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Coexistence of a diffusive predator–prey model with Holling type-II functional response and density dependent mortality

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ABSTRACT

In this paper, we consider a two competitor–one prey diffusive model in which both competitors exhibit Holling type-II functional response and one of the competitors exhibits density dependent mortality rate. First, we study the local and global existence of strong solution by using the C_0 analytic semigroup. Then, we consider the local and global stability of the positive constant equilibrium by using the linearization method and Laypunov functional method, respectively. Furthermore, we derive some results for the existence and non-existence of non-constant stationary solutions when the diffusion rate of a certain species is small or large. The existence of non-constant stationary solutions implies the possibility of pattern formation in this system.

1. Introduction

Continuous models, usually in the form of differential equations, have formed a large part of the traditional mathematical ecology literature. In these models, the key terms specifying the outcome of predator–prey interactions are the functional and numerical response, which reflect the relationship between predators and their prey. In general, two competitor–one prey model has the following form (see [1,3,5,9,11–14,17,19,23,24,31,32] and the references therein):

$$\begin{aligned}
&\frac{du_1}{dt} = ru_1\left(1-\frac{u_1}{K}\right) - au_2 f(u_1) - Au_3 g(u_1) \quad \text{in } \mathbb{R}_+,\\
&\frac{du_2}{dt} = u_2\big(-d + ef(u_1)\big) \quad \text{in } \mathbb{R}_+,\\
&\frac{du_3}{dt} = u_3\big(-D - Gu_3 + Eg(u_1)\big) \quad \text{in } \mathbb{R}_+,\\
&u_i(0) = u_{i0} \geqslant 0, \quad i = 1,2,3,
\end{aligned} \tag{1}$$

where $\mathbb{R}_+ = (0,\infty)$, the parameters r, K, G, a, A, b, B, d, D, e, E are strictly positive, and u_{i0}, $i = 1,2,3$, stand for the initial condition. u_2 and u_3 are the densities of two competitors and u_1 is the density of the prey. The density dependent mortality term for the second species Gu_3^2, referred as a 'closure term', describes either a self-limitation of the consumer, u_3, or the influence of predation. $f(u_1)$ and $g(u_1)$ are the so-called prey-dependent functional response which are taken as follows

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$$\text{(i)}\quad \frac{mu_1}{e+u_1},\qquad \text{(ii)}\quad \frac{mu_1^2}{e+\varepsilon u_1+u_1^2},\qquad \text{(iii)}\quad \frac{mu_1^2}{e+u_1^2},$$

where m, e and ε are positive constants, m denotes the maximal growth rate of the species and e is the half-saturation constant. The model (i) is called the Michaelis–Menten or Holling type-II function, (ii) is called the sigmoidal response function, and (iii) is called the Holling type-III function.

On the other hand, understanding of spatial and temporal behaviors of interaction species in ecological systems is a center issue in population ecology. One aspect of great interest for a model with multispecies interactions is whether the involved species can persist or even stabilize at a coexistence steady state. When the species are homogeneously distributed, this would be indicated by a constant positive solution of an ordinary differential equation system. In the spatially inhomogeneous case, the existence of a non-constant time-independent positive solution, also called stationary pattern, is an indication of the richness of the corresponding partial differential equation dynamics. In recent years, stationary pattern induced by diffusion has been studied extensively, and many important phenomena have been observed (see [4,6–8,15,16, 21–30,34–36] and references therein).

In this paper, we will study the effect of diffusion in the predator–prey system (1) with Holling type-II functional response, i.e., the following reaction–diffusion system:

$$\begin{aligned}
&\frac{\partial u_1}{\partial t}-d_1\Delta u_1=ru_1\left(1-\frac{u_1}{K}\right)-\frac{au_1u_2}{1+bu_1}-\frac{Au_1u_3}{1+Bu_1}\quad \text{in } \Omega\times\mathbb{R}_+,\\
&\frac{\partial u_2}{\partial t}-d_2\Delta u_2=u_2\left(-d+\frac{eu_1}{1+bu_1}\right)\quad \text{in } \Omega\times\mathbb{R}_+,\\
&\frac{\partial u_3}{\partial t}-d_3\Delta u_3=u_3\left(-D-Gu_3+\frac{Eu_1}{1+Bu_1}\right)\quad \text{in } \Omega\times\mathbb{R}_+,\\
&\frac{\partial u_1}{\partial \nu}=\frac{\partial u_2}{\partial \nu}=\frac{\partial u_3}{\partial \nu}=0\quad \text{on } \partial\Omega\times\mathbb{R}_+,\\
&u_i(\cdot,0)=u_{i0}\geqslant 0,\quad i=1,2,3 \text{ in } \Omega,
\end{aligned}\tag{2}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(1\leqslant N\leqslant 3)$ with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial\nu}$ is the outward directional derivative normal to $\partial\Omega$, parameters $r, K, G, a, A, b, B, d, d_1, d_2, d_3, D, e, E$ are strictly positive, and $u_{i0}\in C^{2+\delta}(\bar{\Omega})$ $(i=1,2,3)$ for some $\delta\in(0,1)$, stand for the initial condition.

In order to study the stationary pattern induced by diffusion, we consider the steady state of (2), i.e., the following semi-linear elliptic system:

$$\begin{aligned}
&-d_1\Delta u_1=ru_1\left(1-\frac{u_1}{K}\right)-\frac{au_1u_2}{1+bu_1}-\frac{Au_1u_3}{1+Bu_1}\quad \text{in } \Omega\times\mathbb{R}_+,\\
&-d_2\Delta u_2=u_2\left(-d+\frac{eu_1}{1+bu_1}\right)\quad \text{in } \Omega\times\mathbb{R}_+,\\
&-d_3\Delta u_3=u_3\left(-D-Gu_3+\frac{Eu_1}{1+Bu_1}\right)\quad \text{in } \Omega\times\mathbb{R}_+,\\
&\frac{\partial u_1}{\partial \nu}=\frac{\partial u_2}{\partial \nu}=\frac{\partial u_3}{\partial \nu}=0\quad \text{on } \partial\Omega\times\mathbb{R}_+.
\end{aligned}\tag{3}$$

For convenience, we denote $\Lambda=(r,K,a,A,b,B,d,D,e,E,G)$ and $\mathcal{U}=(u_1,u_2,u_3)$. It is easy to see that (2) has a positive constant steady state $E^*=(u_1^*,u_2^*,u_3^*)$, where

$$u_1^*=\frac{d}{e-bd},\qquad u_2^*=\frac{1+bu_1^*}{a}\left[r\left(1-\frac{u_1^*}{K}\right)-\frac{Au_3^*}{1+Bu_1^*}\right],\qquad u_3^*=\frac{1}{G}\left(-D+\frac{Ed}{Bd+e-bd}\right)\tag{4}$$

provided

$$r\left(1-\frac{u_1^*}{K}\right)-\frac{Au_3^*}{1+Bu_1^*}>0,\qquad e>bd,\qquad Ed>D(Bd+e-bd).\tag{5}$$

Let us start with the formulation of the original initial–boundary problem (2). We define the following product Banach spaces for any $2\leqslant p<\infty$

$$\mathcal{H}=L_p(\Omega)\times L_p(\Omega)\times L_p(\Omega),\qquad \mathcal{E}=W^{1,p}(\Omega)\times W^{1,p}(\Omega)\times W^{1,p}(\Omega).\tag{6}$$

One can check that, by the Lumer–Phillips Theorem and the generation theorem for analytic semigroup [33], the densely defined, sectorial, linear operator

$$\mathcal{A}=\begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}: D(\mathcal{A}) \to \mathcal{H}, \tag{7}$$

where $A_1=-d_1\Delta+1,\ A_2=-d_2\Delta+d,\ A_3=-d_3\Delta+D,\ D(\mathcal{A})=D(A_1)\times D(A_2)\times D(A_3)$ with

$$D(A_1)=D(A_2)=D(A_3)\triangleq\left\{\omega\in W^{2,p}(\Omega)\ \Big|\ \frac{\partial\omega}{\partial\nu}=0 \text{ on } \partial\Omega\right\}, \tag{8}$$

is the generator of an analytic C_0 analytic semigroup $\{e^{-\mathcal{A}t},\ t\geqslant 0\}$, on the Banach space $\mathcal{H}$. By the fact that $W^{1,p}(\Omega)\hookrightarrow L_{2p}(\Omega)$ is a continuous embedding for $N\leqslant 3$, one can check the nonlinear mapping $\Phi(\mathcal{U}):\mathcal{E}\to\mathcal{H}$ is a locally Lipchitz continuous mapping defined on $\mathcal{E}$, where $\mathcal{U}=(u_1,u_2,u_3)$ and $\Phi=(\phi_1,\phi_2,\phi_3)$ with

$$\begin{aligned}
\phi_1&=u_1+ru_1\left(1-\frac{u_1}{K}\right)-\frac{au_1u_2}{1+bu_1}-\frac{Au_1u_3}{1+Bu_1},\\
\phi_2&=\frac{eu_1u_2}{1+bu_1},\\
\phi_3&=u_3\left(-Gu_3+\frac{Eu_1}{1+Bu_1}\right).
\end{aligned} \tag{9}$$

Then the initial–boundary value problem (2) is formulated into an initial value problem as follows

$$\begin{aligned}
&\frac{d\mathcal{U}}{dt}+\mathcal{A}\mathcal{U}=\Phi(\mathcal{U}),\quad t>0,\\
&\mathcal{U}=\mathcal{U}_0=(u_{10},u_{20},u_{30})\in\mathcal{H}.
\end{aligned} \tag{10}$$

By the theory of evolutionary equations [33], one can use the Contraction Mapping Theorem and the Gronwall–Henry inequality [33, Lemma D.4] to prove the local existence and uniqueness of the strong solution $\mathcal{U}(t)$ of the initial value problem (10) and the strong solution has the property

$$\mathcal{U}\in C\big([0,T_{\max});\mathcal{H}\big)\cap C^1\big((0,T_{\max});\mathcal{H}\big)\cap L_2\big([0,T_{\max});\mathcal{E}\big), \tag{11}$$

where $[0,T_{\max})$ is the maximal interval of existence. Furthermore, the mild solution of (10) can be represented as

$$\mathcal{U}(t)=e^{-\mathcal{A}t}\mathcal{U}_0+\int_0^t e^{-\mathcal{A}(t-s)}\Phi\big(\mathcal{U}(s)\big)\,ds. \tag{12}$$

In the remaining of this paper, we shall carry out the detailed analysis to systems (2) and (3). In Section 2, we study the dissipation of system (2), which ensures the strong solution of (2) is global existence, *viz*, $T_{\max}=\infty$. In Section 3, we consider the local and global stability of the steady state E^* by using linearization method and Laypunov functional method, respectively. In Section 4, we give a priori estimates to the positive solution of system (3), which are important to study the non-existence of positive solution of (3) in Section 5 and existence of positive solution of (3) in Section 6.

2. Dissipation of system (2)

In this section, we will discuss the dissipation of system (2) and get the following theorem:

Theorem 2.1. *There exist two positive constants K_1 and K_2, which depend only on $\Lambda,d_1,d_2,d_3,\Omega$ such that any non-negative solution $\mathcal{U}(x,t)=(u_1,u_2,u_3)$ of system* (2) *satisfies*

$$\begin{aligned}
&\limsup_{t\to\infty}\big\|u_1(x,t)\big\|_{C(\bar{\Omega})}\leqslant K,\\
&\limsup_{t\to\infty}\big\|u_2(x,t)\big\|_{C(\bar{\Omega})}\leqslant K_1,\\
&\limsup_{t\to\infty}\big\|u_3(x,t)\big\|_{C(\bar{\Omega})}\leqslant K_2.
\end{aligned} \tag{13}$$

In order to prove Theorem 2.1, we first give the L_1-estimates for the non-negative solutions of system (2), then we derive the L_p-estimates for p large enough by using the L^1-estimates (see [2,10,33]).

Lemma 2.2. *Any non-negative solution $\mathcal{U}(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))$ of system (2) satisfies the following estimate*

$$\|u_1\|_{L_1(\Omega)}+\frac{a}{e}\|u_2\|_{L_1(\Omega)}+\frac{A}{E}\|u_3\|_{L_1(\Omega)} \leqslant e^{-ct}\int_\Omega\left(u_{10}(x)+\frac{a}{e}u_{20}(x)+\frac{A}{E}u_{30}(x)\right)+\frac{K(r+c)^2}{4r}|\Omega|\left(1-e^{-ct}\right), \tag{14}$$

where $c=\min\{d,D\}$. Furthermore, for any $q>1$, there exists a positive constant C which depends only on $\Lambda,q,d_1,d_2,d_3,\Omega$ such that any non-negative solution $\mathcal{U}(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))$ of system (2) satisfies

$$\begin{aligned}
\|u_1\|_{L_q(\Omega)} &\leqslant C\left(\|u_1\|_{L_1(\Omega)}+\frac{a}{e}\|u_2\|_{L_1(\Omega)}+\frac{A}{E}\|u_3\|_{L_1(\Omega)}\right),\\
\|u_2\|_{L_q(\Omega)} &\leqslant C\left(\|u_1\|_{L_1(\Omega)}+\frac{a}{e}\|u_2\|_{L_1(\Omega)}+\frac{A}{E}\|u_3\|_{L_1(\Omega)}\right),\\
\|u_3\|_{L_q(\Omega)} &\leqslant C\left(\|u_1\|_{L_1(\Omega)}+\frac{a}{e}\|u_2\|_{L_1(\Omega)}+\frac{A}{E}\|u_3\|_{L_1(\Omega)}\right).
\end{aligned} \tag{15}$$

Proof. First, by using comparison principle, we get

$$u_1(x,t)\leqslant \max\left\{\max_{x\in\bar\Omega}u_{10}(x),\ K\right\}\triangleq\mathcal{M}. \tag{16}$$

Next, from (2), we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega\left(u_1+\frac{a}{e}u_2+\frac{A}{E}u_3\right)dx &\leqslant \int_\Omega\left(ru_1-\frac{r}{K}u_1^2-\frac{ad}{e}u_2-\frac{AD}{E}u_3\right)dx\\
&=-c\int_\Omega\left(u_1+\frac{a}{e}u_2+\frac{A}{E}u_3\right)dx+\int_\Omega\left((r+c)u_1-\frac{r}{K}u_1^2\right)dx\\
&\leqslant -c\int_\Omega\left(u_1+\frac{a}{e}u_2+\frac{A}{E}u_3\right)dx+\frac{K(r+c)^2}{4r}|\Omega|.
\end{aligned} \tag{17}$$

Integrating the above inequality from 0 to t, we obtain (14).

Finally, we will prove (15) by induction. We already know (15) holds for $q=1$. Next, we assume (15) holds for some $q=\gamma\geqslant 1$ and we will prove (15) holds for $q=2\gamma$. Multiplying the three equations of (2) by $u_1^{2\gamma-1},u_2^{2\gamma-1},u_2^{2\gamma-1}$ respectively, and then integrating the results over Ω, we obtain

$$\begin{aligned}
\frac{1}{2\gamma}\frac{d}{dt}\int_\Omega\left(u_1^{2\gamma}+u_2^{2\gamma}+u_3^{2\gamma}\right)dx &\leqslant -\frac{2\gamma-1}{\gamma^2}\int_\Omega\left(d_1\left|\nabla u_1^\gamma\right|^2+d_2\left|\nabla u_2^\gamma\right|^2+d_3\left|\nabla u_3^\gamma\right|^2\right)dx\\
&\quad+\int_\Omega\left(ru_1^{2\gamma}+eu_1u_2^{2\gamma}+Eu_1u_3^{2\gamma}\right)dx.
\end{aligned} \tag{18}$$

That is,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega\left(u_1^{2\gamma}+u_2^{2\gamma}+u_3^{2\gamma}\right)dx &\leqslant -\frac{2\epsilon(2\gamma-1)}{\gamma}\int_\Omega\left(\left|\nabla u_1^\gamma\right|^2+\left|\nabla u_2^\gamma\right|^2+\left|\nabla u_3^\gamma\right|^2\right)dx\\
&\quad+2\gamma D\int_\Omega\left(u_1^{2\gamma}+u_2^{2\gamma}+u_3^{2\gamma}\right)dx,
\end{aligned} \tag{19}$$

where $\epsilon=\min\{d_1,d_2,d_3\}$, $D=\max\{r,e\mathcal{M},E\mathcal{M}\}$.

By using Nirenberg–Gagliardo's inequality and Young's inequality, we get

$$\begin{aligned}
\int_\Omega u_1^{2\gamma}\,dx &\leqslant \varepsilon\left[\int_\Omega\left|\nabla u_1^\gamma\right|^2dx+\left(\int_\Omega u_1^\gamma\,dx\right)^2\right]+\eta(\varepsilon)\left(\int_\Omega u_1^\gamma\,dx\right)^p,\\
\int_\Omega u_2^{2\gamma}\,dx &\leqslant \varepsilon\left[\int_\Omega\left|\nabla u_2^\gamma\right|^2dx+\left(\int_\Omega u_2^\gamma\,dx\right)^2\right]+\eta(\varepsilon)\left(\int_\Omega u_2^\gamma\,dx\right)^p,
\end{aligned}$$

$$\int_\Omega u_3^{2\gamma}\,dx \leqslant \varepsilon\left[\int_\Omega \left|\nabla u_3^\gamma\right|^2 dx + \left(\int_\Omega u_3^\gamma\,dx\right)^2\right] + \eta(\varepsilon)\left(\int_\Omega u_3^\gamma\,dx\right)^p \tag{20}$$

for any constant $\varepsilon > 0$, where $p > 1$ is some constant, $\eta(\varepsilon)$ is positive constant depending on ε.

Taking $\varepsilon = 2\epsilon(2\gamma-1)/(q(2qD+1))$, then by (19) and (20), there exist two positive constants ℓ_1 and ℓ_2 such that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \left(u_1^{2\gamma}+u_2^{2\gamma}+u_3^{2\gamma}\right)dx &\leqslant -\int_\Omega \left(u_1^{2\gamma}+u_2^{2\gamma}+u_3^{2\gamma}\right)dx + \ell_1\left[\left(\int_\Omega u_1^\gamma\,dx\right)^2 + \left(\int_\Omega u_2^\gamma\,dx\right)^2 + \left(\int_\Omega u_2^\gamma\,dx\right)^2\right]\\ &\quad + \ell_2\left[\left(\int_\Omega u_1^\gamma\,dx\right)^p + \left(\int_\Omega u_2^\gamma\,dx\right)^p + \left(\int_\Omega u_2^\gamma\,dx\right)^p\right]\\ &\leqslant -\int_\Omega \left(u_1^{2\gamma}+u_2^{2\gamma}+u_3^{2\gamma}\right)dx + \mathcal{C}\end{aligned} \tag{21}$$

by using the assumption (i.e., (15) holds for some $q = \gamma \geqslant 1$), where $\mathcal{C}$ is dependent only on $\Lambda, \gamma, p, \ell_1, \ell_2, d_1, d_2, d_3, \Omega$. Integrating the above inequality from 0 to t, we prove (15) holds for $q = 2\gamma$. So, (15) holds by induction. The proof is completed. □

Proof of Theorem 2.1. From the first equation of (2), we know u_1 satisfies

$$\begin{cases}\dfrac{\partial u_1}{\partial t} - d_1\Delta u_1 \leqslant r u_1\left(1-\dfrac{u_1}{K}\right) & \text{in } \Omega\times\mathbb{R}_+,\\[2mm] \dfrac{\partial u_1}{\partial \nu} = 0 & \text{on } \partial\Omega\times\mathbb{R}_+,\\ u_1(x,0) = u_{i0}(x) & \text{in } \Omega.\end{cases} \tag{22}$$

Then the first inequality of (13) holds by comparison principle.

In order to prove the other two inequalities of (13), we exploit the C_0 analytic semigroup theory. Recall A_i $(i=2,3)$ defined in Section 1. Let $\sigma(A_i)$ represent the spectrum of the operator A_i, it is easy to see $\operatorname{Re}\sigma(A_i) > \varrho$ for some constant $\varrho > 0$. From [10,33], we can define two Banach spaces $\mathcal{Y}_i^\alpha = D(A_i^\alpha)$ $(i=2,3)$ with norm $\|\cdot\|_{\mathcal{Y}_i^\alpha} = \|A_i^\alpha\cdot\|_{L_p(\Omega)}$ for $\alpha > 0$ and $p > 1$.

Let us choose $p > \max\{2, N/2\}$ and $N/(2p) < \alpha < 1$, then the following imbedding result holds [10,33]:

$$\mathcal{Y}_i^\alpha \hookrightarrow C^\beta(\bar\Omega) \quad \text{for } 0 \leqslant \beta < 2\alpha - N/p \text{ and } i = 2,3. \tag{23}$$

By (12) and $u_{i0} \in C^{2+\delta}(\bar\Omega) \hookrightarrow \mathcal{Y}_i^\alpha$ $(i=2,3)$, it is easy to see

$$\begin{aligned} u_2(x,t) &= e^{-A_2 t}u_{20}(x) + \int_0^t e^{-(t-\tau)A_2}\frac{e u_1(x,\tau)u_2(x,\tau)}{1+bu_1(x,\tau)}\,d\tau \in \mathcal{Y}_2^\alpha,\\ u_3(x,t) &= e^{-A_3 t}u_{30}(x) + \int_0^t e^{-(t-\tau)A_3}\left[\frac{E u_1(x,\tau)u_3(x,\tau)}{1+Bu_1(x,\tau)} - Gu_3^2(x,\tau)\right]d\tau \in \mathcal{Y}_3^\alpha,\end{aligned} \tag{24}$$

and

$$\begin{aligned}\|u_2\|_{\mathcal{Y}_2^\alpha} &\leqslant \left\|A_2^\alpha e^{-A_2 t}u_{20}(x)\right\|_{L_p(\Omega)} + \int_0^t \left\|A_2^\alpha e^{-(t-\tau)A_2}\right\|\left\|\frac{e u_1(x,\tau)u_2(x,\tau)}{1+bu_1(x,\tau)}\right\|_{L_p(\Omega)} d\tau\\ &\leqslant C_\alpha t^{-\alpha}e^{-\varrho t}\left\|u_{20}(x)\right\|_{L_p(\Omega)} + C_\alpha\zeta_1\left(\|u_1\|_{L_p(\Omega)} + \|u_2\|_{L_p(\Omega)} + \|u_3\|_{L_p(\Omega)}\right)\int_0^t (t-\tau)^{-\alpha}e^{-\varrho(t-\tau)}\,d\tau\\ &\leqslant C_\alpha t^{-\alpha}e^{-\varrho t}\left\|u_{20}(x)\right\|_{L_p(\Omega)} + C_\alpha\zeta_1\left(\|u_1\|_{L_p(\Omega)} + \|u_2\|_{L_p(\Omega)} + \|u_3\|_{L_p(\Omega)}\right)\varrho^{\alpha-1}\Gamma(1-\alpha),\\ \|u_3\|_{\mathcal{Y}_3^\alpha} &\leqslant \left\|A_3^\alpha e^{-A_3 t}u_{30}(x)\right\|_{L_p(\Omega)} + \int_0^t \left\|A_3^\alpha e^{-(t-\tau)A_3}\right\|\left\|\frac{E u_1(x,\tau)u_3(x,\tau)}{1+Bu_1(x,\tau)} - Gu_3^2(x,\tau)\right\|_{L_p(\Omega)} d\tau\end{aligned}$$

$$\leqslant C_\alpha t^{-\alpha} e^{-\varrho t} \left\| u_{30}(x) \right\|_{L_p(\Omega)} + C_\alpha \zeta_2 \left(\|u_1\|_{L_p(\Omega)} + \|u_2\|_{L_p(\Omega)} + \|u_3\|_{L_p(\Omega)} \right) \int_0^t (t-\tau)^{-\alpha} e^{-\varrho(t-\tau)}\, d\tau$$
$$\leqslant C_\alpha t^{-\alpha} e^{-\varrho t} \left\| u_{30}(x) \right\|_{L_p(\Omega)} + C_\alpha \zeta_2 \left(\|u_1\|_{L_p(\Omega)} + \|u_2\|_{L_p(\Omega)} + \|u_3\|_{L_p(\Omega)} \right) \varrho^{\alpha-1} \Gamma(1-\alpha), \tag{25}$$

where C_α is a positive constant depending only on α, ζ_1, ζ_2 are two positive constants depending only on $\Lambda, d_1, d_2, d_3, \Omega$, $\Gamma(r) = \int_0^\infty \tau^{r-1} e^{-\tau}\, d\tau$ denotes the gamma function. From (14), (15), (23) and (25), we can get the last two inequalities of (13). The proof is completed. □

3. Stability analysis of the constant solution E^*

In this section, we study the local and global stability of the steady state $E^* = (u_1^*, u_2^*, u_3^*)$ of (2), which is defined in (4) and (5).

Theorem 3.1 *(Local stability). Assume* (5) *holds. If*

$$\begin{aligned}
&\text{(i)} \quad u_1^* \rho\left(u_1^*\right) + G u_3^* > 0,\\
&\text{(ii)} \quad d_3 u_1^* \rho\left(u_1^*\right) + d_1 G u_3^* \geqslant 0,\\
&\text{(iii)} \quad \left[G u_1^* u_3^* \rho\left(u_1^*\right) + A E u_3^* g\left(u_1^*\right) g'\left(u_1^*\right)\right] d_2 + a e u_2^* f\left(u_1^*\right) f'\left(u_1^*\right) d_3 \geqslant 0,\\
&\text{(iv)} \quad \rho\left(u_1^*\right) > \rho_+^*,
\end{aligned} \tag{26}$$

then the positive equilibrium E^* of system (2) *is locally asymptotically stable, where*

$$\begin{aligned}
&\rho\left(u_1^*\right) = \frac{r}{K} - \frac{a b u_2^*}{(1+b u_1^*)^2} - \frac{A B u_3^*}{(1+B u_1^*)^2},\\
&\rho_+^* = \frac{1}{2}\left[-\left(\frac{a e u_2^*}{G u_3^* (1+b u_1^*)^3} + \frac{A E}{G(1+B u_1^*)^3} + \frac{G u_3^*}{u_1^*} \right) + \sqrt{\triangle \Gamma} \right],\\
&\triangle \Gamma \triangleq \left[\frac{a e u_2^*}{G u_3^* (1+b u_1^*)^3} \right]^2 + \frac{2 a e u_2^*}{G u_3^* (1+b u_1^*)^3} \left[\frac{A E}{G(1+B u_1^*)^3} + \frac{G u_3^*}{u_1^*} \right] + \left[\frac{A E}{G(1+B u_1^*)^3} - \frac{G u_3^*}{u_1^*} \right]^2 > 0,\\
&f\left(u_1^*\right) = \frac{u_1^*}{1+b u_1^*}, \qquad f'\left(u_1^*\right) = \frac{1}{(1+b u_1^*)^2}, \qquad g\left(u_1^*\right) = \frac{u_1^*}{1+B u_1^*}, \qquad g'\left(u_1^*\right) = \frac{1}{(1+B u_1^*)^2}.
\end{aligned} \tag{27}$$

Theorem 3.2 *(Global stability). Assume that the positive equilibrium E^* of system* (2) *is locally stable. If*

$$\frac{r}{K} - \frac{a b u_2^*}{(1+b u_1^*)} - \frac{A B u_3^*}{(1+B u_1^*)} > 0, \tag{28}$$

then E^* is global stable.

Proof of Theorem 3.1. Let $0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 < \cdots$ be the eigenvalue of the operator $-\Delta$ in Ω with the homogeneous Neumann boundary condition, and set

$$\begin{aligned}
&\mathbf{X} = \left\{ (u_1, u_2, u_3) \in \left[C^2(\Omega) \cap C^1(\bar{\Omega})\right]^3 \colon \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial u_3}{\partial \nu} = 0 \text{ on } \partial\Omega \right\},\\
&\mathbf{E}(\mu) = \left\{ \phi \colon -\Delta\phi = \mu\phi \text{ in } \Omega,\ \frac{\partial \phi}{\partial \nu} \text{ on } \partial\Omega \right\}, \quad \text{with } \mu \in \mathbb{R}^1,\\
&\{\phi_{ij}\}_{j=1}^{\dim \mathbf{E}(\mu_i)} \text{ to be an orthonormal basis of } \mathbf{E}(\mu_i),\\
&\mathbf{X}_{ij} = \left\{ \mathbf{c}\phi_{ij} \colon \mathbf{c} \in \mathbb{R}^3 \right\}.
\end{aligned} \tag{29}$$

Then,

$$\mathbf{X} = \bigoplus_{i=0}^{\infty} \mathbf{X}_i, \quad \text{where } \mathbf{X}_i = \bigoplus_{j=1}^{\dim \mathbf{E}(\mu_i)} \mathbf{X}_{ij}. \tag{30}$$

Define

$$\rho(u_1)=\frac{r}{K}+au_2^*\frac{d}{du_1}\left(\frac{f(u_1)}{u_1}\right)+Au_3^*\frac{d}{du_1}\left(\frac{g(u_1)}{u_1}\right), \tag{31}$$

where $f(u_1)=\frac{u_1}{1+bu_1}$, $g(u_1)=\frac{u_1}{1+Bu_1}$.

Let $\bar u_1=u_1-u_1^*$, $\bar u_2=u_2-u_2^*$, $\bar u_3=u_3-u_3^*$, and the linearized system of (2) at (u_1^*,u_2^*,u_3^*) is

$$\begin{aligned}
&\frac{\partial\bar u_1}{\partial t}-d_1\Delta\bar u_1=j_{11}\bar u_1+j_{12}\bar u_2+j_{13}\bar u_3\quad\text{in }\Omega\times\mathbb{R}_+,\\
&\frac{\partial\bar u_2}{\partial t}-d_2\Delta\bar u_2=j_{21}\bar u_1\quad\text{in }\Omega\times\mathbb{R}_+,\\
&\frac{\partial\bar u_3}{\partial t}-d_3\Delta\bar u_3=j_{31}\bar u_1+j_{33}\bar u_3\quad\text{in }\Omega\times\mathbb{R}_+,\\
&\frac{\partial\bar u_1}{\partial\nu}=\frac{\partial\bar u_2}{\partial\nu}=\frac{\partial\bar u_3}{\partial\nu}=0\quad\text{on }\partial\Omega\times\mathbb{R}_+,
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
&j_{11}=-u_1^*\rho\big(u_1^*\big),\qquad j_{12}=-af\big(u_1^*\big)<0,\qquad j_{13}=-Ag\big(u_1^*\big)<0,\\
&j_{21}=eu_2^*f'\big(u_1^*\big)>0,\qquad j_{31}=Eu_3^*g'\big(u_1^*\big)>0,\qquad j_{33}=-Gu_3^*<0.
\end{aligned} \tag{33}$$

Denote

$$L=\begin{bmatrix} d_1\Delta+j_{11} & j_{12} & j_{13}\\ j_{21} & d_2\Delta & 0\\ j_{31} & 0 & d_3\Delta+j_{33}\end{bmatrix}. \tag{34}$$

Then, for each $i\in\{0,1,2,3,\ldots\}$, $\mathbf{X}_i$ is invariant under the operator L, and λ is an eigenvalue of L on $\mathbf{X}_i$ if and only if λ is an eigenvalue of the following matrix

$$A_i=\begin{bmatrix} -d_1\mu_i+j_{11} & j_{12} & j_{13}\\ j_{21} & -d_2\mu_i & 0\\ j_{31} & 0 & -d_3\mu_i+j_{33}\end{bmatrix}. \tag{35}$$

The characteristic equation of A_i is given by

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0, \tag{36}$$

where

$$\begin{aligned}
a_1&=a_1(\mu_i)=(d_1+d_2+d_3)\mu_i-(j_{11}+j_{33})=(d_1+d_2+d_3)\mu_i+u_1^*\rho\big(u_1^*\big)+Gu_3^*,\\
a_2&=a_2(\mu_i)=(d_1d_2+d_1d_3+d_2d_3)\mu_i^2-\big[(d_2+d_3)j_{11}+(d_1+d_2)j_{33}\big]\mu_i+j_{11}j_{33}-j_{13}j_{31}-j_{12}j_{21},\\
a_3&=a_3(\mu_i)=d_1d_2d_3\mu_i^3-(d_1d_2j_{33}+d_2d_3j_{11})\mu_i^2+\big[(j_{11}j_{33}-j_{13}j_{31})d_2-j_{12}j_{21}d_3\big]\mu_i+aeGu_2^*u_3^*f\big(u_1^*\big)f'\big(u_1^*\big).
\end{aligned}$$

Routh–Hurwitz criteria state that all roots of the characteristic Eq. (36) have negative real parts if and only if

$$a_1>0,\qquad a_3>0,\qquad a_1a_2-a_3>0. \tag{37}$$

$a_1>0$ follows from (i) of (26).

(33) shows that $aeGu_2^*u_3^*f(u_1^*)f'(u_1^*)>0$, so we have

$$a_3(\mu_0)>0, \tag{38}$$

and

$$\begin{aligned}
a_3(\mu_i)\ &>\ \mu_i\big[d_1d_2d_3\mu_i^2-(d_1d_2j_{33}+d_2d_3j_{11})\mu_i+\big[(j_{11}j_{33}-j_{13}j_{31})d_2-j_{12}j_{21}d_3\big]\big]\\
&=:\mu_i\chi(\mu_i)\geqslant\mu_1\chi(\mu_i),\quad i\geqslant 1.
\end{aligned}$$

It follows (ii) that $\chi(\tau)$ is increasing in $[0,\infty)$, so by (iii) of (26) and (33) we obtain

$$\begin{aligned}
a_3(\mu_i)&>\mu_1\chi(\mu_i)\geqslant\mu_1\chi(0)\\
&=\mu_1\big[(j_{11}j_{33}-j_{13}j_{31})d_2-j_{12}j_{21}d_3\big]\\
&=\mu_1\big\{\big[Gu_1^*u_3^*\rho\big(u_1^*\big)+AEu_3^*g\big(u_1^*\big)g'\big(u_1^*\big)\big]d_2+aeu_2^*f\big(u_1^*\big)f'\big(u_1^*\big)d_3\big\}\\
&\geqslant 0
\end{aligned} \tag{39}$$

for $i\geqslant 1$. Then, we get $a_3>0$.

Notice that

$$a_1a_2 - a_3 = c_1\mu_i^3 + c_2\mu_i^2 + (c_{11}d_1 + c_{12}d_2 + c_{13}d_3)\mu_i + G\big(u_1^*\big)^2 u_3^* \hbar\big(\rho\big(u_1^*\big)\big), \tag{40}$$

where $c_1, c_2, c_{11}, c_{12}, c_{13} > 0$ if $u_1^*\rho(u_1^*) + Gu_3^* > 0$ and

$$\hbar\big(\rho\big(u_1^*\big)\big) \triangleq \big[\rho\big(u_1^*\big)\big]^2 + \rho\big(u_1^*\big)\left[\frac{aeu_2^* f(u_1^*)}{Gu_1^*u_3^*} f'\big(u_1^*\big) + \frac{AEg(u_1^*)}{Gu_1^*} g'\big(u_1^*\big) + \frac{Gu_3^*}{u_1^*}\right] + \frac{AEu_3^* g(u_1^*)}{(u_1^*)^2} g'\big(u_1^*\big).$$

Thus, we have $a_1a_2 - a_3 > 0$ if $\hbar(\rho(u_1^*)) > 0$.

Observe that for the quadratic form $\hbar(\rho(u_1^*))$ and recall the definition of $\triangle\Gamma$, ρ_+^* and $\hbar(-\frac{Gu_3^*}{u_1^*}) = -\frac{aeu_2^* f(u_1^*)}{Gu_1^*u_3^*} f'(u_1^*) < 0$. Thus, $\rho_+^* > -\frac{Gu_3^*}{u_1^*}$. It follows that $a_1a_2 - a_3 > 0$ if and only if $\rho(u_1^*) > \rho_+^*$. We observe that $\rho_+^* < 0$. Therefore, by Routh–Hurwitz criteria we get Theorem 3.1. The proof is completed. □

Proof of Theorem 3.2. We use the Lyapunov functionals for the proof. Define

$$W(u_1, u_2, u_3) = \alpha \int \frac{u_1 - u_1^*}{u_1}\, du_1 + \beta \int \frac{u_2 - u_2^*}{u_2}\, du_2 + \gamma \int \frac{u_3 - u_3^*}{u_3}\, du_3 \tag{41}$$

and

$$E(t) = \int_\Omega W\big(u_1(x,t), u_2(x,t), u_3(x,t)\big)\, dx, \tag{42}$$

where α, β, γ are positive constants to be determined. Along any trajectory of system (2), we have

$$\begin{aligned}
\frac{dE(t)}{dt} &= \int_\Omega [W_{u_1}u_{1t} + W_{u_2}u_{2t} + W_{u_3}u_{3t}]\, dx \\
&= \int_\Omega \left[\alpha \frac{u_1 - u_1^*}{u_1} d_1\Delta u_1 + \beta \frac{u_2 - u_2^*}{u_2} d_2\Delta u_2 + \gamma \frac{u_3 - u_3^*}{u_3} d_3\Delta u_3\right] dx \\
&\quad + \alpha \int_\Omega \left[r\left(1 - \frac{u_1}{K}\right) - \frac{au_2}{1 + bu_1} - \frac{Au_3}{1 + Bu_3}\right]\big(u_1 - u_1^*\big)\, dx \\
&\quad + \beta \int_\Omega \left[-d + \frac{eu_1}{1 + bu_1}\right]\big(u_2 - u_2^*\big)\, dx + \gamma \int_\Omega \left[-D - Gu_3 + \frac{Eu_1}{1 + Bu_1}\right]\big(u_3 - u_3^*\big)\, dx \\
&= -\int_\Omega \left[d_1\alpha \frac{u_1^*}{u_1^2}|\nabla u_1|^2 + d_2\beta \frac{u_2^*}{u_2^2}|\nabla u_2|^2 + d_3\gamma \frac{u_3^*}{u_3^2}|\nabla u_3|^2\right] dx \\
&\quad + \alpha \int_\Omega \left[-\frac{r}{K} + \frac{abu_2^*}{(1 + bu_1)(1 + bu_1^*)} + \frac{ABu_3^*}{(1 + Bu_1)(1 + Bu_1^*)}\right]\big(u_1 - u_1^*\big)^2\, dx \\
&\quad + \int_\Omega \frac{1}{1 + bu_1^*}\left[-\alpha a + \beta e - \frac{\beta beu_1^*}{1 + bu_1^*}\right]\big(u_1 - u_1^*\big)\big(u_2 - u_2^*\big)\, dx \\
&\quad + \int_\Omega \frac{1}{1 + Bu_1^*}\left[-\alpha A + \gamma E - \frac{\gamma BEu_1^*}{1 + Bu_1^*}\right]\big(u_1 - u_1^*\big)\big(u_3 - u_3^*\big)\, dx - \gamma G \int_\Omega \big(u_3 - u_3^*\big)^2\, dx.
\end{aligned}$$

Choose

$$\alpha = 1, \qquad \beta = \frac{a}{e - bd}, \qquad \gamma = \frac{A(e - bd + Bd)}{E(e - bd)}. \tag{43}$$

Then we have

$$\begin{aligned}
\frac{dE(t)}{dt} &\leqslant \int_\Omega \left[-\frac{r}{K} + \frac{abu_2^*}{(1 + bu_1)(1 + bu_1^*)} + \frac{ABu_3^*}{(1 + Bu_1)(1 + Bu_1^*)}\right]\big(u_1 - u_1^*\big)^2\, dx \\
&\quad - \int_\Omega \frac{AG(e - bd + Bd)}{E(e - bd)}\big(u_3 - u_3^*\big)^2\, dx.
\end{aligned} \tag{44}$$

The coefficient for $(u_3 - u_3^*)^2$ is always negative. The coefficient for $(u_1 - u_1^*)^2$ is

$$-\frac{r}{K} + \frac{abu_2^*}{(1+bu_1)(1+bu_1^*)} + \frac{ABu_3^*}{(1+Bu_1)(1+Bu_1^*)} \leqslant -\frac{r}{K} + \frac{abu_2^*}{1+bu_1^*} + \frac{ABu_3^*}{1+Bu_1^*} < 0. \tag{45}$$

Thus, if (28) is satisfied, then $\frac{dE(t)}{dt} \leqslant 0$ and $\frac{dE(t)}{dt} = 0$ if and only if $u_1 = u_1^*$, $u_2 = u_2^*$, $u_3 = u_3^*$. The proof is completed. □

4. A priori estimates to the positive solution of system (3)

In this section, we will give a priori estimates to the positive solution of system (3) and our results are the following two theorems:

Theorem 4.1. *Any positive solution $\mathcal{U}(x) = (u_1(x), u_2(x), u_3(x))$ of system* (3) *satisfies*

$$\max_{x\in\bar\Omega} u_1(x) \leqslant K, \qquad \max_{x\in\bar\Omega} u_2(x) \leqslant \frac{deK}{d_2a} + \frac{reK}{da}, \qquad \max_{x\in\bar\Omega} u_3(x) \leqslant \frac{DEK}{d_3A} + \frac{rEK}{DA}. \tag{46}$$

Theorem 4.2. *There exist three positive constants*: C_1 (*depending on* $\frac{r}{d_1}, \Omega$), C_2 (*depending on* $\frac{e}{bd_2}, \Omega$), C_3 (*depending on* $\frac{E}{Bd_3}, \Omega$), *such that any positive solution $\mathcal{U}(x) = (u_1(x), u_2(x), u_3(x))$ of system* (3) *satisfies*

$$\frac{\max_{x\in\bar\Omega} u_1(x)}{\min_{x\in\bar\Omega} u_1(x)} \leqslant C_1, \qquad \frac{\max_{x\in\bar\Omega} u_2(x)}{\min_{x\in\bar\Omega} u_2(x)} \leqslant C_2, \qquad \frac{\max_{x\in\bar\Omega} u_3(x)}{\min_{x\in\bar\Omega} u_3(x)} \leqslant C_3. \tag{47}$$

In order to prove Theorems 4.1 and 4.2. Let us first introduce two lemmas. The first lemma that is due to Lou and Ni [20].

Lemma 4.3 *(Maximum principle). Suppose that $g \in (\bar\Omega \times \mathbb{R})$.*

(i) *Assume that $w \in C^2(\omega) \cap C^1(\bar\Omega)$ and satisfies*

$$\Delta w(x) + g\bigl(x, w(x)\bigr) \geqslant 0 \quad in\ \Omega, \qquad \frac{\partial w}{\partial \nu} \leqslant 0 \quad on\ \partial\Omega. \tag{48}$$

If $w(x_0) = \max_{x\in\bar\Omega} w(x)$, then $g(x_0, w(x_0)) \geqslant 0$.

(ii) *Assume that $w \in C^2(\omega) \cap C^1(\bar\Omega)$ and satisfies*

$$\Delta w(x) + g\bigl(x, w(x)\bigr) \leqslant 0 \quad in\ \Omega, \qquad \frac{\partial w}{\partial \nu} \geqslant 0 \quad on\ \partial\Omega. \tag{49}$$

If $w(x_0) = \min_{x\in\bar\Omega} w(x)$, then $g(x_0, w(x_0)) \leqslant 0$.

Next, we state the second lemma that is due to Lin, Ni and Takagi [18].

Lemma 4.4 *(Harnack inequality). Let $w \in C^2(\omega) \cap C^1(\bar\Omega)$ be a positive solution to $\Delta w(x) + c(x)w(x) = 0$, where $c \in C(\bar\Omega)$, satisfying the homogeneous Neumann boundary condition. Then there exists a positive constant C, depending only on $\|c(x)\|_{C(\bar\Omega)}$ and Ω, such that*

$$\max_{x\in\bar\Omega} w(x) \leqslant C \min_{x\in\bar\Omega} w(x). \tag{50}$$

Proof of Theorem 4.1. Let $x_0 \in \bar\Omega$ such that $u_1(x_0) = \max_{x\in\bar\Omega} u_1(x)$, then by Lemma 4.3 we have

$$ru_1(x_0)\left(1 - \frac{u_1(x_0)}{K}\right) - \frac{au_1(x_0)u_2(x_0)}{1+bu_1(x_0)} - \frac{Au_1(x_0)u_3(x_0)}{1+Bu_1(x_0)} \geqslant 0, \tag{51}$$

which implies $u_1(x_0) = \max_{x\in\bar\Omega} u_1(x) \leqslant K$.

Define $y(x) \triangleq d_1eu_1(x) + d_2au_2(x)$, then y satisfies

$$\begin{aligned} &-\Delta y = reu_1\left(1 - \frac{u_1}{K}\right) - adu_2 - \frac{Aeu_1u_3}{1+Bu_1} \quad \text{in } \Omega, \\ &\frac{\partial y}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{52}$$

Let $x_1 \in \bar{\Omega}$ such that $y(x_1) = \max_{x\in\bar{\Omega}} y(x)$, then by Lemma 4.3 we have

$$reu_1(x_1)\left(1 - \frac{u_1(x_1)}{K}\right) - adu_2(x_1) - \frac{Aeu_1(x_1)u_3(x_1)}{1 + Bu_1(x_1)} \geqslant 0, \tag{53}$$

which implies $u_1(x_1) \leqslant K$ and $u_2(x_1) \leqslant \frac{reu(x_1)}{ad} \leqslant \frac{reK}{ad}$. So, by the definition of $y(x)$, we obtain

$$d_2 a \max_{x\in\bar{\Omega}} u_2(x) \leqslant \max_{x\in\bar{\Omega}} y(x) = y(x_1) = deu_1(x_1) + d_2 au_2(x_1) \leqslant deK + \frac{reKd_2}{d}, \tag{54}$$

i.e.,

$$\max_{x\in\bar{\Omega}} u_2(x) \leqslant \frac{deK}{d_2 a} + \frac{reK}{da}. \tag{55}$$

Similarly, we can prove $\max_{x\in\bar{\Omega}} u_3(x) \leqslant \frac{DEK}{d_3 A} + \frac{rEK}{DA}$. The proof is completed. □

Proof of Theorem 4.2. It is easy to see $u_1(x)$ satisfies

$$\begin{aligned} &\Delta u_1 + \frac{c_1(x)}{d_1} u_1 = 0 \quad \text{in } \Omega, \\ &\frac{\partial u_1}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{56}$$

where $c_1(x) = ru_1(1 - \frac{u_1}{K}) - \frac{au_1u_2}{1+bu_1} - \frac{Au_1u_3}{1+Bu_1}$. Since $\|c_1/d_1\|_{C(\bar{\Omega})} \leqslant r/d_1$, by Lemma 4.4, we can get the first inequality of (47). The proof the other two inequalities of (47) are similar. The proof is completed. □

5. Non-existence of non-constant positive solution of system (3)

In Theorem 3.2, the global stability of the constant coexistence steady state implies the non-existence of non-constant positive solution of (3) regardless of diffusions. Several non-existence results of non-constant positive solutions to (3) will presented in this section, and in these results, the diffusion coefficients do play important roles. The mathematical techniques to be employed is the energy method.

Theorem 5.1. *Recall $0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 < \cdots$ represent the eigenvalues of the operator $-\Delta$ in Ω with the homogeneous Neumann boundary condition. Then we have*:

(i) *For any $\varepsilon_1 > 0$, there exists a positive constant D_1^*, depending on $\Lambda, \varepsilon_1, \Omega$, such that* (3) *has no non-constant positive solution provided $\mu_1 d_1 > D_1^*$, $\mu_1 d_2 > \frac{eK}{1+bK} + \varepsilon_1$ and $\mu_1 d_3 > \frac{EK}{1+BK} + \varepsilon_1$.*
(ii) *For any $\varepsilon_2 > 0$, there exists a positive constant D_2^*, depending on $\Lambda, \varepsilon_2, \Omega$, such that* (3) *has no non-constant positive solution provided $\mu_1 d_1 > r + \varepsilon_2$, $\lambda_1 d_2 > D_2^*$ and $\lambda_1 d_3 > D_2^*$.*

Proof. We only prove (i), the proof of (ii) can be accomplished similarly. Let (u_1, u_2, u_3) be a positive solution of (3) and write $\bar{u} = |\Omega|^{-1}\int_\Omega u(x)\,dx$. Then, multiplying the first equation of (3) by $(u_1 - \bar{u}_1)$, integrating over Ω and using Theorems 4.1 and 4.2, we have

$$\begin{aligned} d_1 \int_\Omega \left|\nabla(u_1 - \bar{u}_1)^2\right| dx &= \int_\Omega \left[ru_1 - \frac{r}{K}u_1^2 - \frac{au_1u_2}{1+bu_1} - \frac{Au_1u_3}{1+Bu_1} - r\bar{u}_1 + \frac{r}{K}\bar{u}_1^2 + \frac{a\bar{u}_1\bar{u}_2}{1+b\bar{u}_1} + \frac{A\bar{u}_1\bar{u}_3}{1+B\bar{u}_1}\right](u_1 - \bar{u}_1)\,dx \\ &= \int_\Omega \left[r - \frac{r}{K}(u_1 + \bar{u}_1) - \frac{au_2}{(1+bu_1)(1+b\bar{u}_1)} - \frac{Au_3}{(1+Bu_1)(1+B\bar{u}_1)}\right](u_1 - \bar{u}_1)^2\,dx \\ &\quad - \int_\Omega \left[\frac{a\bar{u}_1}{1+b\bar{u}_1}(u_1 - \bar{u}_1)(u_2 - \bar{u}_2) + \frac{A\bar{u}_1}{1+B\bar{u}_1}(u_1 - \bar{u}_1)(u_3 - \bar{u}_3)\right] dx \\ &\leqslant \left[r + c(\varepsilon)\right] \int_\Omega (u_1 - \bar{u}_1)^2\,dx + \varepsilon \int_\Omega (u_2 - \bar{u}_2)^2\,dx + \varepsilon \int_\Omega (u_3 - \bar{u}_3)^2\,dx \end{aligned} \tag{57}$$

for any $\varepsilon > 0$, where $c(\varepsilon)$ is a positive constant depending on $\varepsilon, \Lambda, \Omega$.

Similarly, by using the second and the third equations of (3), we obtain

$$d_2 \int_\Omega \left|\nabla (u_2-\bar{u}_2)^2\right| dx = \int_\Omega \left[-du_2 + \frac{eu_1u_2}{1+bu_1} + d\bar{u}_2 - \frac{e\bar{u}_1\bar{u}_2}{1+b\bar{u}_1}\right](u_2-\bar{u}_2)\,dx$$
$$= \int_\Omega \left[-d + \frac{e\bar{u}_1}{1+b\bar{u}_1}\right](u_2-\bar{u}_2)^2\,dx + \int_\Omega \frac{eu_2}{(1+bu_1)(1+b\bar{u}_1)}(u_1-\bar{u}_1)(u_2-\bar{u}_2)\,dx$$
$$\leqslant c(\varepsilon)\int_\Omega (u_1-\bar{u}_1)^2\,dx + \left(\frac{eK}{1+bK}+\varepsilon\right)\int_\Omega (u_2-\bar{u}_2)^2\,dx \tag{58}$$

for any $\varepsilon > 0$, where $c(\varepsilon)$ is a positive constant depending on $\varepsilon, \Lambda, \Omega$.

$$d_3 \int_\Omega \left|\nabla (u_3-\bar{u}_3)^2\right| dx = \int_\Omega \left[-Du_3 - Gu_3^2 + \frac{Eu_1u_3}{1+Bu_1} + D\bar{u}_3 + G\bar{u}_3 - \frac{E\bar{u}_1\bar{u}_3}{1+B\bar{u}_1}\right](u_3-\bar{u}_3)\,dx$$
$$= \int_\Omega \left[-D - G(u_3+\bar{u}_3) + \frac{E\bar{u}_1}{1+B\bar{u}_1}\right](u_3-\bar{u}_3)^2\,dx$$
$$+ \int_\Omega \frac{Eu_3}{(1+Bu_1)(1+B\bar{u}_1)}(u_1-\bar{u}_1)(u_3-\bar{u}_3)\,dx$$
$$\leqslant c(\varepsilon)\int_\Omega (u_1-\bar{u}_1)^2\,dx + \left(\frac{EK}{1+BK}+\varepsilon\right)\int_\Omega (u_3-\bar{u}_3)^2\,dx, \tag{59}$$

for any $\varepsilon > 0$, where $c(\varepsilon)$ is a positive constant depending on $\varepsilon, \Lambda, \Omega$.

Hence, adding (57)–(59) and applying Poincaré's inequality, we have

$$\mu_1 d_1 \int_\Omega (u_1-\bar{u}_1)^2\,dx + \mu_1 d_2 \int_\Omega (u_2-\bar{u}_2)^2\,dx + \mu_1 d_3 \int_\Omega (u_3-\bar{u}_3)^2\,dx$$
$$\leqslant \left[r + 3c(\varepsilon)\right]\int_\Omega (u_1-\bar{u}_1)^2\,dx + \left(\frac{eK}{1+bK}+2\varepsilon\right)\int_\Omega (u_2-\bar{u}_2)^2\,dx + \left(\frac{EK}{1+BK}+2\varepsilon\right)\int_\Omega (u_3-\bar{u}_3)^2\,dx. \tag{60}$$

We take $\varepsilon = \varepsilon_1/2$ in (60) such that $\varepsilon_1 = 2\varepsilon$. If $\mu_1 d_2 > \frac{eK}{1+bK} + \varepsilon_1$ and $\mu_1 d_3 > \frac{EK}{1+BK} + \varepsilon_1$, then it is easy to see from (60) that there exists $D_1^* = r + 3c(\varepsilon_1/2)$ such that (3) has only the positive constant solution $(u_1, u_2, u_3) = (u_1^*, u_2^*, u_3^*)$ if $\mu_1 d_1 > D_1^*$. The proof is completed. □

6. Existence of non-constant positive solutions of system (3)

This section is devoted to the existence of non-constant positive solutions of (3) for certain values of diffusion coefficients d_2 and d_3, respectively, while the other parameters are fixed. Our results show that, if the parameters are properly chosen, both the general stationary pattern and more interesting Turing pattern can arise as a result of diffusion. Let $m(\mu_n)$ be the multiplicity of μ_n. Our main findings are the following two theorems.

Theorem 6.1. *Assume*

$$j_{11}j_{33} - j_{13}j_{31} > 0, \qquad j_{33}d_1 + j_{11}d_3 > 0,$$
$$\triangle_1 \triangleq (j_{33}d_1 + j_{11}d_3)^2 - 4d_1d_3(j_{11}j_{33} - j_{13}j_{31}) > 0, \tag{61}$$

where $j_{k,l}$, $k,l = 1,2,3$ are defined in (33)*. If $\mu_2^*(d_2) \in (\mu_i, \mu_{i+1})$ and $\mu_3^*(d_2) \in (\mu_j, \mu_{j+1})$ for some $j > i \geqslant 0$, where $\mu_2^*(d_2)$ and $\mu_3^*(d_2)$ are defined in Proposition* 2*, and the sum $\sum_{n=i+1}^{j} m(\mu_n)$ is odd, then there exists a positive constant $\tilde{D}_2$ such that, if $d_2 \geqslant \tilde{D}_2$,* (3) *admits at least one non-constant positive solution.*

Theorem 6.2. *Assume*

$$j_{11}d_2 > 0, \qquad \triangle_2 = (j_{11}d_2)^2 + 4j_{12}j_{21}d_1d_2 > 0, \tag{62}$$

where $j_{k,l}$, $k,l = 1,2,3$ are defined in (33)*. If $\mu_2^*(d_3) \in (\mu_i, \mu_{i+1})$ and $\mu_3^*(d_3) \in (\mu_j, \mu_{j+1})$ for some $j > i \geqslant 0$, where $\mu_2^*(d_3)$ and $\mu_3^*(d_3)$ are defined in Proposition* 3*, and the sum $\sum_{n=i+1}^{j} m(\mu_n)$ is odd, then there exists a positive constant $\tilde{D}_3$ such that, if $d_3 \geqslant \tilde{D}_3$,* (3) *admits at least one non-constant positive solution.*

In order to prove the above two theorems, we start with some preliminary results. Recall the definition of $\mathcal{U}$, E^* in Section 1 and $\mathbf{X}$ in (29), we denote

$$\mathbf{X}^+ = \{\mathcal{U} \in \mathbf{X} \mid u_i > 0 \text{ on } \bar{\Omega},\ i = 1,2,3\},$$
$$\mathbf{B}(\mathbf{C}) = \big\{\mathcal{U} \in \mathbf{X} \bigm| C^{-1} < u_i < C \text{ on } \bar{\Omega},\ i = 1,2,3\big\}, \quad C > 0. \tag{63}$$

With the diffusion matrix $\mathcal{D} = \operatorname{diag}(d_1, d_2, d_3)$, (3) can be written as

$$-\mathcal{D}\Delta\mathcal{U} = \mathcal{F}(\mathcal{U}) \quad \text{in } \Omega,$$
$$\frac{\partial \mathcal{U}}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{64}$$

where

$$\mathcal{F}(\mathcal{U}) = \begin{pmatrix} ru_1(1 - \frac{u_1}{K}) - \frac{au_1u_2}{1+bu_1} - \frac{Au_1u_3}{1+Bu_1} \\ u_2(-d + \frac{eu_1}{1+bu_1}) \\ u_3(-D - Gu_3 + \frac{Eu_1}{1+Bu_1}) \end{pmatrix}. \tag{65}$$

Then $\mathcal{U}$ is a positive solution to (64) if and only if

$$\mathcal{G}(\mathcal{U}) \triangleq \mathcal{U} - (\mathcal{I} - \Delta)^{-1}\big[\mathcal{D}^{-1}\mathcal{F}(\mathcal{U}) + \mathcal{U}\big] = 0 \quad \text{for } \mathcal{U} \in \mathbf{X}^+, \tag{66}$$

where $(\mathcal{I} - \Delta)^{-1}$ is the inverse of $\mathcal{I} - \Delta$ in $\mathbf{X}$. As $\mathcal{G}(\cdot)$ is a compact perturbation of the identity operator, for any $\mathbf{B} = \mathbf{B}(\mathbf{C})$, the Leray–Schauder degree $\deg(\mathcal{G}(\cdot), 0, \mathbf{B})$ is well defined if $\mathcal{G}(\mathcal{U}) \neq 0$ on $\partial\mathbf{B}$.

We also note that

$$D_{\mathcal{U}}\mathcal{G}\big(E^*\big) = \mathcal{I} - (\mathcal{I} - \Delta)^{-1}\big[\mathcal{D}^{-1}\mathcal{F}_{\mathcal{U}}\big(E^*\big) + \mathcal{I}\big], \tag{67}$$

and recall that if $D_{\mathcal{U}}\mathcal{G}(E^*)$ is invertible, the index of $\mathcal{G}$ at E^* is defined as $\operatorname{index}(\mathcal{G}(\cdot), E^*) = (-1)^\gamma$, where γ is the multiplicity of negative eigenvalues of $D_{\mathcal{U}}\mathcal{G}(E^*)$ [25, Theorem 2.8.1].

For the sake of convenience, we denote

$$H(d_1, d_2, d_3, \mu) = \det\big[\mu\mathcal{I} - \mathcal{D}^{-1}\mathcal{F}_{\mathcal{U}}\big(E^*\big)\big] = \frac{1}{d_1d_2d_3}\det\big[\mu\mathcal{D} - \mathcal{F}_{\mathcal{U}}\big(E^*\big)\big]. \tag{68}$$

By arguments similar to those in [29], it can be shown that the following proposition holds.

Proposition 1. *Suppose that, for all $n \geqslant 0$, the matrix $\mu_n\mathcal{I} - \mathcal{D}^{-1}\mathcal{F}_{\mathcal{U}}(E^*)$ is non-singular. Then*

$$\operatorname{index}\big(\mathcal{G}(\cdot), E^*\big) = (-1)^\gamma, \quad \textit{where } \gamma = \sum_{n \geqslant 0,\ H(d_1,d_2,d_3;\mu_n) < 0} m(\mu_n). \tag{69}$$

To compute $\operatorname{index}(\mathcal{G}(\cdot), E^*)$, we have to consider the sign of $H(d_1, d_2, d_3; \mu)$. Direct calculation gives

$$\det\big[\mu\mathcal{D} - \mathcal{F}_{\mathcal{U}}\big(E^*\big)\big] = A_3(d_2, d_3)\mu^3 + A_2(d_2, d_3)\mu^2 + A_1(d_2, d_3)\mu - \det\big[\mathcal{F}_{\mathcal{U}}\big(E^*\big)\big]$$
$$\triangleq \mathcal{A}(d_2, d_3; \mu), \tag{70}$$

with

$$A_3(d_2, d_3) = d_1d_2d_3, \qquad A_2(d_2, d_3) = -(j_{33}d_1d_2 + j_{11}d_2d_3),$$
$$A_1(d_2, d_3) = (j_{11}j_{33} - j_{13}j_{31})d_2 - j_{12}j_{21}d_3. \tag{71}$$

We first consider the dependence of $\mathcal{A}$ on d_2. Let $\tilde{\mu}_i(d_2; d_3)$, $i = 1,2,3$, be the three roots of $\mathcal{A}(d_2, d_3; \mu) = 0$ satisfying $\operatorname{Re}\tilde{\mu}_1(d_2; d_3) \leqslant \operatorname{Re}\tilde{\mu}_2(d_2; d_3) \leqslant \operatorname{Re}\tilde{\mu}_3(d_2; d_3)$. Since $\det\mathcal{F}_{\mathcal{U}}(E^*) < 0$ and $A_3(d_2, d_3) > 0$, one of $\tilde{\mu}_i(d_2; d_3)$ is real and negative, and the product of the other two is positive.

In addition, we have

$$\lim_{d_2 \to \infty} \mathcal{A}(d_2; d_3)/d_2 = \mu\big[d_1d_3\mu^2 - (j_{33}d_1 + j_{11}d_3)\mu + j_{11}j_{33} - j_{13}j_{31}\big]. \tag{72}$$

Note that if (61) holds, we can establish the following proposition.

Proposition 2. *Assume that* (61) *holds. Then there exists a positive constant D_2^* such that when $d_2 \geqslant D_2^*$, the three roots $\tilde{\mu}_i(d_2; d_3)$, $i = 1, 2, 3$, of $\mathcal{A}(d_2, d_3; \mu)$ are real and satisfy*

$$\begin{aligned}
&\lim_{d_2\to\infty} \tilde{\mu}_1(d_2; d_3) = 0,\\
&\lim_{d_2\to\infty} \tilde{\mu}_2(d_2; d_3) = \frac{1}{2d_1d_3}\left[j_{33}d_1 + j_{11}d_3 - \sqrt{\triangle_1}\,\right] \triangleq \mu_2^*(d_3) > 0,\\
&\lim_{d_2\to\infty} \tilde{\mu}_3(d_2; d_3) = \frac{1}{2d_1d_3}\left[j_{33}d_1 + j_{11}d_3 + \sqrt{\triangle_1}\,\right] \triangleq \mu_3^*(d_3) > 0.
\end{aligned} \tag{73}$$

Moreover, when $d_2 > D_2^*$,

$$\begin{aligned}
&-\infty < \tilde{\mu}_1(d_2; d_3) < 0 < \tilde{\mu}_2(d_2; d_3) < \tilde{\mu}_3(d_2; d_3),\\
&\mathcal{A}(d_2, d_3; \mu) < 0 \quad \text{if } \mu \in \big(-\infty, \tilde{\mu}_1(d_2; d_3)\big) \cup \big(\tilde{\mu}_2(d_2; d_3), \tilde{\mu}_3(d_2; d_3)\big),\\
&\mathcal{A}(d_2, d_3; \mu) > 0 \quad \text{if } \mu \in \big(\tilde{\mu}_1(d_2; d_3), \tilde{\mu}_2(d_2; d_3)\big) \cup \big(\tilde{\mu}_3(d_2; d_3), \infty\big).
\end{aligned} \tag{74}$$

Similarly, we consider d_3 as the parameter, we have the following propositions.

Proposition 3. *Assume that* (62) *holds. Then there exists a positive constant D_3^* such that when $d_3 \geqslant D_2^*$, the three roots $\tilde{\mu}_i(d_3; d_2)$, $i = 1, 2, 3$, of $\mathcal{A}(d_2, d_3; \mu)$ are real and satisfy*

$$\begin{aligned}
&\lim_{d_3\to\infty} \tilde{\mu}_1(d_3; d_2) = 0,\\
&\lim_{d_3\to\infty} \tilde{\mu}_2(d_3; d_2) = \frac{1}{2d_1d_2}\left[j_{11}d_2 - \sqrt{\triangle_2}\,\right] \triangleq \mu_2^*(d_2) > 0,\\
&\lim_{d_3\to\infty} \tilde{\mu}_3(d_3; d_2) = \frac{1}{2d_1d_3}\left[j_{11}d_2 + \sqrt{\triangle_2}\,\right] \triangleq \mu_3^*(d_2) > 0.
\end{aligned} \tag{75}$$

Moreover, when $d_3 > D_3^*$,

$$\begin{aligned}
&-\infty < \tilde{\mu}_1(d_3; d_2) < 0 < \tilde{\mu}_2(d_3; d_2) < \tilde{\mu}_3(d_3; d_2),\\
&\mathcal{A}(d_2, d_3; \mu) < 0 \quad \text{if } \mu \in \big(-\infty, \tilde{\mu}_1(d_3; d_2)\big) \cup \big(\tilde{\mu}_2(d_3; d_2), \tilde{\mu}_3(d_3; d_2)\big),\\
&\mathcal{A}(d_2, d_3; \mu) > 0 \quad \text{if } \mu \in \big(\tilde{\mu}_1(d_3; d_2), \tilde{\mu}_2(d_3; d_2)\big) \cup \big(\tilde{\mu}_3(d_3; d_2), \infty\big).
\end{aligned} \tag{76}$$

Now, we can give the proofs of Theorems 6.1 and 6.2. Since the proof of Theorem 6.2 is similar to the proof of Theorem 6.1, we only give the proof for Theorem 6.1.

Proof of Theorem 6.1. By Proposition 1 and our assumptions, there exists a positive constant $\tilde{D}_2$, such that when $d_2 \geqslant \tilde{D}_2$, (74) holds and

$$\mu_i < \tilde{\mu}_2(d_2; d_3) < \mu_{i+1}, \qquad \mu_j < \tilde{\mu}_3(d_2; d_3) < \mu_{j+1}. \tag{77}$$

According to Theorem 5.1, for $\hat{d}_2$ and $\hat{d}_3$ large enough, there exists $\hat{d}_1$ such that (3) has no non-constant positive solutions when $d_1 \geqslant \hat{d}_1$, $d_2 \geqslant \hat{d}_2$ and $d_3 \geqslant \hat{d}_3$. In addition, since $\det[\mathcal{F}_{\mathcal{U}}](E^*) < 0$ and $\lim_{n\to\infty} \mu_n = \infty$, from (70), we can further choose $\hat{d}_1$, $\hat{d}_2$ and $\hat{d}_3$ to be so large such that

$$H(\hat{d}_1, \hat{d}_2, \hat{d}_3; \mu_n) > 0 \quad \text{for all } n \geqslant 0. \tag{78}$$

Now, we show that for any $d_2 \geqslant \tilde{D}_2$, (3) has at least one non-constant positive solution. The proof, which is accomplished by a contradict argument, is based on the homotopy invariance of the topological degree. Suppose on the contrary that the assertion is not true for some $d_2 = \tilde{d}_2 \geqslant \tilde{D}_2$.

Fix $d_2 = \tilde{d}_2$, let $d_i(t) = td_i + (1-t)\hat{d}_i$, $i = 1, 2, 3$ and define

$$\mathcal{D}(t) = \operatorname{diag}\big[d_1(t), d_2(t), d_3(t)\big].$$

Now we consider the following problem

$$\begin{aligned}
&-\mathcal{D}(t)\Delta\mathcal{U} = \mathcal{F}(\mathcal{U}) \quad \text{in } \Omega,\\
&\frac{\partial \mathcal{U}}{\partial \nu} = 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{79}$$

Then $\mathcal{U}$ is a positive solution of (3) if and only if it is a positive solution of (79) for $t=1$. It is obvious that E^* is the unique constant positive solution of (79) for $0 \leqslant t \leqslant 1$. For any $0 \leqslant t \leqslant 1$, $\mathcal{U}$ is a positive solution of (79) if and only if

$$\mathcal{G}(t;\mathcal{U}) \triangleq \mathcal{U} - (\mathcal{I}-\Delta)^{-1}\big[\mathcal{D}^{-1}(t)\mathcal{F}(\mathcal{U}) + \mathcal{U}\big] = 0 \quad \text{for } \mathcal{U} \in \mathbf{X}^+. \tag{80}$$

Clearly, $\mathcal{G}(1;\mathcal{U}) = \mathcal{G}(\mathcal{U})$. Theorem 5.1 shows that the only positive solution of $\mathcal{G}(0;\mathcal{U})$ is E^*. From direct calculation,

$$D_{\mathcal{U}}\mathcal{G}\big(t;E^*\big) = \mathcal{I} - (\mathcal{I}-\Delta)^{-1}\big[\mathcal{D}^{-1}(t)\mathcal{F}_{\mathcal{U}}\big(E^*\big) + \mathcal{I}\big]. \tag{81}$$

In particular,

$$\begin{aligned} &D_{\mathcal{U}}\mathcal{G}\big(0;E^*\big) = \mathcal{I} - (\mathcal{I}-\Delta)^{-1}\big[\hat{\mathcal{D}}^{-1}\mathcal{F}_{\mathcal{U}}\big(E^*\big) + \mathcal{I}\big], \\ &D_{\mathcal{U}}\mathcal{G}\big(1;E^*\big) = \mathcal{I} - (\mathcal{I}-\Delta)^{-1}\big[\mathcal{D}^{-1}\mathcal{F}_{\mathcal{U}}\big(E^*\big) + \mathcal{I}\big] = D_{\mathcal{U}}\mathcal{G}\big(E^*\big), \end{aligned} \tag{82}$$

where $\hat{\mathcal{D}} = \text{diag}[\hat{d}_1, \hat{d}_2, \hat{d}_3]$. From (68) and (70) we see that

$$H(d_1,d_2,d_3;\mu) = \frac{1}{d_1,d_2,d_3}\mathcal{A}(d_2,d_3;\mu). \tag{83}$$

In view of (74) and (77), it follows from (83) that

$$\begin{aligned} &H(d_1,d_2,d_3;\mu_0) = H(0) > 0, \\ &H(d_1,d_2,d_3;\mu_n) < 0, \quad i+1 \leqslant n \leqslant j, \\ &H(d_1,d_2,d_3;\mu_n) > 0, \quad 1 \leqslant n \leqslant i \text{ and } n \geqslant j+1. \end{aligned} \tag{84}$$

Therefore, zero is not an eigenvalue of the matrix $\mu_i\mathcal{I} - \mathcal{D}^{-1}\mathcal{F}_{\mathcal{U}}(E^*)$ for all $n \geqslant 0$ and

$$\sum_{n\geqslant 0,\, H(d_1,d_2,d_3;\mu_n)<0} m(\mu_n) = \sum_{n=i+1}^{j} m(\mu_n) = \text{an odd number}. \tag{85}$$

Then Proposition 1 shows that

$$\text{index}\big(\mathcal{G}(1;\cdot), E^*\big) = (-1)^\gamma = -1. \tag{86}$$

On the other hand, by (78) and Proposition 1 again, we obtain

$$\text{index}\big(\mathcal{G}(0;\cdot), E^*\big) = (-1)^0 = 1. \tag{87}$$

In view of $\tilde{d}_2 \geqslant \tilde{D}_2$, by Theorems 4.1 and 4.2, there exists a positive constant C, depending on $\tilde{D}_2, d_1, d_3, \hat{d}_1, \hat{d}_2, \hat{d}_3, \Lambda$, such that, for $0 \leqslant t \leqslant 1$, the positive solutions of (79) satisfy $C^{-1} < u_1, u_2, u_3 < C$. Therefore, $\mathcal{G}(t;\mathcal{U}) \neq 0$ on $\partial\mathbf{B}(\mathbf{C})$ for all $0 \leqslant t \leqslant 1$. By the homotopy invariance of the topological degree,

$$\deg\big(\mathcal{G}(1;\cdot), 0, \mathbf{B}(\mathbf{C})\big) = \deg\big(\mathcal{G}(0;\cdot), 0, \mathbf{B}(\mathbf{C})\big). \tag{88}$$

Moreover, under our assumptions, the only positive solution of both $\mathcal{G}(1;\mathcal{U}) = 0$ and $\mathcal{G}(0;\mathcal{U}) = 0$ in $\mathbf{B}(\mathbf{C})$ is E^*, and hence, by (86) and (87),

$$\deg\big(\mathcal{G}(0;\cdot), 0, \mathbf{B}(\mathbf{C})\big) = \text{index}\big(\mathcal{G}(0;\cdot), E^*\big) = (-1)^0 = 1, \tag{89}$$

and

$$\deg\big(\mathcal{G}(1;\cdot), 0, \mathbf{B}(\mathbf{C})\big) = \text{index}\big(\mathcal{G}(1;\cdot), E^*\big) = (-1)^\gamma = -1. \tag{90}$$

This contradicts (88), and the proof is completed. □

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