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Hamiltonian systems of propagation of elastic waves and localized vibrations in the strip plate

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Abstract

In this paper, based on Lagrange–Germanian theory of elastic thin plates, applying the method in Hamiltonian state space, the elastic waves and vibrations when the boundary of the two lateral sides of the strip plate are free of traction are investigated, and the process of analysis and solution are proposed. The existence of all kinds of vibration modes and wave propagation modes is also analyzed. By using eigenfunction expansion method, the dispersion relations of wave-guide modes in the strip plate are derived, and the comparisons with the dispersion relations directly obtained by the classical theory of thin plates are also presented. At last, the results are analyzed and discussed.

Keywords: Thin plates; Wave propagation and localized vibration; Hamiltonian systems; Dispersion relation; Vibration control of spacecraft structure

1. Introduction

The classical theory of thin plates was deduced by Lagrange–German in the 19th century. Because of the simplicity of governing equations, it was widely applied to analyze engineering structures. In the past, semi-inverse method was always applied to solve flexural vibration in plates as well as elastodynamics. Rytwinska and Kwiecinski (1983) applied a semi-inverse method to obtain exact solutions to thin, rigid-ideally plastic plates resting on beams that are capable of deflecting together with the edges of plates. Using the semi-inverse method, Levinson (1985) gave the three-dimensional solution of the free vibration for the free vibration of simply supported, rectangular plates of arbitrary thickness within the linear theory of elastodynamics. Because this method has many limitations, it is difficult to analyze complex boundary problems. However, Hamiltonian systems can solve many boundary problems which cannot be solved by the classical methods.

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There are two frames in mechanical analysis. One is analyzed in the tangent bundles; the other is in the cotangent bundles. The former is the Lagrangian systems, the latter is the Hamiltonian systems. Feng Kang ever suggested applying Hamiltonian formalism when investigating the Finite Element Method in early stage. As strip plate structures are widely used in aerospace and civil engineering, Xu and Datta (1990) analyzed elastic waveguide in these structures. With regard to the problems of elastodynamics and static mechanics in plates, Zhong (1995) and Zhong and Yao (1999) studied them by making use of the method of state space in Hamiltonian systems, and gave many analytical results. Based on Hamiltonian systems, Yao and Yang (2001) presented Saint Venant solutions for the problem of multi-layered composite plane anisotropic plates, a mixed energy variational principle was proposed, and dual equations were also derived in the symplectic space. Applying the dynamical viewpoints and plate theory, Andrew (2003) analyzed the accommodated conditions of beam theory.

In this paper, based on the theory of elastic thin plates of Lagrange–German, applying the solving method in Hamiltonian state space, the solutions of elastic vibrations and waveguide when the boundary of the two lateral sides of strip plate are free of traction are investigated. The existence conditions of vibration modes and waveguide modes are studied. The dispersion relations of waveguide modes are analyzed. At last, the results are analyzed and discussed.

2. Hamiltonian formulas of elastic waves in plates and its solutions

According to the theory of Lagrange–Germanian plates, in the orthogonal coordinates, the expressions of bending moment, torsional moment and shearing force in plates are written as

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right), \quad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right), \quad -M_{xy} = M_{yx} = -D(1-v)\frac{\partial^2 w}{\partial x \partial y},$$
$$Q_x = -D\frac{\partial \nabla^2 w}{\partial x}, \quad Q_y = -D\frac{\partial \nabla^2 w}{\partial y}, \tag{1}$$

where D is the bending stiffness of plates, $D = Eh^3/12(1-v^2)$, w is the lateral displacement of plates.

The density function of strain complementary energy in flexural plates is expressed as

$$U_0^* = \frac{1}{2D(1-v^2)}[M_x^2 + M_y^2 - 2vM_xM_y + 2(1+v)M_{xy}^2]. \tag{2}$$

Applying the generalized variational principles of two kinds of variables, the functional of the hybrid energy variational principle of Hellinger–Reissner in flexural thin plates is (see Hu, 1981)

$$\delta\Pi_2 = \delta\left\{\int_D \Gamma\,\mathrm{d}y + \int_S B\,\mathrm{d}s\right\} = 0, \tag{3}$$

here B is the function about boundary, Γ is the density function of hybrid energy, and its expression is

$$\Gamma = -M_x\frac{\partial^2 w}{\partial x^2} - 2M_{xy}\frac{\partial^2 w}{\partial x \partial y} - M_y\frac{\partial^2 w}{\partial y^2} - \frac{1}{2D(1-v^2)}[M_x^2 + M_y^2 + 2(1+v)M_{xy}^2 - 2vM_xM_y] - \frac{1}{2}\rho h\omega^2 w^2. \tag{4}$$

According to the variational of $\delta\Pi_2/\delta M_y = 0$, one can obtain

$$M_y = -D(1-v^2)\frac{\partial^2 w}{\partial y^2} + vM_x, \quad V_x = Q_x + \frac{\partial M_{xy}}{\partial y}. \tag{5}$$

To apply the Hamiltonian systems, coordinate x is simulated by time variable. Thus, in the state space we can let the generalized displacement variables be (see Greenberg and Marletta, 2000)

$$\boldsymbol{q} = (q_1, q_2)^{\mathrm{T}} = (w, \partial w/\partial x)^{\mathrm{T}} = (w, \varphi_x)^{\mathrm{T}}. \tag{6}$$

So, the generalized velocity is $\dot{\boldsymbol{q}} = \frac{\partial \boldsymbol{q}}{\partial x} = (\dot{q}_1, \dot{q}_2)^{\mathrm{T}} = (\dot{w}, \dot{\varphi}_x)^{\mathrm{T}}$. And that in the phase space, we can let the generalized displacement and the generalized momentum be $\boldsymbol{q} = (q_1, q_2)^{\mathrm{T}}$ and $\boldsymbol{p} = (p_1, p_2)^{\mathrm{T}} = (V_x, -M_x)^{\mathrm{T}}$, respectively.

Substituting Eq. (5) into Eq. (3), the original variational principle can be changed into generalized variational principle in Hamiltonian system

$$\delta\Pi_2^H = \delta\left\{\int_D \Gamma^H \, dy + \int_S B \, ds\right\} = 0, \tag{7}$$

here superscript H denotes the Hamiltonian systems, Γ^H is the density function of hybrid energy in Hamiltonian systems, and is written as

$$\Gamma^H = \boldsymbol{p}^{\mathrm{T}}\dot{\boldsymbol{q}} - H(\boldsymbol{q},\boldsymbol{p}), \tag{8}$$

where $H(\boldsymbol{q},\boldsymbol{p})$ is the Hamiltonian density function of Lagrange–German plate, and is calculated as

$$H(\boldsymbol{q},\boldsymbol{p}) = p_1q_2 + \frac{1}{2D}p_2^2 - \nu p_2\frac{\partial^2 q_1}{\partial y^2} - \frac{1}{2}D(1-\nu^2)\left(\frac{\partial^2 q_1}{\partial y^2}\right)^2 - D(1-\nu)\left(\frac{\partial q_2}{\partial y}\right)^2 + \frac{1}{2}\rho h\omega^2 q_1^2. \tag{9}$$

Thus, the Lagrangian density function of flexural waves in plate structure is

$$L(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{p}^{\mathrm{T}}\dot{\boldsymbol{q}} - H(\boldsymbol{q},\boldsymbol{p}) = \frac{1}{2}D\left(\frac{\partial^2 q_1}{\partial y^2}\right)^2 + \frac{1}{2}D\dot{q}_2^2 + D(1-\nu)\left(\frac{\partial q_2}{\partial y}\right)^2 + \nu D\left(\frac{\partial^2 q_1}{\partial y^2}\right)\dot{q}_2 - \frac{1}{2}\rho h\omega^2 q_1^2. \tag{10}$$

From variational $\delta\Pi_2^H/\delta\boldsymbol{q} = 0$ and $\delta\Pi_2^H/\delta\boldsymbol{p} = 0$, the following equations can be obtained:

$$\begin{aligned}
&\dot{q}_1 = \frac{\partial H(\boldsymbol{q},\boldsymbol{p})}{\partial p_1} = q_2, \quad \dot{q}_2 = \frac{\partial H(\boldsymbol{q},\boldsymbol{p})}{\partial p_2} = \frac{1}{D}p_2 - \nu\frac{\partial^2 q_1}{\partial y^2},\\
&\dot{p}_1 = -\frac{\partial H(\boldsymbol{q},\boldsymbol{p})}{\partial q_1} = D(1-\nu^2)\frac{\partial^4 q_1}{\partial y^4} + \nu\frac{\partial^2 p_2}{\partial y^2} - \rho h\omega^2 q_1,\\
&\dot{p}_2 = -\frac{\partial H(\boldsymbol{q},\boldsymbol{p})}{\partial q_2} = -2D(1-\nu)\frac{\partial^2 q_2}{\partial y^2} - p_1.
\end{aligned} \tag{11}$$

According to Eqs. (11), in phase space, the equation of flexural waves in plates can be derived (see Zhong, 1995)

$$\dot{\mathbf{v}} = H\mathbf{v} = \mu\mathbf{v}, \tag{12}$$

where $\mathbf{v}$ is the state vector of the dynamical systems, and $\mathbf{v} = [\boldsymbol{q}^{\mathrm{T}}, \boldsymbol{p}^{\mathrm{T}}]^{\mathrm{T}}$, H is the Hamiltonian operator matrix of 4×4, and is expressed as

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\nu\partial^2/\partial y^2 & 0 & 0 & 1/D \\ D(1-\nu^2)\partial^4/\partial y^4 - \rho h\omega^2 & 0 & 0 & \nu\partial^2/\partial y^2 \\ 0 & -2D(1-\nu)\partial^2/\partial y^2 & -1 & 0 \end{bmatrix}.$$

The relation between transverse eigenvalue λ and the longitudinal eigenvalue μ is

$$(\mu^2+\lambda^2)^2 - k^4 = (\mu^2+\lambda^2+k^2)(\mu^2+\lambda^2-k^2) = 0, \tag{13}$$

here k is elastic wave number, and $k = (\rho h\omega^2/D)^{1/4}$.

2.1. Determination of zero eigensolution

Zero eigenvalue plays an important role in mechanics of elasticity. To the elastodynamics in the strip plate, when the two lateral sides are free of traction, there will be eigensolution corresponding to zero eigenvalue. Then the equation is

$$H\psi_{(0)} = 0, \quad \psi_{(0)} = [w, \varphi_x, V_x, -M_x]^{\mathrm{T}} = \psi_0^{(1)} + \psi_0^{(2)}. \tag{14}$$

When the two lateral sides are free of traction, the expressions of boundary conditions are

$$D(1-\nu^2)\frac{\partial^2 w}{\partial y^2} - \nu M_x = D(1-\nu^2)\frac{\partial^2 q_1}{\partial y^2} + \nu p_2 = 0,$$
$$\frac{\partial M_y}{\partial y} + 2\frac{\partial M_{xy}}{\partial x} = -(1+2\nu-\nu^2)D\frac{\partial^3 q_1}{\partial y^3} - (2-\nu)\frac{\partial p_2}{\partial y} = 0. \tag{15}$$

From analysis, one can obtain that the basal eigensolutions which are linear independent can be presented as

$$\psi_0^{(1)} = (w, 0, 0, -M_x)^{\mathrm{T}}, \quad \psi_0^{(2)} = (0, \varphi_x, V_x, 0)^{\mathrm{T}}, \tag{16}$$

here $w = w_0^{(1)} = \cos(ky) + \delta\cosh(ky)$, $\delta = \delta(ka) = \cos(ka)/\cosh(ka)$, $\varphi_x = \varphi_{x0}^{(1)} = 0$, $V_x = V_{x0}^{(1)} = 0$, $M_x = M_{x0}^{(1)} = \nu Dk^2[\cos(ky) - \delta\cosh(ky)]$.

Then, the existence condition of transverse vibration modes is to ensure the periodicity in space, namely, the elastic wave number should satisfy the following expression:

$$\tan(ka) + \tanh(ka) = 0. \tag{17}$$

Thus, the eigenvector of zero order in Jordan form is

$$v_0 = \boldsymbol{v}_0^{(1)} + \boldsymbol{v}_0^{(2)} = \psi_0^{(1)} + \psi_0^{(2)}. \tag{18}$$

The physical meanings of $\boldsymbol{v}_0^{(1)}, \boldsymbol{v}_0^{(2)}$ denote a kind of vibration modes which do not propagate in the x-direction, namely, they are homogeneous in the x-direction, and oscillatory standing waves in the y-direction. The transverse displacement and bending moment of $\psi_0^{(1)}$ are w and M_x, respectively. And that the rotor angle and shearing force of $\psi_0^{(2)}$ are φ and V_x, respectively. The value of $\psi_0^{(2)}$ is zero.

Solving the zero eigenvector of the first order $\psi_1^{(1)}$

$$H\psi_1^{(1)} = \psi_0^{(1)}, \tag{19}$$
$$\psi_1^{(1)} = (0, \varphi_x, V_x, 0)^{\mathrm{T}}, \tag{20}$$

here $w = w_1^{(1)} = 0, \varphi_x = \varphi_{x1}^{(1)} = \cos(ky) + \delta\cosh(ky) = w_0^{(1)}$, $M_x = M_{x1}^{(1)} = 0$, $V_x = V_{x1}^{(1)} = D(2-\nu)k^2[\cos(ky) - \delta\cosh(ky)] = \frac{2-\nu}{\nu}M_{x0}^{(1)}$.

The solution of Eq. (14) is

$$\boldsymbol{v}_1^{(1)} = \psi_1^{(1)} + x\psi_0^{(1)} \tag{21}$$

namely,

$$\boldsymbol{v}_1^{(1)} = [xw_0^{(1)}, \varphi_{x1}^{(1)}, V_{x1}^{(1)}, -xM_{x0}^{(1)}]^{\mathrm{T}}. \tag{22}$$

$\boldsymbol{v}_1^{(1)}$ is a kind of vibration modes for non-propagation, whose physical meaning denotes the rotation of rigid body in the xoz plane. After analyzing, one can know that the chain of other secondary eigensolutions breaks off till $\psi_2^{(1)}$.

Solving zero eigenvector of the first order $\psi_1^{(2)}$

$$H\psi_1^{(2)} = \psi_0^{(2)}, \tag{23}$$
$$\psi_1^{(2)} = (w, 0, 0, -M_x)^{\mathrm{T}}, \tag{24}$$

here $w = w_1^{(2)} = 0, M_x = M_{x1}^{(2)} = 0$.

It is obvious that the chain of other secondary eigensolutions breaks off till $\psi_1^{(2)}$.

With respect to zero eigenvalue, the corresponding dynamical modes of it are vibration modes of several orders of transverse vibration. However, the integrations along the y-axis of all mechanical variables are equal to zero in this case.

2.2. Determination of non-zero eigensolution

The symmetrical case is investigated, the eigensolution $\psi_n = (q_n, p_n)^{\mathrm{T}}$ of symmetrical case in flexural vibration of plates can be described as

$$\begin{aligned} &w = A_{11}\cosh(\lambda_1 y) + A_{12}\cosh(\lambda_2 y), \quad \varphi_x = A_{21}\cosh(\lambda_1 y) + A_{22}\cosh(\lambda_2 y), \\ &V_x = A_{31}\cosh(\lambda_1 y) + A_{32}\cosh(\lambda_2 y), \quad -M_x = -A_{41}\cosh(\lambda_1 y) - A_{42}\cosh(\lambda_2 y), \end{aligned} \tag{25}$$

where $\lambda_n^2 = -\mu^2 - k_n^2$, $k_2^2 = -k_1^2 = -k^2$, A_{mn} $(m = 1, 2, \ldots 4,\ n = 1, 2)$ are mode coefficients, and are not all independent. After analyzing, one can find that only two of them are independent (see Zhong and Yao, 1999).

Substituting Eq. (25) into Eq. (12), the relations of mode coefficients can be obtained

$$\begin{aligned} &A_{2n}/A_{1n} = \mu, \quad A_{3n}/A_{1n} = D(\lambda_n^4 + v\mu^2\lambda_n^2 - k^4)/\mu, \\ &A_{4n}/A_{1n} = -D(v\lambda_n^2 + \mu^2) \quad (n = 1, 2). \end{aligned} \tag{26}$$

By satisfying the free boundary condition of the strip plate, the equation that the eigenvalues should satisfy is obtained

$$(\lambda_1^2 + v\mu^2)[\lambda_2^3 + (2 - v)\mu^2\lambda_2]\tanh(\lambda_2 a) - (\lambda_2^2 + v\mu^2)[\lambda_1^3 + (2 - v)\mu^2\lambda_1]\tanh(\lambda_1 a) = 0. \tag{27}$$

The case that wave-guide exists along the positive x-direction is considered. According to Eq. (27), let $\mu = \mathrm{i}\beta$, the dispersion equation in Hamiltonian systems can be derived

$$(\lambda_1^2 - v\beta^2)[\lambda_2^3 - (2 - v)\beta^2\lambda_2]\tanh(\lambda_2 a) - (\lambda_2^2 - v\beta^2)[\lambda_1^3 - (2 - v)\beta^2\lambda_1]\tanh(\lambda_1 a) = 0, \tag{28}$$

here β is the propagating wave number of elastic waves in plates.

Based on the theory of elastic thin plates, the dispersion equation of flexural waves in the strip plate is

$$(\lambda_1^2 - v\beta^2)[\lambda_2^3 - (2 - v)\beta^2\lambda_2]\tanh(\lambda_2 a) - (\lambda_2^2 - v\beta^2)[\lambda_1^3 - (2 - v)\beta^2\lambda_1]\tanh(\lambda_1 a) = 0. \tag{29}$$

3. Analysis and discussion

Through analysis, one can see that in the research of elastodynamics in plates, the concepts such as translation and rotation of rigid body in statistics do not exist in plates, and the non-propagating modes, for example the standing waves which distribute uniformly in the x-direction and oscillate in the y-direction, substitute them. Applying the Hamiltonian formula, one can also obtain the localized vibration modes in plates besides determining the extended modes along the x-direction. However, applying the theory of Lagrange–German plates, we can only obtain the extended modes along the x-direction, and partial localized vibrations may be missed.

Different from the classical analytical method of vibration modes, when applying the Hamiltonian formula, through the dynamical models corresponding to the zero eigenvalue (longitudinal eigenvalue $\mu = 0$), we can see that spatial vibration modes in the y-direction may exist in the strip plate. However, in the research of the present paper, the integrations of all mechanical vector along the y-axis are equal to zero. Considering the propagation of elastic waves, applying the semi-inverse method and the Hamiltonian formula, respectively, the obtained dispersion relations are the same.

For non-zero eigenvalue, the main findings of this work are as follows:

(1) When the imaginary part of this eigenvalue is greater than zero, if the real part is less than zero, it denotes the decay propagating mode in the positive x-direction; if the real part is greater than zero, it denotes the unstable state with exponential increase; if the real part is equal to zero, it denotes the propagating mode in the positive x-direction.
(2) When the imaginary part of this eigenvalue is less than zero, if the real part is less than zero, it denotes the decay propagating mode in the negative x-direction; if the real part is greater than zero, it denotes the state of energy accumulation; if the real part is equal to zero, it denotes the propagating mode in the negative x-direction.

(3) When the imaginary part of this eigenvalue is equal to zero, if the real part is less than zero, it denotes localized vibration; if the real part is greater than zero, it denotes the state of instability.

Anyway, in any case of the above, the vibration frequency of structure and the wave number should satisfy the dispersion relation, namely, the vibration frequency of structure, the wave number of incident waves and the geometric parameters of structure should satisfy a certain relation.

The analytical methods and results of this paper can provide significant references for the analysis and design of vibration control of the aerospace structures and the civil constructions.

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