Nondecreasing Solutions of a Quadratic Integral Equation of Urysohn Type

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Abstract—Applying the technique associated with measures of noncompactness, we prove the existence of nondecreasing solutions of a quadratic integral equation of Urysohn type in the space of real functions defined and continuous on a closed bounded interval. © 2006 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

The technique of measures of noncompactness is often used in several branches of nonlinear analysis. Especially, that technique turns out to be very useful tool in the existence theory for several types of integral equations [1-4].

It is worthwhile mentioning that in applications, the most useful measures of noncompactness are those defined in an axiomatic way [4]. It is caused by the fact that applying such measures of noncompactness and the fixed-point theorem of Darbo type we are able to prove not only the existence of solutions of investigated integral equations but we can also obtain some characterization of those solutions (cf., [4,5]).

In this paper, we use a special measure of noncompactness defined in [6]. The use of that measure enables us to study the solvability of integral equations in the class of nondecreasing functions.

The results obtained in the present paper extend and generalize several ones concerning integral equations of Urysohn type which were intensively studied in mathematical literature (cf., [7-12], for instance).
2. NOTATION AND AUXILIARY RESULTS

Now, we are going to present definitions and basic facts needed further on.

Let $E$ be an infinitely dimensional, real Banach space with the norm $\|\cdot\|$ and the zero element $0$. Denote by $B(x, r)$ the closed ball centered at $x$ and with radius $r$. We write $B_r$ for $B(0, r)$. If $X$ is a nonempty subset of $E$, we denote by $\bar{X}$ and $\text{Conv}X$ the closure and the convex closure of $X$, respectively. Moreover, we use the standard notation $X + Y$ and $\lambda X$ for algebraic operations on sets.

Further on, denote by $\mathcal{M}_E$ the family of nonempty and bounded subsets of $E$ and by $\mathfrak{m}_E$ its subfamily consisting of all relatively compact sets. We accept the following definition [4].

**DEFINITION 2.1.** A function $\mu : \mathcal{M}_E \rightarrow [0, \infty)$ is said to be a measure of noncompactness in $E$ if it satisfies the following conditions,

1. the family $\ker \mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{m}_E$;
2. $X \subseteq Y \implies \mu(X) \leq \mu(Y)$;
3. $\mu(\bar{X}) = \mu(\text{Conv}X) = \mu(X)$;
4. $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$;
5. if $(X_n)$ is a sequence of closed sets from $\mathcal{M}_E$ such that $X_{n+1} \subseteq X_n$, for $n = 1, 2, \ldots$, and if

$$\lim_{n \to \infty} \mu(X_n) = 0,$$

then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ described in 1. is referred to as the kernel of the measure of noncompactness $\mu$. For further details concerning measures of noncompactness and its properties, we refer to [4]. For our purposes, we will only need the following fixed-point theorem of Darbo type [4,13].

**THEOREM 2.1.** Let $Q$ be a nonempty bounded closed convex subset of the space $E$ and let $F : Q \rightarrow Q$ be a continuous operator such that $\mu(FX) \leq k \mu(X)$ for any nonempty subset $X$ of $Q$, where $k \in [0, 1)$ is a constant. Then, $F$ has a fixed point in the set $Q$.

**REMARK 2.1.** Observe that under the assumptions of the above theorem it can be shown that the set $\text{Fix}F$ consisting of all fixed point of $F$ belonging to $Q$ is a member of the family $\ker \mu$ (cf., [4]). This observation enables us to characterize solutions of the studied operator equations.

In the sequel, we will work in the classical Banach space $C[a, b]$ consisting of all real functions defined and continuous on the interval $[a, b]$. For convenience, we take $[a, b] = [0, 1] = I$ and we write $C(I)$ instead of $C[0, 1]$. The space $C(I)$ is equipped with the standard norm $\|x\| = \max \{|x(t)| : t \in I\}$.

In what follows, let us recall the definition of a measure of noncompactness in $C(I)$ which will be used in our investigations. This measure was introduced in the paper [6].

To define the mentioned measure, let us fix a nonempty and bounded subset $X$ of $C(I)$. For $x \in X$ and $\varepsilon > 0$ denote by $\omega(x, \varepsilon)$ the modulus of continuity of the function $x$, i.e.,

$$\omega(x, \varepsilon) = \sup \{|x(t) - x(s)| : t, s \in I, |t - s| \leq \varepsilon\}.$$

Next, let us put

$$\omega(X, \varepsilon) = \sup \{\omega(x, \varepsilon) : x \in X\}, \quad \omega_0(X) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon).$$

Further, let us define the following quantities,

$$d(x) = \sup \{|x(s) - x(t)| - |x(s) - x(t)| : t, s \in I, t \leq s\},$$

$$d(X) = \sup \{d(x) : x \in X\}.$$

Notice that $d(X) = 0$ if and only if all functions belonging to $X$ are nondecreasing on $I$. 
Finally, let us define the function $\mu$ by putting

$$\mu(X) = \omega_0(X) + d(X).$$

It can be proved [6] that the function $\mu$ is a measure of noncompactness in the space $C(I)$. The kernel $\ker\mu$ of this measure contains all nonempty and bounded subsets $X$ of $C(I)$ such that functions from $X$ are equicontinuous and nondecreasing on the interval $I$. Moreover, the measure $\mu$ has also some additional properties (cf., [6]).

### 3. MAIN RESULT

In this section, we will study the nonlinear quadratic integral equation of Urysohn type having the form,

$$x(t) = a(t) + f(t, x(t)) \int_0^1 u(t, s, x(s)) \, ds, \quad t \in I. \quad (3.1)$$

The functions $a(t)$, $f = f(t, x)$ and $u = u(t, s, x)$ appearing in this equation are given while $x = x(t)$ is an unknown function.

Let us mention that the particular case of the above equation having the form,

$$f_0(t) = f(t, \cdot) u(t, s, \cdot) \, ds$$

can be encountered in some models connected with the traffic theory and biology, among others [10].

For further purposes let us also recall that the function $f = f(t, x)$ involved in equation (3.1) generates the operator $F$ defined by the formula,

$$(Fx)(t) = f(t, x(t)), \quad (3.2)$$

where $x = x(t)$ is an arbitrary function defined on the interval $I$. Such an operator is called the superposition operator and has several interesting properties (cf., the monograph [14]).

In what follows, we will investigate the quadratic integral equation (3.1) assuming that the following conditions are satisfied.

(i) $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that there exists a constant $k > 0$ such that

$$|f(t, x) - f(t, y)| \leq k|x - y|$$

for all $t \in I$ and $x, y \in \mathbb{R}$. Moreover, $f : I \times \mathbb{R}_+ \to \mathbb{R}_+.$

(ii) $d(Fx) \leq kd(x)$ for any nonnegative function $x \in C(I)$, where $F$ is the superposition operator defined by (3.2) and $k$ is the same constant as in (i).

(iii) $a \in C(I)$ and $a$ is a nondecreasing and nonnegative on the interval $I$.

(iv) $u : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $u : I \times I \times \mathbb{R}_+ \to \mathbb{R}_+$ and for arbitrarily fixed $s \in I$ and $x \in \mathbb{R}_+$ the function $t \to u(t, s, x)$ is nondecreasing on $I$.

(v) There exists a nondecreasing function $m : \mathbb{R}_+ \to \mathbb{R}_+$ such that $|u(t, s, x)| \leq m(|x|)$ for all $t, s \in I$ and $x \in \mathbb{R}$.

(vi) The inequality,

$$\|a\| + (b + kr)m(r) \leq r$$

has a positive solution $r_0$ such that $km(r_0) < 1$, where $b = \max \{f(t, 0) : t \in I\}$.

Now, we can formulate our main result.
THEOREM 3.1. Under Assumptions (i)–(vi), equation (3.1) has at least one solution \( x = x(t) \) which belong to the space \( C(I) \) and is nondecreasing on the interval \( I \).

PROOF. Let us consider the operator \( U \) defined on the space \( C(I) \) by the formula,

\[
(Ux)(t) = a(t) + f(t, x(t)) \int_0^1 u(t, \tau, x(\tau)) \, d\tau.
\]

Taking into account Assumptions (i), (iii), (iv), and the properties of the superposition operator (cf., [14]), we infer that the function \( Ux \) is continuous on \( I \) for any function \( x \in C(I) \), i.e., the operator \( U \) transforms the space \( C(I) \) into itself.

Moreover, in view of Assumptions (i) and (v), we get

\[
|Ux(t)| \leq |a(t)| + \|f(t, x(t))\| \int_0^1 |u(t, \tau, x(\tau))| \, d\tau
\]

\[
\leq |a(t)| + \|f(t, x(t)) - f(t, 0)\| + \|f(t, 0)\| \int_0^1 m(|x(\tau)|) \, d\tau
\]

\[
\leq |a(t)| + (k \|x(t)\| + f(t, 0)) \int_0^1 m(|x(\tau)|) \, d\tau.
\]

The above inequality yields

\[
\|Ux\| \leq \|a\| + (k \|x\| + b) \int_0^1 m(\|x\|) \, d\tau
\]

\[
\leq \|a\| + (k \|x\| + b)m(\|x\|),
\]

where \( b = \max\{f(t, 0) : t \in I\} \).

Hence, keeping in mind Assumption (vi), we deduce that there exists \( r_0 > 0 \) with \( km(r_0) < 1 \) and such that the operator \( U \) transforms the ball \( B_{r_0} \) into itself.

In the sequel, we will consider the operator \( U \) on the subset \( B_{r_0}^+ \) of the ball \( B_{r_0} \) defined as follows,

\[
B_{r_0}^+ = \{x \in B_{r_0} : x(t) \geq 0 \text{ for } t \in I\}.
\]

Obviously, the set \( B_{r_0}^+ \) is nonempty, bounded, closed, and convex. Hence and in view of Assumptions (i), (iii), and (iv), we infer that \( U \) transforms the set \( B_{r_0}^+ \) into itself.

In what follows, we show that \( U \) is continuous on the set \( B_{r_0}^+ \). To do this, let us fix \( \varepsilon > 0 \) and take arbitrarily \( x, y \in B_{r_0}^+ \) such that \( \|x - y\| \leq \varepsilon \). Then, for \( t \in I \), we derive the following chain of estimates,

\[
|(Ux)(t) - (Uy)(t)| \leq \left| f(t, x(t)) \int_0^1 u(t, \tau, x(\tau)) \, d\tau - f(t, y(t)) \int_0^1 u(t, \tau, x(\tau)) \, d\tau \right|
\]

\[
+ \left| f(t, y(t)) \int_0^1 u(t, \tau, x(\tau)) \, d\tau - f(t, y(t)) \int_0^1 u(t, \tau, y(\tau)) \, d\tau \right|
\]

\[
\leq |f(t, x(t)) - f(t, y(t))| \int_0^1 u(t, \tau, x(\tau)) \, d\tau
\]

\[
+ |f(t, y(t))| \int_0^1 |u(t, \tau, x(\tau)) - u(t, \tau, y(\tau))| \, d\tau
\]

\[
\leq k \|x(t) - y(t)\| \int_0^1 m(x(\tau)) \, d\tau
\]
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\[ + \|[f(t, y(t)) - f(t, 0)] + f(t, 0)\| \int_0^1 |u(t, \tau, x(\tau)) - u(t, \tau, y(\tau))| \, d\tau \]

\[ \leq k \varepsilon m(r_0) + (kr_0 + b) \int_0^1 \beta_{r_0}(\varepsilon) \, d\tau = k \varepsilon m(r_0) + (kr_0 + b)\beta_{r_0}(\varepsilon), \]

where we denoted

\[ \beta_{r_0}(\varepsilon) = \sup \{ |u(t, \tau, x) - u(t, \tau, y)| : t, \tau \in I, \, x, y \in [0, r_0], \, |x - y| \leq \varepsilon \}. \]

Observe that \( \beta_{r_0}(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \) which is a consequence of the uniform continuity of the function \( u \) on the set \( I \times I \times [0, r_0] \).

Next, let us notice that from the above estimates, we derive the following inequality,

\[ \|Ux - Uy\| \leq k \varepsilon m(r_0) + (kr_0 + b)\beta_{r_0}(\varepsilon), \]

which yields the continuity of the operator \( U \) on the set \( B_{r_0}^{+} \).

Now, let us take a nonempty set \( X, \, X \subset B_{r_0}^{+} \). Further, fix arbitrarily \( \varepsilon > 0 \) and choose \( x \in X \) and \( t, s \in I \) such that \( |t - s| \leq \varepsilon \). Then, keeping in mind our assumptions, we obtain

\[ |(Ux)(s) - (Ux)(t)| \leq |a(s) - a(t)| + \left| f(s, x(s)) \int_0^1 u(s, \tau, x(\tau)) \, d\tau - f(t, x(t)) \int_0^1 u(t, \tau, x(\tau)) \, d\tau \right| \]

\[ \leq \omega(a, \varepsilon) + \left| f(s, x(s)) \right| \int_0^1 |u(s, \tau, x(\tau)) - u(t, \tau, x(\tau))| \, d\tau \]

\[ \leq \omega(a, \varepsilon) + |f(s, x(s)) - f(t, x(t))| \int_0^1 |u(s, \tau, x(\tau)) - u(t, \tau, x(\tau))| \, d\tau \]

\[ + |f(t, x(t))| \int_0^1 |u(s, \tau, x(\tau)) - u(t, \tau, x(\tau))| \, d\tau \]

\[ \leq \omega(a, \varepsilon) + \|f(s, x(s)) - f(t, x(s))\| + |f(t, x(t)) - f(t, x(t))| \int_0^1 m(x(\tau)) \, d\tau \]

\[ + \|f(t, x(t)) - f(t, 0)\| + f(t, 0)\| \int_0^1 |u(s, \tau, x(\tau)) - u(t, \tau, x(\tau))| \, d\tau \]

\[ \leq \omega(a, \varepsilon) + \omega_{r_0}(f, \varepsilon) + k \|x(s) - x(t)\| m(r_0) + |k |x(t)| + b| \int_0^1 \gamma_{r_0}(u, \varepsilon) \, d\tau \]

\[ \leq \omega(a, \varepsilon) + \omega_{r_0}(f, \varepsilon) + k \omega(\varepsilon) m(r_0) + (kr_0 + b)\gamma_{r_0}(u, \varepsilon), \]

where we have denoted

\[ \omega_{r_0}(f, \varepsilon) = \sup \{ |f(t, x) - f(s, x)| : t, s \in I, \, |t - s| \leq \varepsilon, \, x \in [0, r_0] \}, \]

\[ \gamma_{r_0}(u, \varepsilon) = \sup \{ |u(s, \tau, x) - u(t, \tau, x)| : t, s, \tau \in I, \, |t - s| \leq \varepsilon, \, x \in [0, r_0] \}. \]
Notice that, in view of the uniform continuity of the function $f$ on the set $I \times [0, r_0]$, we have that $\omega_{r_0}(f, \varepsilon) \to 0$ as $\varepsilon \to 0$. Similarly, since the function $u$ is uniformly continuous on the set $I \times I \times [0, r_0]$ we conclude that $\omega_{r_0}(u, \varepsilon) \to 0$ as $\varepsilon \to 0$. These facts in conjunction with the above estimate allow us to arrive at the following inequality,

$$\omega_0(U_X) \leq km(r_0)\omega_0(X).$$

(3.3)

Further, fix arbitrarily $x \in X$ and $t, s \in I$ with $t \leq s$. Then, taking into account the assumptions of our theorem, we get

$$\left|\left[(U_x)(s) - (U_x)(t)\right] - \left[(U_x)(s) - (U_x)(t)\right]\right|$$

$$\leq \left|\left[a(s) - a(t)\right] - \left[a(s) - a(t)\right]\right|$$

$$+ \left|f(s, x(s)) \int_0^1 u(s, \tau, x(\tau)) d\tau - f(t, x(t)) \int_0^1 u(t, \tau, x(\tau)) d\tau\right|$$

$$- \left[f(s, x(s)) \int_0^1 u(s, \tau, x(\tau)) d\tau - f(t, x(t)) \int_0^1 u(t, \tau, x(\tau)) d\tau\right]$$

$$\leq \left|f(s, x(s)) \int_0^1 u(s, \tau, x(\tau)) d\tau - f(t, x(t)) \int_0^1 u(t, \tau, x(\tau)) d\tau\right|$$

$$+ \left|f(t, x(t)) \int_0^1 u(s, \tau, x(\tau)) d\tau - f(t, x(t)) \int_0^1 u(t, \tau, x(\tau)) d\tau\right|$$

$$- \left[f(s, x(s)) \int_0^1 u(s, \tau, x(\tau)) d\tau - f(t, x(t)) \int_0^1 u(t, \tau, x(\tau)) d\tau\right]$$

$$\leq \left|f(s, x(s)) - f(t, x(t))\right| \int_0^1 u(s, \tau, x(\tau)) d\tau$$

$$+ \left|f(t, x(t)) \int_0^1 u(s, \tau, x(\tau)) d\tau - \int_0^1 u(t, \tau, x(\tau)) d\tau\right|$$

$$- \left[f(s, x(s)) - f(t, x(t))\right] \int_0^1 u(s, \tau, x(\tau)) d\tau$$

$$- f(t, x(t)) \left[\int_0^1 u(s, \tau, x(\tau)) d\tau - \int_0^1 u(t, \tau, x(\tau)) d\tau\right]$$

$$\leq \left|f(s, x(s)) - f(t, x(t))\right| - \left[f(s, x(s)) - f(t, x(t))\right] \int_0^1 u(s, \tau, x(\tau)) d\tau$$
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\[ + f(t, x(t)) \int_0^1 |u(s, \tau, x(\tau)) - u(t, \tau, x(\tau))| \, d\tau \]

\[ - f(t, x(t)) \int_0^1 |u(s, \tau, x(\tau)) - u(t, \tau, x(\tau))| \, d\tau \]

\[ \leq d(Fx) \int_0^1 u(s, \tau, x(\tau)) \, d\tau \leq k d(x) \int_0^1 m(x(\tau)) \, d\tau \leq k d(x) m(r_0). \]

This estimate implies

\[ d(UX) \leq km(r_0) d(X). \tag{3.4} \]

Now, joining (3.3), (3.4), and the definition of the measure of noncompactness \( \mu \) given in Section 2, we get

\[ \mu(UX) \leq km(r_0) \mu(X). \]

Hence, taking into account that \( km(r_0) < 1 \) and applying Theorem 2.1, we infer that equation (3.1) has at least one solution belonging to the space \( C(I) \). Moreover, in view of Remark 2.1 and the description of the kernel of a measure of noncompactness \( \mu \) (cf., Section 2), we deduce that all solutions of the integral equation (3.1) belonging to the set \( B^{+}_{r_0} \) are nondecreasing on the interval \( I \). This completes the proof.

**Remark 3.1.** Observe that assuming additionally that \( a(t) > 0 \), for \( t \in I \), we infer that all solutions of the equation (3.1) which belong to \( B^{+}_{r_0} \) are continuous, nondecreasing, and positive on the interval \( I \).

### 4. Examples and Final Remarks

In this section, we provide a few examples concerning Theorem 3.1 and connected mainly with Assumptions (i), (ii), (v), and (vi) of this theorem.

**Example 4.1.** Assume that the function \( f = f(t, x) \) appearing in Assumptions (i) and (ii) has the form \( f(t, x) = p(t)x + q(t) \), where \( p(t) \) and \( q(t) \) are continuous and nondecreasing on the interval \( I \) and \( p(t) > 0 \) and \( q(t) \geq 0 \), for \( t \in I \).

In this case it is easily seen that the function \( f(t, x) \) satisfies Assumption (i), where the constant \( k \) is defined by the equality,

\[ k = \max[p(t) : t \in I]. \]

Moreover, let us observe that the function \( f(t, x) \) satisfies also Assumption (ii).

Indeed, taking an arbitrary nonnegative function \( x \in C(I) \) and \( t, s \in I \) such that \( t \leq s \), we obtain

\[
\begin{align*}
|(Fx)(s) - (Fx)(t)| - [(Fx)(s) - (Fx)(t)] &= |f(s, x(s)) - f(t, x(t))| - |f(s, x(s)) - f(t, x(t))| \\
&= |p(s)x(s) + q(s) - p(t)x(t) - q(t)| \\
&\quad - |p(s)x(s) + q(s) - p(t)x(t) - q(t)| \\
&\leq |p(s)x(s) - p(s)x(t)| + |p(s)x(t) - p(t)x(t)| \\
&\quad + |q(s) - q(t)| \\
&\quad - |p(s)x(s) - p(s)x(t)| \\
&\quad - |p(s)x(t) - p(t)x(t)| - |q(s) - q(t)|
\end{align*}
\]
Hence, we have that
\[
d(Fx) \leq k d(x).
\]

**Example 4.2.** Now, let us assume that the function \( m \) appearing in Assumptions (v) and (vi) has the form \( m(r) = cr \), where \( c \) is a positive constant such that \( bc < 1 \) and \( (1 - bc)^2 \geq 4kc \|a\| \). Then, the inequality from Assumption (vi) has the form,
\[
\|a\| + (b + kr)cr \leq r.
\]

Notice that the number,
\[
r_0 = \frac{1 - bc - \sqrt{(1 - bc)^2 - 4kc \|a\|}}{2kc}
\]
is a positive solution of the above inequality. Moreover, we have
\[
km(r_0) = \frac{1 - bc - \sqrt{(1 - bc)^2 - 4kc \|a\|}}{2} \leq \frac{1}{2} < 1.
\]

**Example 4.3.** Assume that the function \( m \) from Assumptions (v) and (vi) is given by the formula \( m(r) = r^2 \). Then, the inequality from (vi) has the form,
\[
\|a\| + (b + kr)r^2 \leq r.
\]

It is easy to check that taking the constants \( b \) and \( k \) such that \( k < 2b^2 \) and assuming that \( a = a(t) \) is a nondecreasing function belonging to the space \( C(I) \) such that \( \|a\| < (2b^2 - k)/8b^3 \), we have that \( r_0 = 1/2b \) is a solution of the above inequality. Moreover, \( km(r_0) = k/4b^2 < 1/2 \). We omit the simple calculations.

**Example 4.4.** We show that Assumption (i) is essential for validity of Theorem 3.1.

Indeed, take \( a(t) \equiv 1 \), \( u(t, s, x) \equiv 1 \) and \( f(t, x) = x^2 \). Then, it is easily seen that there are satisfied Assumptions (iii), (iv), and (v) of Theorem 3.1. We show that Assumption (ii) is also satisfied if we restrict ourselves for functions belonging to the set \( B^+_r = \{ x \in B_r : x(t) \geq 0 \text{ for } t \in I \} \), where \( r \) is arbitrarily fixed positive constant.

In fact, for \( x \in B^+_r \) and \( t, s \in I \) with \( t \leq s \), we have
\[
|\langle (Fx)(s) - (Fx)(t) \rangle - [(Fx)(s) - (Fx)(t)]| = |x^2(s) - x^2(t)| - [x^2(s) - x^2(t)]
\]
\[
= |x(s) - x(t)| \cdot |x(s) + x(t)|
\]
\[
= |x(s) + x(t)| \cdot [\{x(s) - x(t)\} - [x(s) - x(t)]]
\]
\[
= |x(s) + x(t)| \cdot 2r \cdot d(x).
\]
Hence,

\[ d(Fx) \leq 2r \, d(x). \]

It is obvious that the function \( f(t, x) = x^2 \) does not satisfy Assumption (i). On the other hand, let us observe that in our case the integral equation (3.1) has the form,

\[ x(t) = 1 + x^2(t) \int_0^1 ds = 1 + x^2(t), \]

and it is not solvable in \( C(I) \).

**EXAMPLE 4.5.** Consider the following quadratic integral equation,

\[ x(t) = t^2 e^{-t} + \frac{t}{1 + t^2} \int_0^1 \arctg \left( \frac{tx^2(s)}{1 + s^2} \right) ds, \tag{4.1} \]

where \( t \in I = [0, 1] \).

We investigate the solvability of this equation on the base of Theorem 3.1.

First of all, observe that we have that \( a(t) = t^2 e^{-t} \), so the function \( a(t) \) satisfies Assumption (iii) with \( \|a\| = 1/e \).

Further notice that \( f(t, x) = tx/(1 + t^2) \). Thus, the function \( f(t, x) \) satisfies Assumption (i) since \( f : I \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) and

\[ |f(t, x) - f(t, y)| \leq \frac{1}{2} |x - y|, \]

for \( x, y \in \mathbb{R} \) and \( t \in I \). Moreover, keeping in mind Example 4.1 it is easy to check that this function satisfies Assumption (ii).

Next, we have that the function \( u(t, s, x) \) involved in equation (4.1) has the form,

\[ u(t, s, x) = \arctg \frac{tx^2}{1 + s^2}. \]

Obviously this function satisfies Assumption (iv).

Moreover, we get

\[ |u(t, s, x)| \leq x^2, \]

so the function \( m(r) \) appearing in Assumption (v) has the form \( m(r) = r^2 \).

Now, let us consider the inequality associated with Assumption (vi), which has the form \( (b = 0, k = 1/2) \),

\[ \frac{1}{e} + \frac{1}{2} r^3 \leq r, \]

or equivalently,

\[ 2r - r^3 \geq \frac{2}{e}. \tag{4.2} \]

Using the standard methods of differential calculus we can verify that the function \( g(r) = 2r - r^3 \) attains its maximum at the point \( r_0 = \sqrt{2/3} \) and \( g(r_0) = (4/3) \sqrt{2/3} > 2/e \). So, the number \( r_0 \) is a positive solution of the inequality (4.2) for which \( km(r_0) = 1/3 < 1 \).

Finally, taking into account all the above established facts and Theorem 3.1 we conclude that the equation (4.1) has at least one solution \( x = x(t) \) defined, continuous and nondecreasing on the interval \( I \). Moreover, \( \|x\| \leq r_0 = \sqrt{2/3} \).
REFERENCES