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Existence of renormalized solutions to nonlinear elliptic equations with a lower-order term and right-hand side a measure ☆,☆☆

M. Francesca Betta [a], Anna Mercaldo [b], François Murat [c], M. Michaela Porzio [d]

[a] *Dipartimento di Matematica, Seconda Università di Napoli, via Vivaldi 43, 81100 Caserta, Italy*
[b] *Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Complesso Monte S. Angelo, via Cintia, 80126 Napoli, Italy*
[c] *Laboratoire Jacques-Louis Lions, Université Paris VI, Boîte courrier 187, 75252 Paris cedex 05, France*
[d] *Facoltà di Scienze Matematiche, Fisiche e Naturali, Università degli Studi del Sannio, via Port'Arsa 11, 82100 Benevento, Italy*

Abstract

In this paper we prove the existence of a renormalized solution for a class of non coercive nonlinear equations whose prototype is:

$$\begin{cases} -\Delta_p u + b(x)|\nabla u|^{\lambda} = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geqslant 2$, Δ_p is the so called p-Laplace operator, $1 < p < N$, μ is a Radon measure with bounded variation on Ω, $0 \leqslant \lambda \leqslant p-1$ and b belongs to the Lorentz space $L^{N,1}(\Omega)$.

Résumé

Dans cet article nous démontrons l'existence d'une solution renormalisée pour une classe d'équations non linéaires non coercives dont le prototype est :

$$\begin{cases} -\Delta_p u + b(x)|\nabla u|^{\lambda} = \mu & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

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E-mail addresses: mariafrancesca.betta@unina2.it (M.F. Betta), mercaldo@unina.it (A. Mercaldo), murat@ann.jussieu.fr (F. Murat), porzio@unisannio.it (M.M. Porzio).

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où Ω est un ouvert borné de $\mathbb{R}^N$, $N \geqslant 2$, Δ_p est le p-Laplacien, $1 < p < N$, μ est une mesure de Radon bornée, $0 \leqslant \lambda \leqslant p-1$ et b appartient à l'espace de Lorentz $L^{N,1}(\Omega)$.

Keywords: Existence; Nonlinear elliptic equations; Noncoercive problems; Measures data

Mots-clés : Existence ; Équations elliptiques non linéaires ; Problèmes non coercifs ; Données mesures

1. Introduction

In this paper we consider a class of problems whose prototype is

$$\begin{cases} -\Delta_p u + b(x)|\nabla u|^{\lambda} = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geqslant 2$, Δ_p is the so called p-Laplace operator, $1 < p < N$, μ is a Radon measure with bounded variation on Ω, $0 \leqslant \lambda \leqslant p-1$ and the coefficient b belongs to the Lorentz space $L^{N,1}(\Omega)$. We are interested in proving an existence result.

This problem has two main features: on the one hand, the right-hand side is a measure (and not an element of the dual space $W^{-1,p'}(\Omega)$); on the other hand, the operator is in general not coercive when the norm of b in $L^{N,1}(\Omega)$ is not small. Those features produce specific difficulties.

Let us begin with the problems induced by the fact that the right-hand side is a measure. For the moment we assume $b = 0$, i.e., that there is no nonlinear term $b(x)|\nabla u|^{\lambda}$.

In the linear case (where $p = 2$), Stampacchia defined in [34] a notion of solution of (1.1) by duality, for which he proved existence and uniqueness; he proved in particular that this solution belongs to $W_0^{1,q}(\Omega)$ for every $q < N/(N-1)$ and satisfies (1.1) in the distributional sense. Stampacchia's duality arguments have been extended to the nonlinear case when $p = 2$ ([30]), but not to the case $p \neq 2$.

The nonlinear case was firstly studied in [8,9] (and then in [14], where a term $b(x)|\nabla u|^{p-1}$ is considered). In these papers the existence of a solution which satisfies the equation in the distributional sense is proven when $p > 2 - 1/N$; this assumption on p ensures that the solution belongs to $W_0^{1,q}(\Omega)$ for every $q < N(p-1)/(N-1)$ (note that $N(p-1)/(N-1) > 1$ when $p > 2 - 1/N$).

There are however two difficulties when one considers this type of solution for (1.1). On the first hand, when p is close to 1, i.e., $p \leqslant 2 - 1/N$, simple examples show that the solution of (1.1) does not in general belong to the space $W_{\mathrm{loc}}^{1,1}(\Omega)$ (take the Dirac mass at the center of a ball Ω). On the other hand, a classical counterexample ([33], see also [32]) shows that such a solution is, in general, not unique.

To overcome these difficulties two equivalent notions of solutions have been introduced, the notion of entropy solution in [1,10] and the notion of renormalized solution in [26,29, 30], in the case where the measure μ belongs to $L^1(\Omega)$ or to $L^1(\Omega) + W^{-1,p'}(\Omega)$; in these papers the existence and uniqueness of such solutions are proven. In [13] these notions of solution have been extended to the case of a general measure with bounded variation, and an existence result is proven and (partial) uniqueness results obtained for "comparable" solutions (see further results about uniqueness in [22]).

Let us finally explain the restriction $1 < p < N$ on p. When $p > N$, Sobolev embedding theorem and a duality argument imply that the space of measures with bounded variation on Ω is a subspace of $W^{-1,p'}(\Omega)$, which reconduces the problem to a classical one, and the counterpart of the results of the present paper can therefore be found in [3,4,15,17] for $p > N$. When $p = N$, existence and uniqueness results of a solution in the distributional sense have been proven in [18,19,21] in the case $b = 0$. We do not consider in the present paper the case $p = N$ which would lead to further technicalities.

Let us now pass to the problems induced by the nonlinear term $b(x)|\nabla u|^{\lambda}$. When μ belongs to the dual space $W^{-1,p'}(\Omega)$, and when $\lambda = p - 1$, the use of the test function u in (1.1) leads to

$$\int_{\Omega} |\nabla u|^p \leqslant \|\mu\|_{W^{-1,p'}(\Omega)} \|u\|_{W_0^{1,p}(\Omega)} + \|b\|_{L^N(\Omega)} \big\| |\nabla u| \big\|_{L^p(\Omega)}^{p-1} \|u\|_{L^{p^*}(\Omega)} \tag{1.2}$$

with $1/p^* = 1/p - 1/N$, which using Sobolev embedding produces an a priori estimate when $\|b\|_{L^N(\Omega)}$ is sufficiently small. When $\|b\|_{L^N(\Omega)}$ is large, Bottaro and Marina developed in [11] a technique which allowed them to prove an a priori estimate and an existence and uniqueness result in the linear case. This existence result was generalized to the nonlinear case in [15]. Similar results were obtained by symmetrization techniques in [2–4,16, 17].

In the present paper, we face both difficulties (right-hand side measure and b large). Our goal is to prove the existence of a renormalized solution for a class of problems whose prototype is (1.1) (see Theorem 2.1, which is proven in Section 3). More precisely, we prove the existence of a renormalized solution of (1.1) when $0 \leqslant \lambda \leqslant p - 1$, when $b \in L^{N,1}(\Omega)$ and when μ is a general measure with bounded variation.

The idea is to consider first the case where $\|b\|_{L^{N,1}(\Omega)}$ is small; in this case the operator is coercive. Hence, using the truncation $T_k(u)$ as a test function in (1.1), we easily obtain that $\|\nabla T_k(u)\|_{(L^p(\Omega))^N}^p \leqslant Mk$ for every $k > 0$, where $M = \|\mu\|_{M_b(\Omega)} + \|b|\nabla u|^{p-1}\|_{L^1(\Omega)}$. We then use the following result of [1] (that we slightly generalize in Appendix A): when the truncations $T_k(v)$ of a function v belong to $W_0^{1,p}(\Omega)$ and satisfy the inequality $\|\nabla T_k(v)\|_{(L^p(\Omega))^N}^p \leqslant Mk$ for all $k > 0$, then v satisfies $\||\nabla v|^{p-1}\|_{L^{N',\infty}(\Omega)} \leqslant C_0 M$. Therefore, one has:

$$\begin{aligned} \big\| |\nabla u|^{p-1} \big\|_{L^{N',\infty}(\Omega)} &\leqslant C_0 M = C_0 \big[\|\mu\|_{M_b(\Omega)} + \big\| b |\nabla u|^{p-1} \big\|_{L^1(\Omega)} \big] \\ &\leqslant C_0 \big[\|\mu\|_{M_b(\Omega)} + \|b\|_{L^{N,1}(\Omega)} \big\| |\nabla u|^{p-1} \big\|_{L^{N',\infty}(\Omega)} \big], \end{aligned}$$

and when $\|b\|_{L^{N,1}(\Omega)}$ is small, we obtain an a priori estimate, which allows one to prove the existence result.

In the case where $\|b\|_{L^{N,1}(\Omega)}$ is not small, we use the technique of Bottaro–Marina, which in some sense allows one to reduce the problem to a finite sequence of problems with $\|b\|_{L^{N,1}(\Omega)}$ small and to prove again the existence of a renormalized solution.

In conclusion, in the present paper we prove the existence of a renormalized solution when μ is a Radon measure with bounded variation and when the lower-order term has a growth like $b(x)\,|\nabla u|^\lambda$, with $0 \leqslant \lambda \leqslant p-1$ and a coefficient b which belongs to the Lorentz space $L^{N,1}(\Omega)$. This seems to be close to the optimal result that one can hope in such a framework.

The present paper has been announced in [5].

In a forthcoming paper [6], we prove uniqueness results for a class of problems whose prototype is a nondegenerated variation of (1.1), in the case where the right-hand side μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$ and where b belongs to some Lebesgue space $L^r(\Omega)$. Let us note the following surprising and unsatisfactory fact: while we prove in the present paper that there exists at least a renormalized solution of (1.1) for $0 \leqslant \lambda \leqslant p-1$, we prove in [6] that the renormalized solution of this problem (or more exactly of its nondegenerated variation), if it exists, is unique when $0 \leqslant \lambda < \lambda^*(N,p)$, where in some cases, $\lambda^*(N,p) > p-1$, while in other cases $\lambda^*(N,p) < p-1$. Therefore, the intervals in λ for which we prove either existence or uniqueness do not coincide in general. The same phenomenon appears in the case where one deals with usual weak solutions for right-hand sides in $W^{-1,p'}(\Omega)$: we prove uniqueness results in this more classical framework in [7].

2. Definitions and main result

In this section, we recall the definition of a renormalized solution for nonlinear elliptic problems with right-hand side a measure (cf. [13]), and we state our existence result. We begin with a few preliminaries about the decomposition of measures (which can be found in [13]) and about Lorentz spaces (see, e.g., [23,27,31]).

In the whole of this paper, Ω is a bounded open subset of $\mathbb{R}^N$, $N \geqslant 2$, and p is a real number, $1 < p < N$, with p' defined by $1/p + 1/p' = 1$.

2.1. Decomposition of measures

We start recalling the definition of p-capacity. The p-capacity $\operatorname{cap}_p(K,\Omega)$ of a compact set $K \subset \Omega$ with respect to Ω is:

$$\operatorname{cap}_p(K,\Omega) = \inf\left\{ \int_\Omega |\nabla\varphi|^p\colon\ \varphi \in C_c^\infty(\Omega),\ \varphi \geqslant \chi_K \right\},$$

where χ_K is the characteristic function of K (we will use the convention that $\inf\emptyset = +\infty$). If $U \subseteq \Omega$ is an open set, then we denote

$$\operatorname{cap}_p(U,\Omega) = \sup\bigl\{\operatorname{cap}_p(K,\Omega)\colon\ K \text{ compact},\ K \subseteq U\bigr\},$$

while the p-capacity of any subset $B \subseteq \Omega$ is defined as:

$$\operatorname{cap}_p(B,\Omega) = \inf\bigl\{\operatorname{cap}_p(U,\Omega)\colon\ U \text{ open},\ B \subseteq U\bigr\}.$$

We denote by $M_b(\Omega)$ the space of all Radon measures on Ω with bounded variation and by $C_b^0(\Omega)$ the space of all bounded, continuous functions on Ω. Thus $\int_\Omega \varphi \,\mathrm{d}\mu$ is well defined for $\varphi \in C_b^0(\Omega)$ and $\mu \in M_b(\Omega)$. Moreover μ^+ and μ^- are the positive and the negative parts of the measure μ, respectively.

Definition 2.1. A sequence $\{\mu_n\}$ of measures in $M_b(\Omega)$ converges in the narrow topology to a measure μ in $M_b(\Omega)$ if

$$\lim_{n\to+\infty} \int_\Omega \varphi \,\mathrm{d}\mu_n = \int_\Omega \varphi \,\mathrm{d}\mu,$$

for every $\varphi \in C_b^0(\Omega)$.

We define $M_0(\Omega)$ as the set of all the measures μ in $M_b(\Omega)$ which are absolutely continuous with respect to the p-capacity, i.e., which satisfy $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ such that $\mathrm{cap}_p(B, \Omega) = 0$. We define $M_s(\Omega)$ as the set of all the measures μ in $M_b(\Omega)$ which are singular with respect to the p-capacity, i.e., which are concentrated in a set $E \subset \Omega$ such that $\mathrm{cap}_p(E, \Omega) = 0$.

An important property of the measures in $M_b(\Omega)$ is the following [20, Lemma 2.1]:

Proposition 2.1. *For every measure in $M_b(\Omega)$ there exists an unique pair of measures (μ_0, μ_s), with $\mu_0 \in M_0(\Omega)$ and $\mu_s \in M_s(\Omega)$, such that $\mu = \mu_0 + \mu_s$.*

The measures μ_0 and μ_s will be called the absolutely continuous part and the singular part of μ with respect to the p-capacity. Actually, for what concerns μ_0, one has the following decomposition result [10, Theorem 2.1]:

Proposition 2.2. *Let μ_0 be a measure in $M_b(\Omega)$. Then μ_0 belongs to $M_0(\Omega)$ if and only if it belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$. Thus if μ_0 belongs to $M_0(\Omega)$, there exists f in $L^1(\Omega)$ and g in $(L^{p'}(\Omega))^N$ such that*

$$\mu_0 = f - \mathrm{div}(g),$$

in the sense of distributions. Moreover, every function $v \in W_0^{1,p}(\Omega)$ is measurable with respect to μ_0 and belongs to $L^\infty(\Omega, \mu_0)$ if v further belongs to $L^\infty(\Omega)$, and one has:

$$\int_\Omega v \,\mathrm{d}\mu_0 = \int_\Omega f v + \int_\Omega g \nabla v, \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

As a consequence of the previous results and the Hahn decomposition theorem we get:

Proposition 2.3. *Every measure μ in $M_b(\Omega)$ can be decomposed as follows*:

$$\mu = \mu_0 + \mu_s = f - \mathrm{div}(g) + \mu_s^+ - \mu_s^-,$$

where μ_0 is a measure in $M_0(\Omega)$, hence can be written as $f - \operatorname{div}(g)$, with $f \in L^1(\Omega)$ and $g \in (L^{p'}(\Omega))^N$, and where μ_s^+ and μ_s^- (the positive and the negative parts of μ_s) are two nonnegative measures in $M_b(\Omega)$, which are concentrated in two disjoint subsets E^+ and E^- of zero p-capacity.

2.2. A few properties of Lorentz spaces

In the present paper, we will use only the following properties of the Lorentz spaces, which are intermediate spaces between the Lebesgue spaces, in the sense that, for every $1 < s < r < \infty$, one has

$$L^{r,1}(\Omega) \subset L^{r,r}(\Omega) = L^r(\Omega) \subset L^{r,\infty}(\Omega) \subset L^{s,1}(\Omega). \tag{2.1}$$

For $1 < r < \infty$, the Lorentz space $L^{r,\infty}(\Omega)$ is the space of Lebesgue measurable functions such that

$$\|f\|_{L^{r,\infty}(\Omega)} = \sup_{t>0} t\big[\operatorname{meas}\big\{x \in \Omega\colon \big|f(x)\big| > t\big\}\big]^{1/r} < +\infty, \tag{2.2}$$

endowed with the norm defined by (2.2). For $1 < q < \infty$, the Lorentz space $L^{q,1}(\Omega)$ is the space of Lebesgue measurable functions such that:

$$\|f\|_{L^{q,1}(\Omega)} = \int_0^{|\Omega|} f^*(t) t^{1/q} \frac{\mathrm{d}t}{t} < +\infty, \tag{2.3}$$

endowed with the norm defined by (2.3). Here f^* denotes the decreasing rearrangement of f, i.e., the decreasing function defined by

$$f^*(t) = \inf\big\{s \geqslant 0\colon \operatorname{meas}\big\{x \in \Omega\colon \big|f(x)\big| > s\big\} < t\big\}, \quad t \in \big[0, |\Omega|\big].$$

For references about rearrangements see, for example, [12,24].

The space $L^{r,\infty}(\Omega)$ is the dual space of $L^{r',1}(\Omega)$, where $1/r + 1/r' = 1$, and one has the generalized Hölder inequality

$$\begin{cases} \forall f \in L^{r,\infty}(\Omega),\ \forall g \in L^{r',1}(\Omega), \\ \left|\displaystyle\int_\Omega fg\right| \leqslant \|f\|_{L^{r,\infty}(\Omega)} \|g\|_{L^{r',1}(\Omega)}. \end{cases} \tag{2.4}$$

2.3. Definition of a renormalized solution and existence result

For $k > 0$, denote by $T_k : \mathbb{R} \to \mathbb{R}$ the usual truncation at level k, that is

$$T_k(s) = \begin{cases} s, & |s| \leqslant k, \\ k \operatorname{sign}(s), & |s| > k. \end{cases}$$

Consider a measurable function $u:\Omega \to \overline{\mathbb{R}}$ which is finite almost everywhere and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. Then there exists (see, e.g., [1], Lemma 2.1) an unique measurable function $v:\Omega \to \overline{\mathbb{R}}^N$ such that

$$\nabla T_k(u) = v\chi_{\{|u|\leqslant k\}} \quad \text{almost everywhere in } \Omega,\ \forall k > 0. \tag{2.5}$$

We define the gradient ∇u of u as this function v, and denote $\nabla u = v$. Note that this definition is different of the definition of the distributional gradient. However, if $v \in (L^1_{\text{loc}}(\Omega))^N$, then $u \in W^{1,1}_{\text{loc}}(\Omega)$ and v is the distributional gradient of u. In contrast there are examples of functions $u \notin L^1_{\text{loc}}(\Omega)$ (and thus such that the gradient of u in the distributional sense is not defined) for which the gradient ∇u is defined in the previous sense (see Remarks 2.10 and 2.11, Lemma 2.12 and Example 2.16 of [13]).

In the present paper, we consider a nonlinear elliptic problem which can formally be written as

$$\begin{cases} -\operatorname{div}\big(a(x,u,\nabla u)\big) + H(x,u,\nabla u) + G(x,u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Here the function $a:\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying:

$$a(x,s,\xi)\xi \geqslant \alpha|\xi|^p, \quad \alpha > 0, \tag{2.7}$$

$$\big|a(x,s,\xi)\big| \leqslant c\big[|\xi|^{p-1} + |s|^{p-1} + a_0(x)\big], \quad a_0(x) \in L^{p'}(\Omega),\ c > 0, \tag{2.8}$$

$$\big(a(x,s,\xi) - a(x,s,\eta), \xi - \eta\big) > 0, \quad \xi \neq \eta, \tag{2.9}$$

for almost every $x \in \Omega$ and for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^N$. Moreover the functions $H:\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $G:\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying:

$$\begin{cases} \big|H(x,s,\xi)\big| \leqslant b_0(x)|\xi|^{p-1} + b_1(x), \\ b_0 \in L^{N,1}(\Omega), \quad b_1 \in L^1(\Omega), \end{cases} \tag{2.10}$$

$$G(x,s)s \geqslant 0, \tag{2.11}$$

$$\begin{cases} \big|G(x,s)\big| \leqslant b_2(x)|s|^r + b_3(x), \\ b_2 \in L^{z',1}(\Omega), \qquad b_3 \in L^1(\Omega), \end{cases} \tag{2.12}$$

for almost every $x \in \Omega$ and for every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, where

$$0 \leqslant r < \frac{N(p-1)}{N-p}, \quad z = \frac{N(p-1)}{N-p}\frac{1}{r} \quad \text{and} \quad \frac{1}{z} + \frac{1}{z'} = 1. \tag{2.13}$$

Finally, μ is a measure in $M_b(\Omega)$ that is decomposed in

$$\mu = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^-, \tag{2.14}$$

according to Proposition 2.3.

Remark 2.1. A special case where the function b_0 satisfies $b_0 \in L^{N,1}(\Omega)$ (as requested in hypothesis (2.10)) is the case where $b_0 \in L^q(\Omega)$ for some $q > N$.

Definition 2.2. We say that u is a renormalized solution of (2.6) if it satisfies the following conditions:

$$u \text{ measurable on } \Omega, \text{ almost everywhere finite}, \ T_k(u) \in W_0^{1,p}(\Omega), \ \forall k > 0; \tag{2.15}$$

$$|u|^{p-1} \in L^{N/(N-p),\infty}(\Omega); \tag{2.16}$$

the gradient ∇u introduced in (2.5), satisfies:

$$|\nabla u|^{p-1} \in L^{N',\infty}(\Omega), \tag{2.17}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int\limits_{n\leqslant u<2n} a(x,u,\nabla u)\nabla u\varphi = \int\limits_{\Omega} \varphi \, \mathrm{d}\mu_s^+, \tag{2.18}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int\limits_{-2n<u\leqslant -n} a(x,u,\nabla u)\nabla u\,\varphi = \int\limits_{\Omega} \varphi \, \mathrm{d}\mu_s^- \tag{2.19}$$

for every $\varphi \in C_b^0(\Omega)$; and finally

$$\int\limits_{\Omega} a(x,u,\nabla u)\cdot\nabla u\, h'(u)v + \int\limits_{\Omega} a(x,u,\nabla u)\cdot\nabla v\, h(u) + \int\limits_{\Omega} H(x,u,\nabla u)h(u)v$$
$$+ \int\limits_{\Omega} G(x,u)h(u)v = \int\limits_{\Omega} fh(u)v + \int\limits_{\Omega} g\nabla u\, h'(u)v + \int\limits_{\Omega} g\nabla v\, h(u) \tag{2.20}$$

for every $v \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ and for every $h \in W^{1,\infty}(\mathbb{R})$ with compact support in $\mathbb{R}$, which are such that $h(u)v \in W_0^{1,p}(\Omega)$.

Since $h(u)v \in W_0^{1,p}(\Omega)$ and since $\operatorname{supp}(h) \subset [-2n, 2n]$ (for a suitable $n > 0$ depending on h), we can rewrite (2.20) as follows:

$$\int\limits_{\Omega} a\big(x, T_{2n}(u), \nabla T_{2n}(u)\big)\cdot\nabla T_{2n}(u)\, h'(u)v + \int\limits_{\Omega} a\big(x, T_{2n}(u), \nabla T_{2n}(u)\big)\cdot\nabla v h(u)$$
$$+ \int\limits_{\Omega} H\big(x, T_{2n}(u), \nabla T_{2n}(u)\big)h(u)v + \int\limits_{\Omega} G\big(x, T_{2n}(u)\big)h(u)v$$
$$= \int\limits_{\Omega} fh(u)v + \int\limits_{\Omega} g\nabla T_{2n}(u)h'(u)v + \int\limits_{\Omega} g\nabla v h(u). \tag{2.21}$$

Let us observe that every integral in (2.21) is well defined in view of (2.7)–(2.14) since $T_k(u) \in W_0^{1,p}(\Omega)$.

Remark 2.2. Conditions (2.16) and (2.17), and the growth conditions (2.10) and (2.12) on H and G imply that for every renormalized solution

$$G(x,u) \in L^1(\Omega) \quad \text{and} \quad H(x,u,\nabla u) \in L^1(\Omega). \tag{2.22}$$

Remark 2.3. We point out that we do not assume that the renormalized solution u belongs to some Lebesgue space $L^r(\Omega)$ with $r \geqslant 1$. Indeed, it can happen that $u \notin L^1_{\mathrm{loc}}(\Omega)$ as showed in Example 2.16 of [13] when $H = G = 0$.

Remark 2.4. If u is a renormalized solution of (2.6), then u is also a distributional solution in the sense that u satisfies:

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla\phi + \int_\Omega H(x,u,\nabla u)\phi + \int_\Omega G(x,u)\phi = \int_\Omega \phi\,\mathrm{d}\mu, \quad \text{for all } \phi \in C_0^\infty(\Omega). \tag{2.23}$$

Indeed if u is a renormalized solution of (2.6), we know that u is measurable and almost everywhere finite in Ω, and that $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$, which allows one to define ∇u in the sense of (2.5). We also know that $|\nabla u|^{p-1}$ then belongs to $L^{N',\infty}(\Omega)$ and $|u|^{p-1} \in L^{N/(N-p),\infty}(\Omega)$, so that $|a(x,u,\nabla u)|$ belongs to $L^{N',\infty}(\Omega)$ by the growth condition (2.8). Taking $\phi \in C_0^\infty(\Omega)$ and h_n defined by:

$$h_n(s) = \begin{cases} 0, & |s| > 2n, \\ \dfrac{2n-|s|}{n}, & n < |s| \leqslant 2n, \\ 1, & |s| \leqslant n, \end{cases} \tag{2.24}$$

and letting n tend to infinity, we obtain (2.23).

Moreover, every renormalized solution u of (2.6) belongs to $W_0^{1,q}(\Omega)$ for every $q < N'(p-1)$ when $p > 2 - 1/N$: indeed, $p > 2 - 1/N$ implies $N'(p-1) > 1$, and therefore the gradient ∇u defined by (2.5), which satisfies (2.17), belongs to $(L^q(\Omega))^N$ for every $q < N'(p-1)$, and is the distributional gradient of u (see Remark 2.10 of [13]).

Remark 2.5. As in [13], Remark 2.20, we observe that a measure $\mu \in M_b(\Omega)$ is not the most general possible right-hand side which can be considered in (2.6). Indeed one can consider the case of the nonlinear elliptic problem

$$\begin{cases} -\mathrm{div}\big(a(x,u,\nabla u)\big) + H(x,u,\nabla u) + G(x,u) = \mu - \mathrm{div}(F) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $F \in (L^{p'}(\Omega))^N$, where the right-hand side now belongs to $M_b(\Omega) + W^{-1,p'}(\Omega)$.

In that case, Theorem 2.1 below continues to hold with the same proof (see Remark 2.20 of [13]) whenever Definition 2.2 above is modified in the following way. On the one hand the requirements (2.18) and (2.19) have to be replaced by the following ones: there exists sequences (s_n^+, t_n^+) and (s_n^-, t_n^-), with

$$s_n^+ < t_n^+, \qquad s_n^+ \to +\infty \quad \text{as } n \to +\infty, \qquad s_n^- < t_n^-, \qquad s_n^- \to +\infty \quad \text{as } n \to +\infty,$$

such that

$$\lim_{n\to+\infty} \frac{1}{t_n^+ - s_n^+} \int\limits_{s_n^+ \leqslant u < t_n^+} a(x,u,\nabla u)\nabla u \varphi = \int\limits_{\Omega} \varphi \, \mathrm{d}\mu_s^+,$$

$$\lim_{n\to+\infty} \frac{1}{t_n^- - s_n^-} \int\limits_{-t_n^- \leqslant u < -s_n^-} a(x,u,\nabla u)\nabla u \varphi = \int\limits_{\Omega} \varphi \, \mathrm{d}\mu_s^-,$$

for every $\varphi \in C_b^0(\Omega)$. On the other hand, g has to be replaced by $g + F$ in the right-hand sides of (2.20) and (2.21).

The main result of the present paper is the following existence result:

Theorem 2.1. *Under assumptions* (2.7)–(2.14)*, there exists at least one renormalized solution u of* (2.6)*.*

3. Proof of Theorem 2.1

In order to prove Theorem 2.1, we begin by approximating the data. The main point is to obtain an a priori estimate of $|\nabla u_n|^{p-1}$ in $L^{N',\infty}(\Omega)$, which will provide an estimate in $L^1(\Omega)$ of the term $H_n(x, u_n, \nabla u_n)$. When the data b_0 is sufficiently small, this is done by using the function $T_k(u_n)$ as a test function, together with a generalization of Lemma 4.2 of [1] (see Lemma A.1 below), which allows one to estimate the norm of $|\nabla u_n|^{p-1}$ in the Lorentz space $L^{N',\infty}(\Omega)$ by means of the norm of $\nabla T_k(u_n)$ in $(L^p(\Omega))^N$. In the general case where b_0 is not small, we use the Bottaro–Marina technique. In the last part of the section, we prove that the terms $H_n(x, u_n, \nabla u_n)$ and $G_n(x, u_n)$ converge strongly in $L^1(\Omega)$, which allows us to reconduce the proof to the stability result proved in [13] when $a(x, s, \xi)$ does not depend on s, and in [28] in the general case.

3.1. Approximation of the data

By Proposition 2.3 the bounded Radon measure μ can be decomposed as

$$\mu = f - \mathrm{div}(g) + \mu_s^+ - \mu_s^-,$$

where $f \in L^1(\Omega)$, $g \in (L^{p'}(\Omega))^N$ and μ_s^+ and μ_s^- (the positive and the negative parts of μ_s) are two nonnegative measures in $M_b(\Omega)$ which are concentrated in two disjoint subsets E^+ and E^- of zero p-capacity.

As in [13], we approximate the measure μ by a sequence μ_n defined as:

$$\mu_n = f_n - \mathrm{div}(g) + \lambda_n^{\oplus} - \lambda_n^{\ominus},$$

where

$$f_n \text{ is a sequence of functions in } L^{p'}(\Omega) \text{ that converges to } f \text{ in } L^1(\Omega) \text{ weakly}, \tag{3.1}$$

$$\begin{cases} \lambda_n^{\oplus} \text{ is a sequence of nonnegative functions in } L^{p'}(\Omega) \text{ that converges to } \mu_s^+ \\ \quad \text{in the narrow topology of measures,} \end{cases} \tag{3.2}$$

and

$$\begin{cases} \lambda_n^{\ominus} \text{ is a sequence of nonnegative functions in } L^{p'}(\Omega) \text{ that converges to } \mu_s^- \\ \quad \text{in the narrow topology of measures.} \end{cases} \tag{3.3}$$

Note that μ_n belongs to $W^{-1,p'}(\Omega)$.

We set:

$$H_n(x,s,\xi) = T_n\big(H(x,s,\xi)\big), \tag{3.4}$$

$$G_n(x,s) = T_n\big(G(x,s)\big). \tag{3.5}$$

Observe that

$$\big|H_n(x,s,\xi)\big| \leqslant \big|H(x,s,\xi)\big| \leqslant b_0(x)|\xi|^{p-1} + b_1(x), \tag{3.6}$$

$$\big|H_n(x,s,\xi)\big| \leqslant n, \tag{3.7}$$

$$G_n(x,s)s \geqslant 0, \tag{3.8}$$

$$\big|G_n(x,s)\big| \leqslant \big|G(x,s)\big| \leqslant b_2(x)|s|^r + b_3(x), \tag{3.9}$$

$$\big|G_n(x,s)\big| \leqslant n. \tag{3.10}$$

Let $u_n \in W_0^{1,p}(\Omega)$ be a weak solution of the following problem:

$$\begin{cases} -\mathrm{div}\big(a(x,u_n,\nabla u_n)\big) + H_n(x,u_n,\nabla u_n) + G_n(x,u_n) = \mu_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.11}$$

i.e.,

$$\begin{cases} u_n \in W_0^{1,p}(\Omega), \\ \int\limits_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v + \int\limits_\Omega H_n(x,u_n,\nabla u_n)v + \int\limits_\Omega G_n(x,u_n)v \\ \quad = \int\limits_\Omega f_n v + \int\limits_\Omega g\nabla v + \int\limits_\Omega \lambda_n^{\oplus} v - \int\limits_\Omega \lambda_n^{\ominus} v, \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{3.12}$$

The existence of a solution u_n of (3.12) is a classical result (see, e.g., [25]). Moreover, such a solution is also a renormalized solution of (3.11).

3.2. *A priori estimate of* $|\nabla u_n|^{p-1}$ *in* $L^{N',\infty}(\Omega)$

This is the main step of the proof of Theorem 2.1.

Theorem 3.1. *Under the hypotheses of Theorem* 2.1, *every solution* u_n *of* (3.12) *satisfies*:

$$\big\| |\nabla u_n|^{p-1} \big\|_{L^{N',\infty}(\Omega)} \leqslant c, \tag{3.13}$$

$$\big\| |u_n|^{p-1} \big\|_{L^{N/(N-p),\infty}(\Omega)} \leqslant c, \tag{3.14}$$

where c *is a positive constant which depends only on* p, $|\Omega|$, N, α, $\|b_0\|_{L^{N,1}(\Omega)}$, $\|b_1\|_{L^1(\Omega)}$, $\|g\|_{(L^{p'}(\Omega))^N}$, $\sup_n \|f_n\|_{L^1(\Omega)}$, $\sup_n[\lambda_n^{\oplus}(\Omega) + \lambda_n^{\ominus}(\Omega)]$, *and on the rearrangement* b_0^* *of* b_0 (*see Remark* 3.1 *at the end of Section* 3).

Proof of Theorem 3.1.

The simple case where $\|b_0\|_{L^{N,1}(\Omega)}$ is small enough.
Using $T_k(u_n)$, $k > 0$, as a test function in (3.12), we obtain:

$$\int\limits_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n) + \int\limits_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n) + \int\limits_\Omega G_n(x,u_n)T_k(u_n)$$
$$= \int\limits_\Omega f_n T_k(u_n) + \int\limits_\Omega g\nabla T_k(u_n) + \int\limits_\Omega \lambda_n^{\oplus} T_k(u_n) - \int\limits_\Omega \lambda_n^{\ominus} T_k(u_n). \tag{3.15}$$

We now evaluate the various integrals in (3.15). From the ellipticity condition (2.7) we have:

$$\int\limits_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n) = \int\limits_{\{|u_n|\leqslant k\}} a(x,u_n,\nabla u_n)\cdot\nabla u_n \geqslant \alpha \int\limits_{\{|u_n|\leqslant k\}} |\nabla u_n|^p$$
$$= \alpha \int\limits_\Omega \big|\nabla T_k(u_n)\big|^p. \tag{3.16}$$

On the other hand, by the growth assumption (2.10) on H, or more exactly (3.6), and by the generalized Hölder inequality (2.4) in the Lorentz spaces, we get:

$$\begin{aligned}
&\left|\int_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n)\right| \\
&\quad\leqslant k\int_\Omega \bigl|H(x,u_n,\nabla u_n)\bigr| \leqslant k\left[\int_\Omega b_0(x)|\nabla u_n|^{p-1}+\int_\Omega b_1(x)\right] \\
&\quad\leqslant k\bigl[\|b_0\|_{L^{N,1}(\Omega)}\bigl\||\nabla u_n|^{p-1}\bigr\|_{L^{N',\infty}(\Omega)}+\|b_1\|_{L^1(\Omega)}\bigr]. \qquad (3.17)
\end{aligned}$$

Using (2.11) or, more exactly (3.8), it results

$$\int_\Omega G_n(x,u_n)T_k(u_n)\geqslant 0. \qquad (3.18)$$

Finally, we have

$$\int_\Omega f_n T_k(u_n)\leqslant k\|f_n\|_{L^1(\Omega)}, \qquad (3.19)$$

$$\int_\Omega g\nabla T_k(u_n)\leqslant \frac{\alpha}{p}\bigl\|\nabla T_k(u_n)\bigr\|^p_{(L^p(\Omega))^N}+\frac{1}{p'\alpha^{1/(p-1)}}\|g\|^{p'}_{(L^{p'}(\Omega))^N}, \qquad (3.20)$$

$$\left|\int_\Omega \lambda_n^\oplus T_k(u_n)\right|\leqslant k\int_\Omega \lambda_n^\oplus = k\lambda_n^\oplus(\Omega), \qquad (3.21)$$

$$\left|\int_\Omega \lambda_n^\ominus T_k(u_n)\right|\leqslant k\lambda_n^\ominus(\Omega). \qquad (3.22)$$

Observe that by (3.1)–(3.3) and the Definition 2.1 of the convergence of measures in the narrow topology,

$$\sup_n\|f_n\|_{L^1(\Omega)}+\sup_n\bigl(\lambda_n^\oplus(\Omega)+\lambda_n^\ominus(\Omega)\bigr)<+\infty. \qquad (3.23)$$

Therefore, from (3.15), using (3.16)–(3.23), we get:

$$\begin{aligned}
\frac{\alpha}{p'}\int_\Omega \bigl|\nabla T_k(u_n)\bigr|^p \leqslant\ & k\bigl[\|b_0\|_{L^{N,1}(\Omega)}\bigl\||\nabla u_n|^{p-1}\bigr\|_{L^{N',\infty}(\Omega)}+\|b_1\|_{L^1(\Omega)} \\
&+\|f_n\|_{L^1(\Omega)}+\lambda_n^\oplus(\Omega)+\lambda_n^\ominus(\Omega)\bigr] \\
&+\frac{1}{p'\alpha^{1/(p-1)}}\|g\|^{p'}_{(L^{p'}(\Omega))^N}. \qquad (3.24)
\end{aligned}$$

Let us define:

$$\begin{cases} M = \dfrac{p'}{\alpha}\Big[\|b_0\|_{L^{N,1}(\Omega)}\big\||\nabla u_n|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \\ \qquad\qquad + \|b_1\|_{L^1(\Omega)} + \sup\limits_n \|f_n\|_{L^1(\Omega)} + \sup\limits_n\big(\lambda_n^{\oplus}(\Omega) + \lambda_n^{\ominus}(\Omega)\big)\Big], \\ M^* = \dfrac{p'}{\alpha}\Big[\|b_1\|_{L^1(\Omega)} + \sup\limits_n \|f_n\|_{L^1(\Omega)} + \sup\limits_n\big(\lambda_n^{\oplus}(\Omega) + \lambda_n^{\ominus}(\Omega)\big)\Big], \\ L = \dfrac{1}{\alpha^{p'}}\|g\|^{p'}_{(L^{p'}(\Omega))^N}. \end{cases} \tag{3.25}$$

We explicitly observe that M, M^* and L are finite.

Inequality (3.24) becomes

$$\int\limits_\Omega \big|\nabla T_k(u_n)\big|^p \leqslant Mk + L, \quad \forall k > 0. \tag{3.26}$$

By Lemma A.1 of Appendix A, we get:

$$\big\||\nabla u_n|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \leqslant C(N,p)\big[M + |\Omega|^{1/N'-1/p'}L^{1/p'}\big],$$

where $C(N,p)$ depends only on N and p. In view of (3.25), this means

$$\begin{aligned}\big\||\nabla u_n|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \leqslant\; & C(N,p)\frac{p'}{\alpha}\|b_0\|_{L^{N,1}(\Omega)}\big\||\nabla u_n|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \\ & + C(N,p)\big[M^* + |\Omega|^{1/N'-1/p'}L^{1/p'}\big].\end{aligned}$$

If $\|b_0\|_{L^{N,1}(\Omega)}$ is small enough, and more exactly, if

$$C(N,p)\frac{p'}{\alpha}\|b_0\|_{L^{N,1}(\Omega)} < 1, \tag{3.27}$$

we immediately obtain:

$$\big\||\nabla u_n|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \leqslant \frac{C(N,p)[M^* + |\Omega|^{1/N'-1/p'}L^{1/p'}]}{1 - C(N,p)(p'/\alpha)\|b_0\|_{L^{N,1}(\Omega)}},$$

i.e., (3.13).

The general case: presentation of the method.

If $\|b_0\|_{L^{N,1}(\Omega)}$ is small enough, i.e., satisfies (3.27), the desired result is proved. In the general case where (3.27) does not hold, we use the technique introduced by Bottaro and Marina (see [11]) for the study of the linear problem with right-hand side in the dual space (this technique was generalized in [15] to the nonlinear problem with right-hand side in the dual space). We adapt here this technique to the problem with right-hand side measure and to a coefficient b_0 in a Lorentz space. The idea is in some sense to decompose b_0 in a finite sum of terms, each of which satisfies (3.27).

We will estimate $|\nabla u_n|^{p-1}$ in $L^{N',\infty}(\Omega)$ by decomposing $|\nabla u_n|^{p-1}$ in a sum of terms of the type

$$|\nabla u_n|^{p-1}\chi_{\{m_{i+1}<|u_n|<m_i\}},$$

where the constants m_i will be conveniently chosen. The values of the constants m_i will actually depend on n, but their number will not: the index i will vary between 0 and I, with I bounded by I^* independent of n.

Actually the proof becomes a little bit more complicated because we need the measure of the set $\{x \in \Omega\colon\ m < |u_n(x)| < m_i\}$ to be continuous with respect to the parameter m (for m_i given). This lead us to define the set Z_n in the following way. As $|\Omega|$ is finite, the set of the constants c such that $|\{x \in \Omega\colon\ |u_n(x)| = c\}| > 0$ is at most countable. Let Z_n^c be the (countable) union of all those sets. Its complementary $Z_n = \Omega \setminus Z_n^c$ is therefore the union of the sets such that $|\{x \in \Omega\colon\ |u_n(x)| = c\}| = 0$. Since for every c,

$$\nabla u_n = 0 \quad \text{a.e. on } \bigl\{x \in \Omega\colon\ \bigl|u_n(x)\bigr| = c\bigr\},$$

and since Z_n^c is at most a countable union, we obtain that

$$\nabla u_n = 0 \quad \text{a.e. on } Z_n^c. \tag{3.28}$$

In the sequel of the proof, we will consider the measure of the set

$$\bigl|Z_n \cap \bigl\{m_{i+1} < |u_n| < m_i\bigr\}\bigr|$$

for m_i and m_{i+1} conveniently chosen. Since the constants c such that the sets $\{|u_n(x)| = c\}$ have a strictly positive measure have been eliminated by considering Z_n, it results that for m_i fixed and $0 < m < m_i$ the function

$$m \to \bigl|Z_n \cap \bigl\{m < |u_n| < m_i\bigr\}\bigr| \quad \text{is continuous.} \tag{3.29}$$

The general case: first step.

Define for $m > 0$ the "remainder" S_m of the truncation T_m, that is

$$S_m(s) = s - T_m(s), \quad \forall s \in \mathbb{R},$$

or, in other terms,

$$S_m(s) = \begin{cases} 0, & |s| \leqslant m, \\ \bigl(|s| - m\bigr)\operatorname{sign}(s), & |s| > m. \end{cases} \tag{3.30}$$

Using in (3.12) the test function $T_k(S_m(u_n))$ with m to be specified later, we obtain:

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k\big(S_m(u_n)\big)+\int_\Omega H_n(x,u_n,\nabla u_n)T_k\big(S_m(u_n)\big)$$
$$+\int_\Omega G_n(x,u_n)T_k\big(S_m(u_n)\big)$$
$$=\int_\Omega f_nT_k\big(S_m(u_n)\big)+\int_\Omega g\nabla T_k\big(S_m(u_n)\big)$$
$$+\int_\Omega \lambda_n^\oplus T_k\big(S_m(u_n)\big)-\int_\Omega \lambda_n^\ominus T_k\big(S_m(u_n)\big). \tag{3.31}$$

As in the first step we have:

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k\big(S_m(u_n)\big)\geqslant \int_{\{m\leqslant|u_n|\leqslant m+k\}} a(x,u_n,\nabla u_n)\nabla u_n$$
$$\geqslant \alpha\int_\Omega \big|\nabla T_k\big(S_m(u_n)\big)\big|^p, \tag{3.32}$$

$$\int_\Omega G_n(x,u_n)T_k\big(S_m(u_n)\big)\geqslant 0, \tag{3.33}$$

$$\int_\Omega f_nT_k\big(S_m(u_n)\big)\leqslant k\|f_n\|_{L^1(\Omega)}, \tag{3.34}$$

$$\int_\Omega g\nabla T_k\big(S_m(u_n)\big)\leqslant \frac{\alpha}{p}\big\|\nabla T_k\big(S_m(u_n)\big)\big\|^p_{(L^p(\Omega))^N}+\frac{1}{p'\alpha^{1/(p-1)}}\|g\|^{p'}_{(L^{p'}(\Omega))^N}, \tag{3.35}$$

$$\left|\int_\Omega \lambda_n^\oplus T_k\big(S_m(u_n)\big)\right|\leqslant k\lambda_n^\oplus(\Omega), \tag{3.36}$$

$$\left|\int_\Omega \lambda_n^\ominus T_k\big(S_m(u_n)\big)\right|\leqslant k\lambda_n^\ominus(\Omega). \tag{3.37}$$

Let us now estimate

$$\left|\int_\Omega H_n(x,u_n,\nabla u_n)T_k\big(S_m(u_n)\big)\right|.$$

Using $S_m(s)=0$ for $|s|\leqslant m$, the growth assumption (3.6), the property (3.28) of Z_n and the generalized Hölder inequality (2.4) in the Lorentz spaces, we have:

$$
\begin{aligned}
\left|\int_\Omega H_n(x,u_n,\nabla u_n)T_k\big(S_m(u_n)\big)\right| &\leqslant k\int_{\{|u_n|>m\}}\left|H_n(x,u_n,\nabla u_n)\right| \\
&\leqslant k\Bigg[\int_{\{|u_n|>m\}} b_0|\nabla u_n|^{p-1}+\int_\Omega b_1\Bigg]=k\Bigg[\int_{Z_n\cap\{|u_n|>m\}} b_0\left|\nabla S_m(u_n)\right|^{p-1}+\int_\Omega b_1\Bigg] \\
&\leqslant k\big[\|b_0\|_{L^{N,1}(Z_n\cap\{u_n>m\})}\big\||\nabla S_m(u_n)|^{p-1}\big\|_{L^{N',\infty}(\Omega)}+\|b_1\|_{L^1(\Omega)}\big]. \qquad (3.38)
\end{aligned}
$$

Combining (3.31)–(3.38) we have, for all $k>0$,

$$\big\|\nabla T_k\big(S_m(u_n)\big)\big\|^p_{(L^p(\Omega))^N}\leqslant M_1k+L,$$

where M_1 is defined by

$$M_1=\frac{p'}{\alpha}\|b_0\|_{L^{N,1}(Z_n\cap\{|u_n|>m\})}\big\|\left|\nabla S_m(u_n)\right|^{p-1}\big\|_{L^{N',\infty}(\Omega)}+M^*,$$

and where M^* and L are defined by (3.25). By Lemma A.1, we get:

$$
\begin{aligned}
\big\|\left|\nabla S_m(u_n)\right|^{p-1}\big\|_{L^{N',\infty}(\Omega)} &\leqslant C(N,p)\big[M_1+|\Omega|^{1/N'-1/p'}L^{1/p'}\big] \\
&= C(N,p)\frac{p'}{\alpha}\|b_0\|_{L^{N,1}(Z_n\cap\{|u_n|>m\})}\big\|\left|\nabla S_m(u_n)\right|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \\
&\quad + C(N,p)\big[M^*+|\Omega|^{1/N'-1/p'}L^{1/p'}\big]. \qquad (3.39)
\end{aligned}
$$

Since the decreasing rearrangements of b_0 and of its restriction $b_0|_{Z_n\cap E}$ to $Z_n\cap E$ satisfy

$$(b_0|_{Z_n\cap E})^*(t)\leqslant(b_0)^*(t),\quad t\in\big[0,|Z_n\cap E|\big], \qquad (3.40)$$

for any measurable set E, we have:

$$
\begin{aligned}
\|b_0\|_{L^{N,1}(Z_n\cap\{|u_n|>m\})} &= \int_0^{|Z_n\cap\{|u_n|>m\}|}\big(b_0|_{Z_n\cap\{|u_n|>m\}}\big)^*(t)t^{1/N}\frac{\mathrm{d}t}{t} \\
&\leqslant \int_0^{|Z_n\cap\{|u_n|>m\}|}(b_0)^*(t)t^{1/N}\frac{\mathrm{d}t}{t}. \qquad (3.41)
\end{aligned}
$$

In the case where

$$C(N,p)\frac{p'}{\alpha}\|b_0\|_{L^{N,1}(Z_n)}=C(N,p)\frac{p'}{\alpha}\int_0^{|Z_n|}(b_0)^*(t)t^{1/N}\frac{\mathrm{d}t}{t}\leqslant\frac{1}{2}, \qquad (3.42)$$

we choose $m = m_1 = 0$. If (3.42) does not hold, we can choose $m = m_1 > 0$ such that

$$C(N,p)\frac{p'}{\alpha}\int_0^{|Z_n\cap\{|u_n|>m_1\}|}(b_0)^*(t)t^{1/N}\frac{\mathrm{d}t}{t}=\frac{1}{2};$$

indeed, the function $m \to |Z_n \cap \{|u_n| > m\}|$ is continuous (see (3.29)), decreasing, and tends to 0 when m tends to ∞. Note that m_1 actually depends on n.

Moreover, if we define δ by

$$C(N,p)\frac{p'}{\alpha}\int_0^{\delta}(b_0)^*(t)t^{1/N}\frac{\mathrm{d}t}{t}=\frac{1}{2}, \tag{3.43}$$

(observe that δ does not depend on n), we have

$$\bigl|Z_n\cap\bigl\{|u_n|>m_1\bigr\}\bigr|=\delta. \tag{3.44}$$

With this choice of $m = m_1$, we obtain from (3.39) that

$$\bigl\|\bigl|\nabla S_{m_1}(u_n)\bigr|^{p-1}\bigr\|_{L^{N',\infty}(\Omega)}\leqslant 2C(N,p)\bigl[M^*+|\Omega|^{1/N'-1/p'}L^{1/p'}\bigr]. \tag{3.45}$$

The general case: second step.

Define for $0 \leqslant m < m_1$ the function $S_{m,m_1}:\mathbb{R}\to\mathbb{R}$ by

$$S_{m,m_1}(s)=\begin{cases} m_1-m, & s>m_1,\\ s-m, & m\leqslant s\leqslant m_1,\\ 0, & -m\leqslant s\leqslant m,\\ s+m, & -m_1\leqslant s\leqslant -m,\\ m-m_1, & s<-m_1.\end{cases} \tag{3.46}$$

We observe that setting $m_0 = +\infty$, the function S_m defined by (3.30) is nothing but the function S_{m,m_0} whose definition is similar to (3.46).

Using in (3.12) the test function $T_k(S_{m,m_1}(u_n))$ with m to be specified later, we obtain:

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k\bigl(S_{m,m_1}(u_n)\bigr)+\int_\Omega H_n(x,u_n,\nabla u_n)T_k\bigl(S_{m,m_1}(u_n)\bigr)\\ &\quad+\int_\Omega G_n(x,u_n)T_k\bigl(S_{m,m_1}(u_n)\bigr)=\int_\Omega f_nT_k\bigl(S_{m,m_1}(u_n)\bigr)+\int_\Omega g\nabla T_k\bigl(S_{m,m_1}(u_n)\bigr)\\ &\quad+\int_\Omega \lambda_n^{\oplus}T_k\bigl(S_{m,m_1}(u_n)\bigr)-\int_\Omega \lambda_n^{\ominus}T_k\bigl(S_{m,m_1}(u_n)\bigr).\end{aligned} \tag{3.47}$$

As in the previous step, we have:

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k\big(S_{m,m_1}(u_n)\big)\geqslant\alpha\int_\Omega\big|\nabla T_k\big(S_{m,m_1}(u_n)\big)\big|^p,\tag{3.48}$$

$$\int_\Omega G_n(x,u_n)T_k\big(S_{m,m_1}(u_n)\big)\geqslant 0,\tag{3.49}$$

$$\int_\Omega f_n T_k\big(S_{m,m_1}(u_n)\big)\leqslant k\|f_n\|_{L^1(\Omega)},\tag{3.50}$$

$$\int_\Omega g\nabla T_k\big(S_{m,m_1}(u_n)\big)$$
$$\leqslant\frac{\alpha}{p}\big\|\nabla T_k\big(S_{m,m_1}(u_n)\big)\big\|^p_{(L^p(\Omega))^N}+\frac{1}{p'\alpha^{1/(p-1)}}\|g\|^{p'}_{(L^{p'}(\Omega))^N},\tag{3.51}$$

$$\left|\int_\Omega\lambda_n^\oplus T_k\big(S_{m,m_1}(u_n)\big)\right|\leqslant k\lambda_n^\oplus(\Omega),\tag{3.52}$$

$$\left|\int_\Omega\lambda_n^\ominus T_k\big(S_{m,m_1}(u_n)\big)\right|\leqslant k\lambda_n^\ominus(\Omega).\tag{3.53}$$

Moreover, using $S_{m,m_1}(s)=0$ for $|s|\leqslant m$ and the growth assumption (3.6), we have:

$$\left|\int_\Omega H_n(x,u_n,\nabla u_n)T_k\big(S_{m,m_1}(u_n)\big)\right|$$
$$\leqslant k\left[\int_{\{|u_n|>m\}}b_0|\nabla u_n|^{p-1}+\int_\Omega b_1\right]$$
$$\leqslant k\left[\int_{\{m<|u_n|<m_1\}}b_0|\nabla u_n|^{p-1}+\int_{\{|u_n|\geqslant m_1\}}b_0|\nabla u_n|^{p-1}+\int_\Omega b_1\right].\tag{3.54}$$

Let us estimate each term of the right-hand side of (3.54). Using the property (3.28) of Z_n and the generalized Hölder inequality (2.4) in the Lorentz spaces, we have:

$$\int_{\{m<|u_n|<m_1\}}b_0|\nabla u_n|^{p-1}$$
$$=\int_{Z_n\cap\{m<|u_n|<m_1\}}b_0\big|\nabla S_{m,m_1}(u_n)\big|^{p-1}$$
$$\leqslant\|b_0\|_{L^{N,1}(Z_n\cap\{m<|u_n|<m_1\})}\big\|\big|\nabla S_{m,m_1}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)}.\tag{3.55}$$

Similarly, for the second term of the right-hand side of (3.54) we have:

$$\int_{\{|u_n|\geqslant m_1\}} b_0|\nabla u_n|^{p-1} = \int_\Omega b_0\big|\nabla S_{m_1}(u_n)\big|^{p-1} \leqslant \|b_0\|_{L^{N,1}(\Omega)}\big\|\big|\nabla S_{m_1}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)}.$$

Therefore, we have:

$$\begin{aligned}
&\bigg|\int_\Omega H_n(x,u_n,\nabla u_n)T_k\big(S_{m,m_1}(u_n)\big)\bigg| \\
&\quad\leqslant k\Big[\|b_0\|_{L^{N,1}(Z_n\cap\{m<|u_n|<m_1\})}\big\|\big|\nabla S_{m,m_1}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \\
&\qquad + \|b_0\|_{L^{N,1}(\Omega)}\big\|\big|\nabla S_{m_1}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)} + \|b_1\|_{L^1(\Omega)}\Big].
\end{aligned}\tag{3.56}$$

Combining (3.47)–(3.56) we have, for all $k>0$

$$\big\|\nabla T_k\big(S_{m,m_1}(u_n)\big)\big\|^p_{(L^p(\Omega))^N} \leqslant M_2k + L,$$

where M_2 is defined by:

$$\begin{aligned}
M_2 = {}& \frac{p'}{\alpha}\|b_0\|_{L^{N,1}(Z_n\cap\{m<|u_n|<m_1\})}\big\|\big|\nabla S_{m,m_1}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \\
&+ \frac{p'}{\alpha}\|b_0\|_{L^{N,1}(\Omega)}\big\|\big|\nabla S_{m_1}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)} + M^*,
\end{aligned}$$

and where M^* and L are defined by (3.25).

By Lemma A.1 we get:

$$\begin{aligned}
&\big\|\big|\nabla S_{m,m_1}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \leqslant C(N,p)\big[M_2 + |\Omega|^{1/N'-1/p'}L^{1/p'}\big] \\
&= C(N,p)\frac{p'}{\alpha}\|b_0\|_{L^{N,1}(Z_n\cap\{m<|u_n|<m_1\})}\big\|\big|\nabla S_{m,m_1}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \\
&\quad + C(N,p)\bigg[\frac{p'}{\alpha}\|b_0\|_{L^{N,1}(\Omega)}\big\|\big|\nabla S_{m_1}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \\
&\qquad + M^* + |\Omega|^{1/N'-1/p'}L^{1/p'}\bigg].
\end{aligned}\tag{3.57}$$

Using (3.40), we have, similarly to (3.41)

$$\|b_0\|_{L^{N,1}(Z_n\cap\{m<|u_n|<m_1\})} = \int_0^{|Z_n\cap\{m<|u_n|<m_1\}|} \left(b_0|_{Z_n\cap\{m<|u_n|<m_1\}}\right)^*(t)t^{1/N}\,\frac{\mathrm{d}t}{t}$$
$$\leqslant \int_0^{|Z_n\cap\{m<|u_n|<m_1\}|} (b_0)^*(t)t^{1/N}\,\frac{\mathrm{d}t}{t}. \tag{3.58}$$

In the case where

$$C(N,p)\frac{p'}{\alpha}\int_0^{|Z_n\cap\{0<|u_n|<m_1\}|} (b_0)^*(t)t^{1/N}\,\frac{\mathrm{d}t}{t} \leqslant \frac{1}{2}, \tag{3.59}$$

we choose $m = m_2 = 0$. If (3.59) does not hold, we can choose $m = m_2 > 0$ such that

$$C(N,p)\frac{p'}{\alpha}\int_0^{|Z_n\cap\{m_2<|u_n|<m_1\}|} (b_0)^*(t)t^{1/N}\,\frac{\mathrm{d}t}{t} = \frac{1}{2};$$

indeed the function $m \to |Z_n \cap \{m < |u_n| < m_1\}|$ is continuous (see (3.29)), decreasing, and tends to 0 when m tends to m_1 and to $|Z_n \cap \{0 < |u_n| < m_1\}|$ when m tends to 0. Note that m_2 actually depends on n and that

$$\left|Z_n \cap \left\{m_2 < |u_n| < m_1\right\}\right| = \delta, \tag{3.60}$$

where δ is defined by (3.43).

With this choice of $m = m_2$, we obtain from (3.57) that

$$\begin{aligned}
&\left\|\left|\nabla S_{m_2,m_1}(u_n)\right|^{p-1}\right\|_{L^{N',\infty}(\Omega)} \\
&\quad\leqslant 2C(N,p)\left[\frac{p'}{\alpha}\|b_0\|_{L^{N,1}(\Omega)}\left\|\left|\nabla S_{m_1}(u_n)\right|^{p-1}\right\|_{L^{N',\infty}(\Omega)}\right. \\
&\qquad\qquad \left. + M^* + |\Omega|^{1/N'-1/p'}L^{1/p'}\right].
\end{aligned} \tag{3.61}$$

The general case: third step.

Define for $0 \leqslant m < m_2$ the function $S_{m,m_2} : \mathbb{R} \to \mathbb{R}$ by

$$S_{m,m_2}(s) = \begin{cases} m_2 - m, & s > m_2, \\ s - m, & m \leqslant s \leqslant m_2, \\ 0, & -m \leqslant s \leqslant m, \\ s + m, & -m_2 \leqslant s \leqslant -m, \\ m - m_2, & s < -m_2. \end{cases} \tag{3.62}$$

Using in (3.12) the test function $T_k(S_{m,m_2}(u_n))$ with m to be specified later, we obtain:

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k\big(S_{m,m_2}(u_n)\big)+\int_\Omega H_n(x,u_n,\nabla u_n)T_k\big(S_{m,m_2}(u_n)\big)\\
&\qquad+\int_\Omega G_n(x,u_n)T_k\big(S_{m,m_2}(u_n)\big)\\
&\quad=\int_\Omega f_nT_k\big(S_{m,m_2}(u_n)\big)+\int_\Omega g\nabla T_k\big(S_{m,m_2}(u_n)\big)\\
&\qquad+\int_\Omega \lambda_n^{\oplus}T_k\big(S_{m,m_2}(u_n)\big)-\int_\Omega \lambda_n^{\ominus}T_k\big(S_{m,m_2}(u_n)\big).
\end{aligned}\tag{3.63}$$

As before we estimate the various terms; in particular we have (as in (3.54)–(3.56)):

$$\begin{aligned}
&\left|\int_\Omega H_n(x,u_n,\nabla u_n)T_k\big(S_{m,m_2}(u_n)\big)\right|\\
&\quad\leqslant k\bigg[\int_{\{m<|u_n|<m_2\}} b_0|\nabla u_n|^{p-1}+\int_{\{m_2\leqslant|u_n|<m_1\}} b_0|\nabla u_n|^{p-1}\\
&\qquad+\int_{\{|u_n|\geqslant m_1\}} b_0|\nabla u_n|^{p-1}+\int_\Omega b_1\bigg]\\
&\quad\leqslant k\big[\|b_0\|_{L^{N,1}(Z_n\cap\{m<|u_n|<m_2\})}\big\|\big|\nabla S_{m,m_2}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)}\\
&\qquad+\|b_0\|_{L^{N,1}(\Omega)}\big\|\big|\nabla S_{m_2,m_1}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)}\\
&\qquad+\|b_0\|_{L^{N,1}(\Omega)}\big\|\big|\nabla S_{m_1}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)}+\|b_1\|_{L^1(\Omega)}\big].
\end{aligned}$$

We deduce (as in (3.57)) that

$$\begin{aligned}
&\big\|\big|\nabla S_{m,m_2}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)}\\
&\quad\leqslant C(N,p)\frac{p'}{\alpha}\|b_0\|_{L^{N,1}(Z_n\cap\{m<|u_n|<m_2\})}\big\|\big|\nabla S_{m,m_2}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)}\\
&\qquad+C(N,p)\bigg[\frac{p'}{\alpha}\|b_0\|_{L^{N,1}(\Omega)}\big\|\big|\nabla S_{m_2,m_1}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)}\\
&\qquad\qquad+\frac{p'}{\alpha}\|b_0\|_{L^{N,1}(\Omega)}\big\|\big|\nabla S_{m_1}(u_n)\big|^{p-1}\big\|_{L^{N',\infty}(\Omega)}\\
&\qquad\qquad+M^*+|\Omega|^{1/N'-1/p'}L^{1/p'}\bigg].
\end{aligned}\tag{3.64}$$

Using (3.40), we have, arguing as in (3.41) and (3.58)

$$\|b_0\|_{L^{N,1}(Z_n\cap\{m<|u_n|<m_2\})} \leqslant \int_0^{|Z_n\cap\{m<|u_n|<m_2\}|} (b_0)^*(t)t^{1/N}\,\frac{\mathrm{d}t}{t}.$$

In the case where

$$C(N,p)\frac{p'}{\alpha}\int_0^{|Z_n\cap\{0<|u_n|<m_2\}|} (b_0)^*(t)t^{1/N}\,\frac{\mathrm{d}t}{t} \leqslant \frac{1}{2}, \tag{3.65}$$

we choose $m = m_3 = 0$. If (3.65) does not hold, we can choose $m = m_3 > 0$ such that

$$C(N,p)\frac{p'}{\alpha}\int_0^{|Z_n\cap\{m_3<|u_n|<m_2\}|} (b_0)^*(t)t^{1/N}\,\frac{\mathrm{d}t}{t} = \frac{1}{2}.$$

Note that m_3 actually depends on n and that

$$\bigl|Z_n \cap \bigl\{m_3 < |u_n| < m_2\bigr\}\bigr| = \delta, \tag{3.66}$$

where δ is defined by (3.43).

With this choice of $m = m_3$, we obtain from (3.64) that

$$\begin{aligned}
&\bigl\|\bigl|\nabla S_{m_3,m_2}(u_n)\bigr|^{p-1}\bigr\|_{L^{N',\infty}(\Omega)} \\
&\quad\leqslant 2C(N,p)\biggl[\frac{p'}{\alpha}\|b_0\|_{L^{N,1}(\Omega)}\bigl\|\bigl|\nabla S_{m_2,m_1}(u_n)\bigr|^{p-1}\bigr\|_{L^{N',\infty}(\Omega)} \\
&\qquad + \frac{p'}{\alpha}\|b_0\|_{L^{N,1}(\Omega)}\bigl\|\bigl|\nabla S_{m_1}(u_n)\bigr|^{p-1}\bigr\|_{L^{N',\infty}(\Omega)} \\
&\qquad + M^* + |\Omega|^{1/N'-1/p'}L^{1/p'}\biggr].
\end{aligned} \tag{3.67}$$

The general case: end of the proof.

We repeat this procedure until the time it stops, i.e., when we arrive to some $i = I$ (which depends on n) for which we have

$$C(N,p)\frac{p'}{\alpha}\int_0^{|Z_n\cap\{0<|u_n|<m_{I-1}\}|} (b_0)^*(t)t^{1/N}\,\frac{\mathrm{d}t}{t} \leqslant \frac{1}{2};$$

then we choose

$$m_I = 0. \tag{3.68}$$

Let us now estimate I. We have

$$|\Omega| \geqslant |Z_n| \geqslant \left|Z_n \cap \left\{|u_n| > m_1\right\}\right| + \left|Z_n \cap \left\{m_2 < |u_n| < m_1\right\}\right| + \left|Z_n \cap \left\{m_3 < |u_n| < m_2\right\}\right| + \cdots + \left|Z_n \cap \left\{m_{I-1} < |u_n| < m_{I-2}\right\}\right|$$

and, in view of (3.44), (3.60) and (3.66), we know that

$$\left|Z_n \cap \left\{|u_n| > m_1\right\}\right| = \left|Z_n \cap \left\{m_2 < |u_n| < m_1\right\}\right| = \cdots = \left|Z_n \cap \left\{m_{I-1} < |u_n| < m_{I-2}\right\}\right| = \delta,$$

where δ is defined by (3.43), and does not depend on n. Therefore, $(I-1)\delta \leqslant |\Omega|$, and

$$I \leqslant I^* \quad \text{with } I^* = 1 + \left[\frac{|\Omega|}{\delta}\right], \tag{3.69}$$

where $[s]$ denotes the integer part of s, defined by $[s] = \inf\{n \in \mathbb{N}\colon\ s \leqslant n\}$.

Observe that I is estimated by the number I^* which does not depend on n, and which depends on b_0^* through the definition of δ.

We define

$$m_0 = +\infty, \qquad S_{m_1,m_0} = S_{m_1},$$

$$\begin{cases} X_i = \left\| \left|\nabla S_{m_i,m_{i-1}}(u_n)\right|^{p-1} \right\|_{L^{N',\infty}(\Omega)} \quad \text{for } 1 \leqslant i \leqslant I, \\ a = 2C(N,p)\dfrac{p'}{\alpha}\|b_0\|_{L^{N,1}(\Omega)}, \\ b = 2C(N,p)\left[M^* + |\Omega|^{1/N'-1/p'}L^{1/p'}\right], \end{cases} \tag{3.70}$$

where M^* and L are given by (3.25), and we observe that

$$X_1 = \left\| \left|\nabla S_{m_1,m_0}(u_n)\right|^{p-1} \right\|_{L^{N',\infty}(\Omega)} = \left\| \left|\nabla S_{m_1}(u_n)\right|^{p-1} \right\|_{L^{N',\infty}(\Omega)}.$$

We have proved (see (3.45), (3.61), (3.67) and (3.69)) that

$$X_1 \leqslant b, \qquad X_2 \leqslant aX_1 + b, \qquad X_3 \leqslant aX_2 + aX_1 + b, \qquad \ldots,$$
$$X_I \leqslant aX_{I-1} + \cdots + aX_1 + b, \quad I \leqslant I^*.$$

It can be proved by induction that

$$X_i \leqslant (a+1)^{i-1} b \quad \text{for } 1 \leqslant i \leqslant I. \tag{3.71}$$

Since $m_I = 0$ (see (3.68)), we have

$$|\nabla u_n|^{p-1} = \sum_{i=1}^{I} |\nabla u_n|^{p-1} \chi_{\{m_i < |u_n| < m_{i-1}\}} = \sum_{i=1}^{I} \bigl|\nabla S_{m_i, m_{i-1}}(u_n)\bigr|^{p-1},$$

and, therefore, using (3.70) and (3.71)

$$\begin{aligned}\bigl\| |\nabla u_n|^{p-1} \bigr\|_{L^{N',\infty}(\Omega)} &\leqslant \sum_{i=1}^{I} \bigl\| \bigl|\nabla S_{m_i, m_{i-1}}(u_n)\bigr|^{p-1} \bigr\|_{L^{N',\infty}(\Omega)} \leqslant \sum_{i=1}^{I} X_i \\ &\leqslant b \sum_{i=1}^{I} (a+1)^{i-1} = b\left(\frac{(a+1)^I - 1}{a}\right) \leqslant \frac{b}{a}\bigl((a+1)^{I^*} - 1\bigr),\end{aligned}$$

i.e., the desired result (3.13).

Let us finally prove the result (3.14). From (3.26) (note that the hypothesis (3.27) that $\|b_0\|_{L^{N,1}(\Omega)}$ is small has not been used at this stage), we deduce that

$$\int_\Omega \bigl|\nabla T_k(u_n)\bigr|^p \leqslant Mk + L, \quad \forall k > 0,$$

where the constants M and L defined by (3.25) are now bounded independently on n in view of (3.13) and (3.23). The result (3.14) then follows from Lemma A.1. □

3.3. *Passing to the limit in the approximated problem*

Using the growth condition (3.6) on H_n, Theorem 3.1 and the generalized Hölder inequality (2.4), we get:[1]

$$\begin{aligned}&\bigl\| H_n(x, u_n, \nabla u_n) \bigr\|_{L^1(\Omega)} \\ &\quad = \int_\Omega \bigl|H_n(x, u_n, \nabla u_n)\bigr| \leqslant \int_\Omega b_0(x) |\nabla u_n|^{p-1} + \int_\Omega b_1(x) \\ &\quad \leqslant \|b_0\|_{L^{N,1}(\Omega)} \bigl\| |\nabla u_n|^{p-1} \bigr\|_{L^{N',\infty}(\Omega)} + \|b_1\|_{L^1(\Omega)} \leqslant C.\end{aligned} \tag{3.72}$$

On the other hand, we deduce from (3.14) and from the definition (2.13) of z that

$$\bigl\| |u_n|^r \bigr\|_{L^{z,\infty}(\Omega)} \leqslant C. \tag{3.73}$$

[1] In (3.72) and in the rest of this section, C denotes a generic constant, which does not depend on n but can vary from line to line.

Using the growth condition (3.9) on G_n, (3.73) and the generalized Hölder inequality (2.4), we get:

$$\begin{aligned}\big\|G_n(x,u_n)\big\|_{L^1(\Omega)} &= \int_\Omega \big|G_n(x,u_n)\big| \leqslant \int_\Omega b_2(x)|u_n|^r + b_3(x)\\ &\leqslant \|b_2\|_{L^{z',1}(\Omega)}\big\||u_n|^r\big\|_{L^{z,\infty}(\Omega)} + \|b_3\|_{L^1(\Omega)} \leqslant C.\end{aligned}$$

Therefore, the solution u_n of (3.11) satisfies:

$$\begin{cases} -\operatorname{div}\big(a(x,u_n,\nabla u_n)\big) = \Phi_n - \operatorname{div}(g) & \text{in } \mathcal{D}'(\Omega),\\ u_n \in W_0^{1,p}(\Omega), \end{cases} \tag{3.74}$$

where $\Phi_n = f_n - H_n(x,u_n,\nabla u_n) - G_n(x,u_n) + \lambda_n^{\oplus} - \lambda_n^{\ominus}$ is bounded in $L^1(\Omega)$.

Using $T_k(u_n)$ as a test function in (3.74), easily yields that for some $\widehat{M}$ and $\widehat{L}$

$$\int_\Omega \big|\nabla T_k(u_n)\big|^p \leqslant \widehat{M}k + \widehat{L}, \tag{3.75}$$

for every $k > 0$ and every n.

Since u_n, which is a weak solution of (3.74), is also a renormalized solution of (3.74), Theorem 3.2 of [13] (when $a(x,s,\xi)$ does not depend on s), or the result of [28] (in the general case), implies that for a subsequence (which we still denote by n) we have:

$$\begin{cases} u_n \to u & \text{almost everywhere in } \Omega,\\ \nabla u_n \to \nabla u & \text{almost everywhere in } \Omega,\\ \nabla T_k(u_n) \rightharpoonup \nabla T_k(u) & \text{in } \big(L^p(\Omega)\big)^N \text{ weakly}, \end{cases} \tag{3.76}$$

for every fixed $k \in \mathbb{N}$, where u is a function which is measurable on Ω, almost everywhere finite, and such that $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k \in \mathbb{N}$, with a gradient ∇u as introduced in (2.5); moreover by Fatou lemma, we deduce from (3.75) that

$$\int_\Omega \big|\nabla T_k(u)\big|^p \leqslant \widehat{M}k + \widehat{L}.$$

Lemma A.1 then implies that $|u|^{p-1} \in L^{N/(N-p),\infty}(\Omega)$ and $|\nabla u|^{p-1} \in L^{N/(N-1),\infty}(\Omega)$.

From (3.76) and the definition (3.4) of H_n, we deduce that

$$H_n(x,u_n,\nabla u_n) \to H(x,u,\nabla u) \quad \text{almost everywhere in } \Omega. \tag{3.77}$$

From a computation similar to (3.72), we obtain that for every measurable set $E \subset \Omega$

$$\int_E \big|H_n(x,u_n,\nabla u_n)\big| \leqslant \|b_0\|_{L^{N,1}(E)} \big\||\nabla u_n|^{p-1}\big\|_{L^{N',\infty}(E)} + \|b_1\|_{L^1(E)}$$
$$\leqslant \|b_0\|_{L^{N,1}(E)} c + \|b_1\|_{L^1(E)}$$

and, therefore, that

$$H_n(x,u_n,\nabla u_n) \quad \text{is equi-integrable,}$$

since, using as in (3.40) the fact that $(b_0|_E)^*(t) \leqslant (b_0)^*(t)$, we have

$$\|b_0\|_{L^{N,1}(E)} = \int_0^{|E|} (b_0|_E)^*(t) t^{1/N} \frac{\mathrm{d}t}{t} \leqslant \int_0^{|E|} (b_0)^*(t) t^{1/N} \frac{\mathrm{d}t}{t}, \tag{3.78}$$

which is small when $|E|$ is small. This implies, together with (3.77), that

$$H_n(x,u_n,\nabla u_n) \to H(x,u,\nabla u) \quad \text{in } L^1(\Omega) \text{ strongly.}$$

Similarly it is easy to prove that

$$G_n(x,u_n) \to G(x,u) \quad \text{in } L^1(\Omega) \text{ strongly.}$$

In view of this results, the solution u_n of (3.11) satisfies:

$$\begin{cases} -\operatorname{div}\big(a(x,u_n,\nabla u_n)\big) = f_n - \Psi_n - \operatorname{div}(g) + \lambda_n^{\oplus} - \lambda_n^{\ominus} & \text{in } \mathcal{D}'(\Omega), \\ u_n \in W_0^{1,p}(\Omega), \end{cases} \tag{3.79}$$

where u_n satisfies (3.76) and

$$\Psi_n = H_n(x,u_n,\nabla u_n) + G_n(x,u_n) \to H(x,u,\nabla u) + G(x,u) \quad \text{in } L^1(\Omega) \text{ strongly,}$$

where $g \in (L^{p'}(\Omega))^N$ and where f_n, $\lambda_n^{\oplus}$ and $\lambda_n^{\ominus}$ satisfy (3.1), (3.2) and (3.3).

Since u_n, which is a weak solution of (3.79), is also a renormalized solution of (3.79), the stability result of [13] (Theorem 3.4) (when $a(x,s,\xi)$ does not depend on s) or of [28] (in the general case) asserts that u is a renormalized solution of

$$\begin{cases} -\operatorname{div}\big(a(x,u,\nabla u)\big) + H(x,u,\nabla u) + G(x,u) = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^- & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which proves Theorem 2.1. □

Remark 3.1. The above proof shows that the renormalized solution u of (2.6), which is obtained as the limit of the subsequence u_n, satisfies the bounds

$$\big\||\nabla u|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \leqslant c,$$

$$\big\||u|^{p-1}\big\|_{L^{N/(N-p),\infty}(\Omega)} \leqslant c,$$

$$\int_\Omega \big|\nabla T_k(u)\big|^p \leqslant \widehat{M}k + \widehat{L}.$$

Moreover, it can be shown, by using in (2.6) the test functions $v = T_k(u)$ and $h = h_n$, with h_n defined by (2.24), that every renormalized solution of (2.6) satisfies the same bounds. In these bounds, the constants c, as said in the statement of Theorem 3.1 and in its proof, depend only on p, $|\Omega|$, N, α, $\|b_0\|_{L^{N,1}(\Omega)}$, $\|b_1\|_{L^1(\Omega)}$, $\|f\|_{L^1(\Omega)}$, $\|g\|_{(L^{p'}(\Omega))^N}$, $\mu_s^+(\Omega)$, $\mu_s^-(\Omega)$, and on the rearrangement b_0^* of b_0.

Let us emphasize that these bounds do not depends only on $\|b_0\|_{L^{N,1}(\Omega)}$, but also on b_0^*: this is due to (3.69), i.e., $I \leqslant I^* = 1 + [|\Omega|/\delta]$, where δ, which is defined by (3.43), depends on b_0^* and not only on $\|b_0\|_{L^{N,1}(\Omega)}$.

Appendix A. A generalization of a result of [1]

In this Appendix we generalize a result of [1].

Lemma A.1. *Assume that Ω is an open subset of $\mathbb{R}^N$ with finite measure and that $1 < p < N$. Let u be a measurable function satisfying $T_k(u) \in W_0^{1,p}(\Omega)$, for every positive k, and such that*

$$\int_\Omega \big|\nabla T_k(u)\big|^p \leqslant Mk + L, \quad \forall k > 0, \tag{A.1}$$

where M and L are given constants. Then $|u|^{p-1}$ belongs to $L^{p^*/p,\infty}(\Omega)$, $|\nabla u|^{p-1}$ belongs to $L^{N',\infty}(\Omega)$ and

$$\big\||u|^{p-1}\big\|_{L^{p^*/p,\infty}(\Omega)} \leqslant C(N,p)\big[M + |\Omega|^{1/p^*}L^{1/p'}\big], \tag{A.2}$$

$$\big\||\nabla u|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \leqslant C(N,p)\big[M + |\Omega|^{1/N'-1/p'}L^{1/p'}\big], \tag{A.3}$$

where $C(N,p)$ is a constant depending only on N and p and where $1/p^* = 1/p - 1/N$.

Remark A.1. This lemma is a generalized version of Lemmas 4.1 and 4.2 of [1], in which $L = 0$. Estimates (A.3) and (A.2) are optimal in the following sense.

When $L=0$,

$$\big\||u|^{p-1}\big\|_{L^{p^*/p,\infty}(\Omega)} \leqslant C(N,p)M \tag{A.4}$$

and

$$\big\||\nabla u|^{p-1}\big\|_{L^{N',\infty}(\Omega)} \leqslant C(N,p)M \tag{A.5}$$

are the best estimates that one can obtain from (A.1). Indeed consider the function

$$u(x)=\frac{1}{|x|^{(N-p)/(p-1)}}-\frac{1}{R^{(N-p)/(p-1)}},$$

when Ω is the ball of radius R centered in 0. Then (A.1) is satisfied, as well as (A.4) and (A.5), while $|u|^{p-1}$ does not belong to $L^{p^*/p+\delta,1}(\Omega)$ for any $\delta>0$, and $|\nabla u|^{p-1}$ does not belong to $L^{N'+\delta,1}(\Omega)$ for any $\delta>0$.

On the other hand, when $M=0$, (A.1) is equivalent to

$$\int_\Omega |\nabla u|^p \leqslant L, \tag{A.6}$$

i.e., u bounded in $W_0^{1,p}(\Omega)$. From Sobolev inequality there exists a constant $S_{N,p}$ which depends only on N and p, such that for every open set $\Omega\subset\mathbb{R}^N$ and $v\in W_0^{1,p}(\Omega)$, one has:

$$\|v\|_{L^{p^*}(\Omega)}^{p^*} \leqslant S_{N,p}\|\nabla v\|_{L^p(\Omega)}^{p^*}. \tag{A.7}$$

Therefore, we deduce from (A.6) that

$$\|u\|_{L^{p^*}(\Omega)}^{p^*} \leqslant S_{N,p}\|\nabla u\|_{L^p(\Omega)}^{p^*} \leqslant S_{N,p}L^{p^*/p},$$

which implies

$$k^{p^*}\operatorname{meas}\big\{|u|>k\big\} \leqslant S_{N,p}L^{p^*/p},$$

for every $k>0$, or equivalently

$$h^{p^*/(p-1)}\operatorname{meas}\big\{|u|^{p-1}>h\big\} \leqslant S_{N,p}L^{p^*/p},$$

for every $h>0$, i.e.,

$$h\operatorname{meas}\big\{|u|^{p-1}>h\big\}^{p/p^*} \leqslant S_{N,p}^{p/p^*}Lh^{1-p'},$$

for every $h>0$. Therefore, for every positive h_0 arbitrarily fixed, we have:

$$\begin{aligned}
&\big\||u|^{p-1}\big\|_{L^{p^*/p,\infty}(\Omega)}\\
&\quad=\sup_{h>0} h\,\mathrm{meas}\big\{|u|^{p-1}>h\big\}^{p/p^*}\\
&\quad=\sup_{0<h\leqslant h_0} h\,\mathrm{meas}\big\{|u|^{p-1}>h\big\}^{p/p^*}+\sup_{h\geqslant h_0} h\,\mathrm{meas}\big\{|u|^{p-1}>h\big\}^{p/p^*}\\
&\quad\leqslant h_0|\Omega|^{p/p^*}+S_{N,p}^{p/p^*}Lh_0^{1-p'}. \qquad\qquad (A.8)
\end{aligned}$$

Taking $h_0=L^{(p-1)/p}/|\Omega|^{(p-1)/p^*}$, which corresponds to take the two terms of the right-hand side of (A.8) of the same order, we obtain

$$\big\||u|^{p-1}\big\|_{L^{p^*/p,\infty}(\Omega)}\leqslant C(N,p)|\Omega|^{1/p^*}L^{1/p'},$$

i.e., (A.2) when $M=0$. This derivation is close to be optimal.

For what concerns (A.3), if $M=0$ we deduce from (A.6), i.e., $|\nabla u|\in L^p(\Omega)$, that for every $\mu\geqslant 0$

$$\mu^p\,\mathrm{meas}\big\{x\in\Omega\colon |\nabla u|>\mu\big\}\leqslant L,$$

i.e., $|\nabla u|\in L^{p,\infty}(\Omega)$, or equivalently

$$\mu\big(\mathrm{meas}\big\{x\in\Omega\colon |\nabla u|^{p-1}>\mu\big\}\big)^{1/p'}\leqslant L^{1/p'},$$

i.e., $|\nabla u|^{p-1}\in L^{p',\infty}(\Omega)$, which implies

$$\begin{aligned}
&\mu\big(\mathrm{meas}\big\{x\in\Omega\colon |\nabla u|^{p-1}>\mu\big\}\big)^{1/N'}\\
&\quad\leqslant \mu\big(\mathrm{meas}\big\{x\in\Omega\colon |\nabla u|^{p-1}>\mu\big\}\big)^{1/p'}|\Omega|^{1/N'-1/p'}\\
&\quad\leqslant |\Omega|^{1/N'-1/p'}L^{1/p'}, \qquad\qquad (A.9)
\end{aligned}$$

i.e., (A.3) when $M=0$. Again, this derivation is close to be optimal.

Estimates (A.2) and (A.3) are in some sense combinations of the two results obtained for $M=0$ and $L=0$. Observe that the dependence of (A.2) and (A.3) with respect to L and M exhibits two different homogeneities (linear in M, of order $L^{1/p'}$ in L).

Proof of Lemma A.1.

Proof of (A.2). Using Sobolev inequality (A.7) we have, for every $k>0$

$$k^{p^*}\operatorname{meas}\{x\in\Omega\colon |u|>k\} \leqslant \int_\Omega |T_k(u)|^{p^*} \leqslant S_{N,p}\big\||\nabla T_k(u)|\big\|_{L^p(\Omega)}^{p^*} \leqslant S_{N,p}(Mk+L)^{p^*/p}, \tag{A.10}$$

or equivalently, for every $h>0$,

$$h^{p^*/(p-1)}\operatorname{meas}\{x\in\Omega\colon |u|^{p-1}>h\} \leqslant S_{N,p}\big(Mh^{1/(p-1)}+L\big)^{p^*/p},$$

i.e.,

$$\operatorname{meas}\{x\in\Omega\colon |u|^{p-1}>h\} \leqslant S_{N,p}\big(Mh^{-1}+Lh^{-p'}\big)^{p^*/p},$$

i.e., for every $h>0$

$$h\big(\operatorname{meas}\{x\in\Omega\colon |u|^{p-1}>h\}\big)^{p/p^*} \leqslant S_{N,p}^{p/p^*}\big(M+Lh^{1-p'}\big).$$

Therefore, we have:

$$\begin{aligned}
&\big\||u|^{p-1}\big\|_{L^{p^*/p,\infty}(\Omega)} \\
&\quad= \sup_{h>0} h\operatorname{meas}\{|u|^{p-1}>h\}^{p/p^*} \\
&\quad= \sup_{0<h\leqslant h_0} \mu\operatorname{meas}\{|u|^{p-1}>h\}^{p/p^*} + \sup_{h\geqslant h_0} h\operatorname{meas}\{|u|^{p-1}>h\}^{p/p^*} \\
&\quad\leqslant h_0|\Omega|^{p/p^*} + S_{N,p}^{p/p^*}\big(M+Lh^{1-p'}\big),
\end{aligned}$$

which, taking $h_0 = L^{(p-1)/p}/|\Omega|^{(p-1)/p^*}$, proves (A.2).

Proof of (A.3). First step. From (A.1) we deduce that for every $\lambda>0$ and every $k>0$

$$\begin{aligned}
&\lambda^p\operatorname{meas}\{x\in\Omega\colon |\nabla u|>\lambda \text{ and } |u|<k\} \\
&\quad\leqslant \int_{\{|u|<k\}} |\nabla u|^p = \int_\Omega |\nabla T_k(u)|^p \leqslant Mk+L,
\end{aligned}$$

i.e., for every $\mu>0$ and every $k>0$

$$\mu^{p/(p-1)}\operatorname{meas}\{x\in\Omega\colon |\nabla u|^{p-1}>\mu \text{ and } |u|<k\} \leqslant Mk+L. \tag{A.11}$$

From (A.11) and (A.10) we obtain that for every $\lambda>0$ and every $k>0$,

$$\begin{aligned}\operatorname{meas}\{x\in\Omega\colon |\nabla u|^{p-1}>\mu\} &\leqslant \operatorname{meas}\{x\in\Omega\colon |\nabla u|^{p-1}>\mu \text{ and } |u|<k\}\\ &\quad+\operatorname{meas}\{x\in\Omega\colon |\nabla u|^{p-1}>\mu \text{ and } |u|>k\}\\ &\leqslant \frac{Mk+L}{\mu^{p'}}+S_{N,p}\frac{(Mk+L)^{p^*/p}}{k^{p^*}}.\end{aligned}$$

Second step. We now write $k=a+b$ with $a>0$, $b>0$. From the inequality $(x+y)^{p^*/p}\leqslant 2^{p^*/p}(x^{p^*/p}+y^{p^*/p})$, we get:

$$\begin{aligned}&\operatorname{meas}\{x\in\Omega\colon |\nabla u|^{p-1}>\mu\}\\ &\quad\leqslant \frac{Ma}{\mu^{p'}}+\frac{Mb}{\mu^{p'}}+\frac{L}{\mu^{p'}}\\ &\qquad+S_{N,p}2^{p^*/p}(a+b)^{p^*/p-p^*}M^{p^*/p}+S_{N,p}2^{p^*/p}(a+b)^{-p^*}L^{p^*/p}\end{aligned}$$

for every $\mu>0$, $a>0$ and $b>0$. Since $(a+b)^{p^*/p-p^*}\leqslant a^{p^*/p-p^*}=a^{-p^*/p'}$ (indeed $p^*/p-p^*=-p^*/p'<0$), and since $(a+b)^{-p^*}\leqslant b^{-p^*}$, we obtain

$$\begin{aligned}\operatorname{meas}\{x\in\Omega\colon |\nabla u|^{p-1}>\mu\} &\leqslant \frac{Ma}{\mu^{p'}}+S_{N,p}2^{p^*/p}M^{p^*/p}a^{-p^*/p'}\\ &\quad+\frac{Mb}{\mu^{p'}}+S_{N,p}2^{p^*/p}L^{p^*/p}b^{-p^*}+\frac{L}{\mu^{p'}}\\ &\leqslant C(N,p)\biggl[\biggl(\frac{M}{\mu^{p'}}a+M^{p^*/p}a^{-p^*/p'}\biggr)\\ &\qquad+\biggl(\frac{M}{\mu^{p'}}b+L^{p^*/p}b^{-p^*}\biggr)+\frac{L}{\mu^{p'}}\biggr]\end{aligned}\tag{A.12}$$

for some constant $C(N,p)$.

Third step. For the rest of the present proof, we will denote by $C(N,p)$ a constant which only depends on N and p, but can vary from line to line.

After choosing

$$a=M^{1/(N-1)}\mu^{(N-p)/((p-1)(N-1))},\qquad b=\biggl(\frac{L^{p^*/p}\mu^{p'}}{M}\biggr)^{1/(p^*+1)}$$

(those are the values which minimize with respect to a and b the right-hand side of (A.12)), inequality (A.12) yields

$$\operatorname{meas}\{x\in\Omega\colon |\nabla u|^{p-1}>\mu\}\leqslant C(N,p)\biggl[\frac{M^{N'}}{\mu^{N'}}+\biggl(\frac{ML^{1/p}}{\mu^{p'}}\biggr)^{p^*/(p^*+1)}+\frac{L}{\mu^{p'}}\biggr].$$

Since

$$\frac{p'p^*}{N'(p^*+1)} - 1 = \left(\frac{p'}{N'} - 1\right)\frac{p^*}{p(p^*+1)},$$

we get

$$\begin{aligned}
&\mu\big(\operatorname{meas}\{x \in \Omega\colon |\nabla u|^{p-1} > \mu\}\big)^{1/N'}\\
&\quad\leqslant C(N,p)\left[M + M^{p^*/(N'(p^*+1))}\left(\frac{L^{1/N'}}{\mu^{p'/N'-1}}\right)^{p^*/(p(p^*+1))} + \frac{L^{1/N'}}{\mu^{p'/N'-1}}\right].
\end{aligned}$$

But it results

$$\frac{p^*}{N'(p^*+1)} + \frac{p^*}{p(p^*+1)} = 1.$$

Therefore, Young inequality yields

$$\begin{aligned}
&\mu\big(\operatorname{meas}\{x \in \Omega\colon |\nabla u|^{p-1} > \mu\}\big)^{1/N'}\\
&\quad\leqslant C(N,p)\left[M + \frac{p^*}{N'(p^*+1)}M + \frac{p^*}{p(p^*+1)}\frac{L^{1/N'}}{\mu^{p'/N'-1}} + \frac{L^{1/N'}}{\mu^{p'/N'-1}}\right]\\
&\quad\leqslant C(N,p)\left(M + \frac{L^{1/N'}}{\mu^{p'/N'-1}}\right),
\end{aligned} \tag{A.13}$$

for every $\mu > 0$.

Fourth step. From (A.13) and $p < N$, we deduce that

$$\begin{aligned}
&\sup_{\mu>0}\mu\big(\operatorname{meas}\{x \in \Omega\colon |\nabla u|^{p-1} > \mu\}\big)^{1/N'}\\
&\quad\leqslant \sup_{0<\mu\leqslant\mu_0}\mu\big(\operatorname{meas}\{x \in \Omega\colon |\nabla u|^{p-1} > \mu\}\big)^{1/N'}\\
&\qquad + \sup_{\mu>\mu_0}\mu\big(\operatorname{meas}\{x \in \Omega\colon |\nabla u|^{p-1} > \mu\}\big)^{1/N'}\\
&\quad\leqslant \mu_0|\Omega|^{1/N'} + \sup_{\mu>\mu_0} C(N,p)\left(M + \frac{L^{1/N'}}{\mu^{p'/N'-1}}\right)\\
&\quad= \mu_0|\Omega|^{1/N'} + C(N,p)M + C(N,p)\frac{L^{1/N'}}{\mu_0^{p'/N'-1}}\\
&\quad\leqslant C(N,p)\left[|\Omega|^{1/N'}\mu_0 + L^{1/N'}\mu_0^{1-p'/N'} + M\right].
\end{aligned} \tag{A.14}$$

By choosing

$$\mu_0 = \left(\frac{L}{|\Omega|}\right)^{1/p'}$$

(this is the value which minimizes the right-hand side of (A.14) with respect to μ_0) we obtain

$$\sup_{\mu>0} \mu\left(\text{meas}\left\{x \in \Omega\colon |\nabla u|^{p-1} > \mu\right\}\right)^{1/N'} \leqslant C(N, p)\left[M + |\Omega|^{1/N'-1/p'} L^{1/p'}\right],$$

which is the desired result. □

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