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# Special linear combinations of orthogonal polynomials

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**Abstract**

The paper lists a number of problems that motivate consideration of special linear combinations of polynomials, orthogonal with the weight $p(x)$ on the interval $(a,b)$. We study properties of the polynomials, as well as the necessary and sufficient conditions for their orthogonality. The special linear combinations of Chebyshev orthogonal polynomials of four kinds with absolutely constant coefficients hold a distinguished place in the class of such linear combinations.


*Keywords:* Orthogonal polynomials; Linear combinations; Linearization coefficients; Bernstein–Szegö polynomials; Weight function; Chebyshev polynomials; Extrema properties

## 1. Introduction

Let $P_n(x)$ be an orthogonal system of polynomials with respect to the weight $p(x)$ on the interval $(a,b)$. Consider the linear combination of these polynomials of the type

$$\sum_{k=n-m}^{n+r} C_{n,k} P_k(x),$$

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where $C_{n,k}$ are real constants depending in general on $n$ and $k$; $m$ and $r$ are absolute constants, $n > m$. We shall call this linear combination a special linear combination of orthogonal polynomials (SLCOP). In particular, for $r = 0$ we shall call the SLCOP $m$-incomplete special linear combination of orthogonal polynomials ($m$-ISLCOP).

Solutions of a large variety of problems lead to SLCOP. We shall list some of the problems.

### *1.1. Christoffel's problem*

Given a sequence $p_n(x)$ of polynomials, orthogonal with the weight $p(x)$ on the interval $(a, b)$, one has to construct a sequence of polynomials $q_n(x)$, orthogonal with the weight $p(x)\sigma_r(x)$ on $(a, b)$, where $\sigma_r(x) = \prod_{j=1}^{s}(x - x_j)^{m_j}$ is a positive on $(a, b)$ polynomial of degree $r = \sum_{j=1}^{s} m_j$, $x_j \overline{\in} (a, b)$.

Solution to this problem is Christoffel's formula [1, p. 30] in the form of a determinant. An expansion of this determinant is SLCOP

$$\sigma_r(x)q_n(x) = c_n p_n(x) + c_{n+1} p_{n+1}(x) + \cdots + c_{n+r} p_{n+r}(x), \tag{1}$$

where the Fourier coefficients $C_n$ are constants depending on $n$ and $r$. Cristoffel's formula realizes the concrete representation $\sigma_r(x)q_n(x)$ as a special expansion with respect to the polynomials $p_n(x)$.

### *1.2. The Peebles–Korous problem*

G. Peebles and J. Korous [2, pp. 75–76] introduced a positive rational multiplier into the weight function of orthogonals with the weight $p(x)$ polynomials on $(a, b)$. They were interested in the boundedness properties of new orthogonal polynomials. During this study they obtained a representation of polynomials $q_n(x)$, orthogonal with the weight $p(x)/\sigma_r(x)$, through the linear combination of the polynomials $P_n(x)$, orthogonal with the weight $p(x)$ on $(a, b)$. The polynomials $q_n(x)$ allow for representation as an $r$-ISLCOP

$$q_n(x) = C_{n-r} P_{n-r}(x) + C_{n-r+1} P_{n-r+1}(x) + \cdots + C_n P_n(x).$$

Concrete realization of this representation in the form of a determinant is given by the Uvarov's formula [3].

### *1.3. The problem of choosing nodal points (nodes) of quadrature formula*

The interest in studying special linear combinations of orthogonal polynomials is also motivated by the fact that many authors (Giraud [4], Tchakaloff [5], Fejer [6], Shohat [7]) studied mechanical quadratures of Gauss-type with nodes in zeros of such linear combinations of orthogonal polynomials. Giraud constructed quadrature formulas with the nodes in the points related to the zeros of $P_{n+1}(x) - P_{n-1}(x)$, where $P_n(x)$ are the Legendre polynomials. Tchakaloff conducted similar study. While considering mechanical quadratures with positive coefficients, Fejer chose nodes to be zeros of linear combination of Legendre polynomials of type $P_n(x) + AP_{n-1}(x) + BP_{n-2}(x)$, where $A$ and $B$ are real numbers, and $B \leqslant 0$. Shohat generalized Fejer's results; he included into the construction of Gauss-type quadrature formulas the zeros of more generic linear combination of arbitrary orthogonal polynomials

$$\Phi_n(x) + A_1\Phi_{n-1}(x) + \cdots + A_{k-1}\Phi_{n-k+1}(x),$$

where $A_1, A_2, \ldots, A_{k-1}$ are absolute constants.

### *1.4. Extrema problems of Zolotarev–Markov type*

We assume the polynomials $\Pi_n(x)$ of degree $n$ to be normalized in such way that the elder coefficient is equal to 1. $\varrho_l(x)$ is a positive polynomial of degree $l$ on the interval $(-1, 1)$.

**Problem 1.1.** Among all the polynomials $\Pi_n(x)$, find the one that deviates the least from zero in the uniform metrics with the weight $1/\sqrt{\varrho_l(x)}$ on the segment $[-1, 1]$. That is, to find the extrema polynomial that delivers

$$\min_{\forall \Pi_n(x)} \max_{x\in[-1,1]} \frac{|\Pi_n(x)|}{\sqrt{\varrho_l}(x)}.$$

**Problem 1.2.** Among all the polynomials $\Pi_n(x)$, find the one that deviates the least from zero in the uniform metrics with the weight $\sqrt{(1-x^2)/\varrho_l(x)}$ on the segment $[-1, 1]$. That is, to find the extrema polynomial that delivers

$$\min_{\forall \Pi_n(x)} \max_{x\in[-1,1]} \frac{|\Pi_n(x)|\sqrt{1-x^2}}{\sqrt{\varrho_l}(x)}.$$

**Problem 1.3.** Among all the polynomials $\Pi_n(x)$, find the one that deviates the least from zero in the uniform metrics with the weight $\sqrt{(1-x)/\varrho_l(x)}$ on the segment $[-1, 1]$. That is, to find the extrema polynomial that delivers

$$\min_{\forall \Pi_n(x)} \max_{x\in[-1,1]} \frac{|\Pi_n(x)|\sqrt{1-x}}{\sqrt{\varrho_l}(x)}.$$

**Problem 1.4.** Among all the polynomials $\Pi_n(x)$, find the one that deviates the least from zero in the uniform metrics with the weight $\sqrt{(1+x)/\varrho_l(x)}$ on the segment $[-1, 1]$. That is, to find the extrema polynomial that delivers

$$\min_{\forall \Pi_n(x)} \max_{x\in[-1,1]} \frac{|\Pi_n(x)|\sqrt{1+x}}{\sqrt{\varrho_l}(x)}.$$

The solution to Problems 1.1 and 1.2 belongs to A.A. Markov and S.N. Bernstein and was improved by A.I. Achiezer [8]. These solutions are quite complex. Problems 1.3 and 1.4 were posed and solved by the author, as was the simpler and uniform solution for all the 4 problems by using Bernstein–Szegö orthogonal polynomials. These solutions are $l$-incomplete linear combinations of the corresponding Chebyshev orthogonal polynomials, as we shall prove in the present paper.

### *1.5. A.O. Gelfond's problem of the polynomials, least deviating from zero with their derivatives*

This problem is a natural generalization of the well-known problem of polynomials, least deviating from zero, that was posed and solved by Chebyshev [8].

Consider the numbers $\sigma_{n,m}$, $n \geqslant m+2$, $m \geqslant 0$,

$$\sigma_{n,m} = \min_{\forall \Pi_n(x)} \max_{0 \leqslant s \leqslant m} \max_{-1 \leqslant x \leqslant 1} \frac{|\Pi_n^{(s)}(x)|}{n(n-1)\dots(n-s+1)},$$

where the minimum is taken over all the polynomials of degree $n$ with real coefficients and the elder coefficient equal to 1. Asymptotic solution to this problem of polynomials, least deviating from zero with their derivatives, led A.O. Gelfond [9] to polynomials $T_{n,k}(x)$, that can be represented as a special linear combination of Chebyshev orthogonal polynomials $T_n(x)$ of the first kind with the elder coefficient equal to 1,

$$T_{n,k}(x) = T_n(x) + C_{n-2,k}T_{n-2}(x) + C_{n-4,k}T_{n-4}(x) + \dots + C_{n-2k}T_{n-2k}(x),$$

where the explicitly calculated coefficients $C_{n,k}$ depend rationally on $n$ and $k$.

### *1.6. The problem of approximate polynomial solution to differential equations*

In the works of Dzyadyk, Ostrovetsky, Romanenko [10] the solutions to the Cauchy problem for the ordinary differential equations is represented by the polynomials that are close to the best approximation. This approximate solution is a special linear combination of the Chebyshev polynomials of the first kind

$$\mathcal{E}_n(x) = \sum_{i=0}^{l} \tau_i T_{n+i}(x).$$

An algorithm for calculating the coefficients $\tau_i$ is constructed. The method is expanded to the Goursa problem for hyperbolic-type equations, non-linear integral equations of Volterra, Urison, Liapunov–Lichtenstein, and approximate solution to the non-linear Cauchy problem for the hyperbolic-type equations. The algorithm of the approximate solutions to the aforementioned problems essentially uses the $l$-incomplete linear combination of the orthogonal Chebyshev polynomials, as the apparatus for the efficient approximation.

### *1.7. The linearization problem for products of orthogonal polynomials*

The formulas for transforming the product of trigonometric functions into their linear combination are well known, for example,

$$\cos m\theta \cos n\theta = \frac{1}{2}\cos(n+m)\theta + \frac{1}{2}\cos(n-m)\theta.$$

The role of such formulas in the applications is well known.

Similarly important is the transformation of the product of orthogonal polynomials into their linear combination. Such formulas for particular classes of orthogonal polynomials were studied by various authors (Adams, Ferrer, Dougll, Hsu, Hylleraas, Vilenkin, Bailey, MacMahon, Gasper, Miller, Hirschman, Davis, Wainger, Askey, Gillis, Even). The survey of these works is given in the work of R. Askey [11]. For the generic classes of orthogonal polynomials $P_n(x)$, satisfying the recurrence relation

$$P_1(x)P_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x),$$

R. Askey proved the following linearization theorem:

$$P_m(x)P_n(x) = \sum_{k=|n-m|}^{n+m} a(k,m,n)P_k(x).$$

If $a_n \geqslant 0$, $b_n \geqslant 0$, $a_{n+1} \geqslant a_n$, $b_{n+1} \geqslant b_n$, $n = 1, 2, \ldots$, then $a(k,m,n) \geqslant 0$.

Therefore, the linearization problem for the products of orthogonal polynomials also leads to the special linear combinations of orthogonal polynomials.

Here we consider the SLCOP as an independent object of study. This paper studies the properties of such linear combinations of orthogonal polynomials. We find the necessary and sufficient conditions for the linear combinations of orthogonal polynomials to be orthogonal polynomials themselves with respect to certain weight. This is important, since when being orthogonal, they become in certain sense optimal among others, similarly to the case where the Gauss quadrature formula is optimal, if its nodes are zeros of orthogonal polynomials. Among the SLCOP we distinguish those that are generated by Bernstein–Szegö orthogonal polynomials (B-SOP). We consider the B-SOP of four kinds. They are distinguished by the fact that the coefficients of their expansion with respect to Chebyshev orthogonal polynomials are absolute constants. Based upon the oscillatory properties of B-SOP [12–14], we give a simple uniform solution to the Zolotarev–Markov problems. We prove the interpretation theorem, which allows to recognize B-SOP according to coefficients of the special linear combination of the Chebyshev polynomials.

## 2. Orthogonality conditions for the $l$-incomplete linear combinations of orthogonal polynomials

The properties of linear combinations of orthogonal polynomials were studied by Shohat [7,12]. We complete this study by ascertaining the nature of the weight function, with respect to which the linear combinations are orthogonal. Our study uses the functions of the II kind as a tool.

Let $\tilde{\omega}_n(x)$ be polynomials, orthogonal with respect to the weight $p(x)$ on the interval $[a,b]$ and normalized in such way that the elder coefficient is equal to 1. Let $\overline{C^n} = \{C_k^{(n)}\}_{k=n-l}^{n}$ be an arbitrary sequence of $(l+1)$-dimensional vectors, where $C_n^{(n)} = 1$ and $C_k^{(n)}$ $(k < n)$ are real numbers, $l$ is a fixed non-negative integer $(n = 0, 1, 2, \ldots)$.

Consider the sequence of polynomials $\tilde{\Omega}_n(x)$ with elder coefficient equal to one, that is represented by the following linear combination of orthogonal polynomials:

$$\tilde{\Omega}_n(x) = \sum_{i=n-l}^{n} C_i^{(n)} \tilde{\omega}_i(x) \quad (n = 0, 1, 2, \ldots). \tag{2}$$

We assume $\tilde{\omega}_{-k}(x) \equiv 0$ for each natural $k$. Let the polynomials $\tilde{\Omega}_n(x)$ be orthogonal with respect to the weight $q(x)$ on the interval $[a, b]$.

We introduce the following notations:

- $Q_n^{(\omega)}(z)$ is a function of the second kind, corresponding to the polynomial $\tilde{\omega}_n(x)$;
- $Q_n^{(\Omega)}(z)$ is a function of the second kind, corresponding to the polynomial $\tilde{\Omega}_n(x)$, namely

$$Q_n^{(\omega)}(z) = \int_a^b \frac{p(x)\tilde{\omega}_n(x)\,dx}{z-x}, \tag{3}$$

$$Q_n^{(\Omega)}(z) = \int_a^b \frac{q(x)\tilde{\Omega}_n(x)\,dx}{z-x} \tag{4}$$

for $\forall z \,\overline{\in}\, [a, b]$.

**Theorem 2.1.** *For the equality $q(x) = p(x)/\varrho_l(x)$ is a non-negative on the interval $[a, b]$ polynomial of the degree $l$, it is necessary and sufficient that the following condition holds*:

$$Q_n^{(\omega)}(z) = \sum_{i=n}^{n+l} d_i^{(n)} Q_i^{(\Omega)}(z), \tag{5}$$

*where $d_i^{(n)}$ are real constants depending on $n$, $i$ and $l$.*

**Proof.** *Necessity*. Let the polynomials $\tilde{\Omega}_n(x)$ be orthogonal on $[a, b]$ with the weight $q(x) = p(x)/\varrho_l(x)$. Then the polynomials $\tilde{\omega}_n(x)$ are orthogonal on $[a, b]$ with weight $p(x) = \varrho_l(x)q(x)$. The Christoffel formula (1) implies

$$\tilde{\omega}_n(x) = \frac{1}{\varrho_l(x)} \sum_{i=n}^{n+l} d_i^{(n)} \tilde{\Omega}_i(x).$$

Multiplying both sides of the equation by $p(x)/(z-x)$ and integrating with respect to $x$ over $[a, b]$ yields

$$\int_a^b \frac{\tilde{\omega}_n(x)p(x)\,dx}{z-x} = \int_a^b \frac{p(x)\,dx}{\varrho_l(x)(z-x)} \sum_{i=n}^{n+l} d_i^{(n)} \tilde{\Omega}_i(x)\,dx = \sum_{i=n}^{n+l} d_i^{(n)} \int_a^b \frac{p(x)\tilde{\Omega}_i(x)\,dx}{z-x},$$

which yields the formula (5), by taking into account (3) and (4).

*Sufficiency*. The condition (5) and the formula (3) imply that $\forall z \,\overline{\in}\, [a, b]$,

$$\int_a^b \frac{p(x)\tilde{\omega}_n(x)\,dx}{z-x} = \sum_{i=n}^{n+l} d_i^{(n)} \int_a^b \frac{q(x)\tilde{\Omega}_i(x)\,dx}{z-x} = \int_a^b \frac{q(x)\sum_{i=n}^{n+l} d_i^{(n)}\tilde{\Omega}_i(x)\,dx}{z-x},$$

that is,

$$\int_a^b \frac{[p(x)\tilde{\omega}_n(x) - q(x)\sum_{i=n}^{n+l} d_i^{(n)} \tilde{\Omega}_i(x)]\,dx}{z-x} \equiv 0$$

for $\forall z \,\overline{\in}\, [a,b]$.

Setting $\varphi(x) = p(x)\tilde{\omega}_n(x) - q(x)\sum_{i=n}^{n+l} d_i^{(n)} \tilde{\Omega}_i(x)$ yields

$$\int_a^b \frac{\varphi(x)\,dx}{z-x} \equiv 0 \quad \text{for } \forall z \,\overline{\in}\, [a,b].$$

We expand the Cauchy kernel into the series

$$\frac{1}{z-x} = \frac{1}{z}\,\frac{1}{1-x/z} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{x}{z}\right)^k = \sum_{k=0}^{\infty}\frac{x^k}{z^{k+1}}.$$

The series converges uniformly for all $x \in [a,b]$ for $|x| > \max\{|a|,|b|\}$. Multiplying both sides of the obtained equality by the summable on $[a,b]$ function $\varphi(x)$ does not violate uniform convergence of the series, and allows to integrate the series memberwise:

$$\int_a^b \frac{\varphi(x)\,dx}{z-x} = \sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\int_a^b \varphi(x)x^k\,dx.$$

Therefore the identity $\int_a^b \frac{\varphi(x)\,dx}{z-x} \equiv 0$ is equivalent to the conditions

$$\int_a^b \varphi(x)x^k\,dx = 0 \quad (k = 0,1,2,\ldots).$$

Hence, the summability of $\varphi(x)$ implies $\varphi(x) \equiv 0$ for all $x \in [a,b]$ [2], therefore

$$p(x)\tilde{\omega}_n(x) - q(x)\sum_{i=n}^{n+l} d_i^{(n)} \tilde{\Omega}_i(x) \equiv 0$$

or

$$\frac{p(x)}{q(x)} = \frac{\sum_{i=n}^{n+l} d_i^{(n)} \tilde{\Omega}_i(x)}{\tilde{\omega}_n(x)}.$$

The left side is independent of $n$; therefore $p(x)/q(x) = \varrho_l(x)$, where

$$\varrho_l(x) = \sum_{i=0}^{l} d_i^{(0)} \tilde{\Omega}_i(x)$$

is a non-negative polynomial of degree $l$. $\square$

**Theorem 2.2.** *If the sequence* $\{\tilde{\Omega}_n(x)\}_{n=0}^{\infty}$ *is orthogonal with some weight* $q(x)$ *on the interval* $[a,b]$*, then*

$$Q_n^{(\omega)}(z) = \sum_{s=n}^{n+l} d_s^{(n)} Q_s^{(\Omega)}(z), \tag{6}$$

*where*

$$d_s^{(n)} = \frac{a_n C_n^{(s)}}{b_s}, \quad a_n = \int_a^b \left[\tilde{\omega}_n(x)\right]^2 p(x)\,dx, \quad b_s = \int_a^b \left[\tilde{\Omega}_s(x)\right]^2 q(x)\,dx. \tag{7}$$

**Proof.** Expanding the Cauchy kernel formally to orthogonal polynomials yields

$$\frac{1}{z-x} \sim \sum_{k=0}^{\infty} \frac{1}{a_k} \tilde{\omega}_k(x) Q_k^{(\omega)}(z),$$
$$\frac{1}{z-x} \sim \sum_{s=0}^{\infty} \frac{1}{b_s} \tilde{\Omega}_s(x) Q_s^{(\Omega)}(z),$$

where $x \in [a,b]$, $z \,\overline{\in}\, [a,b]$.

Irrespectively of the series convergence, the following integral equalities hold:

$$\int_a^b \frac{p(x)\tilde{\omega}_n(x)\,dx}{z-x} = \sum_{k=0}^{\infty} \frac{1}{a_k} Q_k^{(\omega)}(z) \int_a^b p(x)\tilde{\omega}_k(x)\tilde{\omega}_n(x)\,dx,$$
$$\int_a^b \frac{p(x)\tilde{\omega}_n(x)\,dx}{z-x} = \sum_{s=0}^{\infty} \frac{1}{b_s} Q_s^{(\Omega)}(z) \int_a^b p(x)\tilde{\Omega}_s(x)\tilde{\omega}_n(x)\,dx.$$

Hence

$$Q_n^{(\omega)}(z) = \sum_{s=0}^{\infty} \frac{1}{b_s} Q_s^{(\Omega)}(z) \sum_{i=s-l}^{s} C_i^{(s)} \int_a^b \tilde{\omega}_i(x)\tilde{\omega}_n(x)p(x)\,dx$$
$$= \sum_{s=n}^{n+l} \frac{a_n c_n^{(s)}}{b_s} Q_s^{(\Omega)}(z). \qquad \Box$$

**Theorem 2.3.** *In order for the* $l$*-incomplete linear combination of polynomials, orthogonal with the weight* $p(x)$ *on the segment* $[a,b]$, $\Omega_n(x) = \sum_{i=n-l}^{n} c_i^{(n)} \omega_i(x)$, *to be orthogonal with the weight* $q(x)$ *on the segment* $[a,b]$*, it is necessary and sufficient that* $q(x) = p(x)/\varrho_l(x)$*, where* $\varrho_l(x)$ *is a non-negative polynomial on* $[a,b]$ *of degree* $l$*.*

**Proof.** *Necessity*. Let $\Omega_n(x)$ be an orthogonal $l$-incomplete linear combination of polynomials $\omega_i(x)$, orthogonal with the weight $p(x)$. Then by Theorem 2.2, the following condition holds:

$$Q_n^{(\omega)}(z) = \sum_{s=n}^{n+l} d_s^{(n)} Q_s^{(\Omega)}(z), \tag{8}$$

where $d_s^{(n)}$ are calculated by the formula (7). But, under condition (6), Theorem 2.1 implies that the weight function $q(x) = p(x)/\varrho_l(x)$, where $\varrho_l(x)$ is non-negative on the segment $[a, b]$ polynomial of degree $l$.

*Sufficiency*. Let the polynomials $\Omega_n(x)$ of degree $n$ be orthogonal with the weight $p(x)/\varrho_l(x)$ on the segment $[a, b]$. We expand the polynomial $\Omega_n(x)$ with respect to the polynomials $\omega_k(x)$, orthogonal with the weight $p(x)$ on the segment $[a, b]$:

$$\Omega_n(x) = \sum_{i=0}^{n} c_i^{(n)} \omega_i(x), \tag{9}$$

where the Fourier expansion coefficients are calculated by the formula

$$c_i^{(n)} = \frac{1}{a_i} \int_a^b \Omega_n(x) p(x) \omega_i(x)\, dx, \quad \text{where } a_i = \int_a^b p(x)\big[\omega_i(x)\big]^2\, dx.$$

Transform the formula for $c_i^{(n)}$,

$$c_i^{(n)} = \frac{1}{a_i} \int_a^b \Omega_n(x) \frac{p(x)}{\varrho_l(x)} \varrho_l(x) \omega_i(x)\, dx.$$

Since the polynomials $\Omega_n(x)$ are orthogonal on $[a, b]$ with the weight $p(x)/\varrho_l(x)$, $\Omega_n(x)$ are orthogonal with this weight to any polynomial of the degree less than $n$, that is for $l + i < n$. Hence, $c_i^{(n)} = 0$ for $i < n - l$. The polynomial (9) takes the form $l$-ISLCOP

$$\Omega_n(x) = \sum_{i=n-l}^{n} c_i^{(n)} \omega_i(x). \qquad \square$$

## 3. Linearization of the Bernstein–Szegö orthogonal polynomials

Among the classic polynomials orthogonal on the interval $[-1, 1]$, a distinguished place—due to their simplicity, depth of our knowledge about them, and wide applications—is held by the Chebyshev polynomials of four kinds $T_n(x)$, $U_n(x)$, $V_n(x)$, $W_n(x)$, corresponding to the weight functions

$$\mu_1(x) = \frac{1}{\sqrt{1-x^2}}, \quad \mu_2(x) = \sqrt{1-x^2}, \quad \mu_3(x) = \sqrt{\frac{1-x}{1+x}}, \quad \mu_4(x) = \sqrt{\frac{1+x}{1-x}}.$$

We shall denote the Chebyshev polynomials of any definite kind by $\mathcal{P}_n(x)$, and the orthonormal polynomials by $\hat{\mathcal{P}}_n(x)$, where $\mathcal{P}_n(x) = \sqrt{\pi/2}\,\hat{\mathcal{P}}_n(x)$. The Bernstein–Szegö orthogonal polynomials (B-SOP) correspond to the weight functions

$$\omega_k(x) = \frac{\mu_k(x)}{\varrho_l(x)} \quad (k = 1, 2, 3, 4),$$

where $\varrho_l(x)$ is a positive polynomial of degree $l$ on $(-1, 1)$.

We shall differentiate between the B-SOP of four kinds. We shall denote respectively $P_n(x)$ as the I kind, $q_n(x)$ as the II kind, $R_n(x)$ as the III kind, and $S_n(x)$ as the IV kind. We shall use the notation $Q_n(x)$ for B-SOP of the same kind (irrespectively of which one), $\hat{Q}_n(x)$—orthonormal B-SOP, where $Q_n(x) = \sqrt{\pi/2}\,\hat{Q}_n(x)$. In that way for each kind of the Chebyshev polynomials corresponds the same kind of the Bernstein–Szegö polynomials. S.N. Bernstein and G. Szegö had pointed out explicit expressions for the polynomials corresponding to the weights $\omega_k(x)$ $(k = 1, 2, 3)$. Though S.N. Bernstein and G. Szegö did not consider explicitly the polynomials, orthogonal on the interval $[-1, 1]$ with the weight $\omega_4(x)$, it is natural to consider the latter also as Bernstein–Szegö polynomials.

Further on we shall use the following Fejer representation [1] of the non-negative trigonometric polynomial. Every non-negative trigonometric polynomial $g(\theta)$ of degree $l$ can be represented as a square of modulus of a certain polynomial $h(z)$ of degree $l$, for $z = e^{i\theta}$. If this polynomial does not vanish for $|z| < 1$ and is normalized in such a way that $h(0) > 0$, then it is uniquely defined. If $g(\theta)$ is a trigonometric cosine-polynomial with real coefficients, then the coefficients of the polynomial $h(z)$ are real. Let $h(z) = \sum_{j=0}^{l} h_j z^j$ be a polynomial that gives Fejer-normalized representation of the positive polynomial $\varrho_l(x)$.

We expand the Bernstein–Szegö polynomials of each kind into Chebyshev polynomials of the corresponding 'same name' kind. Assuming $x = \cos\theta$, we will start with the B-SOP representation for $n \geqslant l/2$,

$$\begin{aligned}
\hat{P}_n(\cos\theta) &= \sqrt{\frac{2}{\pi}}\Re\big[e^{in\theta}h(e^{i\theta})\big],\\
\hat{q}_n(\cos\theta) &= \sqrt{\frac{2}{\pi}}\frac{1}{\sin\theta}\Im\big[e^{i(n+1)\theta}\overline{h(e^{i\theta})}\,\big],\\
\hat{R}_n(\cos\theta) &= \frac{1}{\sqrt{\pi}}\frac{1}{\sin\frac{\theta}{2}}\Im\big[e^{i(n+1/2)\theta}\overline{h(e^{i\theta})}\,\big],\\
\hat{S}_n(\cos\theta) &= \frac{1}{\sqrt{\pi}}\frac{1}{\cos\frac{\theta}{2}}\Re\big[e^{i(n+1/2)\theta}\overline{h(e^{i\theta})}\,\big].
\end{aligned}$$

For the B-SOP of the first kind, we obtain

$$\begin{aligned}
\hat{P}_n(\cos\theta) &= \sqrt{\frac{2}{\pi}}\Re\big[e^{in\theta}\overline{h(e^{i\theta})}\,\big] = \sqrt{\frac{2}{\pi}}\Re\left[e^{in\theta}\overline{\sum_{j=0}^{l}h_j e^{i\theta j}}\right]\\
&= \sqrt{\frac{2}{\pi}}\Re\left[e^{in\theta}\sum_{j=0}^{l}h_j e^{-i\theta j}\right] = \sqrt{\frac{2}{\pi}}\Re\left[\sum_{j=0}^{l}h_j e^{i(n-j)\theta}\right]
\end{aligned}$$

$$= \sqrt{\frac{2}{\pi}} \sum_{j=0}^{l} h_j \cos(n-j)\theta.$$

Since the Chebyshev polynomials of the first kind on the interval $[-1, 1]$ can be represented as $\hat{T}_n(x) = \sqrt{2/\pi}\cos n\theta$, where $\cos\theta = x$, the following representation of the B-SOP of first kind holds:

$$\hat{P}_n(x) = \sum_{j=0}^{l} h_j \hat{T}_{n-j}(x). \tag{10}$$

For the B-SOP of the second kind, we obtain

$$\hat{q}_n(\cos\theta) = \sqrt{\frac{2}{\pi}} \frac{1}{\sin\theta} \Im\big[e^{i(n+1)\theta}\overline{h(e^{i\theta})}\big] = \sqrt{\frac{2}{\pi}} \frac{1}{\sin\theta} \Im\left[e^{i(n+1)\theta}\overline{\sum_{j=0}^{l} h_j e^{i\theta j}}\right]$$
$$= \sqrt{\frac{2}{\pi}} \frac{1}{\sin\theta} \Im \sum_{j=0}^{l} h_j e^{(n+1-j)\theta} = \sqrt{\frac{2}{\pi}} \frac{1}{\sin\theta} \sum_{j=0}^{l} h_j \sin(n+1-j)\theta.$$

Since the Chebyshev polynomials of the second kind on the interval $[-1, 1]$ can be represented as

$$\hat{U}_n(x) = \sqrt{\frac{2}{\pi}} \frac{\sin(n+1)\theta}{\sin\theta},$$

the following representation holds for the B-SOP of the second kind:

$$\hat{q}_n(x) = \sum_{j=0}^{l} h_j \hat{U}_{n+1-j}(x). \tag{11}$$

For the B-SOP of the third kind, we obtain

$$\hat{R}_n(\cos\theta) = \frac{1}{\sqrt{\pi}} \frac{1}{\sin(\theta/2)} \Im\big[e^{i(n+1/2)\theta}\overline{h(e^{i\theta})}\big]$$
$$= \frac{1}{\sqrt{\pi}} \frac{1}{\sin(\theta/2)} \Im\left[e^{i(n+1/2)\theta}\overline{\sum_{j=0}^{l} h e^{i\theta j}}\right]$$
$$= \frac{1}{\sqrt{\pi}} \frac{1}{\sin(\theta/2)} \Im \sum_{j=0}^{l} h_j e^{i(n+1/2-j)\theta}$$
$$= \frac{1}{\sqrt{\pi}} \frac{1}{\sin(\theta/2)} \sum_{j=0}^{l} h_j \sin\left(n + \frac{1}{2} - j\right)\theta.$$

Since the Chebyshev polynomials of the third kind on the interval $[-1, 1]$ can be represented as

$$\hat{V}_n(x) = \frac{1}{\sqrt{\pi}} \frac{\sin(n+1/2)\theta}{\sin(\theta/2)},$$

the following representation holds for the B-SOP of the third kind:

$$\hat{R}_n(x) = \sum_{j=0}^{l} h_j \hat{V}_{n-j}(x). \tag{12}$$

For the B-SOP of fourth kind, we obtain

$$\begin{aligned}
\hat{S}_n(\cos\theta) &= \frac{1}{\sqrt{\pi}} \frac{1}{\cos(\theta/2)} \Re\bigl[e^{i(n+1/2)\theta}\overline{h(e^{i\theta})}\,\bigr] \\
&= \frac{1}{\sqrt{\pi}} \frac{1}{\cos(\theta/2)} \Re\left[e^{i(n+1/2)\theta}\overline{\sum_{j=0}^{l} h e^{i\theta j}}\right] \\
&= \frac{1}{\sqrt{\pi}} \frac{1}{\cos(\theta/2)} \Re\left[\sum_{j=0}^{l} h_j e^{i(n+1/2-j)\theta}\right] \\
&= \frac{1}{\sqrt{\pi}} \frac{1}{\cos(\theta/2)} \sum_{j=0}^{l} h_j \cos\left(n + \frac{1}{2} - j\right)\theta.
\end{aligned}$$

Since the Chebyshev polynomials of fourth kind on the interval $[-1, 1]$ can be represented as

$$\hat{W}_n(x) = \frac{1}{\sqrt{\pi}} \frac{\cos(n+1/2)\theta}{\cos(\theta/2)},$$

the following representation holds for the B-SOP of the fourth kind:

$$\hat{S}_n(x) = \sum_{j=0}^{l} h_j \hat{W}_{n-j}(x). \tag{13}$$

Combining the formulas (10)–(13) yields that the B-SOP of any kind can be represented as a special $l$-incomplete linear combination of the Chebyshev polynomials of the same kind. The special feature of this representation is that the expansion coefficients $\{h_j\}_{j=0}^{l}$ do not depend upon $n$, they are absolutely constant numbers. This situation is characteristic for the B-SOP.

The following surprisingly simple theorem holds.

**Theorem 3.1.** *The B-SOP of the $i$th kind* $(i = 1, 2, 3, 4)$*, and only them, can be represented as an $l$-incomplete linear combination of the Chebyshev polynomials of the corresponding $i$th kind with absolutely constant coefficients. Namely,*

$$\hat{Q}_n(x) = \sum_{j=0}^{l} h_j \hat{\mathcal{P}}_{n-j}(x),$$

*where $h_j$ are independent of $n$ real coefficients of the polynomial $h(z)$, which gives normalized representation to the weight polynomial $\varrho_l(x)$.*

**Proof.** *Necessity*. Follows from the consideration above, and the formulas (10)–(13).

*Sufficiency*. Let $\Omega_n(x) = \sum_{k=0}^{l} h_k \hat{\mathcal{P}}_{n-k}(x)$ be an orthonormal on the interval $[-1, 1]$ $l$-incomplete linear combination of the orthonormal on $[-1, 1]$ Chebyshev polynomials of certain $i$th kind. According to Theorem 2.3, the weight, with which these polynomials are orthogonal, is $\mu_j(x)/\varrho_l(x)$, where $\mu_j(x)$ is the Chebyshev weight of the $i$th kind. The weight polynomial $\varrho_l(x)$ is reconstructed by the coefficients $h_j$ $(j = 0, 1, \dots, l)$. Namely, from these coefficients one can construct the Fejer-normalized polynomial $h(z) = \sum_{j=0}^{l} h_j z^j$, and $\varrho_l(x) = |h(z)|^2$, where $z = e^{i\theta}$, $x = \cos\theta$.

Thus, the given orthogonal $l$-incomplete linear combination of Chebyshev polynomials of the $i$th kind is a B-SOP of the $i$th kind. □

## 4. B-SOP and localization of zeros of Fejer polynomial

Posing and solving many of the applied and theoretical problems leads to $l$-incomplete linear combinations of Chebyshev polynomials. The results, obtained in terms of such linear combinations, allow for interpretation in terms of Bernstein–Szegö polynomials, as the following theorem shows.

**Theorem 4.1.** *Let the following assumptions hold*:

(1) $\mathcal{P}_n(x)$ *are classic Chebyshev polynomials of one of the four kinds, orthonormal on the interval* $[-1, 1]$;
(2) $\varphi_n(x)$ *is an $l$-incomplete linear combination of the polynomials $\mathcal{P}_n(x)$ with absolutely constant real coefficients*

$$\varphi_n(x) = h_0\mathcal{P}_n(x) + h_1\mathcal{P}_{n-1}(x) + \cdots + h_l\mathcal{P}_{n-l}(x), \tag{14}$$

$\forall l < n \in \mathbb{N}$, $h_0 \neq 0 \neq h_l$;
(3) *The coefficients $h_k$ $(k = 0, 1, \dots, l)$ are such that the polynomial of a complex variable $z$,*

$$h_l(z) = h_l z^l + h_{l-1} z^{l-1} + \cdots + h_1 z + h_0,$$

*either does not have any zeros for $|z| < 1$, or all its zeros are inside of the unit circle.*

*Then, either $\varphi_n(x)$, or*

$$\varphi_n^*(x) = h_l\mathcal{P}_n(x) + h_{l-1}\mathcal{P}_{n-1}(x) + \cdots + h_0\mathcal{P}_{n-l}(x) \tag{15}$$

*are Bernstein–Szegö polynomials of the corresponding kind.*

**Proof.** If the polynomial $h_l(z)$ of degree $l$ does not have zeros for $|z| \leqslant 1$, then it gives Fejer-normalized representation of the positive on $(-1, 1)$ cosine-polynomial

$$\left|h_l(e^{i\theta})\right|^2 = \varrho_l(\cos\theta) = \varrho_l(x) \quad \text{for } x = \cos\theta.$$

We construct the Bernstein–Szegö orthonormal polynomials $Q_n(x)$ of the corresponding kind with respect to the weight $\omega_k(x) = \mu_k(x)/\varrho_l(x)$. The coefficients of the polynomial

$h_l(z)$ are the Fourier–Chebyshev coefficients of the $Q_n(x)$ expansion by the Chebyshev polynomials. The $Q_n(x)$ expansion coincides with the expansion of the polynomial $\varphi_n(x)$. This implies $\varphi_n(x) = Q_n(x)$. If $h_l(z)$ has all its zeros inside the unit circle, then the polynomial $h_l^*(z) = \sum_{k=0}^{l} h_{l-k} z^k$ does not have zeros for $|z| \leqslant 1$. Indeed assuming $z = 1/\zeta$, all the zeros for $|z| < 1$ transform into the exterior of the unit circle $|\zeta| > 1$. In addition,

$$h_l(z) = h_0 + h_1 z + \cdots + h_l z^l = z^l \left( h_l + h_{l-1}\frac{1}{z} + \cdots + h_0 \frac{1}{z^l} \right)$$
$$= \frac{1}{\zeta^l}\left(h_l + h_{l-1}\zeta + \cdots + h_0 \zeta^l\right).$$

This shows that the polynomial $h_l^*(z) = h_l + h_{l-1}z + \cdots + h_0 z^l$ does not have zeros inside $|z| < 1$. In addition, $|h_l(e^{i\theta})| = |h_l^*(e^{-i\theta})|$. Therefore, $h_l^*(z)$ gives Fejer-normalized to the same positive polynomial $\varrho_l(x)$ as $h_l(z)$ does. This reduces the consideration to the previous case. □

**Corollary.** *Under the conditions* (1)–(3) *of Theorem* 4.1*, the $l$-incomplete linear combination of Chebyshev orthogonal polynomials $\varphi_n(x)$ has $n$ simple zeros in the interior of the interval* $(-1, 1)$.

We apply Theorem 4.1 to the result of Galeev [15] on solving the Zolotarev's problem.

Among all the polynomials of degree $n$ with given two elder coefficients 1 and $-\sigma$, find minimal in the norm $L_{[-1,1]}$, that is, the one with minimal deviation from zero in the $L_{[-1,1]}$ metrics. In other words, one is looking for

$$\min_{(a_0,a_1,\ldots,a_{n-2})\in\mathbb{R}^{n-1}} \int_{-1}^{1} \left| x^n - \sigma x^{n-1} + \sum_{k=0}^{n-2} a_k x^k \right| dx.$$

The extrema polynomial in this problem, which Galeev obtained, has the form

$$Z_n(x,\sigma) = \begin{cases} (x-\sigma)U_{n-1}(x) & \text{for } |\sigma| \geqslant 1, \\ U_n(x) - \sigma U_{n-1}(x) + \frac{\sigma^2}{4} U_{n-2}(x) & \text{for } |\sigma| < 1, \end{cases}$$

where $U_n(x)$ are Chebyshev polynomials of the second kind, orthogonal with the weight $\sqrt{1-x^2}$ on the interval $[-1, 1]$.

For $|\sigma| < 1$, $Z_n(x,\sigma)$ is a 2-incomplete linear combination of the polynomials $U_n(x)$. The polynomial

$$h_2(z) = \frac{\sigma^2}{4} z^2 - \sigma z + 1 = \left( \frac{\sigma}{2} z - 1 \right)^2$$

vanishes only for $z_0 = 2/\sigma$. Since $|\sigma| < 1$, we have $|z_0| > 2$; hence $h_2(z)$ does not have zeros for $|z| \leqslant 1$ and gives a normalized representation to the polynomial

$$\varrho_2(x)\left|h_2(e^{i\theta})\right|^2 = \left( \sigma x - \frac{\sigma^2}{4} - 1 \right)^2.$$

This implies that the extrema polynomial in the Zolotarev's problem in the metrics $L[-1,1]$ for $|\sigma| < 1$ is a Bernstein–Szegö orthogonal polynomial of second kind, corresponding to the weight function

$$\omega(x) = \frac{\sqrt{1-x^2}}{(\sigma x - \sigma^2/4 - 1)^2}.$$

In Theorem 4.1 (interpretation) the condition (3) (either the polynomial $h_l(z)$ does not have zeros inside the circle $|z| \leqslant 1$, or all its zeros are inside the unit circle) is essential. We shall obtain a practical criterion for the condition (3) to hold. Such a criterion could be obtained from the known criteria of Rauss–Horvitz, Nyqwist and others by transforming the left semi-plane onto the unit circle.

Let $h_l(\zeta)$ be the polynomial

$$h_l(\zeta) = h_l\zeta^l + h_{l-1}\zeta^{l-1} + \cdots + h_1\zeta + h_0, \quad h_l \neq 0 \neq h_0.$$

We transform conformally the left semi-plane $\Re z < 0$ onto the unit circle $|z| < 1$ with help of the function $\zeta = \frac{z+1}{z-1}$. The polynomial $h_l(\zeta)$ transforms into the rational function $R(z) = h_l\bigl(\frac{z+1}{z-1}\bigr) = \frac{g_l(z)}{(z-1)^l}$, where $g_l(z) = \sum_{k=0}^{l} h_k(z+1)^k(z-1)^{l-k}$. If $\zeta_s$ is a zero of the polynomial $h_l(\zeta)$ of multiplicity $\alpha_s$, that is inside the unit circle so that $|\zeta_s| < 1$, $h_l^{(r)}(\zeta_s) = 0$ for $r = 0, 1, \ldots, \alpha_s - 1$ and $h_l^{(\alpha_s)}(\zeta_s) \neq 0$, then $\zeta_s$ is a zero of the polynomial $g_l(z)$ of the same multiplicity $\alpha_s$, lying in the left semi-plane, so that $\Re z_s < 0$ and $g_l^{(r)}(z_s) = 0$ for $r = 0, 1, \ldots, \alpha_s - 1$ and $g_l^{(\alpha_s)}(z_s) \neq 0$, where $\zeta_s = \frac{z_s+1}{z_s-1}$. The total number of zeros of $h_l(\zeta)$ inside the unit circle is equal to the total number of zeros of $g_l(z)$ in the left semi-plane. We apply the argument principle when determining the number of zeros of $g_l(z)$ in the left semi-plane, hence also the number of zeros of the polynomial $h_l(z)$ inside the unit circle.

We introduce notations:

- $n_\lambda$ is the number of zeros (including multiplicities) of the polynomial $g_l(z)$ in the left semi-plane.
- $\nu$ is the number of rotations around the origin of the image of the imaginary axis under the mapping $\omega = g_l(z)$.

The argument principle implies

$$n_\lambda = \frac{l}{2} + \nu,$$

where $l$ is the degree of the polynomials $g_l(z)$ and $h_l(\zeta)$.

Thus we arrive at the following results:

(1) The number of zeros $h_l(\zeta)$ inside the unit circle is equal to

$$n_\lambda = \frac{l}{2} + \nu.$$

(2) In order for the polynomial $h_l(\zeta)$ not to have zeros inside the unit circle, it is necessary and sufficient that

$$\nu = -\frac{l}{2}.$$

(3) In order for the polynomial $h_l(\zeta)$ to have all its zeros inside the unit circle, it is necessary and sufficient that

$$\nu = \frac{l}{2}.$$

## 5. Solution to the Zolotarev–Markov problems

By using the B-SOP properties, we shall show elementary and uniform solution to the aforementioned extrema problems of the Zolotarev–Markov type. We will need the following fundamental oscillatory properties of the B-SOP for this purpose that were found by the author in [13,14]. We formulate these properties as four theorems.

**Theorem 5.1.** *The B-SOP of the first kind $P_n(x)$ are the snakes for the pair of functions $\pm\sqrt{\varrho_\ell(x)}$, with the e-points in zeros of the B-SOP of the second kind $q_{n-1}(x)$ and in $\pm 1$, namely,*

$$P_n\big[x_\nu^{(n-1)}(q)\big] = (-1)^{n-\nu}\sqrt{\varrho_\ell\big[x_\nu^{(n-1)}(q)\big]}, \quad \nu = 0, 1, 2, \ldots, n. \tag{16}$$

**Theorem 5.2.** *The B-SOP of the second kind are the snakes for the pair of functions $\pm\sqrt{\varrho_\ell(x)/(1-x^2)}$, with the e-points in zeros of the B-SOP of the first kind $P_{n+1}(x)$, namely,*

$$q_n\big[x_\nu^{(n+1)}(P)\big] = (-1)^{n+1-\nu}\sqrt{\frac{\varrho_\ell(x)}{1-x^2}}\bigg|_{x=x_\nu^{(n+1)}(P)}, \quad \nu = 1, 2, \ldots, n+1. \tag{17}$$

**Theorem 5.3.** *The B-SOP of the third kind are the snakes for the pair of functions $\pm\sqrt{\varrho_\ell(x)/(1-x)}$ with the e-points in zeros of the B-SOP of the fourth kind $S_n(x)$ and in $x = -1$, namely,*

$$R_n\big[x_\nu^{(n)}(S)\big] = (-1)^{n-\nu}\sqrt{\frac{\varrho_\ell[x_\nu^{(n)}(S)]}{1 - x_\nu^{(n)}(S)}}, \quad \nu = 0, 1, 2, \ldots, n. \tag{18}$$

**Theorem 5.4.** *The B-SOP of the fourth kind $S_n(x)$ are the snakes for the pair of functions $\pm\sqrt{\varrho_\ell(x)/(1+x)}$ with the e-points in zeros of the B-SOP of the third kind $R_n(x)$ and in $x = 1$, namely,*

$$S_n\big[x_\nu^{(n)}(R)\big] = (-1)^{n+1-\nu}\sqrt{\frac{\varrho_\ell[x_\nu^{(n)}(R)]}{1 + x_\nu^{(n)}(R)}}, \quad \nu = 1, 2, \ldots, n+1. \tag{19}$$

Complete proofs of these theorems are given in the author's paper [14].

We prove that the solution to the Zolotarev–Markov extrema problem number $i$ $(i = 1, 2, 3, 4)$ is the B-SOP of the same $i$th kind.

The equality (16) implies

$$\min_{\forall \Pi_n(x)} \max_{-1\leqslant x\leqslant 1} \frac{|\Pi_n(x)|}{\sqrt{\varrho_l(x)}} = \max_{-1\leqslant x\leqslant 1} \frac{|P_n(x)|}{\sqrt{\varrho_l(x)}} = \frac{|P_n(x)|}{\sqrt{\varrho_l(x)}}\bigg|_{x=x_\nu^{(n-1)}(q)} = 1.$$

The equality (17) implies

$$\min_{\forall \Pi_n(x)} \max_{-1\leqslant x\leqslant 1} \left|\Pi_n(x)\right|\sqrt{\frac{1-x^2}{\varrho_l(x)}} = \max_{-1\leqslant x\leqslant 1} \left|q_n(x)\right|\sqrt{\frac{1-x^2}{\varrho_l(x)}}$$
$$= \left|q_n(x)\right|\sqrt{\frac{1-x^2}{\varrho_l(x)}}\Bigg|_{x=x_\nu^{(n+1)}(P)} = 1.$$

The equality (18) implies

$$\min_{\forall \Pi_n(x)} \max_{-1\leqslant x\leqslant 1} \left|\Pi_n(x)\right|\sqrt{\frac{1-x}{\varrho_l(x)}} = \max_{-1\leqslant x\leqslant 1} \left|R_n(x)\right|\sqrt{\frac{1-x}{\varrho_l(x)}}$$
$$= \left|R_n(x)\right|\sqrt{\frac{1-x}{\varrho_l(x)}}\Bigg|_{x=x_\nu^{(n)}(S)} = 1.$$

The equality (19) implies

$$\min_{\forall \Pi_n(x)} \max_{-1\leqslant x\leqslant 1} \left|\Pi_n(x)\right|\sqrt{\frac{1+x}{\varrho_l(x)}} = \max_{-1\leqslant x\leqslant 1} \left|S_n(x)\right|\sqrt{\frac{1+x}{\varrho_l(x)}}$$
$$= \left|S_n(x)\right|\sqrt{\frac{1+x}{\varrho_l(x)}}\Bigg|_{x=x_\nu^{(n)}(R)} = 1.$$

These equalities together with the Theorems 5.1–5.4 show that the corresponding functions

$$\frac{P_n(x)}{\sqrt{\varrho_l(x)}}, \quad q_n(x)\sqrt{\frac{1-x^2}{\varrho_l(x)}}, \quad R_n(x)\sqrt{\frac{1-x}{\varrho_l(x)}}, \quad S_n(x)\sqrt{\frac{1+x}{\varrho_l(x)}}$$

have in the $n+1$ $e$-points of B-SOP of the corresponding kind on the interval $[-1, 1]$ the same absolute deviations with alternating signs. Hence, the $e$-points constitute for them the Chebyshev alternance. According to Chebyshev's theorem [8], the B-SOP of the $i$th kind are the solutions to the corresponding $i$th problem $(i = 1, 2, 3, 4)$ of Zolotarev–Markov.

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