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On the Resonance Problem with Nonlinearity which has Arbitrary Linear Growth

PAVEL DRÁBEK

Technical University of Plzeň, Department of Mathematics, Nejedlého sady 14,306 14 Plzeň, Czechoslovakia

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This paper deals with the nonlinear two-point boundary value problem at resonance. Even nonlinearities g with an arbitrary linear growth in $+\infty$ (resp. $-\infty$) may be considered but only on the cost of the corresponding bound on their linear growth at $-\infty$ (resp. $+\infty$). It generalizes the previous results in this direction obtained by M. Schechter, J. Shapiro, and M. Snow (*Trans. Amer. Math. Soc.* **241** (1978), 69–78), L. Cesari and R. Kannan (*Proc. Amer. Math. Soc.* **88** (1983), 605–613), and S. Ahmad (*Proc. Amer. Math. Soc.* **93** (1984), 381–384). © 1987 Academic Press, Inc.

1. INTRODUCTION

We are studying the solvability of the boundary value problem (BVP)

$$u'' + u + g(x, u) = h(x), \qquad u(0) = u(\pi) = 0, \tag{1.1}$$

where $h \in L^1(0, \pi)$ and g is a Caratheodory function. If g does not depend on x, if the numbers

$$g_{+\infty} = \liminf_{s \to +\infty} g(s) \qquad \text{and} \qquad g^{-\infty} = \limsup_{s \to -\infty} g(s)$$

are finite and if

$$g^{-\infty} \int_0^\pi \sin x \, dx < \int_0^\pi h(x) \sin x \, dx < g_{+\infty} \int_0^\pi \sin x \, dx \tag{1.2}$$

holds then it follows from a slight modification of the theorem due to Landesman and Lazer [7] that (1.1) has a solution (see, e.g., Fučík [6]). This result has been generalized in subsequent papers by Schechter *et al.* [8], Cesari and Kannan [3], and Ahmad [1]. These improvements are due to the fact that the numbers $g_{+\infty}, g^{-\infty}$ are allowed to be infinite

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provided that the function g does not grow too quickly at $\pm\infty$. More precisely, it is proved in [8] that (1.2) is a sufficient condition for the solvability of (1.1) if there exist real numbers c and $q>0$ such that

$$g(s)/s \leqslant c \qquad \text{for } |s| \geqslant q, \tag{1.3}$$

g is odd, nondecreasing, and $c<0.24347$. This result was improved in [3]: $c<0.433$ suffices and g need not be odd. In the recent paper [1] there are no hypotheses concerning the monotonicity of g and (1.3) is assumed to be satisfied with some $c<3$. Since the BVP

$$u''+u+3u=\sin 2x, \qquad u(0)=u(\pi)=0$$

has no solution the condition $c<3$ is sharp.

It follows from the proof of the result [1] that (1.3) with $c<3$ and (1.2) are sufficient for the solvability of the problem (1.1) because $\lambda_2-\lambda_1=3$, where λ_1 and λ_2 is the first and the second eigenvalue, respectively, of

$$u''+\lambda u=0, \qquad u(0)=u(\pi)=0; \tag{1.4}$$

i.e., the distance between λ_1 and λ_2 determines the rate of the linear growth of the nonlinearity g.

The purpose of this paper is to show that we can consider also nonlinearity g with an arbitrary linear growth at $+\infty$ (resp. $-\infty$) but with the corresponding bound on their linear growth at $-\infty$ (resp. $+\infty$). To prove the result we use some properties of the "generalized eigenvalue problem"

$$u''+au^+-bu^-=0, \qquad u(0)=u(\pi)=0, \tag{1.5}$$

instead of the properties of (1.4) (here $u^\pm := (|u|\pm u)/2$ are the positive and the negative part of the function u). For example, our Theorem 3.1 implies that for any positive integer m BVP,

$$u''+u+mu^+-(m)^{-1/2}u^- = h(x), \qquad u(0)=u(\pi)=0,$$

has a solution for arbitrary h from $L^1(0,\pi)$.

Let us note that in contrast to the previous results our nonlinearity g may depend also on the variable x.

2. Some Properties of the Generalized Spectrum

In this section we present the results on the BVPs for the second-order ODEs which will be used throughout the rest of the paper. The first one is the following lemma by Fučík [6, Lemma 42.2] concerning the "generalized spectrum" of (1.5).

LEMMA 2.1. *Let $(a, b) \in \mathbb{R}^2$. The BVP (1.5) has a nontrivial solution if and only if*

$$(a, b) \in C_0^* \cup \left[\bigcup_{k=1}^{\infty} (C_k \cup C_k^*) \right],$$

where

$$C_0^* := \{(a, b) \in \mathbb{R}^2 : (a-1)(b-1) = 0\},$$
$$C_k := \{(a, b) \in \mathbb{R}^2 : a > k^2, b > 0, b^{1/2} = ka^{1/2}/(a^{1/2} - k)\},$$
$$C_k^* := \{(a, b) \in \mathbb{R}^2 : a > k^2, b > 0, b^{1/2} = (k+1)\, a^{1/2}/(a^{1/2} - k)\}$$
$$\cup \{(a, b) \in \mathbb{R}^2 : a > (k+1)^2, b > 0, b^{1/2} = ka^{1/2}(a^{1/2} - k - 1)\},$$

k is an integer.

Remark 2.1. The proof of this assertion may be found in [6]. The figure describing the structure of the sets C_k, C_k^* is drawn in [4].

Remark 2.2. The proof of Lemma 2.1 contains some useful information about the nontrivial solutions of (1.5). Namely, the nontrivial solution $u_{a,b}$ of (1.5) corresponding to $(a, b) \in C_k$, $k = 1, 2, ...$ (resp. $(a, b) \in C_k^*$, $k = 0, 1, 2, ...$) has precisely $2k - 1$ (resp. $2k$) zero points in $]0, \pi[$. The distance between two successive zero points is either $\pi a^{-1/2}$ or $\pi b^{-1/2}$ if $u_{a,b}$ is positive or negative, respectively, between these zero points.

The following assertion is most important in the proof of our result.

LEMMA 2.2. *Let $g_\pm$ be two functions in $L^\infty(0, \pi)$. Assume that there exists a real number $a > 1$ such that*

$$g_+(x) \leqslant a - \varepsilon \qquad \text{and} \qquad g_-(x) \leqslant a/(a^{1/2} - 1)^2 - \varepsilon \tag{2.1}$$

hold with some (arbitrary small) $\varepsilon > 0$. Then the nonlinear Dirichlet BVP

$$u'' + g_+(x)\, u^+ - g_-(x)\, u^- = 0, \qquad u(0) = u(\pi) = 0, \tag{2.2}$$

has either only a trivial solution or a nontrivial solution which is strictly negative or strictly positive in $]0, \pi[$.

Remark 2.3. Note that we have $(a, b) \in C_1$ if $b = a/(a^{1/2} - 1)^2$.

Proof of Lemma 2.2. Based essentially on the shooting method, let us suppose that (2.2) has a nontrivial solution u. Using (2.1) we can compare the zero points of u and $u_{a,b}$ (the nontrivial solution of (1.5) with $(a, b) \in C_1$; see Remark 2.3). This comparison proves that u has only one zero point in $]0, \pi]$ (cf. [4, Lemma 2.2]). In order to fulfil the boundary condition in π the nontrivial solution u must be either strictly negative or strictly positive in $]0, \pi[$.

3. Main Result

Let us consider BVP (1.1). We shall suppose that the right-hand side h is an element of the Banach space $X := L^1(0, \pi)$, equipped with the usual norm $\|\cdot\|$, and g is a Caratheodory function (i.e., $g(\cdot, s)$ is measurable for all s and $g(x, \cdot)$ is continuous for a.e. $x \in [0, \pi]$). Let

$$|g(x, s)| \leqslant p_1(x) + p_2 |s| \tag{3.1}$$

for a.e. $x \in [0, \pi]$ and for all $s \in \mathbb{R}$, with some $p_1 \in X$, $p_2 \in \mathbb{R}$, $p_2 \geqslant 0$.

We shall consider only such a function g that

$$\text{(g)}\; g^{-\infty}(x) = \limsup_{s \to -\infty} g(x, s) \qquad \text{and} \qquad g_{+\infty} = \liminf_{s \to +\infty} g(x, s)$$

are bounded from above and from below, respectively, for a.e. $x \in [0, \pi]$.

A solution u of (1.1) is a continuously differentiable function $u\colon [0, \pi] \to \mathbb{R}$ such that u' is absolutely continuous, u satisfies boundary conditions, and the equation (1.1) holds a.e. in $[0, \pi]$.

THEOREM 3.1. *Let us suppose that there exists some real number $a > 1$ such that*

$$\limsup_{s \to +\infty} \frac{g(x, s)}{s} \leqslant a - 1 - 2\varepsilon, \tag{3.2}$$

$$\limsup_{s \to -\infty} \frac{g(x, s)}{s} \leqslant a/(a^{1/2} - 1)^2 - 1 - 2\varepsilon \tag{3.3}$$

hold for a.e. $x \in [0, \pi]$ with some small $\varepsilon > 0$. Moreover, let (g) *hold and*

$$\int_0^\pi g^{-\infty}(x) \sin x \, dx < \int_0^\pi h(x) \sin x \, dx < \int_0^\pi g_{+\infty}(x) \sin x \, dx \tag{3.4}$$

for a.e. $x \in [0, \pi]$. Then (1.1) *is solvable.*

Proof. The idea of the proof is analogous to that used in [1] and it is based on the well-known continuation method of Leray and Schauder. Consider the linear operator $K\colon X \to X$ defined by $Ke :=$ the unique solution u of the linear BVP: $u'' + (1 + d)\, u = e$ and $u(0) = u(\pi) = 0$, with $0 < d < \min\{\varepsilon, 3\}$. It is easy to see that K is a well-defined, completely continuous operator. The standard regularity argument for ODEs proves that K maps bounded sets in $L^1(0, \pi)$ onto relatively compact sets in $C(0, \pi)$.

Since the Nemytski's operator induced by g is continuous, $X \to X$ and maps bounded sets into bounded sets, the map

$$H: [0, 1] \times X \to X,$$
$$H(\tau, u) := u - K(h + \tau(du - g(\cdot, u)))$$

is a homotopy of completely continuous perturbations of the identity. Our aim is to prove that

(r) there exist $r > 0$ such that $\|u\| < r$ for any $(\tau, u) \in [0, 1] \times X$ satisfying $H(\tau, u) = 0$.

Let us assume the contrary. Then there is a sequence $(\tau_n, u_n) \in [0, 1] \times X$ verifying $H(\tau_n, u_n) = 0$ and $\|u_n\| > n$, for every $n \geq 0$. Hence the normalized sequence $v_n := u_n/\|u_n\|$ verifies

$$v_n = K(h\|u_n\|^{-1} + \tau_n(dv_n - \|u_n\|^{-1} g(\cdot, u_n))). \tag{3.5}$$

According to (3.1) the sequence $g_n := \|u_n\|^{-1} g(\cdot, u_n)$ is bounded in X. Therefore, passing if necessary to subsequences, we can assume that $v_n \to v$ uniformly on $[0, \pi]$ (we use that K is a completely continuous operator). But in this case (3.1) implies that

$$|g_n(x)| \leq |p_1(x)| \, \|u_n\|^{-1} + p_2|v_n| \leq p(x),$$

for all n, with some $p \in X$. Hence

$$\int_{x_1}^{x_2} |g_n(x)| \, dx \to 0 \qquad \text{for} \quad |x_1 - x_2| \to 0 \tag{3.1'}$$

uniformly with respect to n. Therefore $\{g_n\}_{n=1}^{\infty}$ is weakly sequentially compact (see [5, IV. Corollary 8.11]), i.e., there is some $f \in X$ such that $\{g_n\}_{n=1}^{\infty}$ converges weakly to f (passing if necessary to subsequences). Simultaneously we obtain from here that $\lim |v_n'(x_1) - v_n'(x_2)| = 0$, for $|x_1 - x_2| \to 0$, uniformly with respect to n. Indeed, it is sufficient to realize that (3.5) is equivalent to

$$v_n'' + v_n + d(1 - \tau_n) v_n + \tau_n g_n = \|u_n\|^{-1} h, \qquad v_n(0) = v_n(\pi) = 0,$$

and to take into account (3.1'). We also claim that $\|v_n''\|$ is bounded independently of n. Since by Rolle's theorem, v_n' must vanish somewhere in $]0, \pi[$, the sequence $\{v_n'\}_{n=1}^{\infty}$ is both equicontinuous and uniformly bounded on $[0, \pi]$. Therefore, by using the Arzela–Ascoli theorem we may also assume that $v_n' \to v'$ uniformly on $[0, \pi]$. Of course, we may assume $\tau_n \to \tau \in [0, 1]$. Since every bounded linear map is continuous as well as weakly continuous, we can pass to the weak limit in (3.5) and we get

$$v = K(\tau \, dv - \tau f). \tag{3.6}$$

Note that with respect to (g)

$$\liminf_{s \to \pm\infty} \frac{g(x, s)}{s} \geqslant 0 \tag{3.7}$$

for a.e. $x \in [0, \pi]$. Then it is a direct consequence of (3.2), (3.3), and (3.7) (by using Lebesgue's theorem and Fatou's lemma) that

$$f(x) = p_+(x)\, v^+(x) - p_-(x)\, v^-(x) \qquad \text{a.e. on } [0, \pi], \tag{3.8}$$

with the functions $p_\pm$ from $L^\infty(0, \pi)$ verifying

$$0 \leqslant p_+(x) \leqslant a - 1 - 2\varepsilon, \qquad 0 \leqslant p_-(x) \leqslant a/(a^{1/2} - 1)^2 - 1 - 2\varepsilon \tag{3.9}$$

a.e. on $[0, \pi]$ (cf. [4]). But (3.9) implies that the functions $g_\pm(x) = \tau p_\pm(x) + 1 + d(1 - \tau)$ satisfy the assumption (2.1). Hence we obtain from (3.6), (3.8), and Lemma 2.2 that the function v (note that $\|v\| = 1$) does not change sign in $]0, \pi[$. Assuming that $v(x) > 0$ in $]0, \pi[$ we arrive at a contradiction with (3.4) (the alternative case $v(x) < 0$ in $]0, \pi[$ will also lead to a contradiction with (3.4)). The operator equation $H(\tau_n, u_n) = 0$ is equivalent to

$$u_n'' + u_n + (1 - \tau_n)\, du_n + \tau_n g(x, u_n) = h(x), \qquad u_n(0) = u_n(\pi) = 0, \tag{3.10}$$

for a.e. $x \in [0, \pi]$. Multiplying (3.10) by $\sin x$ and integrating by parts, we obtain

$$\int_0^\pi [d(1 - \tau_n)\, u_n + \tau_n g(x, u_n)] \sin x\, dx = \int_0^\pi h(x) \sin x\, dx. \tag{3.11}$$

Since $v'(0) > 0$, $v'(\pi) < 0$ and $v_n' \to v'$ uniformly on $[0, \pi]$ as $n \to \infty$, it follows that $u_n(x) > 0$ on $]0, \pi[$, for large n. Consequently the sequence $z_n(x) = d(1 - \tau_n)\, u_n(x) + \tau_n g(x, u_n)$ is bounded from below a.e. in $[0, \pi]$ independently of n (see (g)) and $u_n(x) \to \infty$ uniformly on compact subintervals of $]0, \pi[$. Hence it follows from (3.4) that

$$\int_0^\pi h(x) \sin x\, dx < \int_0^\pi \left[\liminf_{n \to +\infty} z_n(x) \right] \sin x\, dx. \tag{3.12}$$

On the other hand, Fatou's lemma and (3.11) imply

$$\int_0^\pi \left[\liminf_{n \to +\infty} z_n(x) \right] \sin x\, dx \leqslant \liminf_{n \to +\infty} \int_0^\pi z_n(x) \sin x\, dx$$
$$= \int_0^\pi h(x) \sin x\, dx,$$

which contradicts (3.12). This proves (r), and we can apply Leray–Schauder's continuation principle. We conclude that

$$\deg(H(1, u); B_r(0), 0) = \deg(H(0, u); B_r(0), 0), \tag{3.13}$$

where $B_r(0)$ denotes the ball in X centered at the origin and with the radius $r > 0$. Assuming r large enough also the homotopy

$$\bar{H}: [0, 1] \times X \to X,$$
$$\bar{H}(\sigma, u) := u - K((1-\sigma) h),$$

does not vanish for $(\sigma, u) \in [0, 1] \times \partial B_r(0)$. Note that $\bar{H}(0, u) = H(0, u)$, for all $u \in X$. Hence (3.13) yields

$$\deg(H(1, u); B_r(0), 0) = \deg(\bar{H}(1, u); B_r(0), 0).$$

But $u \to \bar{H}(1, u)$ is a linear one-to-one map and so we have

$$\deg(H(1, u); B_r(0), 0) \neq 0.$$

This means that

$$u = K(h - g(\cdot, u) + du)$$

has a solution which is a solution of (1.1) by a standard regularity argument. This proves the theorem.

Remark 3.1. Let us suppose (instead of (g)) that

(g′) $g^{+\infty}(x) = \limsup_{s \to +\infty} g(x, s)$ and $g_{-\infty}(x) = \liminf_{s \to -\infty} g(x, s)$

are bounded from above and from below, respectively, a.e. in $[0, \pi]$.

Then

$$\limsup_{s \to \pm\infty} \frac{g(x, s)}{s} \leqslant 0 \qquad \text{and} \qquad \liminf_{s \to \pm\infty} \frac{g(x, s)}{s} \geqslant -p_2, \tag{3.14}$$

with respect to (3.1) for any function g satisfying (g′). Replacing $g^{-\infty}(x)$ and $g_{+\infty}(x)$ in (3.4) by $g^{+\infty}(x)$ and $g_{-\infty}(x)$, respectively, Theorem 3.1 will hold. Indeed, in this case the limit functions $p_{\pm}$ (see the proof of Theorem 3.1) satisfy $-p_2 \leqslant p_{\pm} \leqslant 0$ because of (3.14). Choose $d < 0$ in the definition of K. Then $g_{\pm}$ fulfil the hypotheses of Lemma 2.2 with an arbitrary $a > 1$, and the proof can be performed in an analogous way. It means that (3.4) is a sufficient condition for the solvability of (1.1) even if $g(\cdot, s)$ has an arbitrary rate of the linear growth at $\pm\infty$ (cf. [3]).

Remark 3.2. As was already pointed out in the Introduction, [1] implies that the rate of the linear growth of g at $\pm\infty$ is related to the spectrum of (1.4) and hence to the nodal properties of the corresponding eigenfunctions. In this case (1.2) is a sufficient condition for the solvability of (1.1) since the nonlinearity g is not at resonance with the second eigenvalue of (1.4). Our result is exactly in the same spirit. Nonlinearity g is not at resonance with the second "generalized eigenvalue"

$$C_1 = \{(a, b) \in \mathbb{R}^2 : a > 1, b^{1/2} = a^{1/2}/(a^{1/2} - 1)\}$$

of (1.5).

Remark 3.3. Some nonresonance problems are studied in [2, 4] using the description of the "generalized spectrum" of BVPs of the type (1.5).

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