Lagrangians adapted to submersions and foliations

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**A B S T R A C T**

Lagrangians related to submersions and foliations, which are analogous to Riemannian submersions and Riemannian foliations respectively are studied in the paper. One prove that a bundle-like Lagrangian, a transverse hyperregular Lagrangian, a hyperregular Lagrangian foliated cocycle or a geodesic orthogonal property are equivalent data for a foliation. A conjecture of E. Ghys is proved in a more general setting than that of Finslerian foliations: a foliation that has a transverse positively definite Lagrangian is a Riemannian foliation. One extend also a result of Miernowski and Mozgawa on Finslerian foliations, proving that the natural lift to the normal bundle of a Lagrangian foliation that has a transverse positively definite Lagrangian is a Riemannian foliation.

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Following the ideas of E. Ghys' Appendix E of [10], Miernowski and Mozgawa posed in [8, Theorem 3.2] the question if any Finslerian foliation (see [4, 5, 7, 8, 13]) is a Riemannian foliation. We give here a solution to this problem proving more: a foliation that allows a positive allowed transverse Lagrangian is a Riemannian foliation (Theorem 4). The result is proved in [4] for a Finslerian foliation on a compact manifold, using a different method.

In [8, Theorem 3.2] the authors prove also that the natural lift of a Finslerian foliation to its normal bundle is a Riemannian foliation. Using different arguments, we extend this result, proving that the same conclusion holds for a foliation that allows a strict positively definite transverse Lagrangian (Theorem 5).

Connections between the foliations studied here on the one hand and the classes of foliations studied in [11], [18] or [19] and the semi-sprays considered in [17] on the other, would probably be interesting tasks for future investigations.

Lagrangian epimorphisms on vector bundles are considered and studied in the first section. One consider the general case of an epimorphism, since this case is involved in the study of the vertical bundle of a submersion and it is a natural setting to consider together bundle-like and transverse Lagrangians.

The various Lagrangian data on foliations, analogous to Riemannian data on Riemannian foliations, are discussed in the second section. The results on positive transverse Lagrangians, related to Miernowski and Mozgawa’s paper on Finslerian foliations, are also proved here.

We use a local calculus, that is, in our opinion, a geodesic way to understand easily a lot of geometric facts about Lagrangians.

1. Lagrangians, epimorphisms and submersions

Let \( E \xrightarrow{\pi} M \) and \( E'' \xrightarrow{\pi''} M \) be two vector bundles over the same base and \( f : E \to E'' \) be an epimorphism. Then the restriction of the differential \( f_* \) to the vertical bundle \( VE \subset TE \) defines an \( f\)-(epi)morphism of vector bundle \( F = f^*f : VE \equiv \pi^*E \to VE'' \equiv (\pi'')^*E''. \) The vertical bundle and the horizontal bundle of \( f \) are the vector subbundles \( Vf = \ker f \subset VE \) and \( HF = (\ker F)^1 \subset VE \) respectively, where the orthogonal is taken according to the scalar product given by the Hessian of \( f \), provided that \( g \) is non-degenerate on \( \ker f \). A Whitney sum decomposition \( VE = Vf \oplus HF \) follows in this case. (Notice that all the scalar products considered in the paper are not necessarily positive definite.)

The annihilator of the vector subbundle \( E' = \ker f \subset E \) is a vector subbundle \( (E')^0 \xrightarrow{\pi_0} M \), \( (E')^0 \subset E^* \), defined by the linear forms in \( E^* \) which are null on the vectors in \( f \).

We use in the sequel local coordinates adapted to the epimorphism \( f \) : \( (x^i) \) on \( M \), \( (x^i, y^a, \bar{y}^b) \) on \( E \) and \( (x^i, \bar{y}^b) \) on \( E'' \), such that \( f \) has the local form \( (x^i, y^a, \bar{y}^b) \to (x^i, \bar{y}^b) \). If \( (x^i, p_a, p_{\bar{a}}) \) and \( (\bar{y}^b, p_{\bar{b}}) \) are local coordinates on \( E^* \) and \( (E')^0 \) respectively, then the local coordinates on \( E' \) are \( (x^i, y^a, 0) \) and on \( (E')^0 \) are \( (x^i, 0, p_{\bar{a}}) \).

In the sequel we are interested to see when given an epimorphism \( f : E \to E'' \), under which conditions a Lagrangian \( L : E \to \mathbb{R} \) can give rise to a Lagrangian \( L'' : E'' \to \mathbb{R} \).

Let us consider the Lagrangian \( L \) and the epimorphism \( f \) be given. We say that the Lagrangian \( L \) is regularly related to the epimorphism \( f \) if the following conditions are fulfilled:

1. \( L^{-1}(\mathbb{R}) \cap (E')^1 \subset E \) is a submanifold and the canonical projection \( E \xrightarrow{\pi} M \) induces a fibered submanifold, denoted by \( (E')^1 \xrightarrow{\pi_1} M \);
2. the restriction \( f_{|_{(E')^1}} : (E')^1 \to E''_{a} \) is a diffeomorphism.

Using local coordinates adapted to the epimorphism \( f \), the coordinates of the points in \( (E')^1 \) have the form \( (x^i, y^a = Q^a(x^i, y^b, \bar{y}^\bar{b}), \bar{y}^\bar{b}) \), and the construction of \( (E')^1 \) gives that

\[
\frac{\partial L}{\partial y^a}(x^i, Q^a(x^i, y^b, \bar{y}^\bar{b}), \bar{y}^\bar{b}) = 0.
\]

Condition (R2) implies the condition

\( (*) \) the matrix \( g_{ab}(x^i, y^a, \bar{y}^\bar{b}) = \frac{\partial^2 L}{\partial y^a \partial \bar{y}^\bar{b}}(x^i, y^a, \bar{y}^\bar{b}) \) is non-degenerate along \( (E')^1 \).

It follows that the local form of \( f_{|_{(E')^1}} \) is

\[
(x^i, Q^a(x^i, \bar{y}^\bar{b}), \bar{y}^\bar{b}) \to (x^i, \bar{y}^\bar{b}).
\]

Conversely, if the condition \( (*) \) is fulfilled, then the implicit mapping theorem (see [6, Chapter 1, Section 5]) implies that Eq. (1) has locally a unique solution \( Q^a(x^i, \bar{y}^\bar{b}) \), thus \( f_{|_{(E')^1}} \) is a local diffeomorphism. We can summarize the above discussion as follows.

**Proposition 1.** Let the Lagrangian \( L \) and the epimorphism \( f \) be given. Then \( L \) is regularly related to \( f \) iff condition \( (*) \) holds and \( f_{|_{(E')^1}} \) is a (global) diffeomorphism.
In the case when \( L \) is regularly related to \( f \), one obtains an inverse diffeomorphism \( S = \frac{1}{f_\#} : E^\prime \rightarrow \{E\}' \) and we define \( L^\prime = L \circ S : E^\prime \rightarrow \mathbb{R} \), called the induced Lagrangian on \( E^\prime \) (by \( L \) and \( f \)). Using local coordinates:

\[
L''(x^i, \bar{y}^j) = (x^i, Q^a(x^i, \bar{y}^j), \bar{y}^j).
\]

(3)

Notice that the map \( S \) can be also be regarded as well as a section of the fibered manifold with projection \( E \rightarrow E' \). In the case when \( L = G \) is the Finslerian defined by a scalar product \( g \), the induced Lagrangian is given by the restriction of the scalar product to \( \{E\}' \). We are going to see that it is true also in the Lagrangian case.

We say that an epimorphism \( f : E \rightarrow E'' \) is a weak Lagrange epimorphism if there is a regular Lagrangian \( L \) on \( E \) called an adapted Lagrangian, and a section \( S : E' \rightarrow E \) of the epimorphism \( f \) such that denoting \( L = S(E') \rightarrow E \) the inclusion and by \( f \) the induced epimorphism of vertical bundles, then the restriction \( F''_f : f'' \rightarrow V E'' \) is an \( f \)-isomorphism which is an isomorphism on fibers according to the vertical Hessians of the regular Lagrangians \( L_{|S(E')} \) and \( L'' = L \circ S \) on \( E'' \) respectively.

We say that an epimorphism \( f : E \rightarrow E'' \) is a strong Lagrange epimorphism (or a Lagrange epimorphism for short) if there are some regular Lagrangians \( L \) on \( E \) and \( L'' \) on \( E'' \), called adapted Lagrangians, such that the induced epimorphism \( F : V E \rightarrow f'' \) is an isomorphism on fibers according to the vertical Hessians. Any Riemannian epimorphism \( f \) is a strong Lagrange epimorphism and any Lagrange epimorphism is also a weak one, since always there is a section \( S : E' \rightarrow E \), linear on fibers.

If \( f : E \rightarrow E'' \) is an epimorphism, we say that a Lagrangian \( L : E \rightarrow \mathbb{R} \) on \( E \) is vertical quadratic if its restriction to the subbundle \( ker f \subset E \) comes from a scalar product; that means that the restriction of \( L \) to \( Vf \) has the form \( L_{|f} = G + \varphi \), where \( L \) is the Lagrangian defined by a scalar product on the fibers of \( Vf \) and \( \varphi : M \rightarrow \mathbb{R} \) is differentiable. Using local coordinates, \( L(x^i, y^j, 0) = \frac{1}{2} y^a y^b g_{ab}(x^i) + \varphi(x^i) \). We say also that two vector subbundles \( E_1, E_2 \subset E \) are supplementary if there is a Whitney sum reduction \( E = E_1 \oplus E_2 \).

**Theorem 1.** Consider an epimorphism \( f : E \rightarrow E'' \), a Lagrangian \( L \) on \( E \) that is regularly related to \( f \) and a vector subbundle \( N \subset E \), supplementary to \( ker f \). Then there is a Lagrangian \( L'' \) on \( E'' \) such that:

1. The epimorphism \( f \) is weak according to the Lagrangians \( L \) and \( L'' \).
2. There is a Lagrangian \( L \) on \( E \) that is vertical quadratic, such that \( f \) is a Lagrange epimorphism according to the Lagrangians \( L \) and \( L'' \) respectively and also \( \ker(f)^{-1} = N \); \( L \) is regular (hyperregular) iff \( L'' \) is regular (hyperregular).

**Proof.** We use local coordinates. We denote by \( \{a_0 = \frac{a_1 a_2}{a_2 a_3} \} \) and \( \{a_4 = \frac{a_0 a_2}{a_2} \} \) the local bases of sections in the vector bundles \( V E \) and \( V E'' \) respectively. We denote also by \( g \) the \( (pseudometric) \) tensor defined by the Hessians of \( L \) on \( E \) and \( Vf \), and by \( \{b_{ab} \} = \{b_{ab}, b_{a0} = b_{b0}, b_{a2} \} \) its components using the above base. Notice that \( \{a_0 \} \) is a local base of sections in \( Vf \) and \( \{a_0 = b_{a0} b_{a2} = b_{a2} \} \) is a local base of sections in \( \ker(f) \), where \( (g_{ab}) = (g_{ab})^{-1} \) as matrices. It is easy to see that we have \( g(a_0, a_2) = g_{a0} - g_{a2} b_{a2} b_{b0} \). The local form of \( f_{\#}(E')^{-1} \) is given by \( \{a_4 = \frac{a_0 a_2}{a_2} \} \). Differentiating \( L \) and \( E'' \), and to respect to \( y^i \), we obtain \( \frac{\partial^2 Q^a}{\partial y^i y^j} = -\frac{\partial^2 g}{\partial y^i y^j} \), where \( (g_{ab}) = (g_{ab})^{-1} \) on \( \{a_4 = \frac{a_0 a_2}{a_2} \} \) and \( \frac{\partial^2 g}{\partial y^i y^j} = \frac{g_{ab}}{a_4} g_{ab} \). Let \( L \) be the epimorphism induced by \( f \) on vertical bundles. The local correspondence between bases of sections by means of \( f_{\#}(E')^{-1} \) is \( s_0 = s_0 - g_{a0} b_{a2} s_0 \). We have \( g(s_0, s_0) = g_{a0} - g_{a2} b_{a2} b_{b0} = g_{b0} = g(s_0, s_0) \). Using a linear algebra computation, we have \( (g_{ab} - g_{a2} b_{a2} b_{b0}) = (g_{ab})^{-1} \), thus \( 1 \) follows.

In order to prove \( 2 \), let \( g \) be a scalar product, non-degenerated on the fibers of \( ker f \), such that the orthogonal of \( ker f \) is \( N \) and denote by \( P : E \rightarrow (ker f)^{-1} \) the orthogonal projection according to \( g \). Then the Lagrangian \( L : E \rightarrow \mathbb{R} \), defined by

\[
\bar{L}(x, y) = g(P(y), P(y)) + L'(f_0(x, f(y'))),
\]

is hyperregular, vertical quadratic and it defines a Lagrange epimorphism as required by \( 1 \).

If \( (x^i, y^j, \bar{y}^k) \) and \( (\bar{x}^i, \bar{y}^j) \) are local coordinates on \( E \) and \( E'' \), respectively, then let us denote by \( L''(x^i, \bar{y}^j) \) the local form of \( L'' \), and \( \{g_{ab} = g_{ab} = g_{a0}, g_{a2} \} \) the local components of a scalar product \( g \) on the fibers of \( E \) such that \( (ker f)^{-1} = N \), according to the metric \( g \). Then the local form of the Lagrangian \( L : TM \rightarrow \mathbb{R} \) given by formula (4) is \( \bar{L}(x^i, y^j, \bar{y}^j) = \frac{1}{2} g_{ab}(x^i) y^a y^b + L'(x^i, \bar{y}^j) \), where \( \bar{y}^j = y^j - g_{ab} g_{ab}(x^i) y^b \). The restriction \( \bar{L}_{|ker f} \) has the form \( \bar{L}(x^i, y^j, 0) = \frac{1}{2} g_{ab}(x^i) y^a y^b + L'(x^i, 0) \), thus it comes from a scalar product. Since \( \frac{\partial^2 Q^a}{\partial y^i y^j} = g_{ab}(y^a - \bar{y}^a g_{ab}(x^i) y^b g_{ab}(y^b)) \), it follows that \( Q''(x^i, y^j) = g_{ab}(x^i) g_{ab}(x^i) y^b \) and \( (ker f)^{-1} = N \). \( \square \)

Notice that, in general, \( (ker f)^{-1} \subset E \) is not the total space of a vector subbundle.

The case of epimorphisms of vector bundles over different bases reduces to that of vector bundles over the same base.

Let \( f_0 : M \rightarrow M'' \) be a surjective submersion, \( E \rightarrow M \) and \( E'' \rightarrow M'' \) be vector bundles and \( f : E \rightarrow E'' \) an \( f_0 \)-morphism of vector bundles. We say that \( (f, f_0) \) is a Lagrangian epimorphism according to Lagrangians \( L : E \rightarrow \mathbb{R} \) and \( L'' : E'' \rightarrow \mathbb{R} \) respectively, if \( f_{\#} f \) is a Lagrangian epimorphism of vector bundles over the same base \( M \), according to the Lagrangians \( L \) and \( L'' = f'' L'' \) respectively. The Lagrangian \( L : E \rightarrow \mathbb{R} \) is projectable on a Lagrangian \( L'' : E'' \rightarrow \mathbb{R} \) if it is regularly related to the epimorphism \( f_{\#} f : E \rightarrow f_{\#} E'' \) and the Lagrangian \( L'' : f_{\#} E'' \rightarrow \mathbb{R} \) given by Theorem 1 has the form \( L'' = f'' L'' \).
If \( L : E \rightarrow M \) and \( L_1 : E_1 \rightarrow M_1 \) are two Lagrangians, then we say that a vector bundle morphism \( E \xrightarrow{(f_0,f_1)} E_1 \) is a Lagrange isometry if \( L_1 = L \circ f \).

Taking into account of the definition of \( L'' \), based on formulas (3) and (1), the following statement can be proved by a straightforward verification.

**Proposition 2.** Let us consider \( D_{1,2} \xrightarrow{v} M, D_{1,2} \xrightarrow{w} M'' \) and \( E_1 \xrightarrow{\pi_1} M_1 \) vector bundles, \( E \xrightarrow{(f_0,f_1)} E_1 \) and \( E'' \xrightarrow{\varphi \circ f_0 = f_1} E_1'' \) epimorphisms, \( \varphi : M \rightarrow M'' \) a diffeomorphism such that \( f_0 = \varphi \circ f_0 \) and there is an isomorphism of vector bundles \( E'' \xrightarrow{\psi} \varphi^* E_1'(i.e. \text{an isomorphism } E'' \xrightarrow{(\varphi,\psi)} E_1') \) such that \( f_1 = \psi \circ f \).

Then if a hyperregular Lagrangian \( L : E \rightarrow M \) is projectable on \( E'' \) on a Lagrangian \( L'' : E'' \rightarrow M'' \), then it is also projectable on \( L_1'' \) on a Lagrangian \( L_1'' : E'' \rightarrow M'' \) and \( (\varphi, \psi) \) is a Lagrange isometry of the Lagrangians \( L'' \) and \( L_1'' \).

In particular, if \( E'' = E_1'' \), then \( (\varphi, \psi) \) is an isometry of the Lagrangian \( L'' \).

A special case of epimorphisms over different bases is that of submersions. We use effectively this situation for Lagrangian foliations that are studied in the next section. Since a submersion is a particular foliation (a simple foliation), all the results proved for Lagrangian foliations are valid for proper Lagrangian submersions.

It is the case when \( E = TM, E'' = TM'' \). \( f_0 = u : M \rightarrow M'' \) is a submersion and \( f = u \circ \tau : TM \rightarrow TM'' \) is the differential of \( u \). In this case, considering \( (u, u_\tau) \), a Lagrange epimorphism is called a Lagrange submersion, a weak Lagrange epimorphism is called a weak Lagrange submersion and a Lagrangian \( L : TM \rightarrow M'' \) is a projectable Lagrangian on \( TM'' \) if the induced Lagrangian \( u^* TM'' \) is projectable on a Lagrangian \( L'' \) on \( M'' \). Using local coordinates, it means that the function \( L'' \) given by \( L'(x, \sigma, y) = L(x, \sigma, y) \) (see (3)) does not depend on \( x^1 \).

If \( u : M \rightarrow M'' \) is a submersion, then we say that a Lagrangian \( L : TM \rightarrow M'' \) is projectable on \( L'' \) on a Lagrangian \( L'' \) if the canonical Lagrangian \( u^* TM'' \) on \( u^* TM'' \) has the form \( L'' = (u \tau)^* L'' \), where \( L'' \) is a Lagrangian on \( M'' \).

Let \( u : M \rightarrow M'' \) be a Lagrangian. \( L'' \) is a Lagrangian on \( M'' \) and \( g \) be a scalar product on the fibers of \( TM \). If \( (x^1, x^2, \ldots, x^\ell) \) are local coordinates on \( U \subset M \) and \( (y^1, y^2, \ldots, y^\ell) \) are local coordinates on \( U \subset M \) and \( \pi(U) = U \), then if \( L'(x^1, y^1) \) is the local form of \( L'', \) then the local form of the Lagrangian \( L \) on \( E \) is \( L(x^1, x^2, \ldots, x^\ell, y^1) = \frac{1}{2} g_{x^1x^1}(x^w, x^w) y^1 y^1 + L'(x^1, y^1) \), where \( y^1 = y^1 - \bar{g}^{uv}(x^u, x^v) g_{v^u}(x^w, x^w) y^v \), \( (\bar{g}^{uv}) = (g_{uv})^{-1} \). This formula is a local coordinate transcription of formula (4). (See [3].)

## 2. Lagrangians and foliations

We define concepts and basic facts on foliations from [10], but some different notations. We denote, relatively to \( \mathcal{F} : \mathcal{X}_{\text{loc}}(\mathcal{F}) \) the set of locally tangent vector fields to \( \mathcal{F} ; B^0_{\text{loc}}(\mathcal{F}) \) the set of locally basic real smooth functions \( f \) on \( M; B_{\text{loc}}(\mathcal{F}) \) the set of locally foliated vector fields \( X \) on \( M; TM/\mathcal{F} = NF \) and \( \tilde{\Pi} : TM \rightarrow NF \) is the canonical vector bundle projection (called here the transverse epimorphism).

On the domain \( U \) of a chart in the foliated atlas, we denote by \( \pi_U : U \rightarrow \bar{U} \) the canonical projection which has the fibers the leaves of \( \mathcal{F}_U \) and also by \( \Pi_U = (\pi_U)_\ast : TU \rightarrow \bar{U} \). We say that a linear form \( \varphi \) on the fibers of \( NF \) is basic if \( \varphi(X, Y) \in B^0_{\text{loc}}(\mathcal{F}) \) for every basic local sections \( X \) and \( Y \). Obviously, we can consider in a similar way the definition of a basic covariant tensor on \( NF \), of any order. A foliation \( (M, \mathcal{F}) \) lifts naturally to a foliation \( (NF, NF_\mathcal{F}) \) on the normal bundle, such that the natural projection \( NF \xrightarrow{\text{nat}} M \) is a covering which is restricted from leaves to leaves. In order to consider transverse Lagrangians \( L : NF \rightarrow \mathbb{R} \) that are continuous on \( NF \) and differentiable on \( NF_\ast = NF \setminus \{0\} \) (or \( L : N_1 F \rightarrow \mathbb{R} \), where \( N_1 F \subset NF \) is a fibered submanifold), one must restrict the foliation \((NF, NF_\mathcal{F})\) to \((NF_\mathcal{F}, NF_\mathcal{F})\) on \( NF_\ast = NF \setminus \{0\} \).

The basic Hessian of a Lagrangian \( L : NF \rightarrow \mathbb{R} \) on \( NF \) is the symmetric bilinear form on the fibers of \( \mathcal{F}_\mathcal{F}^{\perp} NF \), defined by \( \text{Hess}(L)(X, Y) = (\pi_{\mathcal{F}})_\ast (\mathcal{F}_U)_\ast \), for \( X \) and \( Y \) local induced basic sections on an open \( \mathcal{F}_U \) then extended by linearity for \( X, Y \in \mathcal{F} \), \( Y \in \Gamma(\mathcal{F}_U) \), and then on \( NF \). Using local coordinates, \( g_{x^w}(x^w, x^w, y^w) = \sum_{i,j} \alpha_i \beta_j (x^w, x^w, y^w) \) are the local coefficients of the basic Hessian.

A Riemannian foliation can be given using different but equivalent definitions (see [10] or [16]). We are concerned to extend some of them to foliations that are related to special Lagrangians.

Let us translate this property to the Lagrangian case. We say that a Lagrangian \( L : TM \rightarrow \mathbb{R} \) is a bundle-like Lagrangian for \( \mathcal{F} \) if it is regularly related to the transverse epimorphism \( \Pi : TM \rightarrow NF \) and the Lagrangian \( L'' : NF \rightarrow \mathbb{R} \) given by Theorem 1 is hyperregular and a basic function for the lifted normal foliation \((NF_\mathcal{F}, NF_\mathcal{F})\). It is easy to see that a bundle-like metric for a Riemannian foliation (see [10, Section 2.3] or [14, Chapter IV, Section 4]) gives a bundle-like Lagrangian.

We give below a geometric interpretation of a bundle-like Lagrangian condition, closed to the bundle-like metric condition for a scalar product. Since \( L \) is regularly related to \( \Pi \), then \( L^{-1}(\ker \Pi^\perp_\mathcal{F}) \subset TM \) is the total space of a fibered submanifold \( TF^{-1} \subset TM \). In general it is not a vector subbundle, but using the proof of Theorem 1 one can prove the following statement.

**Proposition 3.** If \( L : TM \rightarrow \mathbb{R} \) is a bundle-like Lagrangian for \( \mathcal{F} \) and \( N \subset TM \) is a vector subbundle supplementary to \( TF \), then a Lagrangian \( L : TM \rightarrow \mathbb{R} \) given by Theorem 1 is also bundle-like for \( \mathcal{F} \), it is hyperregular, the restriction \( L_{TF} \) comes from a scalar product and the subbundle \( TF^{-1} \subset TM \) defined by \( L \) is \( N_\perp \).
Notice that if the vertical Hessian of $L$ is strict positively definite, then the Lagrangian $L$ is regularly related to $\bar{T}$ iff the restriction $\bar{T}|_{\mathcal{F}^{\perp}_{-1}} : \mathcal{T}\mathcal{F}^{\perp} \to NF_{\ast}$ is a diffeomorphism.

**Proposition 4.** Let $L : TM \to \mathbb{R}$ be a hyperregular Lagrangian that is regularly related to $\bar{T}$. Then $L$ is bundle-like for the foliation $(M, \mathcal{F})$ iff for any local foliated vector field $X$ on a $U \subset M$ that is also a local section of $\mathcal{T}\mathcal{F}^{\perp}$ on $U$, non-zero in every point, then $U \ni x \to L(x, X_{x}) \in \mathbb{R}$ is a basic function on $U$.

**Proof.** We use local coordinates. The local form of a Lagrangian $L' : NF \to \mathbb{R}$ given by Theorem 1 is $L' = L(x^u, x^\theta, y^\theta)$; it is bundle-like for the lifted normal foliation $(NF_{\ast}, NF_{\ast})$ iff $\frac{\partial L}{\partial y^u}(x^u, x^\theta, Q^u(x^u, x^\theta, y^\theta), y^\theta) = 0$. A local section $x$ of $\mathcal{T}\mathcal{F}^{\perp}$ on $U$, also a local basic foliated field, has the form $(x^u, x^\theta) \mapsto (x^u, x^\theta, Q^u(x^u, x^\theta, y^\theta), s^u(x^\theta))$. The condition that $U \ni x \to L(x, X_{x}) \in \mathbb{R}$ be a local foliated vector field $X$ on $U \subset M$ and also a local section of $\mathcal{T}\mathcal{F}^{\perp}$ on $U$ reads $\frac{\partial L}{\partial y^u}(x^u, x^\theta, Q^u(x^u, x^\theta, s^u(x^\theta)), s^u(x^\theta)) = 0$. The conclusion follows, provided that the local foliated vector field $X$ is arbitrary taken and non-zero in every point. \(\square\)

A transverse Lagrangian to a foliation $(M, \mathcal{F})$ is a Lagrangian $L'' : NF \to \mathbb{R}$ that is a basic function for the lifted normal foliation $(NF_{\ast}, NF_{\ast})$.

**Proposition 5.** A foliation allows a hyperregular transverse Lagrangian $L''$ iff it allows a bundle-like Lagrangian $L$.

**Proof.** Let $L' : NF \to \mathbb{R}$ be a hyperregular transverse Lagrangian and the transverse epimorphism $\bar{T} : TM \to NF$. Using 2) of Theorem 1, we can obtain a hyperregular Lagrangian $L : TM \to \mathbb{R}$ that is a bundle-like Lagrangian. Conversely, giving a bundle-like Lagrangian $L'$, the existence of $L''$ follows from the definition of $L$. \(\square\)

A foliated cocycle $(U_i, \pi_i, \gamma_{ij})_{i,j \in I}$ for a foliation $(M, \mathcal{F})$ is given by a differentiable manifold $S$ (a transverse model) such that $(U_i, \pi_i)$ is an open cover of $M$, $(\pi_i : U_i \to S)_{i \in I}$ are submersions such that the leaves of $\mathcal{F}_{| U_i}$ and $\pi_i$ are the same and for every $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, then $\gamma_{ij}$ is a local diffeomorphism of open subsets in $S$ and for each $x \in U_i \cap U_j$, $f_i(x) = (\gamma_{ij} \circ f_j)(x)$. A foliated cocycle is Riemannian if $\gamma_{ij}$ are local Riemannian isometries (see [14, Chapter IV, Section 4], or [3, Chapter III, Section 1.4]). Analogously, to the Finslerian case (as in [8] or [7]) it is Lagrangian if there is a Lagrangian $L : TS \to \mathbb{R}$ on the transverse model such that for every $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, then $\gamma_{ij}$ is a local Lagrangian isometry of $L$.

**Proposition 6.** A foliation $(M, \mathcal{F})$ allows a Lagrangian foliated cocycle iff it allows a transverse Lagrangian $L'' : NF \to \mathbb{R}$.

**Proof.** If $(U_i, \pi_i, \gamma_{ij})_{i,j \in I}$ is a Lagrangian foliated cocycle, then one defines a transverse Lagrangian $L'' : NF \to \mathbb{R}$ using formula $L''(x, \bar{T}x, \gamma_{ij}(x)) = L(f_i(x), f_j(x))$, where $x \in U_i$ and $X_x \in T_x M$. One proves by straightforward verification that the definition of $L''$ does not depend on $X$ and $i$, as well as that $L''$ is a transverse Lagrangian. Using the same formula that relates $L''$ and $L$, one can prove the converse fact, that given $L''$ on can recover $L$, using a foliated cocycle for $\mathcal{F}$. \(\square\)

The second order equations on a manifold are defined by a semi-spray (see [6,9,15]). We say that a submanifold $N \subset TM$ is a geodesic submanifold for a semi-spray provided that it enjoys the property that if an integral curve of the semi-spray intersects $N$, then it is entirely enclosed in $N$.

Let us consider a hyperregular Lagrangian $L : TM \to M$ that is bundle-like for a foliation $(M, \mathcal{F})$. The Lagrangian $L'' : NF \to \mathbb{R}$, given by 2) of Theorem 1 (for $L$ and the transverse epimorphism $\bar{T} : TM \to NF$) has the local form $L''(x^u, x^\theta, y^\theta) = L(x^u, x^\theta, Q^u(x^u, x^\theta, y^\theta), y^\theta)$. The condition that $L''$ be a basic function becomes $\frac{\partial L}{\partial y^u}(x^u, x^\theta, Q^u(x^u, x^\theta, y^\theta), y^\theta) = 0$; it is the condition that $L$ be a bundle-like Lagrangian for the foliation $\mathcal{F}$.

**Theorem 2.** Given a hyperregular bundle-like Lagrangian $L$, the submanifold $\mathcal{T}\mathcal{F}^{\perp} \subset TM$ is a geodesic submanifold according to the semi-spray of $L$ iff $L$ is a bundle-like Lagrangian for $\mathcal{F}$.

**Proof.** We use local coordinates. The tangent spaces of $\mathcal{F}^{\perp}$ in $TM$ is generated by the local vectors in $\mathcal{X}(TM)$:

\[
\frac{\partial}{\partial x^u} = \frac{\partial}{\partial x^u} + \frac{\partial Q^v}{\partial x^u} \frac{\partial}{\partial y^v}, \quad \frac{\partial}{\partial \theta^u} = \frac{\partial}{\partial \theta^u} + \frac{\partial Q^v}{\partial \theta^u} \frac{\partial}{\partial y^v}, \quad \frac{\partial}{\partial y^u} = \frac{\partial}{\partial y^u} + \frac{\partial Q^v}{\partial y^u} \frac{\partial}{\partial y^v}.
\]

The local form of the semi-spray $S : TM \to TTM$ is

\[
(x^u, x^\theta, y^\theta) \mapsto (x^u, x^\theta, y^u, y^\theta, y^\phi, y^\phi, S^u(x^u, x^\theta, y^\theta, y^\phi), S^\theta(x^u, x^\theta, y^\theta, y^\phi), S^\phi(x^u, x^\theta, y^\theta, y^\phi)),
\]

where
Theorem 4.

The idea to a positively allowed Lagrangian, in order to construct a transverse Riemannian metric to a foliation that allows leaves it intersects.

A Riemannian metric such that any geodesic that is perpendicular in a point to a leaf, it remains perpendicular to all the leaves.

According to Proposition 3, if a foliation allows a bundle-like Lagrangian, then it allows also a bundle-like Lagrangian of the form

\[ L''(x^u, \bar{x}^\bar{u}, y^v, \bar{y}^\bar{v}) = g_{uv}^w \left( \frac{\partial L}{\partial x^u} - \frac{\partial^2 L}{\partial y^v / \partial x^w} y^w \right), \]

(\(g_{ij}\))

\[ (g_{ij})^{-1} = (g^{ij}) = \left( \begin{array}{cc} g^{u\bar{v}} & g^{\bar{v}v} \\ g^{u\bar{v}} & g^{\bar{v}v} \end{array} \right). \]

The local form of \( S \) can be written in the form

\[ S = y^u \frac{\delta}{\delta x^u} + \bar{y}^\bar{u} \frac{\delta}{\delta \bar{x}^\bar{u}} + S^v \frac{\delta}{\delta y^v} + \frac{1}{2} \left( S^u - y^v \frac{\partial Q^u}{\partial x^v} - \bar{y}^\bar{v} \frac{\partial Q^\bar{u}}{\partial \bar{x}^\bar{v}} - S^\bar{u} \frac{\partial Q^\bar{u}}{\partial \bar{x}^\bar{v}} \right) \frac{\partial}{\partial y^v}. \]

One restrict in that follows to \( T F^{-1}_{-1} \), where \( y^v = Q^v \).

The partial derivatives of the functions \( Q^u \) can be obtained differentiating Eq. (1) with respect to the variables \( x^u \), \( \bar{x}^\bar{u} \) and \( y^v \) respectively. One obtain:

\[ y^v \frac{\partial Q^u}{\partial x^v} + \bar{y}^\bar{v} \frac{\partial Q^\bar{u}}{\partial \bar{x}^\bar{v}} + S^v \frac{\partial Q^v}{\partial y^v} = \tilde{g}^{uw} \frac{\partial L}{\partial x^w} - \tilde{g}^{u\bar{w}} \frac{\partial^2 L}{\partial y^v / \partial x^w} y^w = \tilde{g}^{uw} \frac{\partial L}{\partial x^w} + S^u, \]

where \( (\tilde{g}^{uv}) = (g_{uv})^{-1} \). It follows that restricting to \( T F^{-1}_{-1} \), we have

\[ S = y^u \frac{\delta}{\delta x^u} + \bar{y}^\bar{u} \frac{\delta}{\delta \bar{x}^\bar{u}} + S^v \frac{\delta}{\delta y^v} + \tilde{g}^{uw} \frac{\partial L}{\partial x^w} (x^u, \bar{x}^\bar{u}, Q^v, y^v) \frac{\partial}{\partial y^v}. \]

Thus \( S \) is tangent to \( T F^{-1}_{-1} \) iff \( \frac{\partial}{\partial y^v}(x^u, \bar{x}^\bar{u}, Q^v, y^v) = 0 \), so the conclusion follows. \( \square \)

According to Proposition 3, if a foliation allows a bundle-like Lagrangian, then it also allows a bundle-like Lagrangian \( L \) that is quadratic when it is restricted to \( T F \); in this case \( T F^{-1}_{-1} \) is a vector subbundle of \( TM \). A more particular form of this case is that of a Riemannian foliation, when we obtain the well-known result: a foliation is Riemannian iff there is a Riemannian metric such that any geodesic that is perpendicular to a leaf, it remains perpendicular to all the leaves it intersects.

Notice that the result of Theorem 2 can be improved as follows, using in the proof the same idea.

Theorem 3. Let \( L'' : NF \rightarrow \mathbb{R} \) be a regular Lagrangian, and let \( L : TM \rightarrow \mathbb{R} \) be given by 2) of Theorem 1, using a scalar product on \( TM \), non-degenerated on \( T F \). Then the vector subbundle \( T F^{-1}_{-1} \subset TM \) is a geodesic submanifold according to the semi-spray of \( L \) iff \( L'' \) is a transverse Lagrangian for \( F \).

Notice that Theorems 2 and 3 can be easily adapted to the cases when the domain of \( L'' \) is an open fibered submanifold of \( NF \).

We say that a transverse Lagrangian \( L'' : NF \rightarrow \mathbb{R} \) is positively allowed if the following two conditions hold:

1) \( L'' \) is positive definite (i.e. its basic Hessian is positively defined) and \( L''(x, y) \geq 0 = L''(x, 0), (\forall)x \in M \) and \( y \in NF_x \);
2) the transverse Lagrangian has the property that there is a smooth function \( \phi : M \rightarrow (0, \infty) \), basic for the foliation \( F \) on \( M \), such that for every \( x \in M \) there is \( y \in NF_x \) such that \( L''(x, y) = \phi(x) \).

For example, a transverse positively definite Finslerian \( F'' : NF \rightarrow \mathbb{R} \) is always positively allowed, where \( \phi \) can be any positive constant function on \( M \). The Euler theorem on homogeneous functions implies that if \( F'' \) is differentiable (on \( NF \)), then it is quadratic on velocities, i.e. it comes from a transverse Riemannian metric; thus the interesting (non-trivial) case occurs when \( F'' \) is continuous and it is differentiable on \( NF_x \).

In [15] one show how to consider different volumes form on \( M \), related to a (positively definite) Finslerian. We translate the idea to a positively allowed Lagrangian, in order to construct a transverse Riemannian metric to a foliation that allows a positively allowed transverse Lagrangian.

Theorem 4. If a foliation \( (M, F) \) allows a positively allowed transverse Lagrangian \( L'' : NF \rightarrow \mathbb{R} \), then the foliation is Riemannian, i.e. there is a transverse Riemannian metric to the foliation.

Proof. Let us consider a local submersion \( \pi : U \rightarrow \bar{U} \) such that its fibers are included in the leaves of \( F \), where \( U \) and \( \bar{U} \) are domains of local coordinates adapted to foliation. A transverse Lagrangian \( L'' : NF \rightarrow \mathbb{R} \) projects on a locally Lagrangian \( \bar{L}'' \) on \( \bar{U} \). We can construct a transverse metric on \( \bar{U} \), by averaging the vertical Hessian of \( L'' \), using a measure that in the Finsler case is the Busseemann–Hausdorff measure (see [15, Section 5.1]). Let us consider, in every point \( \bar{x}(\bar{x}^d) \in \bar{U} \), according to some local coordinates on \( \bar{U} \), the compact subset \( B_x = \{ (y^b, \bar{y}^\bar{b}) \in \mathbb{R}^d : \phi(\bar{x}) \leq \bar{L}''(\bar{x}, y^b, \bar{y}^\bar{b}) \leq \phi(\bar{x}^d) \} \subset \mathbb{R}^d \). Let us denote by \( \text{vol}(B_x) \) the euclidean volume of \( B_x \), according the usual euclidean structure of \( \mathbb{R}^d \). Let the coordinates \( \bar{x}^d \) be changed according to the rule \( \bar{x}^d = \bar{J}(\bar{x}) \bar{x}^d \), \( \bar{J}(\bar{x}) = (3\bar{x}^d / \partial \bar{x}^d) \) be the Jacobian matrix of coordinates and \( B_{\bar{x}} \) be corresponding to the new coordinates. Then it is easy to see that \( \text{vol}(B_{\bar{x}}) = \text{vol}(B_x) \det \bar{J}(\bar{x}) \) and taking \( f : T \bar{U} \rightarrow \mathbb{R} \) differentiable on
non-null vectors, then \( \int_{B_{\xi}} f(\tilde{x}, y^\nu) \, dv = (\det J(\tilde{x})) \int_{B_{\xi}} f(\tilde{x}, y^\nu) \, d\nu \), thus \( F(\tilde{x}) = (\int_{B_{\xi}} f(\tilde{x}, y^\nu) \, d\nu) / \text{vol}(B_{\xi}) \) does not depend on local coordinates and is differentiable on \( \tilde{U} \). It is easy to see that the local functions \( g_{\tilde{g}g}(\tilde{x}) = \int_{B_{\xi}} \frac{\delta^2}{\delta y^\nu \delta y^\mu}(\tilde{x}, y^\nu) \, d\nu / \text{vol}(B_{\xi}) \) define a transverse Riemannian metric to the foliation. □

In the case when the transverse Lagrangian \( L \) is a Finslerian (i.e. it is 2-homogeneous), we obtain an extension of a result proved in [4], in the case when \( M \) is compact.

**Corollary 1.** If a foliation allows a positively definite transverse Finslerian \( F'' : NF \to \mathbb{R} \), then the foliation is Riemannian, i.e. there is a transverse Riemannian metric to the foliation.

Let us consider the transverse vector bundle \( N^2 F \pi_{\xi}^{\xi} \mathcal{F} \) of the transverse foliation \( NF \pi_{\xi}^{\xi} \mathcal{F} \to M \). Lifting once again the foliation \( (NF, \mathcal{F}) \) to its transverse vector bundle, one obtains a foliation \( (N^2 F, N^2 \mathcal{F}) \). The transverse part of this new foliation has as local model the tangent bundle of second order of the transverse model of the initial foliation \( (\mathcal{F}, F) \). Using local coordinates \((x^i, x^\alpha)\) on \( M \), adapted to the foliation, there are some local coordinates \((x^i, x^\alpha, y^\beta, X^\alpha, Y^\beta)\) on \( N^2 F \), such that, additionally, the coordinates \((y^\beta)\) change according to the rule \( y^\beta = \frac{\partial x^\beta}{\partial x^i} y^i \) and also some coordinates \((x^u, x^\theta, y^\theta, X^\theta, Y^\theta)\) on \( N^2 F \), such that, additionally, the coordinates \((X^\alpha, Y^\beta)\) change according to the rules \( X^\alpha = \frac{\partial x^\alpha}{\partial x^i} X^i \), \( Y^\beta = \frac{\partial x^\beta}{\partial x^i} X^i \frac{\partial x^i}{\partial x^\theta} Y^\theta \).

Thus the transverse model of \((N^2(F_{\xi}), N^2(F_{\xi}))\) is \( T\tilde{U} \).

It is well known [see for example [9]] that a regular Finslerian \( L : TM \to \mathbb{R} \) gives rise to a canonically scalar product \( h \) on the fibers of the tangent vector bundle \( TTM \to TM \); then \( h \) is positively definite iff the vertical Hessian of \( L \) is positively definite. In local coordinates, this scalar product has the local form

\[
h(X^i \frac{\partial}{\partial x^i} + Y^j \frac{\partial}{\partial y^j}, Z^k \frac{\partial}{\partial x^k} + W^h \frac{\partial}{\partial y^h}) = g_{ij} X^i Z^j + g_{jh} Y^j W^h.
\]

where \( g_{ij} = \frac{\partial^2}{\partial x^i \partial x^j} - \frac{\partial}{\partial x^i} N^j \frac{\partial}{\partial y^j} \) and \( N^j_i = -\frac{\partial}{\partial y^j} \) are the coefficients of the non-linear connection corresponding to Kern semi-spray of \( L \). (\( x^i, y^j \) \( \rightarrow \) \( (x^i, y^j, -2S_i^j((x^i, y^j)), 2S^i_j((x^i, y^j)) \)), where \( S^i_j((x^i, y^j)) = \frac{1}{2} N^j_i \left( \frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^j} y^j \right) \).

The vertical and horizontal vector subbundles \( VTM, HTM \subset TTM \) are orthogonal according to this scalar product. Taking into account of the construction of \( h \), it is easy to see that if the vertical Hessian of \( L \) is positively definite, then \( h \) is (positively definite) Riemannian metric.

**Theorem 5.** If a foliation has a regular transverse Lagrangian \( L'' \), then the transverse foliation \( (NF, \mathcal{F}) \) has a transverse scalar product \( h \).

If \( L'' \) is positively definite, then \( h \) is a transverse Riemannian metric, thus \( (NF, \mathcal{F}) \) is a Riemannian foliation.

**Proof.** Let us consider a transverse Lagrangian \( L'' : NF \to \mathbb{R} \) to \((M, \mathcal{F})\) and a local submersion \( \pi : U \to \tilde{U} \) such that its fibers are included in the leaves of \( \mathcal{F} \). A natural vector bundle map \( \pi_F : NF_{\xi} \to T\tilde{U} \) is induced, such that \( \pi_F = \pi \circ \Pi \), where \( \Pi : TM \to NF_{\xi} \) is the natural projection. There is a regular Finslerian \( L_{\tilde{U}} : \tilde{U} \to \mathbb{R} \) on \( \tilde{U} \) that corresponds to \( L'' \), following the definition of \( L'' \) and a scalar product \( g_{\tilde{g}T}\tilde{U} \) on the fibers of the vector bundle \( T\tilde{U} \to \tilde{U} \) that corresponds to \( L_{\tilde{U}} \). These \( g_{\tilde{g}T}\tilde{U} \) induce the local transverse scalar product \( (\pi'' \to g_{\tilde{g}T}\tilde{U}) \) on \( NF_{\xi} \) that glue together into a global defined, transverse scalar product to the foliation \( (NF, \mathcal{F}) \). The second statement is a simple consequence of the first one. □

In the case when \( L'' \) is a transverse (positively definite) Finslerian, we obtain the result proved in [8, Theorem 3.2] following a different way. Notice also that if the Lagrangian \( L'' \) is differentiable on \( NF \), then the transverse scalar product \( h \) can be taken to be transverse to the foliation \( (NF, \mathcal{F}) \).

**References**