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Finite Splitness and Finite Projectivity

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INTRODUCTION

Let R be a ring, Let A, C be left R-modules and $f: A \to C$ an epimorphism. f is called pure if $\mathrm{Hom}(M, f)$: $\mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, C)$ is an epimorphism for all finitely presented left R-modules M. Let B be the kernel of f. Then by a theorem of Fieldhouse and Warfield f is pure if and only if B is pure in A in the sense that the natural homomorphism $N \otimes_R B \to N \otimes_R A$ derived from the inclusion map $B \to A$ is a monomorphism for all right R-modules N, and moreover, by Cohn's theorem, this is equivalent to the condition that if a finite system of linear equations $\sum_{j=1}^{n} a_{ij} x_j = b_i$ $(i = 1, 2,..., m)$ with $a_{ij} \in R$ and $b_i \in B$ has a solution $x_1, x_2,..., x_n$ in A then it has a solution in B, i.e., there exist $y_1, y_2,..., y_n \in B$ such that $\sum_{j=1}^{n} a_{ij} y_j = b_i$ $(i = 1, 2,..., m)$. In the present paper, we attempt to generalize this situation by replacing the class of finitely presented modules with the class of finitely generated modules to yield a series of some meaningful results. We call the epimorphism f finitely split if $\mathrm{Hom}(M, f)$ is an epimorphism for all finitely generated left R-modules M, or what is the same thing, if for any finitely generated submodule C_0 of C there exists a homomorphism φ: $C_0 \to A$ such that $f \circ \varphi$ is the identity map of C_0. In terms of the kernel B of f, this is equivalent to the condition that B is a direct summand of every submodule A' of A such that $A' \supset B$ and the factor module A'/B is finitely generated—we say, in this case, that B is finitely split in A. We can show that this is also equivalent to the following condition: If a (finite or infinite) system of linear equations $\sum_{j=1}^{n} a_{ij} x_j = b_i$ $(i \in I)$ with $a_{ij} \in R$ and $b_i \in B$ has a solution $x_1, x_2,..., x_n$ in A then it has a solution in B.

Let M be a left R-module. M is called pure-projective if every pure epimorphism onto M splits, while M is called pure-injective if M is a direct

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summand of every pure extension module of M. It is known that M is pure-projective if and only if M is a direct summand of a direct sum of finitely presented modules, while it is a theorem of Warfield that M is pure-injective if and only if M is algebraically compact in the sense that, given a row-finite $I \times J$ matrix $[a_{ij}]$ over R and an element $m_i \in M$ for each $i \in I$, the system of linear equations $\sum_{j \in J} a_{ij} x_j = m_i$ $(i \in I)$ is solvable with x_j's in M whenever it is finitely solvable in M. We now call M finitely pure-projective if every pure epimorphism onto M is finitely split, while M is called finitely pure-injective if M is finitely split in every pure extension of M. We show that M is finitely pure-projective if and only if, for each finitely generated submodule M_0 of M, there exists a finitely presented module E and homomorphisms $\varphi: M_0 \to E$ and $\psi: E \to M$ such that $\psi \circ \varphi$ is the identity map of M_0, while M is finitely pure-injective if and only if M satisfies the above condition of the algebraic compactness only for all matrices $[a_{ij}]$ of finite columns.

A left R-module M is called flat if for any right R-module A and its submodule B, the natural homomorphism $B \otimes_R M \to A \otimes_R M$ derived from the inclusion map $B \to A$ is a monomorphism. It is known that M is flat if and only if every epimorphism onto M is pure. In connection with this condition, we define M to be finitely projective if every epimorphism onto M is finitely split. Thus every projective module is finitely projective, and every finitely projective module is flat. If R is left Noetherian then every flat left R-module is finitely projective. Also we can show that this is true even if R is a Prüfer ring. So there naturally arises a problem to characterize those rings over which every flat module is finitely projective. In this connection, we consider a factor module F/G, where F is a free left R-module on a countable basis $u_1, u_2, \ldots$, and G a submodule of F generated by $u_1 - a_1 u_2$, $u_2 - a_2 u_3, \ldots$, with a given infinite sequence $a_1, a_2, \ldots$, in R; such a factor module played a crucial role in his investigation of perfect rings by H. Bass. It is well known that F/G is flat. We prove that the module F/G is finitely projective if and only if the ascending chain $l(a_n) \subset l(a_n a_{n+1}) \subset \cdots$ of left annihilators terminates for $n = 1, 2, \ldots$. From this follows that if every flat left R-module is finitely projective then the ascending chain $l(a_1) \subset l(a_1 a_2) \subset \cdots$ terminates for every infinite sequence $a_1, a_2, \ldots$, in R. But we do not know as yet whether or not the converse is true.

Gruson and Raynaud, Garfinkel, as well as Zimmermann–Huisgen introduced the concept of locally projective modules and developed important theories on this. M is called locally projective if, for any epimorphism $f: M' \to M$ and a finitely generated submodule M_0 of M, there is a homomorphism $h: M \to M'$ such that $f \circ h$ induces the identity map on M_0. Thus every locally projective module is finitely projective. Zimmermann–Huisgen, however, shows that every flat left R-module is locally projective if and only if R is left perfect. In view of this, we know that finitely projective module is not always locally projective, because there is certainly a left Noetherian ring which is not left perfect.

Throughout this paper, R means a ring with unit element 1 and R-modules are all unital. If M is an R-module then, for any index set I, M^I and $M^{(I)}$ mean, respectively, the I-times direct product and the I-times direct sum of M. We regard each element of M^I or $M^{(I)}$ as a vector with entries in M, and we regard it as a row vector or a column vector according to the context. For convenience, we denote by (x_i) the row vector and by $[x_i]$ the column vector whose ith entry is x_i for each $i \in I$. If n is a positive integer, we define M^n to be M^I $(=M^{(I)})$, where $I = \{1, 2,..., n\}$.

1. M-Purity and μ-Purity

Let M be a left R-module. Let A, C be left R-modules and $f: A \to C$ an epimorphism. f is called *M-pure* if $\mathrm{Hom}(M, f)$: $\mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, C)$ is an epimorphism, or in other words, for each homomorphism $\psi: M \to C$ there exists a homomorphism $\varphi: M \to A$ such that $f \circ \varphi = \psi$. Let A' be a left R-module and let there be given homomorphisms $g: A \to A'$ and $h: A' \to C$ such that $h \circ g = f$. Then it is easily seen that f is M-pure whenever both g and h are M-pure epimorphism and conversely h is an M-pure epimorphism whenever f is M-pure. On the other hand, it is well known that f splits, i.e., the kernel of f is a direct summand of A if and only if f is C-pure, and this is also equivalent to the condition that f is M-pure for all left R-modules M.

Let I, J be two index sets, and let $\mu = [a_{ij}]$ be a row-finite $I \times J$ matrix over R. For each row vector $(r_i) \in R^{(I)}$ the product $(r_i)\,\mu = (\Sigma_i r_i a_{ij})$ is in $R^{(J)}$, and the mapping $(r_i) \mapsto (r_i)\,\mu$ gives a left R-homomorphism $\mu: R^{(I)} \to R^{(J)}$. The cokernel of this homomorphism is denoted by $\mathrm{Cok}(\mu)$. For a left R-module M, we say that μ is a *defining* matrix of M (or μ defines M) if $\mathrm{Cok}(\mu) \cong M$, i.e., if there is an exact sequence

$$R^{(I)} \xrightarrow{\ \mu\ } R^{(J)} \xrightarrow{\ \theta\ } M \longrightarrow 0,$$

where θ is an epimorphism. As is well known, to the epimorphism θ there corresponds a system of generators $[u_j] \in M^J$ of M such that $\theta(s_j) = (s_j)[u_j] = \sum s_j u_j$ for every row vector $(s_j) \in R^{(J)}$. The exactness of the above sequence implies then $\sum s_j u_j = 0$ if and only if $(s_j) \in R^{(I)}\mu$.

Let M be any left R-module. Let $[u_j \mid j \in J]$, for an index set J, be a system of generators of M. Then the mapping $(s_j) \mapsto (s_j)[u_j]$ gives an epimorphism $R^{(J)} \to M$. Let $[\mu_i \mid i \in I]$, for an index set I, be a system of generators of the kernel of this epimorphism, and let μ be the (row-finite) $I \times J$ matrix whose ith row is μ_i for each $i \in I$. Then the mapping

$(r_i) \mapsto (r_i)\mu = \Sigma r_i \mu_i$ gives an epimorphism from $R^{(I)}$ onto the kernel. Thus we have an exact sequence

$$R^{(I)} \xrightarrow{\mu} R^{(J)} \longrightarrow M \longrightarrow 0,$$

and so μ is a defining matrix of M. The matrix depends on the choice of generators $[u_j]$ and $[\mu_i]$, and therefore defining matrices of M are not necessarily unique. It is however clear that M is finitely generated or cyclic if and only if M has a defining matrix of finite columns or of single columns, respectively, while M is finitely presented if and only if M has a defining matrix of finite rows and columns, i.e., a finite matrix.

Let $\mu = [a_{ij}]$ be any row-finite $I \times J$ matrix over R and V a left R-module. By a *system of linear equations for μ in V* we mean a system of linear equations of the form $\Sigma_j a_{ij} x_j = v_i$ for $i \in I$, where $[v_i]$ is a given vector in V^I. Let A be a left R-module and B a submodule of A. We say that B is *μ-pure* in A (or A is a μ-pure extension of B) if a system of linear equations for μ in B is solvable in B whenever it is solvable in A, or in other words, if, given a vectors $[x_j] \in A^J$ and $[b_i] \in B^I$ satisfying $\mu[x_j] = [b_i]$, there exists a vector $[y_j] \in B^J$ such that $\mu[y_j] = [b_i]$. It is easy to see that if A' is a module between A and B, i.e., $A \supset A' \supset B$, then B is μ-pure in A whenever A' is μ-pure in A and B is μ-pure in A', and conversely B is μ-pure in A' whenever B is μ-pure in A.

PROPOSITION 1. *Let M be a left R-module and μ a defining matrix of M. Let A, C be left R-modules and $f: A \to C$ an epimorphism with kernel B. Then f is M-pure if and only if B is μ-pure in A.*

Proof. Let μ be an $I \times J$ matrix. Then there is a system of generators $[u_j] \in M^J$ of M such that $(r_j)[u_j] = \Sigma r_j u_j = 0$ for $(r_j) \in R^{(J)}$ if and only if $(r_j) \in R^{(I)}\mu$. Suppose that f is M-pure. Let $[x_j] \in A^J$ and $[b_i] \in B^I$ satisfy $\mu[x_j] = [b_i]$. Then we have $\mu[f(x_j)] = [f(b_i)] = 0$. Thus we know that $\Sigma r_j u_j = 0$ always implies $\Sigma r_j f(x_j) = 0$, which means the existence of a homomorphism $\psi: M \to C$ such that $\psi(u_j) = f(x_j)$ for all $j \in J$. Since f is M-pure, there exists a homomorphism $\varphi: M \to A$ such that $f \circ \varphi = \psi$. Put $y_j = x_j - \varphi(u_j)$ for each $j \in J$. Then $y_j \in A$ and $f(y_j) = f(x_j) - f(\varphi(u_j)) = f(x_j) - \psi(u_j) = 0$, so that $y_j \in B$. Moreover $\mu[y_j] = \mu[x_j] - \mu[\varphi(u_j)] = [b_i] - \varphi(\mu[u_j]) = [b_i]$ since $\mu[u_j] = 0$.

Conversely, suppose that B is μ-pure in A. Let $\psi: M \to C$ be a homomorphism. Since f is an epimorphism there is an $x_j \in A$ such that $f(x_j) = \psi(u_j)$ for each $j \in J$. Let $[b_i] = \mu[x_j] \in A^I$. Then $[f(b_i)] = \mu[f(x_j)] = \mu[\psi(u_j)] = \psi(\mu[u_j]) = 0$, which implies that $b_i \in B$ for all $i \in I$. Since B is μ-pure in A, there exists $y_j \in B$ for each $j \in J$ such that $\mu[y_j] = [b_i]$. It follows then $\mu[x_j - y_j] = \mu[x_j] - u[y_j] = 0$, and this implies that there is a homomorphism $\varphi: M \to A$ such that $\varphi(u_j) = x_j - y_j$. Then we have $f(\varphi(u_j)) = f(x_j) - f(y_j) = f(x_j) = \psi(u_j)$, which implies $f \circ \varphi = \psi$ since $[u_j]$ generates M. ∎

Remark. The above proof for Proposition 1 may be regarded as a generalization of that of Warfield [13, Proposition 3].

Consider again a left R-module A and a submodule B of A. Let N be a right R-module. B is called *weakly N-pure* in A if $N \otimes \kappa: N \otimes_R B \to N \otimes_R A$ is a monomorphism, where κ is the inclusion map $B \to A$. On the other hand, let v be a column-finite $I \times J$ matrix over R. We call B *weakly v-pure* in A if, for any column vectors $[x_j] \in A^{(J)}$ and $[b_i] \in B^{(I)}$ such that $v[x_j] = [b_i]$, there exists a $[y_j] \in B^{(J)}$ such that $v[y_j] = [b_i]$. We note here that if v is a finite matrix, i.e., if both I and J are finite then B is weakly v-pure in A if and only if B is v-pure in A. Now let N be given. By left–right analogy there corresponds a defining matrix v of N, which is a column-finite, say $I \times J$ matrix. Also, we can find a system of generators $(v_i) \in N^I$ of N so that we have an exact sequence of right R-modules

$$R^{(J)} \to R^{(I)} \to N \to 0,$$

where the first map is given by the mapping $[r_j] \mapsto v[r_j]$, for $[r_j] \in R^{(J)}$, and the second map is defined by associating each $[s_i] \in R^{(I)}$ with $(v_i)[s_i] = \Sigma v_i s_i$. For a left R-module A, we have, by tensoring over R, the following exact sequence

$$R^{(J)} \otimes_R A \to R^{(I)} \otimes_R A \to N \otimes_R A \to 0.$$

But, as is well known, the first and the second terms are naturally identified with $A^{(J)}$ and $A^{(I)}$, respectively, and thus we have the exact sequence

$$A^{(J)} \to A^{(I)} \to N \otimes_R A \to 0,$$

thereby, as is checked easily, the first map is the left multiplication of v and the second map is given by the mapping $[a_i] \mapsto \sum v_i \otimes a_i$.

PROPOSITION 2. *Let N be a right R-module and v a defining matrix of N. Let A be a left R-module and B a submodule of A. Then B is weakly N-pure in A if and only if B is weakly v-pure in A.*

Proof. We have clearly the following commutative diagram:

$$\begin{array}{ccccccc}
A^{(J)} & \to & A^{(I)} & \to & N \otimes_R A & \to & 0 \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle N \otimes \kappa} & & \\
B^{(J)} & \to & B^{(I)} & \to & N \otimes_R B & \to & 0,
\end{array}$$

where the lower exact sequence is defined in the same way as the upper sequence and the first and the second vertical maps mean the natural inclusion maps. A standard diagram chase then completes the proof. ∎

Now B is called *pure* in A (or A is a pure extension of B) if B is weakly N-pure in A for all right R-modules N. From Proposition 2 it follows that B is pure in A if and only if B is weakly ν-pure in A for all column-finite matrices ν over R. But this is equivalent to the condition that B is μ-pure in A only for all finite matrices μ over R. For, let $\nu = [a_{ij}]$ be any column-finite matrix and let $[x_j] \in A^{(J)}$ and $[b_i] \in B^{(I)}$ satisfy $\nu[x_j] = [b_i]$, i.e., $\sum_{j \in J} a_{ij} x_j = b_i$ for all $i \in I$. Let J_0 be a finite subset of J such that $x_j = 0$ whenever $j \notin J_0$. Since ν is column-finite, there is a finite subset I_0 of I such that $a_{ij} = 0$ whenever $j \in J_0$ and $i \notin I_0$. It follows that $b_i = 0$ whenever $i \notin I_0$ and we have $\sum_{j \in J_0} a_{ij} x_j = b_i$ for all $i \in I_0$. If we denote by ν_0 the $I_0 \times J_0$ submatrix of ν and if we assume that B is ν_0-pure in A, then there exists $[y_j] \in B^{J_0}$ such that $\sum_{j \in J_0} a_{ij} y_j = b_i$ for all $i \in I_0$. If we define $y_j = 0$ for each $j \in J$ not in J_0 then we have a vector $[y_j] \in B^{(J)}$ which clearly satisfies $\sum_{j \in J} a_{ij} y_j = b_i$ for all $i \in I$. Thus B is weakly ν-pure in A.

On the other hand, an epimorphism $f: A \to C$ is called *pure* if f is M-pure for all finitely presented left R-modules M. According to Proposition 1, this is equivalent to the condition that the kernel B of f is μ-pure in A for all finite matrices μ over R. Thus we have the following theorem of Cohn, Fieldhouse, and Warfield.

THEOREM. *Let A, C be left R-modules and $f: A \to C$ an epimorphism with kernel B. Then the following conditions are equivalent*:

(1) *f is pure,*

(2) *B is pure in A,*

(3) *B is μ-pure in A for all finite matrices μ over R.*

Remark. The equivalence of (2) and (3) was established by Cohn [3, Theorem 2.4], while the equivalence of (1) and (2) was proved independently by Fieldhouse [4, Corollary to Theorem 7.1] and Warfield [13, Proposition 3]. Our proof for Cohn's theorem sketched above apparently does not use the fact that every module is a direct limit of finitely presented modules.

2. Finite Splitness

Let A, C be left R-modules and $f: A \to C$ an epimorphism. We call f *finitely split* if f is M-pure for all finitely generated R-modules M, while f is called *singly split* if f is M-pure for all cyclic left R-modules M. Observing the fact that every homomorphic image of finitely generated (or cyclic) module is also finitely generated (or cyclic), we can see that f is finitely (or singly) split if and only if for each finitely generated (or cyclic) submodule C_0 of C there exists a homomorphism $\varphi: C_0 \to A$ such that $f \circ \varphi$ is the identity map of C_0. Clearly every finite split epimorphism is both pure and singly split.

Next let B be a submodule of A. We say that A is a *finite* (*or single*) *extension* of B if the factor module A/B is finitely generated (or *cyclic*), i.e., there is a finitely generated (or *cyclic*) submodule A_0 of A such that $A = A_0 + B$. We shall say that B is *finitely* (or *singly*) *split* in A if, for every submodule A' of A which is a finite (or single) extension of B, B is a direct summand of A'.

THEOREM 3. *Let A, C be left R-modules and $f: A \to C$ an epimorphism with kernel B. Then the following conditions are equivalent*:

(1) *f is finitely* (*or singly*) *split.*

(2) *B is μ-pure in A for all matrices μ of finite* (*or single*) *column*(*s*) *over R.*

(3) *B is finitely* (*or singly*) *split in A.*

(4) *If A_0 is a finitely generated* (*or cyclic*) *submodule of A then there is a homomorphism $A_0 \to B$ which fixes $A_0 \cap B$ element-wise.*

Proof. The equivalence of (1) and (2) is an immediate consequence of Proposition 1. As is well known, there is a one-to-one correspondence between submodules A' of A containing B and submodules C_0 of C by associating C_0 with its inverse image by f, and, if A' corresponds to C_0, $A'/B \cong C_0$ and so A' is a finite (or single) extension of B if and only if C_0 is finitely generated (or cyclic), while B is a direct summand of A' if and only if the restriction of f to A' is a split epimorphism $A' \to C_0$, i.e., there is a $\varphi: C_0 \to A$ such that $f \circ \varphi$ is the identity map of C_0. From these facts follows the equivalence of (1) and (3). The condition (3) is also equivalent to the condition that, for every finitely generated (or cyclic) submodule A_0 of A, B is a direct summand of $A_0 + B$. But this means that there exists a homomorphism $g: A_0 + B \to B$ such that g fixes B element-wise. Therefore the restriction of g to A_0 fixes $A_0 \cap B$ element-wise. Suppose conversely that such a homomorphism $h: A_0 \to B$ exists. Then, for any element $a + b$ with $a \in A_0$, $b \in B$, the element $h(a) + b$ depends only on $a + b$, because $a + b = a' + b'$ with $a' \in A_0$, $b' \in B$ implies $a - a' = b' - b \in A_0 \cap B$ and therefore $h(a) - h(a') = h(a - a') = b' - b$. So, by associating $a + b$ with $h(a) + b$, we have clearly a homomorphism $A_0 + B \to B$ which fixes B element-wise. Thus the equivalence of (3) and (4) are proved. ∎

By Theorem 3 we know that every finitely split submodule is a pure submodule. As for singly split submodules, we have

PROPOSITION 4. *Every singly split submodule B of an R-module A is essentially closed in A.*

Proof. Let A' be such that $A \supset A' \supset B$ and A' is an essential extension of B. Let a be any element of A'. Then $Ra + B$ is a single extension of B and so B is a direct summand of $Ra + B$. Since $Ra + B (\subset A')$ is essential over B, it follows $Ra + B = B$, i.e., $a \in B$. Thus B is essentially closed in A. ∎

If we combine Proposition 4 with Goodearl [6, Corollary 2.15, p. 49], we have

PROPOSITION 5. *If A is quasi-injective R-module then every singly split submodule of A is a direct summand of A.*

Let P be a left-R-module. P is called *pure-projective* if every pure epimorphism (in the category of all left R-modules) is P-pure. As proved in [4, Theorem 7.4] or [13, Corollary 3], *the following conditions are equivalent*:

(1) *P is pure-projective,*

(2) *every pure epimorphism onto P splits,*

(3) *P is a direct summand of a direct sum of finitely presented modules.*

PROPOSITION 6. *The following conditions are equivalent*:

(1) *Every pure epimorphism (in the category of all left R-modules) is finitely split.*

(2) *Every pure epimorphism (in the category of all left R-modules) is singly split.*

(3) *R is left Noetherian.*

Proof. (1) ⇒ (2) is clear. Assume (2), and let L be any left ideal of R. Then every pure epimorphism onto the cyclic left R-module R/L splits, i.e., R/L is pure-projective and hence finitely presented. In view of Stenström [12, Proposition 3.2, p. 11)], this implies that L is finitely generated. Thus R is left Noetherian. Conversely, assume (3). Then every finitely generated left R-module is finitely presented, and therefore every pure epimorphism is finitely split because of the definition of pure epimorphisms. ∎

We now call P *finitely* (or *singly*) *pure-projective* if every pure epimorphism onto P is finitely (or singly) split. Clearly every pure-projective module is finitely pure-projective, and every finitely pure-projective module is singly pure-projective.

PROPOSITION 7. *The following conditions on a left R-modules P are equivalent*:

(1) *P is finitely* (*or singly*) *pure-projective.*

(2) *For each finitely generated* (*or cyclic*) *submodule P_0 of P there exist a finitely presented left-R-module E and homomorphisms $\varphi: P_0 \to E$ and $\psi: E \to P$ such that $\psi \circ \varphi$ is the identity map of P_0.*

(3) *Given a pure epimorphism $f: A \to C$, a homomorphism $h: P \to C$ and a finitely generated* (*or cyclic*) *submodule P_0 of P, there exists a homomorphism $g: P_0 \to A$ such that $f \circ g$ coincides with the restriction of h to P_0.*

Proof. By [13, Proposition 1] there is a direct sum $\Sigma \oplus E_i$ of finitely presented modules E_i which has a pure epimorphism f onto P (see the statement just before [13, Corollary 3]). Assume (1). Then f is finitely (or singly) split. Thus for any finitely generated (or cyclic) submodule P_0 of P there exists a homomorphism $\varphi: P_0 \to \Sigma \oplus E_i$ such that $f \circ \varphi = 1$, the identity map of P_0. The image $\varphi(P_0)$ is finitely generated and hence contained in a suitable finite partial sum of $\Sigma \oplus E_i$. If we denote this finite sum by E then E is finitely presented and the restriction ψ of f to E satisfies $\psi \circ \varphi = 1$. Next assume (2). Suppose that $f: A \to C$, $h: P \to C$, and P_0 are the same as given in (3). Then there are a finitely presented module E and homomorphisms $\varphi: P_0 \to E$, $\psi: E \to P$ such that $\psi \circ \varphi = 1$. Since f is a pure epimorphism, f is E-pure and so there exists a homomorphism $\varepsilon: E \to A$ such that $f \circ \varepsilon = h \circ \psi$. If we put $g = \varepsilon \circ \varphi: P_0 \to A$ then $f \circ g = f \circ \varepsilon \circ \varphi = h \circ \psi \circ \varphi = h \circ 1$ is the restriction of h to P_0. Finally, (1) is the particular case of (3) where $C = P$ and h is the identity map. ∎

COROLLARY 8. *Let P, P' be left R-modules and let there be a finitely* (*or singly*) *split epimorphism $P \to P'$. If P is finitely* (*or singly*) *pure-projective then so is P' too.*

Proof. Let $f: P \to P'$ be a finitely (or singly) split epimorphism and let P'_0 be any finitely generated (or cyclic) submodule of P'. Then there is a homomorphism $h: P'_0 \to P$ such that $f \circ h = 1$. If we assume that P is finitely (or singly) pure-projective then by Proposition 7 there correspond to the finitely generated (or cyclic) submodule $h(P'_0)$ of P a finitely presented module E and homomorphisms $\varphi: h(P'_0) \to E$, $\psi: E \to P$ such that $\psi \circ \varphi = 1$.

Then $\varphi \circ h: P'_0 \to E$ and $f \circ \psi: E \to P'$ satisfy $f \circ \psi \circ \varphi \circ h = f \circ 1 \circ h = f \circ h = 1$. Thus P' is finitely (or singly) pure-projective again by Proposition 7. ∎

PROPOSITION 9. *Every pure submodule of a finitely (or singly) pure-projective module is finitely (or singly) pure-projective too.*

Proof. Let P be a finitely (or singly) pure-projective left R-module and N a pure submodule of P. Let N_0 be a finitely generated (or cyclic) submodule of N. Then by Proposition 7 there exist a finitely presented module E and homomorphisms $\varphi: N_0 \to E$, $\psi: E \to P$ such that $\psi \circ \varphi = 1$. Let μ be a finite, say $m \times n$ matrix over R which is a defining matrix of E. Then there are generators $e_1, e_2, ..., e_n$ of E such that $(r_j)[e_j] = r_1 e_1 + r_2 e_2 + \cdots + r_n e_n = 0$ for $(r_j) \in R^n$ if and only if $(r_j) \in R^m \mu$. So we have in particular $\mu[e_j] = 0$. Let $x_j = \psi(e_j)$ for $j = 1, 2, ..., n$. Then $x_1, x_2, ..., x_n$ are in P and satisfy $\mu[x_j] = 0$. Let next $v_1, v_2, ..., v_l$ be generators of N_0. (If N_0 is cyclic we may assume $l = 1$.) Let for each i, $\varphi(v_i) = \sum_{j=1}^{l} r_{ij} e_j$ with some $r_{ij} \in R$. Then we have $v_i = \psi(\varphi(v_i)) = \sum_j r_{ij} \psi(e_j) = \sum_j r_{ij} x_j$ for $j = 1, 2, ..., l$. Thus the vector $[x_j] \in P^n$ satisfies two systems linear equations $\mu[x_j] = 0$ and $[r_{ij}][x_j] = [v_i]$. Since N is pure in P, there must exist $y_1, y_2, ..., y_n$ in N such that $\mu[y_j] = 0$ and $[r_{ij}][y_j] = [v_i]$. The first equality implies the existence of a homomorphism $\eta: E \to N$ such that $\eta(e_j) = y_j$ for $j = 1, 2, ..., n$. The second equality implies then $v_i = \sum_j r_{ij} y_j = \sum_j r_{ij} \eta(e_j) = \eta(\sum_j r_{ij} e_j) = \eta(\varphi(v_i))$ for $i = 1, 2, ..., l$. Since $v_1, v_2, ..., v_l$ are generators of N_0, this means that $\eta \circ \varphi = 1$. Thus N is finitely (or singly) pure-projective again by Proposition 7. ∎

Let Q be a left R-module. Q is called *pure-injective* if, for any left R-module A and a pure submodule B of A, every homomorphism $B \to Q$ can be extended to a homomorphism $A \to Q$. On the other hand, Q is called *algebraically compact* if, for any row-finite matrix μ over R, a system of linear equations for μ in Q is solvable in Q whenever it is finitely solvable in Q, or in other words, if, for a row-finite $I \times J$ matrix $\mu = [a_{ij}]$ over R and a vector $[q_i]$ in Q^I, the system of linear equations $\sum_j a_{ij} x_j = q_i$ for $i \in I$ has a solution $[x_j]$ in Q^J whenever, for each finite subset I_0 of I, there exists a vector $[x_j^0]$ in Q^J such that $\sum_j a_{ij} x_j^0 = q_i$ for all $i \in I_0$. Also, we call a left R-module C *compact* if there is a compact Hausdorff topology on C making it a topological group and such that the left multiplications by elements of R are continuous. Warfield proved that *every R-module can be embedded as a pure submodule in a compact R-module* [13, Lemma 1], and that *the following conditions are equivalent*: (1) *Q is pure-injective*, (2) *Q is a direct summand of a compact R-module*, (3) *Q is algebraically compact* [13, Theorem 2].

Now, as a dual of the notion of finitely (or singly) pure-projective modules, we define Q to be *finitely* (or *singly*) *pure-injective* if Q is finitely (or singly) split in every pure extension of Q. Clearly, this is equivalent to the condition that Q is a direct summand of every finite (or single) pure extension of Q. On the other hand, we call Q *finitely* (or *singly*) *compact* if Q satisfies the condition of the algebraic compactness for all matrices of finite (or single) column(s) (instead of all row-finite matrices).

THEOREM 10. *The following conditions on a left R-module Q are equivalent*:

(1) *Q is finitely* (*or singly*) *pure-injective.*

(2) *Q is a finitely* (*or singly*) *split submodule of a compact R-module.*

(3) *Q is finitely* (*or singly*) *compact.*

(4) *For any R-module B and a finite* (*or single*) *pure extension A of B, every homomorphism $B \to Q$ can be extended to a homomorphism $A \to Q$.*

Proof. (1) $\Rightarrow$ (2) is an immediate consequence of the Warfield theorem that Q has a compact pure extension.

(2) $\Rightarrow$ (3): Assume that Q is a finitely (or singly) split submodule of a compact R-module C. Let μ be a matrix of finite (or single) column(s) over R and suppose that there is given a system of linear equations for μ in Q which is finitely solvable in Q. Since C is algebraically compact, the system is solvable in C. But Q is μ-pure in C by Theorem 3. Therefore the system is solvable in Q. Thus Q is finitely (or singly) compact.

(3) $\Rightarrow$ (1): Assume that Q is finitely (or singly) compact. Let M be a finite (or single) pure extension of Q. Let μ be a matrix of finite (or single) column(s) which defines the finitely generated (or cyclic) R-module M/Q. Consider a system of linear equations for μ in Q which is solvable in M. Since Q is pure in M, it is finitely solvable in Q and therefore is solvable in Q. Thus we know that Q is μ-pure in M. According to Proposition 1, this is equivalent to that natural epimorphism $M \to M/Q$ is M/Q-pure, and this means that Q is a direct summand of M.

(4) $\Rightarrow$ (1) is clear; indeed, (1) can be regarded as a special case of (4).

(2) $\Rightarrow$ (4): Assume that Q is finitely (or singly) split in a compact extension C. Let B be an R-module and $g\colon B \to Q$ a homomorphism. Let A be a finite pure extension of B. Since C is pure-injective, g can be extended to a homomorphism $f\colon A \to C$. If we consider f modulo B then we have an epimorphism $A/B \to (f(A)+Q)/Q$ and therefore $(f(A)+Q)/Q$ is finitely generated (or cyclic). Since Q is finitely (or singly) split in C, Q must be a direct summand of $f(A)+Q$. This means that there is a homomorphism $h\colon f(A)+Q \to Q$ which fixes Q element-wise. Then it is clear that $h \circ f\colon A \to Q$ is an extension of g. ∎

COROLLARY 11. *Every finitely (or singly) split submodule of a finitely pure-injective module is finitely pure-injective too.*

Proof. Let Q be a finitely (or singly) pure-injective R-module and Q' a finitely (or singly) split submodule of Q. By Theorem 10, Q is a finitely (or singly) split submodule of a compact R-module C. Then Q' is also finitely (or singly) split in C. Therefore Q' is finitely (or singly) pure-injective again by Theorem 10. ▮

3. FINITELY PROJECTIVE MODULES

Let M be a left R-module. M is called *flat* if for any right R-module A and a submodule B of A the natural homomorphism $\kappa \otimes M: B \otimes_R M \to A \otimes_R M$ is a monomorphism, where κ is the inclusion map $B \to A$. The flatness for M is equivalent to the condition that every epimorphism onto M is pure [12, Proposition 11.1, p. 37]. We now define M to be *finitely projective* or *singly projective* if every epimorphism onto M is finitely split or singly split, respectively. Clearly, M is finitely projective if and only if M is flat and finitely pure-projective.

PROPOSITION 12. *The following conditions on an R-module M are equivalent:*

(1) *M is finitely (or singly) projective.*

(2) *There exist a projective R-module P and a finitely (or singly) split epimorphism $P \to M$.*

(3) *For each finitely generated (or cyclic) submodule M_0 of M, there exist a projective R-module P and homomorphisms $\varphi: M_0 \to P$, $\psi: P \to M$ such that $\psi \circ \varphi = 1$, the identity map of M_0.*

(4) *Given an epimorphism $f: A \to C$, a homomorphism $h: M \to C$ and a finitely generated (or cyclic) submodule M_0 of M, there exists a homomorphism $g: M_0 \to A$ such that $f \circ g$ is the restriction of h to M_0.*

Proof. $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are clear. Assume (3), and let f, h, and M_0 be as in (4). Since P is projective, there is a homomorphism $\pi: P \to A$ such that $f \circ \pi = h \circ \psi$. If we put $g = \pi \circ \varphi: M_0 \to A$ then $f \circ g = f \circ \pi \circ \varphi = h \circ \psi \circ \varphi = h \circ 1$ is the restriction of h to M_0. (1) is the particular case of (4). ▮

COROLLARY 13. *Let M, M' be R-modules and let there be a finitely (or singly) split epimorphism $M \to M'$. If M is finitely (or singly) projective then so is M' too.*

PROPOSITION 14. *Every pure submodule of a finitely (or singly) projective module is finitely (or singly) projective.*

This can be proved in the similar manner as (indeed somewhat simplier than) in the proof of Proposition 9, or we can deduce this from Proposition 9 by combining with the known proposition that every pure submodule of a flat module is flat.

PROPOSITION 15. *Let R be a left Noetherian ring. Then every flat left R-module is finitely projective.*

Proof. This is an immediate consequence of Proposition 6.

PROPOSITION 16. *Let R be an integral domain. Then an R-module is singly projective if and only if it is torsion-free.*

Proof. Let M be a left R-module. Then there is a free left R-module F which has an epimorphism $f\colon F \to M$. Suppose that M is singly projective. Let u be any nonzero element of M. Then there is a homomorphism $\varphi\colon Ru \to F$ such that $f \circ \varphi = 1$, i.e., $f(\varphi(u)) = u$. It follows then $\varphi(u) \neq 0$. Since R is an integral domain, F is torsion-free. Therefore, if $ru = 0$ for an $r \in R$ then $r\varphi(u) = \varphi(ru) = 0$ and so $r = 0$. This shows that M is torsion-free. Conversely, suppose M is torsion-free. Let again u be a non-zero element of M. Then Ru is isomorphic to R and so is projective. Therefore the restriction of f to the inverse image of Ru by f must be a split epimorphism, which means that there exists a homomorphism $\varphi\colon Ru \to F$ such that $f \circ \varphi = 1$. Thus f is singly split. ∎

COROLLARY 17. *Let R be an integral domain. Then every flat R-module is singly projective.*

Proof. This follows from Proposition 16 and the fact that every flat R-module is torsion-free, as proved in [12, Example 1, p. 35].

A commutative integral domain is called a *Prüfer ring* if every finitely generated ideal is projective. If R is a Prüfer ring then every torsion-free R-module is flat by [11, Theorem 4.23]; and thus the three concepts of torsion-free modules, flat modules and singly projective modules coincide.

PROPOSITION 18. *Let R be a Prüfer ring. Then every flat R-module is finitely projective.*

Proof. Let M be a flat R-module, or equivalently, a torsion-free R-module. Let M_0 be a finitely generated submodule of M. By [13, Proposition 5; 11, Theorem 4.22], M_0 is projective. Let $f\colon A \to M$ be any epimorphism. Then its restriction to the inverse image of M_0 by f is a split epimorphism, i.e., there exists a homomorphism $\varphi: M_0 \to A$ such that $f \circ \varphi = 1$. Thus f is finitely split. ∎

Now let R be an arbitrary ring and F a free left R-module with a countable free basis $u_1, u_2, u_3, \ldots$. Let $a_1, a_2, a_3, \ldots$, be an infinite sequence in R. Let $v_1 = u_1 - a_1 u_2$, $v_2 = u_2 - a_2 u_3$, $v_3 = u_3 - a_3 u_4, \ldots$, and let G be the submodule of F generated by $v_1, v_2, v_3, \ldots$. These $v_1, v_2, v_3, \ldots$, are linearly independent over R and so form a free basis for G [1, Lemma 28.1]. The submodule G and the factor module F/G were considered by Bass in [2], and we shall call them the *Bass submodule of F* and the *left Bass factor module* over R belonging to the sequence $a_1, a_2, a_3, \ldots$. It was proved that F/G is flat [2, Lemma 1.1; 1, p. 316].

Let x be an element of F. Then it is uniquely expressed as $x = \sum_{i=1}^{\infty} r_i u_i$ with $r_i \in R$ and $r_i = 0$ for all but a finite number of i. For each i, we define

$$[x, i] = r_1 a_1 a_2 \cdots a_{i-1} + r_2 a_2 \cdots a_{i-1} + \cdots + r_{i-1} a_{i-1} + r_i.$$

Thus $[x, 1] = r_1$, $[x, 2] = r_1 a_1 + r_2$, and generally we have

$$[x, i] = [x, i-1]\, a_{i-1} + r_i.$$

(Of course, $[x, i]$ depends on the sequence $a_1, a_2, \ldots$.)

LEMMA 19. *Let x be an element of F. Then*

(i) *$x \in Ru_1 + Ru_2 + \cdots + Ru_n$ if and only if $[x, m] = [x, n]\, a_n \cdots a_{m-1}$ for all $m > n$;*

(ii) *$x \in Rv_1 + Rv_2 + \cdots + Rv_n$ if and only if $[x, m] = 0$ for all $m > n$, and in this case we have*

$$x = [x, 1]\, v_1 + [x, 2]\, v_2 + \cdots + [x, n]\, v_n.$$

Proof. Let $x = \sum_{i=1}^{\infty} r_i u_i$ with $r_i \in R$. Then $x \in Ru_1 + Ru_2 + \cdots + Ru_n$ means that $r_{n+1} = r_{n+2} = \cdots = 0$, or equivalently, $[x, n+1] = [x, n]\, a_n$, $[x, n+2] = [x, n+1]\, a_{n+1}, \ldots$. Substituting the first equality in the second, we have $[x, n+2] = [x, n]\, a_n a_{n+1}$. Then substituting this equality in the third, we have $[x, n+3] = [x, n]\, a_n a_{n+1} a_{n+2}$, and so on. Suppose conversely that $[x, n+1] = [x, n]\, a_n$, $[x, n+2] = [x, n]\, a_n a_{n+1}$, $[x, n+3] = [x, n]_n\, a_n a_{n+1} a_{n+2}, \ldots$. Then clearly we have $[x, n+2] = [x, n+1]\, a_{n+1}$, $[x, n+3] = [x, n+2]\, a_{n+2}, \ldots$, i.e., $r_{n+1} = r_{n+2} = r_{n+3} = \cdots = 0$. This proves (i). Next suppose that $x \in Rv_1 + Rv_2 + \cdots + Rv_n$, i.e., $x = s_1 v_1 + s_2 v_2 + \cdots + s_n v_n$ for some $s_i \in R$. Since $v_i = u_i - a_i u_{i+1}$ for each i, we have $x = s_1(u_1 - a_1 u_2) + s_2 (u_2 - a_2 u_3) + \cdots + s_n(u_n - a_n u_{n+1}) = s_1 u_1 + (s_2 - s_1 a_1)\, u_2 + \cdots + (s_n - s_{n-1} a_{n-1})\, u_n - s_n a_n u_{n+1}$. So, by comparing coefficients, we have $s_1 = r_1 = [x, 1]$, $s_2 = s_1 a_1 + r_2 = [x, 1]\, a_1 + r_2 = [x, 2], \ldots, s_n =$

$s_{n-1}a_{n-1}+r_n=[x, n-1]\,a_{n-1}+r_n=[x, r]$, and $0=x_na_n+r_{n+1}=[x, n]\,a_n+r_{n+1}=[x, n+1]$. Since $x\in Ru_1+Ru_2+\cdots+Ru_{n+1}$, we have $[x, m]=[x, n+1]\,a_{n+1}\cdots a_{m-1}$ for all $m\geqslant n+1$ by (i). Since, however, $[x, n+1]=0$, it follows that $[x, m]=0$ for all $m>n$. Conversely, suppose that $[x, m]=0$ for all $m>n$. Then $r_{n+1}=[x, n+1]-[x, n]\,a_n=-[x, n]\,a_n$, while $r_m=[x, m]-[x, m-1]\,a_{m-1}=0$ if $m>n+1$. This implies that

$$\begin{aligned}
x&=r_1u_1+r_2u_2+\cdots+r_nu_n-[x, n]\,a_nu_{n+1}\\
&=[x, 1]\,u_1+([x, 2]-[x, 1]\,a_1)\,u_2+\cdots\\
&\quad+([x, n]-[x, n-1]\,a_{n-1})\,u_n-[x, n]\,a_nu_{n+1}\\
&=[x, 1](u_1-a_1u_2)+[x, 2](u_2-a_2u_3)+\cdots+[x, n](u_n-a_nu_{n+1})\\
&=[x, 1]\,v_1+[x, 2]\,v_2+\cdots+[x, n]\,v_n.
\end{aligned}$$

This proves (ii). ∎

Lemma 20. *Let* $x\in Ru_1+Ru_2+\cdots+Ru_n$ *and* $m\geqslant n$. *Then* $x\in Rv_1+Rv_2+\cdots+Rv_m$ if and only if $[x, n]\,a_n\cdots a_m=0$.

Proof. From Lemma 19(i), it follows that $[x, m']=[x, n]\,a_n\cdots a_{m'-1}$ for all $m'>m(\geqslant n)$. On the other hand, by Lemma 19(ii), $x\in Rv_1+Rv_2+\cdots+Rv_m$ if and only if $[x, m']=0$ for all $m'>m$. This is equivalent to saying that $[x, n]\,a_n\cdots a_{m'-1}=0$ for all $m'>m$. However, this condition is clearly equivalent to the mere condition that $[x, n]\,a_n\cdots a_m=0$. ∎

Let $a\in R$. We denote by $l(a)$ the left annihilator of a in R, i.e., the left ideal of R consisting of those $r\in R$ for which $ra=0$.

Lemma 21. *Let* $x\in Ru_1+Ru_2+\cdots+Ru_n$. *Then* $x\in G$ *if and only if* $[x, n]\in\bigcup_{k=0}^{\infty} l(a_n\cdots a_{n+k})$.

Proof. This follows from Lemma 20 and the fact that $G=\bigcup_{k=0}^{\infty}(Rv_1+Rv_2+\cdots+Rv_{n+k})$. ∎

Proposition 22. *The following conditions are equivalent*:

(1) *There exists a homomorphism* $Ru_1+Ru_2+\cdots+Ru_n\to G$ *which fixes* $(Ru_1+Ru_2+\cdots+Ru_n)\cap G$ element-wise.

(2) *There exists a homomorphism* $Ru_n\to G$ *which fixes* $Ru_n\cap G$ *element-wise.*

(3) *The ascending chain* $l(a_n)\subset l(a_na_{n+1})\subset l(a_na_{n+1}a_{n+2})\subset\cdots$, *of left annihilators terminates.*

Remark. As was virtually shown in the proof of Theorem 3, the conditions (1) and (2) of Proposition 22 are equivalent to the conditions that G is a direct summand of $Ru_1+Ru_2+\cdots+Ru_n+G$ and a direct summand of Ru_n+G, respectively.

Proof. (1) $\Rightarrow$ (2) is clear. To prove (2) $\Rightarrow$ (3), assume (2). Let $h\colon Ru_n\to G$ be a homomorphism that fixes $Ru_n\cap G$ element-wise. Then $h(u_n)$ is in G and therefore it is in $Rv_1+Rv_2+\cdots+Rv_m$ for some $m\geqslant n$. Take an arbitrary element r from $\bigcup_{k=0}^{\infty} l(a_n\cdots a_{n+k})$, and let $x=ru_n$. Then $h(x)=rh(u_n)$ is in $Rv_1+Rv_2+\cdots+Rv_m$. On the other hand, we have $[x,n]=r$ and so by Lemma 21, x is in G whence in $Ru_n\cap G$. Thus we know that $x=h(x)\in Rv_1+Rv_2+\cdots+Rv_m$. By Lemma 20 we have $r=[x,n]\in l(a_n\cdots a_m)$. This implies that $\bigcup_{k=0}^{\infty} l(a_n\cdots a_{n+k})=l(a_n\cdots a_m)$, which means that condition (3) holds.

Next assume (3), i.e., $\bigcup_{k=0}^{\infty} l(a_n\cdots a_{n+k})=l(a_n\cdots a_m)$ for some $m\geqslant n$. For each $i=1,2,\dots,m$, let

$$w_i=v_i+a_iv_{i+1}+a_ia_{i+1}v_{i+2}+\cdots+a_ia_{i+1}\cdots a_{m-1}v_m,$$

and let $w_{m+1}=0$. Then we have $w_i-a_iw_{i+1}=v_i$ for $i=1,2,\dots,m$. Since $u_1,u_2,\dots,u_{m+1}$ are linearly independent over R, we can well-define a homomorphism $f\colon Ru_1+Ru_2+\cdots+Ru_{m+1}\to G$ by $f(u_i)=w_i$ $(i=1,2,\dots,m+1)$. Since $v_i=u_i-a_iu_{i+1}$, $v_1v_2,\dots,v_m$ are in $Ru_1+Ru_2+\cdots+Ru_{m+1}$ and $f(v_i)=f(u_i)-a_if(u_{i+1})=w_i-a_iw_{i+1}=v_i$ for $i=1,2,\dots,m$. Thus f fixes $Rv_1+Rv_2+\cdots+Rv_m$ element-wise. Let $x\in(Ru_1+\cdots+Ru_n)\cap G$. Then by Lemma 21 we have $[x,n]\in\bigcup_{k=0}^{\infty} l(a_n\cdots a_{n+k})=l(a_n\cdots a_m)$, i.e., $[x,n]\,a_n\cdots a_m=0$. Therefore $x\in Rv_1+Rv_2+\cdots+Rv_m$ by Lemma 20, which implies that $f(x)=x$. Thus the restriction of f to $Ru_1+Ru_2+\cdots+Ru_n$ satisfies the condition (1). ∎

Now we have the following theorem:

THEOREM 23. *Let $M=F/G$ be the left Bass factor module over R belonging to a sequence $a_1,a_2,a_3,\dots$, in R. Then the following conditions are equivalent:*

(1) *M is finitely projective.*

(2) *M is singly projective.*

(3) *The ascending chain $l(a_n)\subset l(a_na_{n+1})\subset l(a_na_{n+1}a_{n+2})\subset\cdots$ of left annihilators terminates for $n=1,2,3,\dots$.*

Proof. (1) $\Rightarrow$ (2) is clear. Assume (2). Then the natural epimorphism $F\to F/G$ is singly split. By Theorem 3, there exists, for each n, a homomorphism $Ru_n\to G$ that fixes $Ru_n\cap G$ element-wise. But this is, according to Proposition 22, equivalent to the condition that the ascending chain $l(a_n) \subset l(a_n a_{n+1}) \subset \cdots$, terminates for each n. Thus (2) $\Rightarrow$ (3) is proved. Next, assume (3). Let F_0 be any finitely generated submodule of F. Then there is a sufficiently large n such that $F_0 \subset Ru_1 + Ru_2 + \cdots + Ru_n$. By Proposition 22 there exists a homomorphism $Ru_1 + Ru_2 + \cdots + Ru_n \to G$ that fixes $(Ru_1 + Ru_2 + \cdots + Ru_n) \cap G$ element-wise. Then clearly the restriction of this homomorphism to F_0 gives a map $F_0 \to G$ that fixes $F_0 \cap G$ element-wise. Thus the natural epimorphism $F \to F/G$ is finitely split by Theorem 3. Since F is projective, this implies that F/G is finitely projective by Proposition 12. ∎

From Theorem 23 we have immediately

THEOREM 24. *Every left Bass factor module over R is finitely projective (or singly projective) if and only if, for every infinite sequence $a_1, a_2, a_3, \ldots$ in R, the ascending chain $l(a_1) \subset l(a_1 a_2) \subset l(a_1 a_2 a_3) \subset \cdots$ terminates.*

COROLLARY 25. *If every flat left R-module is singly projective then for every infinite sequence $a_1, a_2, a_3, \ldots,$ in R the ascending chain $l(a_1) \subset l(a_1 a_2) \subset l(a_1 a_2 a_3) \subset \cdots$ terminates*

Remark 1. There arises a problem to find out a characterization of those rings R over which every flat left module is finitely projective. Propositions 15 and 18 show that left Noetherian rings and Prüfer rings are examples of such a type of rings. Also, left perfect rings give an obvious example, because every flat left module over a left perfect ring is projective by a theorem of Bass. In view of Theorem 24 and Corollary 25, one might conjecture that every flat left R-module is finitely projective if (and only if) the ascending chain $l(a_1) \subset l(a_1 a_2) \subset l(a_1 a_2 a_3) \subset \cdots$ terminates for every infinite sequence $a_1, a_2, a_3, \ldots,$ in R. Indeed, the above three kind of rings clearly satisfy this condition. If this conjecture is true then it follows in particular that every flat module over an integral domain is finitely projective. We also point out that if R satisfies the above condition on termination of ascending chains then R has no infinite number of orthogonal idempotents $\neq 0$ and R is a finite direct sum of indecomposable left ideals. Further, it should be mentioned that if every flat left R-module is finitely projective then in particular every finitely generated flat left R-module is projective, while it is known that commutative integral domains and semi-perfect rings enjoy the last condition [12, Example, p. 39; 8, Exercise 10, p. 136].

Remark 2. As a dual of the finite (or single) projectivity, we may consider a module M—finitely (or singly) injective module, so to speak—which is finitely (or singly) split in every extension module of M. However, by applying Proposition 5 to an injective extension of M, it turns out that such a module M is always injective.

Now in connection with Theorem 23 the following may be of interest.

THEOREM 26. *Let $M = F/G$ be the left Bass factor module over R belonging to an infinite sequence $a_1, a_2, a_3, \ldots$, in R. Then the following conditions are equivalent*:

(1) *M is projective, i.e., G is a direct summand of F.*

(2) *The descending chain $a_n R \supset a_n a_{n+1} R \supset a_n a_{n+1} a_{n+2} R \supset \cdots$ of principal right ideals of R terminates for $n = 1, 2, 3, \ldots$.*

(3) *There exists a row-finite matrix $[c_{ij}]$ of countable rows and columns over R such that, for each $i, j = 1, 2, 3, \ldots$,*

$$a_i c_{i+1,j} = \begin{cases} c_{ij} - 1, & i = j \\ c_{ij}, & i \neq j. \end{cases}$$

Proof. Assume (1). Then the descending chain $a_1 R \supset a_1 a_2 R \supset a_1 a_2 a_3 R \supset \cdots$, terminates by Bass [2, Lemma 1.3] or Anderson and Fuller [1, Lemma 28.2, p. 313]. Let n be any positive integer, and let F_n, G_n be the submodule of F generated by $u_n, u_{n+1}, \ldots$, and $v_n, v_{n+1}, \ldots$, respectively. Thus F_n and G_n are free left R-modules with free bases $u_n, u_{n+1}, \ldots$, and $u_n, u_{n+1}, \ldots$, respectively, and G_n is the Bass submodule of F_n belonging to the sequence $a_n, a_{n+1}, \ldots$. Since G_n is a direct summand of G, it is a direct summand of F. Since further $G_n \subset F_n \subset F$, it follows that G_n is a direct summand of F_n. Therefore, by applying again [2, Lemma 1.3] to F_n/G_n, we know that the descending chain $a_n R \supset a_n a_{n+1} R \supset \cdots$, terminates.

The equivalence of (1) and (3) is virtually established in the proof of [1, Lemma 28.2].

In order to prove (2) $\Rightarrow$ (3), assume (2). For each $i \geq 1$, let $n(i)$ ($\geq i$) be the minimal integer such that

$$a_i \cdots a_{n(i)} R = a_i \cdots a_{n(i)} a_{n(i)+1} R = \cdots$$

and denote this right ideal by K_i. Thus

$$K_{i+1} = a_{i+1} \cdots a_{n(i+1)} R = a_{i+1} \cdots a_{n(i+1)} a_{n(i+1)+1} R = \cdots.$$

Left-multiplying a_i, we have

$$a_i K_{i+1} = a_i a_{i+1} \cdots a_{n(i+1)} R = a_i a_{i+1} \cdots a_{n(i+1)} a_{n(i+1)+1} R = \cdots.$$

This implies that $n(i) \leq n(i+1)$ and $K_i = a_i K_{i+1}$. Thus we know that $n(1) \leq n(2) \leq n(3) \leq \cdots$, and $K_i = a_i K_{i+1} = a_i a_{i+1} K_{i+2} = \cdots = a_i a_{i+1} \cdots a_n K_{n+1}$ for every $n > i$.

(i) Let j be an integer such that $j > n(1)$. Let h be the maximal integer such that $n(h) < j$, i.e., $n(h) \leq j - 1$ (since generally $n(i) \geq i$ for every i, such an h exists). Then $-a_h \cdots a_{j-1} \in a_h \cdots a_{j-1} R = K_h = a_h \cdots a_{j-1} a_j K_{j+1}$, so there is a $c_j \in K_{j+1}$ such that $-a_h \cdots a_{j-1} = a_h \cdots a_{j-1} a_j c_j$. Define $c_{j+1,j} = c_j$, $c_{j,j} = a_j c_j + 1$ $(= a_j c_{j+1,j} + 1)$ and $c_{ij} = a_i \cdots a_{j-1} c_{j,j}$ for each $i < j$. Since $c_j \in K_{j+1} = a_{j+1} K_{j+2}$, there is a $c \in K_{j+2}$ such that $c_j = a_{j+1} c$. Define then $c_{j+2,j} = c$. Similarly, we can define $c_{j+3,j}$ as an element of K_{j+3} such that $c_{j+2,j} = a_{j+2} c_{j+3,j}$ because of $K_{j+2} = a_{j+2} K_{j+3}$, and then define $c_{j+4,j} \in K_{j+4}$ by $c_{j+3,j} = a_{j+3} c_{j+4,j}$. Continuing in this way, we can define $c_{i,j}$ for every $i \geqslant 1$, and it is easy to see that these $c_{i,j}$ satisfy the following equalities:

$$a_j c_{j+1,j} = c_{j,j} - 1, \qquad a_i c_{i+1,j} = c_{i,j} \quad \text{for } i \neq j.$$

Moreover $c_{ij} = 0$ if $i \leqslant h$. For, since $c_{j+1,j}$ $(= c_j)$ was defined by $-a_h \cdots a_{j-1} = a_h \cdots a_j c_{j+1,j}$ as above, we have $-a_i \cdots a_{j-1} = a_i \cdots a_j c_{j+1,j}$ (by left-multiplying $a_i \cdots a_{h-1}$) if $i \leqslant h$. On the other hand, since $c_{jj} = a_j c_{j+1,j} + 1$ and $c_{ij} = a_i \cdots a_{j-1} c_{jj}$ as above (because $i \leqslant h \leqslant n(h) < j$), we have $a_i \cdots a_j c_{j+1,j} = a_i \cdots a_{j-1}(c_{jj} - 1) = c_{ij} - a_i \cdots a_{j-1}$. Thus we have $c_{ij} = 0$.

(ii) Let i be a positive integer. Let j be any integer such that $j > n(i)$. Then of course $j > n(1)$ and so c_{ij} is well defined as in (i). Let h be the maximal integer such that $n(h) < j$ as defined in (i). Then $i \leqslant h$, and therefore $c_{ij} = 0$ as seen above. Thus $c_{ij} = 0$ for all but a finite number of j.

(iii) Let $j \leqslant n(1)$. Then define $c_{ij} = 0$ or $= 1$ or $= a_i \cdots a_{j-1}$ accordingly as $i > j$ or $i = j$ or $i < j$. Then it is also easy to check that these c_{ij} satisfy the same equalities $a_j c_{j+1,j} = c_{jj} - 1$, $a_i c_{i+1,j} = c_{ij}$ $(i \neq j)$ as in (i).

Thus we have defined c_{ij} for all positive integers i, j for which the matrix $[c_{ij}]$ is row-finite and satisfies (3). ∎

Let A, C be left R-modules and $f: A \to C$ an epimorphism. We call f *locally split* if for each $c \in C$ there exists a homomorphism $\varphi: C \to A$ such that $f(\varphi(c)) = c$. Let next B a submodule of A. B is called *locally split* in A if for each $b \in B$ there exists a homomorphism $h: A \to B$ such that $h(b) = b$. We know, however, by using the method in the proof of [11, Theorem 3.29, p. 61], that if $f: A \to C$ is a locally split epimorphism then, for every finitely generated submodule C_0 of C, there exists a homomorphism $\varphi: C \to A$ such that the restriction of $f \circ \varphi$ to C_0 is the identity map of C_0, while if B is a locally split submodule of A then, for every finitely generated submodule B_0 of B, there exists a homomorphism $h: A \to B$ such that the restriction of h to B_0 is the identity map of B_0. Thus it is clear that *every locally split epimorphism is finitely split.* The notion of locally split submodules was considered by Ramamurthi and Rangaswamy [10] by the name of strongly pure submodule. Indeed, every locally split submodule is a pure submodule; but a locally split submodule is not always a finitely split submodule. Moreover, it is to be pointed out that, unlike the purity or the finite splitness, the local splitness of an epimorphism does not necessarily imply the local splitness of its kernel, and conversely.

Let M be a left R-module. M is called *locally projective* if every epimorphism onto M is locally split, while M is called *locally injective* if M is locally split in every extension of M. Clearly, every locally projective module is finitely projective and every locally injective module is absolutely pure. The concept of locally projective modules was originally introduced by Zimmermann–Huisgen [14], and also by Gruson and Raynaud [7] and Garfinkel [5] by the names of flat strict Mittag–Leffler modules and universally torsionless modules respectively, while the concept of locally injective modules was introduced by Ramamurthi and Rangaswamy [10] and called finitely injective modules or strongly absolutely pure (SAP) modules. It is shown in [14], that *the local projectivity of M is equivalent to either of the following conditions*:

(1) *For each finitely generated submodule M_0 of M, there exist a finite number of homomorphisms φ_i: $M \to R$ and the same number of $v_i \in M$ such that $\Sigma\varphi_i(x)\, u_i = x$ for all $x \in M_0$*,

(2) *Given an epimorphism f: $A \to C$, a homomorphism h: $M \to C$ and a finitely generated submodule M_0* of M, *there exists a homomorphism g: $M \to A$ such that the restrictions of $f \circ g$ and h to M_0 coincide*, while it is proved in [10] that *M is locally injective if and only if, for any module A and a finitely generated submodule A_0, every homomorphism $A_0 \to M$ can be extended to a homomorphism $A \to M$.*

Now, refining the well known theorem of Bass [2, Theorem P; 1, Theorem 28.4] that every flat left R-module is projective if and only if R is left perfect, Zimmermann–Huisgen proved in [15, Proposition 33, p. 61] that *every locally projective left R-module is projective if and only if R is left perfect*. From this follows in particular that every finitely projective left R-module is projective if and only if R is left perfect. On the other hand, Zimmermann–Huisgen pointed out in [15, p. 60] that *every flat left R-module is locally projective if and only if R is left perfect*. However, this theorem does not remain true if we replace the local projectivity by the finite projectivity, because according to Propositions 15 and 18 every flat left R-module is finitely projective whenever R is either left Noetherian or a Prüfer ring but these types of rings are not necessarily left perfect.

As a dual of the above Bass' theorem, Megibben proved in [9, Theorem 3] that *every absolutely pure left R-module is injective if and only if R is left Noetherian*, and this was generalized in [10, (3.10)(b)] so that *every locally injective left R-module is injective if and only if R is left Noetherian*. This theorem may be regarded as a dual of the first theorem of

Zimmermann–Huisgen. We propose to dualize her second theorem and particularly ask the question: If every absolutely pure left R-module is locally injective, is R left Noetherian?

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Note added in proof. (1) The author has recently proved that if every flat left module over a ring R is finitely projective then the same holds for any subring of R. In particular, every flat module over a commutative integral domain is finitely projective, and thus Proposition 18 turns out superfluous. This and related results will appear in a forthcoming paper. (2) It has been pointed out to the author that finitely projective modules and finitely pure-projective modules were virtually considered by Clarks (Ph.D. Thesis, Kent State University, 1976), Goodearl (*Pacific J. Math.* **43**, 1972) and Jones (*Comm. Algebra* **9**, 1981), and indeed the above modules coincide with f-projective modules and R-Mittag–Leffler modules, respectively.

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