical range* of T on X by

$$W(T) = \bigcup_\alpha W_\alpha(T).$$

The argument used for the smooth case is easily adapted to prove that $\sigma(T) \subset \overline{W(T)}$ is still valid and so we can conclude that Theorem 2 holds without the hypothesis that X is smooth.

In this connection Lumer has shown [4] that $W(T)$ is real (or positive) if and only if some $W_\alpha(T)$ is real (or positive). Thus $T = T^*$ (or $T \geqslant 0$) has intrinsic meaning and with these conventions we can state the following corollary:

COROLLARY. *If $A > 0$, $B \geqslant 0$ and $C = C^*$, then $\sigma(AB)$ is positive and $\sigma(AC)$ is real.*

4. NONLINEAR OPERATORS ON A HILBERT SPACE

Our final application is more delicate. Here we let H be a real or complex Hilbert space and let $\mathscr{A}$ be the collection of maps from H to itself which are continuous and which send bounded sets into bounded sets. Clearly $\mathscr{A}$ is an algebra with unit. We take the numerical range of $A \in \mathscr{A}$ to be

$$W(A) = \left\{ \frac{\langle Ax_1 - Ax_2, x_1 - x_2 \rangle : x_1 \neq x_2}{\| x_1 - x_2 \|^2} \right\}.$$

There are two possible definitions of the spectrum of $A \in \mathscr{A}$, namely, $\sigma(A)$, and $\sigma_1(A)$ defined respectively as the complements of the sets

$$\rho(A) = \{\lambda : (A - \lambda)^{-1} \in \mathscr{A}\}$$
$$\rho_1(A) = \{\lambda : (A - \lambda)^{-1} \text{ exists and is Lipschitzian}\}.$$

(By definition, B is Lipschitzian if

$$\| Bx_1 - Bx_2 \| \leqslant M \cdot \| x_1 - x_2 \|$$

for some constant $M > 0$ and all x_1, x_2.)

It is easy to see that $\sigma(A) \subset \sigma_1(A)$. Moreover, a theorem of Zarantonello [5] asserts that, with $W(A)$ as defined above, we have the inclusion

$$\sigma_1(A) \subset \overline{W(A)}.$$

Taking $\sigma(A)$ as the definition of the spectrum of A and applying Theorem 1, we get the following result:

THEOREM 3. *Let A and B be bounded and continuous on H. If $0 \notin \overline{W(A)}$, then for each $\lambda \notin \overline{W(B)}/\overline{W(A)}$ the mapping $A^{-1}B - \lambda$ has a bounded, continuous inverse defined on H.*

Taking $\sigma_1(A)$ as the definition of the spectrum of A we get:

THEOREM 4. *Let B be bounded and continuous, let A be Lipschitzian and suppose $0 \notin \overline{W(A)}$. Then for each λ outside the set $\overline{W(B)}/\overline{W(A)}$ the transformation $A^{-1}B - \lambda$ has a Lipschitzian inverse defined on H.*

PROOF. If $0 \notin \sigma_1(B - \lambda A)$, then $(B - \lambda A)^{-1}$ exists and is Lipschitzian. Hence the product $(B - \lambda A)^{-1}A$ is also Lipschitzian. Since however

$$(A^{-1}B - \lambda)(B - \lambda A)^{-1}A = A^{-1}(B - \lambda A)(B - \lambda A)^{-1}A = 1,$$

this implies that $A^{-1}B - \lambda$ has a Lipschitzian inverse and so $\lambda \notin \sigma_1(A^{-1}B)$. In other words,

$$\lambda \in \sigma_1(A^{-1}B) \Rightarrow 0 \in \sigma_1(B - \lambda A)$$

and the remainder of the proof is as before.

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