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Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes ☆

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Abstract

For a couple of lifetimes (X_1, X_2) with an exchangeable joint survival function $\bar{F}$, attention is focused on notions of bivariate aging that can be described in terms of properties of the level curves of $\bar{F}$. We analyze the relations existing among those notions of bivariate aging, univariate aging, and dependence. A goal and, at the same time, a method to this purpose is to define axiomatically a correspondence among those objects; in fact, we characterize notions of univariate and bivariate aging in terms of properties of dependence. Dependence between two lifetimes will be described in terms of their survival copula. The language of copulæ turns out to be generally useful for our purposes; in particular, we shall introduce the more general notion of *semicopula*. It will be seen that this is a natural object for our analysis. Our definitions and subsequent results will be illustrated by considering a few remarkable cases; in particular, we find some necessary or sufficient conditions for Schur-concavity of $\bar{F}$, or for IFR properties of the one-dimensional marginals. The case characterized by the condition that the survival copula of (X_1, X_2) is Archimedean will be considered in some detail. For most of our arguments, the extension to the case of $n>2$ is straightforward.

AMS 2000 subject classifications: 60K10; 60E15; 62N05

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1. Introduction

Univariate notions of aging, such as IFR, DFR, NBU, constitute a well-established core of reliability theory: the definitions given in the, by-now classical, literature are very clear and provide the basis for many useful results, which apply when dealing with the analysis of a single unit or of several units with stochastically independent lifetimes (see, e.g., [6]).

On the contrary, multivariate definitions of aging, i.e. definitions valid when dealing with several dependent lifetimes, are rather controversial. In fact, starting from univariate notions, several types of multivariate extensions can be defined; furthermore, the non-trivial interactions between aging and dependence contribute to make the analysis quite challenging.

More specifically, the study of the relations between multivariate aging and dependence, or between multivariate aging and univariate aging of marginal distributions, or among multivariate aging, dependence, and univariate aging of marginals, is a very intricate issue.

Of course, such relations heavily depend on the specific definition of multivariate aging that is taken into consideration. For example it is shown in the literature that some (“dynamical-type”) definitions of positive multivariate aging do imply “corresponding” conditions of positive dependence (see, e.g. [1,21]). Furthermore, many notions of multivariate aging imply “corresponding” univariate aging properties for the unidimensional marginals.

The present paper will be entirely devoted to analyze the relations between dependence and suitable notions of aging. In Section 4, we shall in fact single out a class of concepts of multivariate aging; later, we shall show that such definitions allow us to obtain results in the desired direction. The discussion of a few examples and applications will clarify the meaning and the interest of those definitions and of the corresponding results. The emphasis, though, is on the nature of the relations between aging and dependence, rather than on the “philosophical” meaning of the aging concepts considered.

A goal and, at the same time, a method in this paper, is to provide convenient axiomatic definitions of “correspondence” among notions of multivariate aging, univariate aging and stochastic dependence. We shall see that the language of copulæ will turn out to be very useful, in that respect.

We point out that we are interested in notions of positive (negative) multivariate aging which can be compatible with negative (positive) dependence and with negative (positive) aging of one-dimensional marginals. Furthermore, it should be said that our treatment is confined to the class of exchangeable pairs of random lifetimes. In the last section, we shall sketch possible ways of extending the analysis to the case of non-exchangeability.

Thus, to summarize, we consider exchangeable bivariate laws which describe the lifetimes of two similar, dependent items and focus on three main features, namely dependence, univariate aging (in short, 1-aging) and bivariate aging (2-aging). As mentioned above, the first two notions are well established, whereas bivariate aging can be given different interpretations.

In [8], we considered the approach that can be briefly described as follows: a bivariate exchangeable law is said to have positive bivariate aging if, conditionally on a same history of survivals, the law of the residual lifetime of the younger component dominates in some stochastic order sense the corresponding law of the elder component (see also [23] for a wider discussion on such approach). This approach allows one to reinterpret some multivariate notions considered in the literature and to define some new notions, starting from well-established univariate definitions.

In [10], we noticed that some of the notions of bivariate aging considered in [8] could be described in terms of the behavior of the level curves of the joint survival function and that, in turn, the latter behavior could be described in terms of dependence of a suitable function.

The approach developed in [10] will be a starting point for the treatment to be developed in the present paper. As a further related feature, here we try and describe univariate aging in terms of dependence of a suitable bivariate law, expanding on ideas presented in [2].

Such an approach led us to introduce the concept of semicopula. In a few words, a semicopula is a bivariate function that shares all the properties of copulæ, except that it need not be 2-increasing. We will associate three different semicopulæ, K, B, C, to a same bivariate exchangeable law. K denotes the usual survival copula, and dependence properties of the bivariate law can be described by requiring K to belong to special families of copulæ. We shall see in the sequel that properties of univariate and bivariate aging will be described in a similar way by imposing that C or B belong to suitable families of semicopulæ.

In order to motivate the results which will be proved, we make some preliminary considerations:

(a) Positive bivariate aging can coexist with several forms of dependence and univariate aging. Nonetheless, the knowledge of dependence (expressed, e.g. by the survival copula, as we shall discuss below) and of the marginal (which determines univariate aging) completely defines the joint law, and hence its bivariate aging. Thus *bivariate aging is the result of the interplay between dependence and univariate aging*.
(b) *Positive dependence plays in favor of positive bivariate aging*: Take an exchangeable law with standard exponential marginals, i.e. with no univariate aging. Then the only source of bivariate aging is dependence. In fact, if the components are independent we are obviously indifferent between the younger and the elder component, whereas if there is positive dependence we are more hopeful in the future of the younger component (rather than in the future of the elder one), because its outlook is strengthened by the long surviving of the other component.
(c) *Positive univariate aging plays in favor of positive bivariate aging*: In fact, consider an independent law. Then trivially we prefer the younger component to the elder iff there is positive univariate aging.

Actually, under suitable, rather natural, conditions to be defined below, we shall prove results in the following vein:

(1) *Positive dependence and positive* 1-*aging imply positive* 2-*aging*: In fact, both positive dependence and positive 1-aging play in favor of positive 2-aging.
(2) *Positive* 2-*aging and negative* 1-*aging imply positive dependence*: In fact, despite the fact that 1-aging is negative, we have positive 2-aging. Hence at least dependence must be "favorable"
(3) *Positive* 2-*aging and negative dependence imply positive* 1-*aging*: Despite negative dependence, we have positive 2-aging. Hence, at least 1-aging must be "favorable".

Corollaries of the above general statements yield, for example, conditions of negative dependence which ensure that a law with Schur-concave survival function (that can be seen as a notion of bivariate IFR) has IFR marginals. Furthermore, we find suitable conditions of positive dependence that, combined with the IFR property of the marginals, imply Schur-concavity.

Statements (1)–(3) above give some compatibility conditions among univariate aging, bivariate aging and dependence. One may think that other implications of the same type could be proved by applying the same kind of techniques, but this does not appear to be true. See Remark 6.8 below.

The paper will be structured as follows: in Section 2 we give some definitions and we fix notation. In particular, we define the concept of semicopula and introduce a triple of semicopulæ associated with a same bivariate, exchangeable, survival function. Furthermore, some basic aspects about the relations among these three semicopulæ are pointed out and some other definitions related with the concept of semicopula are mentioned. In Section 3 we focus on some notions of positive and negative dependence that will be formally extended also to semicopulæ. In Section 4, we give axiomatic definitions of 1- and 2-aging in terms of dependence of suitable semicopulæ; this setup will be used in Section 5 to provide our main results concerning relations, implications and compatibility conditions among dependence, 1- and 2-aging. Sections 6 and 7 deal with examples and applications. In particular, as mentioned above, results related to Schur-concavity will be provided. Furthermore, aging and dependence for the class of the so-called TTE—time-transformed exponential models will be studied in some detail. The paper will end with a section devoted to trace some final comments and remarks; in particular, we list some natural questions that can be given direct answers in terms of our results and sketch some aspects related to their extensions to the case of $n>2$ lifetimes.

2. Definitions and notation

Let $\mathcal{G}$ be the class of one-dimensional, continuous survival functions which are positive and strictly decreasing on $\mathbb{R}_+$ and that take the value 1 at 0. We denote by $\mathcal{F}$ the class of two-dimensional exchangeable survival functions on $\mathbb{R}_+^2$ whose one-dimensional marginals are in $\mathcal{G}$.

Consider now exchangeable random lifetimes X_1, X_2 and denote by $\bar{F}$ their joint survival function:

$$\bar{F}(\mathbf{x}) = P\{X_1 > x_1, X_2 > x_2\}.$$

We denote by $\bar{G}$ the one-dimensional marginal survival function of X_1, X_2, namely,

$$\bar{G}(x) = \bar{F}(x, 0). \tag{1}$$

We also denote by F the distribution function corresponding to $\bar{F}$.

Throughout the paper, we assume $\bar{F} \in \mathcal{F}$, i.e. $\bar{G} \in \mathcal{G}$.

2.1. *Copul*æ *and semicopul*æ

Let $\hat{\mathcal{K}}$ be the family of bivariate copulæ, namely, the family of functions $C : [0,1]^2 \to [0,1]$ such that

$$C(0, v) = C(u, 0) = 0, \quad 0 \leqslant u, \;\; v \leqslant 1, \tag{2}$$

$$C(1, v) = v, \quad C(u, 1) = u, \;\; 0 \leqslant u, v \leqslant 1, \tag{3}$$

$$C(u, v) \quad \text{is increasing in each variable,} \tag{4}$$

$$C(u, v) + C(u', v') - C(u, v') - C(u', v) \geqslant 0, \quad 0 \leqslant u \leqslant u' \leqslant 1, \quad 0 \leqslant v \leqslant v' \leqslant 1. \tag{5}$$

Thus, a copula is the restriction to the unit square of a distribution function with uniform marginals on $[0,1]$.

Let $\hat{\mathcal{X}}$ be the family of functions which satisfy (2)–(4), but which need not satisfy the rectangular inequality (5). We shall call these functions *extended semicopul*æ.

Denote by $\mathcal{H}$ the space of continuous, strictly increasing functions $h : [0,1] \to [0,1]$ such that $h(0) = 0, h(1) = 1$. Clearly, $h \in \mathcal{H}$ if and only if $h^{-1} \in \mathcal{H}$. Thus, denoting by $\circ$ the composition operator in $\mathcal{H}$, $(\mathcal{H}, \circ)$ is a group, whose identity is here denoted by h_0:

$$h_0(x) = x, \quad \forall x \in [0,1].$$

For $C \in \hat{\mathcal{X}}$ and $h \in \mathcal{H}$, let now

$$\Psi_h C(u, v) \equiv h^{-1}(C(h(u), h(v))). \tag{6}$$

Furthermore, for $\mathcal{D} \subset \mathcal{H}$ and $\mathcal{E} \subset \hat{\mathcal{X}}$, let

$$\Psi_{\mathcal{F}}(\mathcal{E}) = \{\Psi_h C | h \in \mathcal{D}, C \in \mathcal{E}\}.$$

It is a simple matter to check that $\Psi_{\mathcal{H}}(\hat{\mathcal{X}}) = \hat{\mathcal{X}}$.

Let also $\hat{\mathcal{S}} = \Psi_{\mathcal{H}}(\hat{\mathcal{K}})$. We shall call $\mathcal{H}$-*semicopul*æ the elements of $\hat{\mathcal{S}}$.

Finally, let $\mathcal{K}, \mathcal{S}, \mathcal{X}$ denote the families of elements of $\hat{\mathcal{K}}, \hat{\mathcal{S}}, \hat{\mathcal{X}}$, respectively, which are *exchangeable* (or symmetric, commutative), namely, which satisfy

$$C(u, v) = C(v, u), \quad 0 \leqslant u, v \leqslant 1. \tag{7}$$

In the sequel, we shall be interested mainly in the family $\mathcal{S}$ of exchangeable $\mathcal{H}$-semicopulæ. It is clear that we can write

$$\mathcal{S} = \Psi_{\mathcal{H}}(\mathcal{K}) = \{\Psi_h C | h \in \mathcal{H}, C \in \mathcal{K}\}. \tag{8}$$

We shall call the elements of $\mathcal{S}$ simply *semicopulæ*. Notice, in particular, that a semicopula is continuous in each variable, whereas an extended semicopula need not have this property.

Extended semicopulæ are dealt in [13], where their mathematical properties are studied in detail. Here, we notice only that $\hat{\mathcal{X}}$ strictly includes the family of the so-called quasicopulæ, namely, of those functions which satisfy (2)–(4) and a Lipschitz condition. For example (see [13,24]), $C_h(u,v) = h^{-1}(h(u)h(v))$ is a quasicopula (and a copula) if and only if $h \in \mathcal{H}$ and $-\log h$ is convex, whereas it is a simple matter to check that C_h is a semicopula for every $h \in \mathcal{H}$.

Furthermore, we notice also that $\hat{\mathcal{X}}$ strictly includes the family of t-norms, or triangular norms, namely, the family of those functions which satisfy (2)–(4), (7) and which are *associative*, namely,

$$C(C(u,v),z) = C(u, C(v,z)), \quad u,v,z \in [0,1].$$

For example, any non-associative copula is an extended semicopula but not a t-norm.

For the description of aging, a relevant family of semicopulæ is

$$\mathcal{A} = \Psi_{\mathcal{H}}(\{S_0\}) = \{\Psi_h S_0 | h \in \mathcal{H}\},$$

where

$$S_0(u,v) = uv \tag{9}$$

denotes the copula of independence. The elements of $\mathcal{A}$ are of the form

$$C_h(u,v) := h^{-1}(h(u)h(v)) = \phi^{-1}(\phi(u) + \phi(v)) =: A_\phi(u,v), \tag{10}$$

where $h \in \mathcal{H}$ is called the multiplicative generator and $\phi = -\log h$ the additive generator. The semicopulæ in $\mathcal{A}$ are indeed Archimedean t-norms (see [20] for definition and properties of Archimedean t-norms), and we shall use also the term *Archimedean semicopulæ*.

Note that, as we mentioned before, an Archimedean semicopula is a copula if and only if $-\log h$ is convex. See [18,14], for further details about copulæ, and [16] for t-norms.

We shall see in the next Section 2.2 the role of the mappings Ψ_h in our setting and corresponding interest for semicopulae. Here, we confine ourselves to the next proposition, which shows, in particular, that the family $\mathcal{S}$ is closed under the operation Ψ_h, for every $h \in \mathcal{H}$, i.e.

$$h \in \mathcal{H}, \;\; C \in \mathcal{S} \Rightarrow \Psi_h C \in \mathcal{S}.$$

Note that a similar property does not hold for the family $\mathcal{K}$ of copulæ, since in general we have: $h \in \mathcal{H}, \;\; C \in \mathcal{K} \not\Rightarrow \Psi_h C \in \mathcal{K}$.

Proposition 2.1. *For $h, h' \in \mathcal{H}$, and $C \in \hat{\mathcal{X}}$, it is*

$$(\Psi_h \circ \Psi_{h'})C = \Psi_{h' \circ h}C. \tag{11}$$

Furthermore, $\Psi_{\mathcal{H}}(\mathcal{A}) = \mathcal{A}$, $\Psi_{\mathcal{H}}(\mathcal{S}) = \mathcal{S}$ *and* $\Psi_{\mathcal{H}}(\hat{\mathcal{S}}) = \hat{\mathcal{S}}$.

Proof. We can write, for $0 \leqslant u, v \leqslant 1$,

$$\begin{aligned}(\Psi_h \circ \Psi_{h'})C(u,v) &= h^{-1}[\Psi_{h'}C(h(u), h(v))] \\ &= h^{-1}[h'^{-1}[C(h'(h(u)), h'(h(v)))]] \\ &= \Psi_{h' \circ h}C(u,v).\end{aligned} \tag{12}$$

The other statements follow immediately. In fact, if $C_{h'} \in \mathcal{A}$, then $\Psi_h C_{h'} = C_{h' \circ h} \in \mathcal{A}$, and if $S = \Psi_{h'}K$, with $K \in \mathcal{K}$, then $\Psi_h S = \Psi_{h' \circ h}K \in \mathcal{S}$. The proof in the case of $\hat{\mathcal{S}}$ is similar. $\square$

Remark 2.2. Let $\mathcal{U} \equiv \{\Psi_h, h \in \mathcal{H}\}$ and let $*$ be the composition in $\mathcal{U}$. Then, by (12), we see that $(\mathcal{U}, *)$ is a (non-commutative) group, isomorphic to $(\mathcal{H}, \circ)$.

Remark 2.3. It is natural to consider on $\mathcal{S}$ the equivalence relation $\approx$ defined by setting

$$S' \approx S'' \Leftrightarrow S'' = \Psi_h S' \quad \text{for some } h \in \mathcal{H}.$$

The set $\mathcal{A}$ is one of the equivalence classes induced by $\approx$. For any other equivalence class $\mathcal{B}$, it is $\Psi_{\mathcal{H}}(\mathcal{B}) = \mathcal{B}$, as well. It is worthwhile noting that the maximal copula $C(u,v) = \min\{u, v\}$ forms by itself an equivalence class.

2.2. *Semicopulæ associated to a joint survival function*

Given a survival function $\bar{F} \in \mathcal{F}$, we shall be interested mainly in three semicopulæ:

(1) The *survival copula* $K = K_{\bar{F}}$ given by

$$K(u,v) = \bar{F}(\bar{G}^{-1}(u), \bar{G}^{-1}(v)). \tag{13}$$

(2) The *multivariate aging function* (see [10]) $B = B_{\bar{F}}$ given by

$$B(u,v) = \exp\{-\bar{G}^{-1}(\bar{F}(-\log u, -\log v))\}. \tag{14}$$

(3) The Archimedean semicopula with (additive) generator given by $\bar{G}^{-1}$, the inverse of the one-dimensional survival function, namely,

$$A_{\bar{G}^{-1}}(u,v) = \bar{G}(\bar{G}^{-1}(u) + \bar{G}^{-1}(v)) = \Gamma^{-1}(\Gamma(u)\Gamma(v)) = C_\Gamma(u,v), \tag{15}$$

where, here and in the sequel, $\Gamma : [0,1] \to [0,1]$ is the function defined by

$$\Gamma(x) := \exp\{-\bar{G}^{-1}(x)\}, \quad \Gamma(0) = 0. \tag{16}$$

Clearly, Γ belongs to the set $\mathcal{H}$ and $A_{\bar{G}^{-1}} = C_\Gamma = \Psi_\Gamma S_0$.

Here are some simple considerations

- K is a copula; in fact, $\bar{G}(X_1)$ and $\bar{G}(X_2)$ are random variables with uniform distribution on $[0, 1]$, and K is their joint distribution function:

  $$K(u, v) = \mathbb{P}(\bar{G}(X_1) \leqslant u, \bar{G}(X_2) \leqslant v).$$

- For $\bar{F} \in \mathcal{F}$, we have, recalling (6),

  $$B = \Psi_{\Gamma^{-1}} K, \tag{17}$$

  $$K = \Psi_\Gamma B. \tag{18}$$

  Eq. (17) shows, in particular, that B is a semicopula, and that K and B belong to the same equivalence class.
- B is not necessarily a copula (see, e.g. Section 7 below). The interest in introducing the multivariate aging function B lies in the fact that B is a semicopula which describes the level curves of $\bar{F}$, regardless of what the marginal of $\bar{F}$ is. In fact, (x_1, x_2) and (x_1', x_2') belong to the same level curve of $\bar{F}$ (i.e. $\bar{F}(x_1, x_2) = \bar{F}(x_1', x_2')$) if and only if (e^{-x_1}, e^{-x_2}) and $(e^{-x_1'}, e^{-x_2'})$ belong to the same level curve of B.

In this respect, we note that, as mentioned in the Introduction, some notions of bivariate aging can be characterized in terms of level curves of $\bar{F}$.

- The semicopula $A_{\bar{G}^{-1}} = C_\Gamma$ will be used in Section 4 to describe univariate aging.
- $A_{\bar{G}^{-1}}$ is a copula if and only if $\bar{G}$ is convex.
- Clearly, $\bar{F}$ specifies $\bar{G}, K$ and B, in view of (1), (13), and (14). Conversely, each of the pairs $(K, \bar{G})$ and $(B, \bar{G})$ completely specifies $\bar{F}$. In fact, it is easy to verify that

  $$\bar{F}(\mathbf{x}) = K(\bar{G}(x_1), \bar{G}(x_2)), \tag{19}$$

  $$\bar{F}(\mathbf{x}) = \bar{G}(-\log B(e^{-x_1}, e^{-x_2})). \tag{20}$$

3. Some relevant families of semicopulæ

We shall consider in this section several families of semicopulæ which will play a relevant role in the description of aging and dependence. In most cases, the properties satisfied by the semicopulæ in the families under consideration can be viewed as formal extensions of some notions of positive and negative dependence to semicopulæ. For this reason, we shall often use the term *dependent families*. We confine ourselves to a few "dependence" concepts, which will provide the examples that we shall use to illustrate our method.

Define first

$$\mathcal{P}^1_+ := \{S \in \mathcal{S} | S(u,v) \geqslant uv\},$$

$$\mathcal{P}^1_- := \{S \in \mathcal{S} | S(u,v) \leqslant uv\}.$$

Clearly, copulæ in $\mathcal{P}^1_+$ are positively quadrant dependent (PQD), whereas copulæ in $\mathcal{P}^1_-$ are negatively quadrant dependent (NQD). We refer to [14] or [12] for details on PQD and other dependence concepts, such as TP_2, LTD, etc.

We shall consider subfamilies $\mathcal{P}_+$ of $\mathcal{P}^1_+$, and we shall refer to them as positively dependent families of semicopulæ. It should be observed that axiomatic definitions *à la* Kimeldorf and Sampson ([15]) cannot be applied automatically in this context. For the moment, we take the concept of a positive dependent family as a primitive one. We shall see in Section 5 which properties are actually needed for our approach.

Analogous considerations hold for negatively dependent semicopulæ. It should be pointed out that it is not clear what a canonical way of associating a negative dependent family of semicopulæ to a positively dependent one could be. In many special cases, though, there is an obvious answer (for example, it is clear that NQD corresponds to PQD). Hypotheses 3 and 4 below deal with this issue, and specify the conditions concerning the relations between positively and negatively dependent families that are needed in our setup.

We shall consider some subfamilies of $\mathcal{P}^1_+$. First of all, the family $\mathcal{P}^1_+$ itself. Then,

$$\mathcal{P}^2_+ := \left\{ S \in \mathcal{S} \left| \frac{S(u',v)}{u'} \leqslant \frac{S(u,v)}{u}, \forall\ 0 < u < u' \leqslant 1, \forall\ 0 \leqslant v \leqslant 1 \right.\right\}, \tag{21}$$

$$\mathcal{P}^3_+ := \{S \in \mathcal{S} | S(us,v) \geqslant S(u,sv), \forall\ 0 \leqslant v \leqslant u \leqslant 1, 0 < s < 1\}. \tag{22}$$

Copulæ in $\mathcal{P}^2_+$ are left tail decreasing (LTD). The family $\mathcal{P}^3_+$ has some relations with Schur-concavity, as we shall see below. The families $\mathcal{P}^2_-$ and $\mathcal{P}^3_-$ are defined in the obvious way.

We shall also consider the following family of Archimedean semicopulæ:

$$\mathcal{P}^4_+ = \left\{ A_\phi \in \mathcal{A} \left| \frac{\phi(v)}{\phi'(v)} \geqslant v \log v, \forall v \in (0,1) \right.\right\}.$$

Copulæ in this family are positively K dependent (PKD). See [2] for further details. The family $\mathcal{P}^4_-$ is defined in the obvious way, reversing the inequality.

An obvious but necessary remark is that the copula of independence S_0 belongs to $\mathcal{P}^j_+$ and to $\mathcal{P}^j_-$, for any $j = 1, \ldots, 4$.

4. Definitions of aging in terms of dependence

We first give some definitions in terms of families of semicopulæ. Some examples and remarks with possible motivations will follow.

Definition 4.1. Let $\mathcal{P} = \{\mathcal{P}_+, \mathcal{P}_-\}$, where $\mathcal{P}_+ \subset \mathcal{P}^1_+$ is a family of positively dependent semicopulæ, and $\mathcal{P}_- \subset \mathcal{P}^1_-$ is a family of negatively dependent semicopulæ. Let also $\bar{F} \in \mathcal{F}$.

(1) We say that $\bar{F}$ is $\mathcal{P}$-positively dependent if $K \in \mathcal{P}_+$.
(2) We say that $\bar{F}$ has $\mathcal{P}$-positive bivariate aging if $B \in \mathcal{P}_+$.
(3) We say that $\bar{F}$ has $\mathcal{P}$-negative univariate aging if $C_\Gamma \in \mathcal{P}_+$.

The corresponding definitions of negative dependence, negative 2-aging and positive 1-aging are obtained replacing $\mathcal{P}_+$ with $\mathcal{P}_-$.

Concerning these definitions, it is worthwhile noticing the following points:

- The survival copula K is a natural object for the description of the dependence of $\bar{F}$.
- There exists a natural relation between bivariate aging and *dependence* of the function B (see also the discussion and examples presented in [10]).
- We describe an aging property of $\bar{G}$ by means of a condition on the semicopula C_Γ. The idea of describing a *univariate* property of $\bar{G}$ by means of a *bivariate* condition turns out to be particularly adequate for the present approach; the relations between *negative* aging of $\bar{G}$ and *positive* dependence of C_Γ were already established and studied in [2] and related ideas can be also found in [23] and [19]. More on this issue below.

Hereafter, the concepts above will be illustrated by considering in detail the cases of the dependent families listed in Section 3. First it is convenient to focus attention on three remarkable limiting cases.

Example 4.1. Recalling Eq. (9), the condition $K = S_0$ is equivalent to stochastic independence of X_1, X_2 (no dependence).

The condition $C_\Gamma = S_0$ means that the one-dimensional marginal distributions of $\bar{F}$ are standard exponential (no 1-aging).

Finally, the condition $B = S_0$ is equivalent to the condition

$$\bar{F}(x_1, x_2) = \bar{G}(x_1 + x_2). \tag{23}$$

Functions satisfying (23) are both Schur-concave and Schur-convex, and have been called *Schur-constant* in Bayesian reliability (see, e.g. [4,23]; see also [17] for general aspects of the concept of Schur-concavity). The interest of this property in reliability lies in the fact that it is equivalent to the requirement that, for every $\tau > 0$ and

$x_1, x_2 \geqslant 0$, the following identity holds:

$$P\{X_1 > x_1 + \tau | X_1 > x_1, X_2 > x_2\} = P\{X_2 > x_2 + \tau | X_1 > x_1, X_2 > x_2\}. \tag{24}$$

This corresponds to a condition of no 2-aging. In fact, the relation above states that, conditionally on a same history of survivals, the marginal laws of the residual lifetimes $(X_1 - x_1)$, $(X_2 - x_2)$, of the younger and of the elder component are identical. It can be shown that the two residual lifetimes are actually exchangeable. See also the brief discussion in Example 4.4 below.

Example 4.2. Let $\mathcal{P}_+ \equiv \mathcal{P}^1_+$, $\mathcal{P}_- \equiv \mathcal{P}^1_-$. The condition $K \in \mathcal{P}_+$ is equivalent to saying that (X_1, X_2) is *PQD*, i.e.

$$P\{X_1 > x_1, X_2 > x_2\} \geqslant P\{X_1 > x_1\} P\{X_2 > x_2\}$$

or

$$\bar{F}(x_1, x_2) \geqslant \bar{G}(x_1)\bar{G}(x_2);$$

the inequalities are reversed for $K \in \mathcal{P}_-$.

In view of Eq. (14), it is easy to check that $B \in \mathcal{P}_+$ means

$$\bar{F}(x_1, x_2) \geqslant \bar{G}(x_1 + x_2), \quad \forall x_1, x_2 \geqslant 0.$$

In turn, the latter inequality can be rewritten as

$$P\{X_1 > x + \tau | X_1 > x\} \leqslant P\{X_2 > \tau | X_1 > x\}, \tag{25}$$

that can be interpreted as a bivariate notion of NBU (new better than used); in fact (see [9]), consider a new item with lifetime X_2 and a used one with lifetime X_1: conditionally on the knowledge of the age of the used item, survival probabilities of the new item are greater than the survival probabilities of the used one. Similarly, $B \in \mathcal{P}_-$ can be interpreted as a bivariate notion of NWU (new worse than used).

Finally, note that $C_\Gamma \in \mathcal{P}_+$ is a condition of negative univariate aging. In fact, by using the arguments in [2], it is easy to check that it is equivalent to the fact that the lifetimes X_1, X_2 are marginally NWU.

Example 4.3. Consider now the case $\mathcal{P}_+ \equiv \mathcal{P}^2_+$, $\mathcal{P}_- \equiv \mathcal{P}^2_-$. In view of (13), the condition $K \in \mathcal{P}^2_+$ (K LTD copula) means that $P\{X_2 > x_2 | X_1 > x_1\}$ is an increasing function of x_1, for any given $x_2 > 0$.

The condition $C_\Gamma \in \mathcal{P}^2_+$ is a condition of negative 1-aging; in fact, it is easily seen to be equivalent to the marginal DFR property of X_1, X_2 (see [2]).

Example 4.4. The condition $C_\Gamma \in \mathcal{P}^3_+$ corresponds (see Proposition 6.1) to the marginal DFR property of X_1, X_2; in other words, it is equivalent to $C_\Gamma \in \mathcal{P}^2_+$.

As far as the condition $B \in \mathcal{P}^3_+$ is concerned, we shall see in Lemma 4.2 that it is equivalent to Schur-concavity of the joint survival function $\bar{F}$:

$$\bar{F}(x_1 + \tau, x_2 - \tau) \geqslant \bar{F}(x_1, x_2), \quad \forall x_1 < x_2, \; 0 \leqslant \tau \leqslant x_2 - x_1. \tag{26}$$

The condition of Schur-concavity of $\bar{F}$ can be seen as a significant notion of positive 2-aging. In fact, it has been considered in, e.g. [4,5,7,22] as a multivariate notion of IFR. Note that such condition holds for the case of two i.i.d. IFR lifetimes and for the case of two lifetimes that are conditionally i.i.d. IFR. More generally, as it is easy to see, it is equivalent to the following condition: for $\tau > 0$ and $0 \leqslant x_1 < x_2$,

$$P\{X_1 > x_1 + \tau | X_1 > x_1, X_2 > x_2\} \geqslant P\{X_2 > x_2 + \tau | X_1 > x_1, X_2 > x_2\}, \tag{27}$$

i.e. given the joint survival data $\{X_1 > x_1, X_2 > x_2\}$, survival probabilities are greater for the younger component than for the elder. A limiting case of Schur-concavity for $\bar{F}$ is the condition of no-aging (24) considered in Example 4.1. For further details, see [23].

Lemma 4.2. *A survival function $\bar{F} \in \mathcal{F}$ is Schur-concave if and only if $B_{\bar{F}} \in \mathcal{P}^3_+$.*

Proof. We begin by observing that it is easy to check that the class $\mathcal{P}^3_+$ can be characterized also in the following way:

$$\mathcal{P}^3_+ = \left\{ S \in \mathcal{S} | S(u, v) \leqslant S\left(us, \frac{v}{s}\right),\ \forall 0 \leqslant v \leqslant u \leqslant 1, s \geqslant \frac{v}{u} \right\}. \tag{28}$$

Next, from (20) note that $\bar{F}$ is Schur-concave if and only if

$$\bar{G}[-\log B(e^{-x-\tau}, e^{-y+\tau})] \geqslant \bar{G}[-\log B(e^{-x}, e^{-y})] \tag{29}$$

for $x < y, \tau < y - x$. Since $\bar{G}$ is a decreasing function, (29) becomes

$$B(e^{-x}e^{-\tau}, e^{-y}e^{\tau}) \geqslant B(e^{-x}, e^{-y})$$

and, since $e^{-x} > e^{-y}$, $e^{-\tau} > \frac{e^{-y}}{e^{-x}}$, the proof is completed by putting

$$u = e^{-x}, \quad v = e^{-y}, \quad s = e^{-\tau}. \qquad \square$$

The conditions $K \in \mathcal{P}^4_+$, $B \in \mathcal{P}^4_+$, $C_\Gamma \in \mathcal{P}^4_+$ will be analyzed in detail in Section 7. Here are some further considerations about Definition 4.1.

Remark 4.3. Definition 4.1, point (3), shows that to every notion of dependence there corresponds a notion of univariate aging. However, one should not think that to every notion of dependence there corresponds a different notion of 1-aging. In fact, 1-aging can be defined through Archimedean semicopulæ, and different notions of positive dependence may agree on the set of Archimedean semicopulæ. For example, Proposition 6.1 below shows that the same notion of 1-aging, namely, DFR, corresponds to (at least) three notions of dependence.

Remark 4.4. Let

$$\bar{H}(x, y) = \bar{G}(x + y).$$

The function $\bar{H}$ is Schur-constant, and it is a bivariate survival function iff $\bar{G}$ is convex. Clearly, $\bar{H}$ has the same 1-marginal as $\bar{F}$, namely, $\bar{G}$. It is easy to check that the Archimedean (semi)copula C_Γ is the survival (semi)copula corresponding to $\bar{H}$, namely, $K_{\bar{H}} = C_\Gamma$.

As mentioned before, the fact that negative aging of $\bar{G}$ corresponds to positive dependence of C_Γ has been studied in [2]; besides what reported in this respect in Examples 4.2–4.4 above, it was also shown in [2] that $\bar{G}$ is DFRA iff C_Γ is PKD.

We can also illustrate this general fact as follows, focusing attention on a typical special case of positive dependence: Let $\bar{G}$ be convex and let $(\tilde{X}, \tilde{Y}) \sim \bar{H}$. Let us compare, for $a > 0$,

$$\mathbb{P}(\tilde{Y} > y + t | \tilde{X} > x, \tilde{Y} > y) \quad \text{and} \quad \mathbb{P}(\tilde{Y} > y + t | \tilde{X} > x + a, \tilde{Y} > y).$$

If $(\tilde{X}, \tilde{Y})$ are positively dependent, e.g. if we require that the second term be bigger than the first one, then

$$\frac{\bar{H}(x+a, y+t)}{\bar{H}(x+a, y)} = \frac{\bar{G}(x+y+a+t)}{\bar{G}(x+y+a)} \geqslant \frac{\bar{G}(x+y+t)}{\bar{G}(x+y)} = \frac{\bar{H}(x, y+t)}{\bar{H}(x, y)}. \tag{30}$$

But (30) states that $\bar{G}$ has negative univariate aging (it says it is DFR).

Remark 4.5. Let $\bar{M} = \Gamma \circ \bar{F}$, i.e.

$$\bar{M}(x, y) = \Gamma(\bar{F}(x, y)).$$

The function $\bar{M}$ has all the properties of a survival function, except possibly for the fact that it need not satisfy the rectangular inequality. Correspondingly, the function $K_{\bar{M}}$ defined formally as in (13) need not be a copula, but it is in general a semicopula. It is not hard to check that $\bar{M}$ has standard exponential marginals (hence, it displays no 1-aging), and has the same bivariate aging as $\bar{F}$, namely, $B_{\bar{F}} = B_{\bar{M}}$. Furthermore, $\bar{M}$ satisfies $K_{\bar{M}} = B_{\bar{M}}$, which is consistent with our previous assertion that in absence of 1-aging, the only source of 2-aging is dependence.

Thus, we may rephrase the definition of $\mathcal{P}$-positive bivariate aging as follows: $\bar{F}$ has $\mathcal{P}$-positive 2-aging iff the function $\bar{M}$ with the same level sets as $\bar{F}$ and standard exponential marginals has a survival (semi)copula in $\mathcal{P}_+$.

5. Relations among 1-aging, 2-aging, and dependence: some conditions and basic results

We define here a basic notion of closure which will be needed in the sequel, and we single out some properties of families of semicopulæ needed to state and prove the main results of this note. Later, we shall show that the families $\mathcal{P}_+^j, \mathcal{P}_-^j, j = 1, \ldots, 4$, considered so far satisfy these properties.

For $\mathcal{P}\subset\mathcal{S}$, let $\mathcal{H}_{\mathcal{P}} = \{h\in\mathcal{H}|C_h\in\mathcal{A}\cap\mathcal{P}\}$. Recall that C_h was defined in (10) and that $\mathcal{A}$ denotes the family of Archimedean semicopulæ; recall also that we denote by h_0 the identity mapping on $[0, 1]$; thus $C_{h_0}(u, v) = S_0(u, v) = uv$.

Definition 5.1. A family $\mathcal{P}\subset\mathcal{S}$, containing S_0 (i.e. such that $h_0\in\mathcal{H}_{\mathcal{P}}$), is said to be Ψ-closed if

$$\Psi_{\mathcal{H}_{\mathcal{P}}}(\mathcal{P}) = \mathcal{P}. \tag{31}$$

Thus, a family containing the independent copula is $\mathcal{P}$-closed if the following implication holds: if $S\in\mathcal{P}$ and if h is such that $C_h\in\mathcal{P}$, then $\Psi_h S\in\mathcal{P}$.

Clearly, for every semicopula C we have $\Psi_{h_0}C = C$. Hence $\Psi_{\mathcal{H}_{\mathcal{P}}}(\mathcal{P})\supset\mathcal{P}$ always holds, and in order to verify the Ψ-closure of a family $\mathcal{P}$ it is enough to check that the opposite inclusion holds.

5.1. Main hypotheses

For what follows, we need to consider the following conditions on a family $\mathcal{P} = \{\mathcal{P}_+, \mathcal{P}_-\}$, where $\mathcal{P}_+\subset\mathcal{P}^1_+$ and $\mathcal{P}_-\subset\mathcal{P}^1_-$.

Hypothesis 1. *The family $\mathcal{P}_+$ is Ψ-closed.*

Hypothesis 2. *The family $\mathcal{P}_-$ is Ψ-closed.*

The following two conditions require a relation between $\mathcal{P}_+$ and $\mathcal{P}_-$.

Hypothesis 3. *$h\in\mathcal{H}_{\mathcal{P}_+}$ if and only if $h^{-1}\in\mathcal{H}_{\mathcal{P}_-}$.*

Hypothesis 4. *The following implication holds*:

$$S\in\mathcal{P}_-, \Psi_h S\in\mathcal{P}_+ \Rightarrow h\in\mathcal{H}_{\mathcal{P}_+}.$$

5.2. Relations among dependence, 1- and 2-aging

We are now ready to state and prove some results about the relations among dependence, 1- and 2-aging.

Proposition 5.2. *Let $\mathcal{P} = \{\mathcal{P}_+, \mathcal{P}_-\}$, and let $\bar{F}\in\mathcal{F}$.*

(1) *Assume Hypotheses* 1 and 3. *If $\bar{F}$ has $\mathcal{P}$-positive* 1*-aging and $\mathcal{P}$-positive dependence*, *then it has $\mathcal{P}$-positive* 2*-aging.*
(2) *Assume Hypotheses* 2 *and* 3. *If $\bar{F}$ has $\mathcal{P}$-negative* 1*-aging and $\mathcal{P}$-negative dependence*, *then it has $\mathcal{P}$-negative* 2*-aging.*

Proof. We shall prove only the first claim, since the second one can be dealt with in a similar way. The assumptions mean that $C_\Gamma \in \mathcal{P}_-$ and $K \in \mathcal{P}_+$. By Hypothesis 3, $C_{\Gamma^{-1}} \in \mathcal{P}_+$. It follows immediately that $B = \Psi_{\Gamma^{-1}} K \in \mathcal{P}_+$, since $\mathcal{P}_+$ is Ψ-closed. $\square$

Proposition 5.3. *Let* $\mathcal{P} = \{\mathcal{P}_+, \mathcal{P}_-\}$, *and let* $\bar{F} \in \mathcal{F}$:

(1) *Assume Hypothesis* 1. *If* $\bar{F}$ *has* $\mathcal{P}$-*positive* 2-*aging and* $\mathcal{P}$-*negative* 1-*aging*, *then it is* $\mathcal{P}$-*positively dependent*.
(2) *Assume Hypothesis* 2. *If* $\bar{F}$ *has* $\mathcal{P}$-*negative* 2-*aging and* $\mathcal{P}$-*positive* 1-*aging*, *then it is* $\mathcal{P}$-*negatively dependent*.

Proof. Again, we shall prove only the first claim. The assumptions mean that $B \in \mathcal{P}_+$ and $C_\Gamma \in \mathcal{P}_+$. Hence we have $K = \Psi_\Gamma B \in \mathcal{P}_+$. $\square$

Proposition 5.4. *Let* $\mathcal{P} = \{\mathcal{P}_+, \mathcal{P}_-\}$, *and let* $\bar{F} \in \mathcal{F}$:

(1) *Assume Hypotheses* 3 *and* 4. *If* $\bar{F}$ *has* $\mathcal{P}$-*negative dependence and* $\mathcal{P}$-*positive* 2-*aging*, *then it has* $\mathcal{P}$-*positive* 1-*aging*.
(2) *Assume Hypothesis* 4. *If* $\bar{F}$ *has* $\mathcal{P}$-*negative* 2-*aging and* $\mathcal{P}$-*positive dependence*, *then it has* $\mathcal{P}$-*negative* 1-*aging*.

Proof. Consider the first claim. We have: $K \in \mathcal{P}_-$ and $B = \Psi_{\Gamma^{-1}} K \in \mathcal{P}_+$. By Hypothesis 4, we have $C_{\Gamma^{-1}} \in \mathcal{P}_+$, and hence $C_\Gamma \in \mathcal{P}_-$, by Hypothesis 3.

The second claim is similar. We have: $B \in \mathcal{P}_-$ and $K = \Psi_\Gamma B \in \mathcal{P}_+$. Then $C_\Gamma \in \mathcal{P}_+$. $\square$

6. Examples and applications

In this section we shall consider some of the specific conclusions that can be drawn by applying Propositions 5.2–5.4 to the families of semicopulæ considered so far.

6.1. Examples of families which satisfy the main hypotheses

6.1.1. Families of Archimedean semicopulæ

We begin our presentation of examples with the analysis of the families $\mathcal{P}_+^j \cap \mathcal{A}$, $j = 1, \dots, 4$. Most of the results in this section could be derived as special cases of next section, in which we will deal with $\mathcal{P}_+^j$, $j = 1, 2, 3$, but we decided to present them in this order for several reasons. In fact, the proofs in the case of Archimedean semicopulæ are simpler and more straightforward; moreover, the family $\mathcal{P}_+^4 = \mathcal{P}_+^4 \cap \mathcal{A}$, is not covered in next section, and the proofs for $\mathcal{P}_+^4$ are analogous to those for $\mathcal{P}_+^j \cap \mathcal{A}$, $j = 1, 2, 3$. Finally, the proof that $\mathcal{P}_+^j \cap \mathcal{A}$, $j = 1, 2, 3$, satisfy Hypothesis 3 will be taken verbatim into next section.

First, it is interesting to note that different dependent families may agree on the set of Archimedean semicopulæ. In fact, the following proposition shows in particular that $\mathcal{P}^2_+ \cap \mathcal{A} = \mathcal{P}^3_+ \cap \mathcal{A}$.

Proposition 6.1. *Let ϕ be an additive generator of an Archimedean semicopula, and let $h = e^{-\phi}$ be the corresponding multiplicative generator. The following are equivalent*:

(1) ϕ^{-1} *is log-convex, i.e. it is a DFR survival function.*
(2) C_h *is* TP_2.
(3) $C_h \in \mathcal{P}^2_+$, *i.e. it is LTD.*
(4) $C_h \in \mathcal{P}^3_+$.

Proof. The equivalence of (1) and (2) can be established by considering the following chain of double implications: ϕ^{-1} is log-convex iff

$$\log \phi^{-1}(x+y) - \log \phi^{-1}(x+y') \geqslant \log \phi^{-1}(x'+y) - \log \phi^{-1}(x'+y'),$$
$$\forall x < x', \ y < y',$$

iff

$$\phi^{-1}(x+y)\phi^{-1}(x'+y') \geqslant \phi^{-1}(x'+y)\phi^{-1}(x+y'),$$

i.e. $\bar{H}(x,y) := \phi^{-1}(x+y)$ is TP_2. This is easily seen to be equivalent to the TP_2 property of A_ϕ, i.e. of C_h.

The equivalence between (1) and (3) was proved in [2]. Finally, we establish the equivalence of (1) and (4). Write $\phi^{-1} = \exp\{-\gamma\}$. Then, observing that $\phi(x) = \gamma^{-1}(-\log x)$, we note that ϕ^{-1} is log-convex iff γ^{-1} is convex iff, for every $0 < v \leqslant u \leqslant 1$ and every $0 < s < 1$,

$$\gamma^{-1}(-\log u - \log s) - \gamma^{-1}(-\log u) \leqslant \gamma^{-1}(-\log v - \log s) - \gamma^{-1}(-\log v),$$

iff

$$\phi(us) - \phi(u) \leqslant \phi(vs) - \phi(v),$$

iff

$$\phi^{-1}(\phi(us) + \phi(v)) \geqslant \phi^{-1}(\phi(u) + \phi(vs)). \qquad \square$$

For the remaining part of this section, we shall not refer to $\mathcal{P}^2_+ \cap \mathcal{A}$ anymore, since it coincides with $\mathcal{P}^3_+ \cap \mathcal{A}$.

Next, we show that, in the case of families of Archimedean semicopulæ, the main hypotheses are not independent of one another.

Proposition 6.2. *Let $\mathcal{P} = \{\mathcal{P}_+, \mathcal{P}_-\}$, where $\mathcal{P}_+ \subset \mathcal{P}^1_+$ and $\mathcal{P}_- \subset \mathcal{P}^1_-$ are families of Archimedean semicopulæ. If $\mathcal{P}$ satisfies Hypotheses* 1 *and* 3, *then it satisfies also Hypotheses* 2 *and* 4.

Proof. Let us begin with Hypothesis 2. Let $C_g, C_h \in \mathcal{P}_-$. Then

$$\Psi_h C_g = C_{g \circ h} = C_{(h^{-1} \circ g^{-1})^{-1}}.$$

Now, $C_{h^{-1}}, C_{g^{-1}} \in \mathcal{P}_+$. Hence $C_{h^{-1} \circ g^{-1}} = \Psi_{g^{-1}} C_{h^{-1}} \in \mathcal{P}_+$, and the conclusion follows.

Next, let us consider Hypothesis 4. Let $C_g \in \mathcal{P}_-$ and let $\Psi_h C_g = C_{g \circ h} \in \mathcal{P}_+$. Then $C_{g^{-1}} \in \mathcal{P}_+$ and

$$C_h = C_{g^{-1} \circ g \circ h} = \Psi_{g \circ h} C_{g^{-1}} \in \mathcal{P}_+. \quad \square$$

Before we turn our attention to the specific families under consideration, we need to state the following Proposition 6.3, which was proved essentially in [2]. It establishes relations between certain conditions of dependence of an Archimedean (semi)copula and suitable properties of the generator. See also [10]. Here, we state the result in terms of multiplicative generators.

Given a multiplicative generator h, let

$$\eta_h(x) := -\log h(e^{-x}) = \phi(e^{-x}).$$

The following properties are easy to verify and shall be needed in the sequel:

$$\eta_{h' \circ h} = \eta_{h'} \circ \eta_h,$$

$$\eta_h^{-1}(x) = \eta_{h^{-1}}(x).$$

Incidentally, observe also the following: if $\phi = \bar{G}^{-1}$ and $\bar{G}(x) = \exp\{-R(x)\}$, then $\Gamma(x) = \exp\{-R^{-1}(-\log x)\}$ and $\eta_\Gamma = R^{-1}$.

Proposition 6.3. *Let C_h be an Archimedean semicopula with multiplicative generator h.*

(1) *$C_h \in \mathcal{P}^1_+ \cap \mathcal{A}$ if and only if η_h is superadditive.*
(2) *$C_h \in \mathcal{P}^2_+ \cap \mathcal{A} = \mathcal{P}^3_+ \cap \mathcal{A}$ if and only if η_h is convex.*
(3) *$C_h \in \mathcal{P}^4_+$ if and only if η_h is star-shaped.*

Similar statements hold for the corresponding negatively dependent families.

We are now ready to show that the families $\mathcal{P}^j \cap \mathcal{A},\ j = 1, 3, 4$ satisfy Hypotheses 1 and 2.

Proposition 6.4. *The families $\mathcal{P}^1_+ \cap \mathcal{A}$, $\mathcal{P}^3_+ \cap \mathcal{A}$ and $\mathcal{P}^4_+$ are Ψ-closed.*

Proof. Let h, h' be two multiplicative generators. If $\eta_{h'}$ and η_h are increasing and superadditive (respectively, star-shaped, convex), then so is $\eta_{h' \circ h} = \eta_{h'} \circ \eta_h$, since these families of functions are closed under composition. The conclusion follows recalling that $\Psi_h C_{h'} = C_{h' \circ h}$. $\square$

Next, we deal with Hypothesis 3.

Proposition 6.5. *For $j = 1, 3, 4$, we have*

$$C_h \in \mathcal{P}_+^j \cap \mathcal{A} \;\Leftrightarrow\; C_{h^{-1}} \in \mathcal{P}_-^j \cap \mathcal{A}.$$

Proof. The fact that the following statements are equivalent yields the desired result.

(1) $C_h \in \mathcal{P}_+^j \cap \mathcal{A}$, with $j = 1$ (respectively, $j = 3, 4$),
(2) η_h is increasing and superadditive (respectively, convex, star-shaped),
(3) $\eta_h^{-1} = \eta_{h^{-1}}$ is increasing and subadditive (respectively, concave, anti-star-shaped),
(4) $C_{h^{-1}} \in \mathcal{P}_-^j \cap \mathcal{A}$, with $j = 1$ (respectively, $j = 3, 4$). $\square$

6.1.2. The families $\mathcal{P}^1$, $\mathcal{P}^2$, $\mathcal{P}^3$

In this section we shall see that all the families considered so far satisfy the main hypotheses.

Proposition 6.6. *The families $\mathcal{P}_+^1$, $\mathcal{P}_+^2$, $\mathcal{P}_+^3$, $\mathcal{P}_-^1$, $\mathcal{P}_-^2, \mathcal{P}_-^3$ are Ψ-closed.*

Proof. We prove the result only for the positive dependent families. The other case is similar

Let $S, C_h \in \mathcal{P}_+^1$. Then

$$\Psi_h S(u, v) = h^{-1}(S(h(u), h(v))) \geqslant h^{-1}(h(u)h(v)) = C_h(u, v) \geqslant uv.$$

Next, let $S, C_h \in \mathcal{P}_+^2$. For $u < u'$ we have

$$\begin{aligned}\frac{\Psi_h S(u', v)}{u'} &= \frac{1}{u'} h^{-1}(S(h(u'), h(v))) = \frac{1}{u'} h^{-1}\left(h(u')\frac{S(h(u'), h(v))}{h(u')}\right) \\ &\leqslant \frac{1}{u'} h^{-1}\left(h(u')\,\frac{S(h(u), h(v))}{h(u)}\right) = \frac{1}{u'} h^{-1}(h(u')h(\tilde{u})) \\ &= \frac{C_h(u', \tilde{u})}{u'},\end{aligned}$$

where $\tilde{u}$ is such that

$$h(\tilde{u}) = \frac{S(h(u), h(v))}{h(u)}.$$

Hence, we get

$$\frac{\Psi_h S(u',v)}{u'} \leqslant \frac{C_h(u',\tilde{u})}{u'} \leqslant \frac{C_h(u,\tilde{u})}{u} = \frac{1}{u} h^{-1}\left(h(u)\frac{S(h(u),h(v))}{h(u)}\right)$$
$$= \frac{1}{u} h^{-1}(S(h(u),h(v))) = \frac{\Psi_h S(u,v)}{u}.$$

Next, let $S, C_h \in \mathcal{P}^3_+$. Let $0 \leqslant v \leqslant u \leqslant 1$ and $0 < s < 1$. First, it is easy to check that $C_h \in \mathcal{P}^3_+$ if and only if

$$x \mapsto \frac{h(xs)}{h(x)}$$

is increasing. Hence

$$\Psi_h S(us,v) = h^{-1}(S(h(us),h(v))) = h^{-1}\left(S\left(h(u)\frac{h(us)}{h(u)}, h(v)\right)\right)$$
$$\geqslant h^{-1}\left(S\left(h(u), \frac{h(us)}{h(u)}h(v)\right)\right)$$
$$\geqslant h^{-1}\left(S\left(h(u), \frac{h(vs)}{h(v)}h(v)\right)\right) = \Psi_h S(u,vs). \qquad \square$$

As far as Hypothesis 3 is concerned, the desired result has already been proved in Section 6.1.1. Hence, we turn our attention to Hypothesis 4.

Proposition 6.7. *The families* $\mathcal{P}^1$, $\mathcal{P}^2$, $\mathcal{P}^3$ *satisfy Hypothesis* 4.

Proof. *The case* $j = 1$. Let S be NQD and let $\Psi_h S$ be PQD. Then

$$uv \leqslant h^{-1}(S(h(u),h(v)) \leqslant h^{-1}(h(u)h(v)) = C_h(u,v).$$

The case $j = 2$. Let $0 < u < u' \leqslant 1$, and let $0 \leqslant v \leqslant 1$. Let also $\tilde{v}$ be such that

$$h(v) = \frac{S(h(u),h(\tilde{v}))}{h(u)},$$

i.e. such that $C_h(u,v) = \Psi_h S(u,\tilde{v})$. Then

$$\frac{C_h(u',v)}{u'} = \frac{1}{u'} h^{-1}(h(u')h(v)) = \frac{1}{u'} h^{-1}\left(h(u')\frac{S(h(u),h(\tilde{v}))}{h(u)}\right)$$
$$\leqslant \frac{1}{u'} h^{-1}\left(h(u')\frac{S(h(u'),h(\tilde{v}))}{h(u')}\right) = \frac{1}{u'}\Psi_h S(u',\tilde{v})$$
$$\leqslant \frac{1}{u}\Psi_h S(u,\tilde{v}) = \frac{C_h(u,v)}{u},$$

as we wanted to prove. Observe that the first inequality is due to the fact that $S \in \mathcal{P}^2_-$, and the second one to the fact that $\Psi_h S \in \mathcal{P}^2_+$.

The case $j=3$. Let $0<v\leqslant u\leqslant 1$ and $0<s<1$. We assume

$$S(us,v)\leqslant S(u,sv),$$

$$h^{-1}(S(h(us),h(v)))\geqslant h^{-1}(S(h(u),h(sv))).$$

Write $\tilde{v}:=h(v)$, $\tilde{u}:=h(u)$ and $\tilde{s}:=h(us)/h(u)$. Then, recalling that h^{-1} is increasing, the two assumptions together yield:

$$S(\tilde{u},\tilde{s}\tilde{v})\geqslant S(\tilde{u}\tilde{s},\tilde{v})=S(h(us),h(v))\geqslant S(\tilde{u},h(sv)).$$

It follows that

$$h(sv)\leqslant \tilde{s}\tilde{v}=\frac{h(us)h(v)}{h(u)},$$

and hence

$$h^{-1}(h(us)h(v))\geqslant h^{-1}(h(u)h(sv)),$$

as we wanted to prove. □

6.2. Applications

The results in Section 6.1 guarantee that Propositions 5.2–5.4 can be applied to each of the cases $\mathcal{P}^j$ $(j=1,\ldots,4)$. Taking into account the considerations contained in Examples 4.2–4.4, here we spell out some of the specific conclusions, that are implied by those propositions, relatively to the cases $j=1,2,3$. The case $\mathcal{P}^4$ will be considered in detail in the next section.

In each example we consider a pair of exchangeable lifetimes X_1,X_2.

Example 6.1. Let X_1,X_2 have NBU marginal distribution and PQD dependence, i.e. $C_\Gamma\in\mathcal{P}^1_-$ and $K\in\mathcal{P}^1_+$. Then, by Proposition 5.2, $B\in\mathcal{P}^1_+$; the latter is equivalent to inequality (25), which has a natural interpretation as a bivariate NBU property for (X_1,X_2) (see Example 4.2).

Similarly we can obtain that the opposite inequality

$$P\{X_1>x+\tau|X_1>x\}\geqslant P\{X_2>\tau|X_1>x\}$$

is implied by the conditions that X_i $(i=1,2)$ is NWU and the joint distribution is NQD.

If (25) holds, and if the NQD property of (X_1,X_2) is assumed, then, by Proposition 5.4, we obtain that X_i is NBU.

Example 6.2. Let X_1,X_2 have IFR marginal distribution, and let the survival copula be LTD, i.e. $P\{X_2>x_2|X_1>x\}$ is an increasing function of x (see Example 4.3). This means that $C_\Gamma\in\mathcal{P}^2_-$ and $K\in\mathcal{P}^2_+$. Then, by applying Proposition 5.2, we get the condition $B \in \mathcal{P}^2_+$, i.e. by the definition given in (21), B is such that

$$\frac{B(u',v)}{u'} \leqslant \frac{B(u,v)}{u}, \quad \forall 0<u<u' \leqslant 1, \ \forall 0 \leqslant v \leqslant 1.$$

Taking into account Eq. (14), we obtain that the following inequality holds:

$$\bar{G}^{-1}[\bar{F}(x',y)] - x' \leqslant \bar{G}^{-1}[\bar{F}(x,y)] - x, \ \ x<x'.$$

This inequality amounts to the Lipschitz condition for the function

$$h(x,y) := \bar{G}^{-1}[\bar{F}(x,y)] = -\log B(e^{-x}, e^{-y}),$$

which has been considered in [7].

Next example, in particular, shows sufficient conditions for a joint distribution with IFR marginals to have a Schur-concave joint survival function.

Example 6.3. Recalling Lemma 4.2, we know that Schur-concavity of the joint survival function is equivalent to the condition $B \in \mathcal{P}^3_+$. We also know that the condition $C_\Gamma \in \mathcal{P}^3_-$ amounts to the IFR property for the marginal distribution of X_1, X_2. Notice now that, by the definition of $\mathcal{P}^3_+$, the condition $K \in \mathcal{P}^3_+$ becomes

$$\bar{F}(\bar{G}^{-1}(su), \bar{G}^{-1}(v)) \geqslant \bar{F}(\bar{G}^{-1}(u), \bar{G}^{-1}(sv)), \quad 0 \leqslant v \leqslant u \leqslant 1, \ 0<s<1. \tag{32}$$

By applying Proposition 5.2 we can get the following conclusion: *Let X_1, X_2 have IFR marginal distribution and let inequality* (32) *hold. Then $\bar{F}$ is Schur-concave*, (i.e. inequalities (26) and (27) hold). Note that condition (27) is stronger than (25). In fact, the latter only requires that the inequality in (27) hold for $x_2 = 0$.

Example 6.4. From Proposition 5.4 we obtain sufficient conditions on a Schur-concave survival function in order to have IFR 1-marginals. In fact, let $\bar{F} \in \mathcal{F}$ be Schur-concave. If $K_{\bar{F}} \in \mathcal{P}^3_-$, i.e. if

$$\bar{F}(\bar{G}^{-1}(su), \bar{G}^{-1}(v)) \leqslant \bar{F}(\bar{G}^{-1}(u), \bar{G}^{-1}(sv)), \quad 0 \leqslant v \leqslant u \leqslant 1, \ 0<s<1.$$

then the 1-marginal $\bar{G}$ is IFR.

Remark 6.8. In this remark, we turn our attention to some of the implications that do not hold. Here are some examples:

- Positive 1-aging and positive 2-aging need not imply positive dependence. In fact, a counterexample is given in [11], Remark 18, where it is shown that

  $$\bar{F}(x,y) = \exp\{1 - e^{x^2+y^2}\}$$

  displays positive 1-aging, positive 2-aging and negative dependence.
- Positive 2-aging need not generally imply positive 1-aging; in particular positive 2-aging and positive dependence need not imply positive 1-aging. This can be easily

seen, e.g., by taking into account the well known fact that a mixture of IFR distributions need not be IFR: let $T_1,\ T_2$ be conditionally i.i.d. IFR given a parameter Θ; then the joint survival function of $T_1,\ T_2$ is Schur-concave (and hence it has $\mathcal{P}^3$-positive 2-aging). It is easy to find conditional distributions (of T_i given Θ) and a marginal distribution for Θ, such that the marginal distribution of T_i is not IFR. For example assume that the law of Θ is a standard exponential, and that the conditional marginal law of $T_1,\ T_2$ is a Weibull $(\theta, 2)$ (which is an IFR law). Then $T_1,\ T_2$ are positively dependent, and their joint (conditional and unconditional) survival function is Schur-concave, i.e. we have positive bivariate aging. However, one can check that the unconditional law is not even NBU; see [11], Remark 7, for details.

7. Time-transformed exponential (TTE) models

In this section we focus our attention on joint exchangeable probability distributions for (X_1, X_2), characterized by the following conditions:

- the survival copula is Archimedean, i.e. $K \in \mathcal{K} \cap \mathcal{A}$.
- $\bar{G} \in \mathcal{G}$.

Here we shall denote the generic element of $\mathcal{K} \cap \mathcal{A}$ as

$$K(u, v) = W[W^{-1}(u) + W^{-1}(v)],$$

where W is a one-dimensional convex survival function belonging to $\mathcal{G}$.

By using (19), we see that the joint survival function is of the form

$$\bar{F}(x_1, x_2) = W[R(x_1) + R(x_2)], \tag{33}$$

where $R : [0, +\infty) \to [0, +\infty)$ is the increasing function defined by $R \equiv W^{-1} \circ \bar{G}$, $\bar{G}$ being the one-dimensional marginal survival function of $\bar{F}$.

In terms of W and R, the marginal survival function is $\bar{G} = W \circ R$.

From (14) it is immediately seen that the corresponding aging function B is an Archimedean semicopula, with additive generator given by $\phi(x) = R(-\log x)$; then B turns out to be a copula if and only if $R \circ (-\log)$ is convex, and, in particular, if R is convex.

To designate distributions with joint survival functions of the form (33), we use the term TTE *models* and the symbol $TTE(W, R)$.

Of course the class of TTE models is a quite restricted one; yet, it contains the remarkable models listed hereafter:

(1) The case of i.i.d. lifetimes, described by the relation $W(x) = \exp\{-x\}$.
(2) The Schur-constant case; this is obtained for $R(x) = x$.
(3) *Proportional hazard models*, of the type

$$\bar{F}(x_1, x_2) = \int_0^{+\infty} \exp\{-\theta[R(x_1) + R(x_2)]\}\, d\Pi(\theta).$$

Π being a probability distribution over $[0, +\infty)$. Here it is

$$W(x) = \int_0^{+\infty} \exp\{-\theta \cdot x\} \, d\Pi(\theta).$$

Of course case (1) is a special case of (3) (Π is a degenerate distribution).

Models with an Archimedean survival copula have been dealt with extensively in the literature; see e.g. [12,14,18]. Their interest in the applications lies, in particular, in that they constitute a natural generalization of frailty models.

The statistical interest of models with Archimedean copulæ and variables that are not necessarily exchangeable is also discussed in [3].

A few different characterizations of the class of TTE models can be given (see also [4,7]); for the purposes of the present paper it is of interest to report the following characterizations, obtained in [10].

Proposition 7.1. (1) *The TTE class is the family of all exchangeable survival functions $\bar{F}$ such that there exists a law with independent marginals which has the same level curves* (*hence, the same multivariate aging function B*) *as $\bar{F}$.*

(2) *The TTE class is the family of all exchangeable survival functions $\bar{F}$ such that there exists a Schur-constant law with the same dependence structure* (*i.e., same copula*) *as $\bar{F}$.*

The arguments developed in the previous sections can be used, in particular, to analyze the relations among dependence, 1- and 2-aging for TTE models. The specific results take a particularly simple form; this will be illustrated by the examples listed below.

In the following two examples we consider a family $\mathcal{P} \equiv (\mathcal{P}_+, \mathcal{P}_-)$ of Archimedean semicopulæ.

Example 7.1. Let X_1, X_2 be two i.i.d lifetimes with survival function $\bar{G}$. Then (X_1, X_2) has $\mathcal{P}$-positive (negative) bivariate aging if and only if it has $\mathcal{P}$-positive (negative) 1-aging.

The next example is related to Remark 4.4.

Example 7.2. Let X_1, X_2 be two lifetimes with a Schur-constant survival function

$$\bar{F}(x_1, x_2) = \bar{G}(x_1 + x_2).$$

Then $K \in \mathcal{P}_+$ ($K \in \mathcal{P}_-$) if and only if $\bar{F}$ displays $\mathcal{P}$-negative 1-aging ($\bar{F}$ displays $\mathcal{P}$-positive 1-aging).

In the forthcoming example we make use of Proposition 6.1 and of Lemma 4.2.

Example 7.3. Consider a TTE model and take $\mathcal{P}_+ \equiv \mathcal{P}^3_+$. Since K is an Archimedean copula, the condition $K \in \mathcal{P}_+$ means, by Proposition 6.1, that K is TP_2 or, equivalently, that $\bar{F}$ is TP_2.

We consider now the family $\mathcal{P}_-$ of RR_2 semicopulæ, i.e. semicopulæ such that

$$S(u', v')S(u'', v'') \leqslant S(u', v'')S(u'', v'), \quad u' \leqslant u'', v' \leqslant v''.$$

$K \in \mathcal{P}_-$ is equivalent to $\bar{F}$ being RR_2. It is immediate to check that if $C_h \in \mathcal{P}_+$, then $C_{h^{-1}} \in \mathcal{P}_-$.

By recalling that $\mathcal{P}_+$ is Ψ-closed, we can apply Propositions 5.2, 5.3, 5.4 to obtain the following implications:

- If the joint survival function $\bar{F}$ is TP_2 and $X_1,\ X_2$ are IFR, then $\bar{F}$ is Schur-concave.
- If $\bar{F}$ is Schur-concave and $X_1,\ X_2$ are DFR, then $\bar{F}$ is TP_2.
- If $\bar{F}$ is Schur-concave and RR_2, then $X_1,\ X_2$ are IFR.

We can also obtain the same implications directly, by taking into account that, for the present case of a $TTE(W, R)$ model, the following holds:

- K is TP_2 iff $\log(W)$ is convex (i.e. iff the inverse of the generator of K, namely, W, is DFR);
- $\bar{F}$ is Schur-concave iff R is convex (i.e. iff the inverse of the generator of B, namely, $e^{-R^{-1}}$, is DFR);
- $\bar{G}$ is IFR iff $-\log(W \circ R)$ is convex.

The above statements point out that dependence and bivariate aging can be described in terms of univariate aging; one could show that this is true in general for TTE models.

Notice also that Schur-concavity of $\bar{F}$ (i.e. convexity of R) implies, as mentioned above, that the aging function B turns out to be a copula.

Example 7.4. Let $\mathcal{P}_+ \equiv \mathcal{P}^4_+,\ \mathcal{P}_- \equiv \mathcal{P}^4_-$. First we note that, by taking into account the definitions of IFRA and DFRA distributions (see, e.g. [6]), and by taking into account the third claim of Proposition 6.3, we can obtain: $C_\Gamma \in \mathcal{P}_+$ if and only if X_1, X_2 are DFRA, and $C_\Gamma \in \mathcal{P}_-$ if and only if X_1, X_2 are IFRA.

In view of this and in the spirit of Definition 4.1, we can interpret the condition $B \in \mathcal{P}_+$ as a notion of bivariate IFRA and the condition $B \in \mathcal{P}_-$ as a notion of bivariate DFRA (see also [10]).

From Section 6.1 we can, in particular, derive that $\mathcal{P}_+, \mathcal{P}_-$ satisfy Hypotheses 1–4 of Section 5.1. Then, Propositions 5.2, 5.3, 5.4 hold, and we are in a position to conclude as follows:

- If K is a PKD copula and X_1, X_2 are IFRA, then (X_1, X_2) is bivariate IFRA.
- If (X_1, X_2) is bivariate IFRA and X_1, X_2 are DFRA, then K is PKD.
- If (X_1, X_2) is bivariate IFRA and K is NKD, then X_1, X_2 are IFRA.

Example 7.5. It can be checked that the joint survival function in a proportional hazard model is TP_2. Furthermore, we know that, if R is concave, two conclusions follow:

(1) The law with survival function $\exp\{-\theta R(x)\}$ is DFR, for each $\theta > 0$.
(2) $\bar{F}$ is Schur-convex, i.e. it has $\mathcal{P}^3$-negative bivariate aging.

By Proposition 5.4, it follows that the marginal

$$\bar{G}(x) = \int_0^\infty \exp\{-\theta R(x)\}\, d\Pi(\theta)$$

is DFR. Thus, we find a particular case of the well-known result that mixtures of DFR laws are DFR.

8. Conclusions and final remarks

As mentioned in the Introduction, the main purpose of this paper is to analyze the intricate relations between dependence and aging. Our main results in this direction have been obtained in Section 5, in terms of the definitions given in Section 4.

Those results allowed us to answer some questions that motivated the present work. Typical such questions are about the relations among Schur-concavity, marginal IFR property and dependence of a joint law.

More precisely, we know that if two lifetimes are i.i.d. (i.e. exchangeable with $K = K_0$, the independent copula) and IFR, then their joint survival functions is Schur-concave. One may think that the same applies for K displaying a suitable degree of positive dependence. In this note we find sufficient conditions on the positive dependence of K that guarantee Schur-concavity of $\bar{F}$, namely, $K \in \mathcal{P}^3_+$.

Similarly, one may wonder what degree of negative dependence guarantees that the marginals of a law with Schur-concave survival function display positive aging. Here we answer this question: $K \in \mathcal{P}^3_-$.

We also show that these sufficient conditions on the survival copula can be expressed in terms of more familiar dependence concepts, in the case of TTE models.

The framework that allows us to answer the above questions involves a description of notions of univariate and bivariate aging in terms of dependence. More specifically, 2-aging is represented by the dependence of the law with survival $\Gamma \circ \bar{F}$, and 1-aging of $\bar{G}$ is represented in terms of the dependence of the Schur-constant law with $\bar{G}$ as marginal survival. A certain symmetry emerges, and a relevant role is played by the three "reference" cases, namely, the independent laws (with $K(u,v) = K_0(u,v) = uv$), the Schur-constant laws (with $B(u,v) = B_0(u,v) = uv$), and the laws with standard exponential marginals (with $A_{\bar{G}^{-1}}(u,v) = uv$).

A natural relation emerges between notions of positive and negative dependence, namely, the relation specified in the main hypotheses. It should be observed that this relation allows us, in view also of Proposition 6.1, to couple "different" notions of dependence, such as LTD and RR_2. As a further example, it is perfectly legitimate to take $\mathcal{P} \equiv (\mathcal{P}_+^2, \mathcal{P}_-^3)$ in our analysis.

As we just repeated, we described 2- and 1-aging in terms of dependence. In the case of TTE models, it is possible to reverse the perspective, and define dependence and 2-aging in terms of 1-aging. This could allow us to provide a more direct proof of results like Propositions 5.2–5.4; this has been briefly sketched at the end of Example 7.3.

Throughout this paper, we confined our attention to exchangeable vectors of lifetimes. In a sense, it is the very spirit of the notions of aging considered here that led us to the assumption of exchangeability.

As mentioned, our main aim is to analyze the nature of basic relations between aging and dependence; we feel that, in this respect, the above limitation is not very severe.

As far as applications to various real-world problems are concerned, the extension of our arguments to the non-exchangeable case is an open field.

The first step to extend our analysis to such a case dwells, however, on appropriate extensions of our definitions of multivariate aging.

One possible way of tackling the latter problem relies on the following two facts:

(a) Relevant aspects of properties of multivariate aging can be described in terms of the behavior of the vector of order statistics.
(b) For any arbitrary vector of lifetimes we can find one and only one vector of exchangeable lifetimes such that the probability laws of the two associate vectors of order statistics coincide.

Then we might say that a (non-exchangeable) vector has a multivariate aging property if this is the case for the "corresponding" exchangeable vector (see also [23]).

A further natural extension of our analysis concerns the case of $n > 2$ exchangeable lifetimes. Most of the definitions and most of the basic properties can be translated to this more general setup with little effort. However, many applications to specific cases become involved when the dimension exceeds $n = 2$. For example, even the basic notion of PQD has no meaning in higher dimension, and should be replaced by PUOD and PLOD (positive upper/lower orthant dependence). We plan to deal with the case of an arbitrary number of exchangeable lifetimes in another paper.

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