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Effectively open real functions

Martin Ziegler[*,1]

University of Paderborn, Germany

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Abstract

A function f is continuous iff the *pre*-image $f^{-1}[V]$ of any open set V is open again. Dual to this topological property, f is called *open* iff the *image* $f[U]$ of any open set U is open again. Several classical open mapping theorems in analysis provide a variety of sufficient conditions for openness.

By the main theorem of recursive analysis, computable real functions are necessarily continuous. In fact they admit a well-known characterization in terms of the mapping $V \mapsto f^{-1}[V]$ being *effective*: given a list of open rational balls exhausting V, a Turing Machine can generate a corresponding list for $f^{-1}[V]$. Analogously, *effective openness* requires the mapping $U \mapsto f[U]$ on open real subsets to be effective.

The present work combines real analysis with algebraic topology and Tarski's quantifier elimination to effectivize classical open mapping theorems and to establish several rich classes of real functions as effectively open.

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1. Introduction

Computability theory over the reals started by investigating single numbers [Tur36]. When real functions were later considered it turned out that continuity was a necessary condition for computability. A function $f : X \to Y$ between topological spaces is continuous iff, for any open

* Fax: +49 5251 606482.
E-mail address: ziegler@uni-paderborn.de.

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set $V \subset Y$, its *pre*-image $f^{-1}[V] \subseteq X$ is open again. In the case of open $X \subseteq \mathbb{R}^n$ and $Y = \mathbb{R}^m$ this means that, for any countable union of m-dimensional open rational Euclidean balls

$$V = \bigcup_j B(\mathbf{y}_j, r_j), \quad \mathbf{y}_j \in \mathbb{Q}^m, \quad r_j \in \mathbb{Q}_{>0}, \quad B(\mathbf{y}, r) := \{\mathbf{u} \in \mathbb{R}^m : |\mathbf{y} - \mathbf{u}| < r\},$$

$U := f^{-1}[V] \subseteq \mathbb{R}^n$ is also a countable union of n-dimensional open rational Euclidean balls $B(\mathbf{x}_\ell, s_\ell)$. Moreover, f is computable in the sense of [Grz57,PER89,Ko91] iff the mapping $V \mapsto f^{-1}[V]$ on hyperspaces of open subsets is effective in that, given a list of (centers $\mathbf{x}_k$ and radii r_k of) open rational Euclidean balls $B(\mathbf{x}_k, r_k) \subseteq \mathbb{R}^m$ exhausting V, one can compute a corresponding list of open rational Euclidean balls $B(\mathbf{y}_\ell, s_\ell) \subseteq \mathbb{R}^n$ exhausting $f^{-1}[V]$; cf. Lemma 6.1.7 in [Wei00].

So to speak ‘dual’ to continuity is openness: the function f is *open* if, rather than its pre-image, its image $f[U] \subseteq Y$ is open for any open set $U \subseteq X$. While for example any constant f lacks the latter property, conditions sufficient for its presence are given by a variety of well-known Open Mapping Theorems for instance in Functional Analysis, Complex Calculus, Real Analysis, or Algebraic Topology.

The classical duality of continuity and openness raises the question whether and to what extent it carries over to the computable setting. For the first two aforementioned theorems, effectivized versions (in the sense of Recursive Analysis) have been established respectively in [Bra01,Her99]; see Theorem 2. It is indeed natural to consider, similarly to continuity and computability, also *effective* openness in the following sense:

Definition 1. Let $X \subseteq \mathbb{R}^n$ be r.e. open, that is a union of certain open rational balls $B(\mathbf{z}_j, t_j)$ whose centers $\mathbf{z}_j$ and radii t_j form computable rational sequences; cf. [Wei00, Definition 5.1.15.3].

Call an open function $f : X \to \mathbb{R}^m$ effectively open if, from any two lists $(\mathbf{x}_j)_{j\in\mathbb{N}}$ in $\mathbb{Q}^n$ and $(r_j)_{j\in\mathbb{N}}$ in $\mathbb{Q}_{>0}$, a Turing Machine can compute two similar lists $(\mathbf{y}_\ell)_\ell$ in $\mathbb{Q}^m$ and $(s_\ell)_\ell$ in $\mathbb{Q}_{>0}$ such that $f\left[\bigcup_j B(\mathbf{x}_j, r_j)\right] = \bigcup_\ell B(\mathbf{y}_\ell, s_\ell)$.

In the convenient language of *Type*-2 *Theory of Effectivity* [Wei00], this amounts to the mapping $U \mapsto f[U]$ on open Euclidean subsets being $(\theta^n_< \to \theta^m_<)$-computable.[2] Here, $\theta^d_<$ denotes a canonical representation for the hyperspace $\mathfrak{O}^d$ of open subsets of $\mathbb{R}^d$; cf. Definition 5.1.15 in [Wei00].

Apart from its natural duality to continuity and computability, openness and effective openness arise in the foundation of CAD/CAE [EL02] in connection with regular sets—i.e., roughly speaking, full-dimensional but not necessarily convex [KS95] ones—as essential prerequisites for computations thereon; cf. Proposition 1.1(d)–(f) and Section 3.1 in [Zie04].

The present work proves several rich and important classes of functions to be effectively open and thus applicable to such problems. Our claims proceed in analogy to those of classical Open Mapping Theorems. An example due to Hertling illustrates the idea:

Theorem 2. (a) *Let $f : \mathbb{C} \to \mathbb{C}$ be complex differentiable and non-constant. Then f is open.*

(b) *Let f furthermore be computable. Then it is effectively open.*

[2] The ‘official’ syntax due to [Wei00] reads as “$(\theta^n_<, \theta^m_<)$”-computability, that is, with a comma; however we favor the suggestive arrow.

(c) *Claim* (b) *holds even uniformly in f, that is, the mapping* $(f, U) \mapsto f[U]$ *with domain*

$$\{(f, U) \mid f : \mathbb{C} \to \mathbb{C} \text{ complex differentiable non-constant}, U \subseteq \mathbb{C} \text{ open}\}$$

is $\left([\varrho^2 \to \varrho^2] \times \theta^2_< \to \theta^2_<\right)$*-computable.*

Proof. (a) is well-known in Complex Analysis; see, e.g., [Rud74, pp. 231–233]. For (b) and (c), cf. Corollary 4.4 and Theorem 4.3 in [Her99], respectively. □

Here, ϱ^2 denotes the Cauchy representation for the set $\mathbb{C}$ of complex numbers, identified with $\mathbb{R}^2$; and $[\varrho^n \to \varrho^m]$ is a natural representation for continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^m$; see Definitions 4.1.17 and 6.1.1 in [Wei00].

1.1. Overview

In the spirit of the above result, we present in Section 2 several classical Open Mapping Theorems from Real Analysis and Algebraic Topology; and in Sections 3 and 4 according effectivizations. More precisely, proof-mining reveals several classes of computable open functions on Euclidean space to be effectively open. We focus on claims similar to Theorem 2(b), that is, for fixed *f* but uniformly in *U*.

Section 5 takes a different approach in devising 'from scratch' proofs that computable open semi-algebraic functions are effectively open. Here, arguments are based on Algebra and exploit Tarski's Quantifier Elimination—quite surprisingly regarding that the latter usually pertains to algebraic models of real computation [BCSS98] due to its reliance on equality as decidable a primitive!

Section 6 finally investigates the general relation between computability and effective openness. We conclude in Section 7 with a strengthening of [Zie04, Theorem 3.9] based on the results from Section 4.

A mathematical publication usually cannot be read simply once from the start to the end. This is due to a proof generally resembling, rather than a straight line, a tree (more precisely: a directed acyclic graph) with prerequisites and axioms in the leafs and arrows (implications) directed to the final claim in the root, intermediate results and lemmas located in between. Applying a lemma only after it has been proven, a logic purist's presentation thus would start with the leafs of that tree and proceed *bottom-up*; whereas a *top-down* presentation would first formulate and motivate the claims, beginning with the most central ones, and postpone proofs. In the present work I have tried to compromise between both approaches.

2. Classical open mapping theorems

We start with a characterization of open functions resembling that of continuous ones. Throughout this work, all balls are considered in the Euclidean sense, that is, *not* implicitly restricted to *X*.

Lemma 3. *Let* $X \subseteq \mathbb{R}^n$ *be open, and denote* $\overline{B}(\mathbf{x}, s) := \{\mathbf{v} \in \mathbb{R}^n : |\mathbf{v} - \mathbf{x}| \leqslant s\}$.

(a i) *A function* $f : X \to \mathbb{R}^m$ *is continuous iff the mapping*

$$\mathrm{Moc}_f : X \times \mathbb{N} \to \mathbb{R}, \quad (\mathbf{x}, k) \mapsto \sup\left\{s \geqslant 0 : f\left[\overline{B}(\mathbf{x}, s) \cap X\right] \subseteq B\left(f(\mathbf{x}), 2^{-k}\right)\right\} \tag{1}$$

is strictly positive;

(a ii) *equivalently*: *to any* $(\mathbf{x}, k) \in X \times \mathbb{N}$, *there exists an* $\ell \in \mathbb{N}$ *such that*

$$f\left[B(\mathbf{x}, 2^{-\ell}) \cap X\right] \subseteq B\left(f(\mathbf{x}), 2^{-k}\right). \tag{2}$$

(b i) *A function* $f : X \to \mathbb{R}^m$ *is open iff the mapping*

$$\mathrm{Moo}_f : X \times \mathbb{N} \to \mathbb{R}, \quad (\mathbf{x}, k) \mapsto \sup\left\{s \geqslant 0 : \overline{B}\,(f(\mathbf{x}), s) \subseteq f\left[B(\mathbf{x}, 2^{-k}) \cap X\right]\right\} \tag{3}$$

is strictly positive;

(b ii) *equivalently*: *to any* $(\mathbf{x}, k) \in X \times \mathbb{N}$, *there exists an* $\ell \in \mathbb{N}$ *such that*

$$B\left(f(\mathbf{x}), 2^{-\ell}\right) \subseteq f\left[B(\mathbf{x}, 2^{-k}) \cap X\right]. \tag{4}$$

Both the function Moc_f according to Eq. (1) as well as any mapping $\mathrm{moc} : X \times \mathbb{N} \to \mathbb{N}$ satisfying Eq. (2) for $\ell = \mathrm{moc}(\mathbf{x}, k)$ are known as *the* or *a*, respectively, (local) *modulus of continuity* of *f*; cf., e.g., [Haz00] or [Wei00, Definition 6.2.6]. The apparent similarity suggests the following:

Definition 4. Moo_f according to Eq. (3) is *the modulus of openness* of *f*; call some mapping $\mathrm{moo} : X \times \mathbb{N} \to \mathbb{N}$ *a modulus of openness* of *f* if Eq. (4) holds for $\ell = \mathrm{moo}(\mathbf{x}, k)$.

In contrast to a modulus of *continuity*, one of openness does suffice to be positive or defined on a dense subset only:

Example 5. $f : \mathbb{R} \ni x \mapsto |x - \pi|$ lacks openness but $\mathrm{Moo}_f : \mathbb{Q} \times \mathbb{N} \to \mathbb{R}$ is strictly positive.

Proof of Lemma 3. Observe that $f[A] \subseteq B \Leftrightarrow A \subseteq f^{-1}[B]$.

(a i) Let Moc_f be strictly positive and $V \subseteq \mathbb{R}^m$ open. To show that $f^{-1}[V]$ is open again, let $\mathbf{x} \in f^{-1}[V]$ be arbitrary. As $\mathbf{y} := f(\mathbf{x}) \in V$ and *V* is open, $B(\mathbf{y}, 2^{-k}) \subseteq V$ for some $k \in \mathbb{N}$. Then for $s := \mathrm{Moc}_f(\mathbf{x}, k)/2$, the open set $U := B(\mathbf{x}, s) \cap X$ satisfies

$$\mathbf{x} \in U \subseteq f^{-1}[f[U]] \overset{(1)}{\subseteq} f^{-1}\left[B\left(\mathbf{y}, 2^{-k}\right)\right] \subseteq f^{-1}[V]$$

that is, an entire open ball around $\mathbf{x}$ lying within $f^{-1}[V]$.
Conversely let *f* be continuous, $\mathbf{x} \in X$ and $k \in \mathbb{N}$. Therefore, the pre-image $U := f^{-1}[V]$ of $V := B\left(f(\mathbf{x}), 2^{-k}\right)$ is open and contains $\mathbf{x}$. In particular $\overline{B}(\mathbf{x}, s) \subseteq U$ for some $s > 0$ and $\mathrm{Moc}_f(\mathbf{x}, k) \geqslant s$ is strictly positive.

(a ii) If $\mathrm{Moc}_f(\mathbf{x}, k) > 0$, then $\ell := \lceil \log_2(1/s) \rceil$ for any $0 < s < \mathrm{Moc}_f(\mathbf{x}, k)$ with $s < 1$. Conversely, (2) yields $s := 2^{-\ell-1}$ as a positive lower bound to $\mathrm{Moc}_f(\mathbf{x}, k)$.

(b i) If *f* is open, then its image $f[U]$ of the open set $U := B(\mathbf{x}, 2^{-k}) \cap X \neq \emptyset$ is open again and thus contains, around the point $f(\mathbf{x}) \in f[U]$, some non-empty ball $\overline{B}\,(f(\mathbf{x}), s)$ entirely; hence $\mathrm{Moo}_f(\mathbf{x}, k) \geqslant s > 0$.
Conversely let *U* denote an open subset of *X*. To any $\mathbf{y} \in f[U]$, consider $\mathbf{x} \in U$ with $\mathbf{y} = f(\mathbf{x})$ and $k \in \mathbb{N}$ such that $B(\mathbf{x}, 2^{-k}) \subseteq U$. Then $s := \mathrm{Moo}_f(\mathbf{x}, k)/2$ satisfies

$$B(\mathbf{y}, s) \overset{(3)}{\subseteq} f[B(\mathbf{x}, 2^{-k}) \cap X] \subseteq f[U].$$

Therefore $f[U]$ is open.

(b ii) Follows as in (a ii). In particular it holds $f[U] = \bigcup_{\mathbf{x}\in U} B\left(f(\mathbf{x}), 2^{-\mathrm{moo}(\mathbf{x},k_\mathbf{x})}\right)$ for open $U \subseteq X$ whenever $k_\mathbf{x} \in \mathbb{N}$ satisfies $B(\mathbf{x}, 2^{-k_\mathbf{x}}) \subseteq U$. $\square$

Many famous classical theorems give sufficient conditions for a real function to be open. Several such claims are collected in the following:

Fact 6. *Let $X \subseteq \mathbb{R}^n$ be open.*

(a) *Suppose continuous $f : X \to \mathbb{R}$ has no local extrema* (*i.e.*, *to any open $U \subseteq X$ and $\mathbf{x} \in U$, there exist $\mathbf{x}_-, \mathbf{x}_+ \in U$ such that $f(\mathbf{x}_-) < f(\mathbf{x}) < f(\mathbf{x}_+)$*); *then f is open.*
(b) *Any affinely linear mapping $\mathbb{R}^n \ni \mathbf{x} \mapsto A \cdot \mathbf{x} + \mathbf{b} \in \mathbb{R}^m$ is open iff it is surjective.*
(c) *Any continuously differentiable* (‘C^1’) *$f : X \to \mathbb{R}^m$ is open, provided its* Jacobian *$f'(\mathbf{x}) = \left((\partial_i f_j)_{ij}\right)(\mathbf{x})$ has rank m for all $\mathbf{x} \in X$.*
(d) *Whenever continuous $f : X \to \mathbb{R}^n$ satisfies* local injectivity (*i.e.*, *to each $\mathbf{x} \in X$ there exists $\varepsilon > 0$ such that the restriction $f|_{B(\mathbf{x},\varepsilon)}$ is injective*), *then it is open.*

Claim (d) generalizes Domain Invariance from Algebraic Topology where often injectivity is presumed globally. Regarding a converse of Claim (c) for $n \leqslant m$, if f' has rank $< m$ on a non-empty open set U, then $f[U]$ cannot be open by virtue of

Fact 7 (*Morse–Sard theorem*). *Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ C^1. If $n \leqslant m$, then the set of* critical values $\{f(\mathbf{x}) : \mathbf{x} \in U \wedge \operatorname{rank} f'(\mathbf{x}) < m\} \subseteq \mathbb{R}^n$ *has Lebesgue measure zero.*

Proof. See e.g. [Mil97]. The requirement $n \leqslant m$ is essential [Whi35]! $\square$

Proof of Fact 6.

(a) Exploit the one-dimensional range and apply the Intermediate Value Theorem: an open ball $B := B(\mathbf{x}, 2^{-k}) \subseteq \mathbb{R}^n$ is connected; hence if $B \subseteq X$, then $f[B] \subseteq \mathbb{R}$ is connected as well and thus a real interval. As furthermore *f* has by prerequisite no local extrema, any $y \in f[B]$ is accompanied by $y_-, y_+ \in f[B]$ such that $y_- < y < y_+$. This implies $(y_-, y_+) \subseteq f[B]$ and reveals that $f[B]$ is an open set. Open $U \subseteq X$ being a union of balls B_i, $f[U] = \bigcup_i f[B_i]$ is open, too.
(b) Follows from (c), as the Jacobian of $f(\mathbf{x}) = A \cdot \mathbf{x} + \mathbf{b}$ is $A \in \mathbb{R}^{m\times n}$ (independent of $\mathbf{x}$) and $\operatorname{rank}(A) = m$ is equivalent to *f* being surjective.
(d) See for example [Dei85, Theorem 4.3] where (for $r = \varepsilon$) the proof proceeds by showing that the *topological degree* $d(\Omega, f, \mathbf{y})$ of *f* with respect to domain $\Omega := B(\mathbf{x}, r)$ is non-zero for all $\mathbf{y}$ in some *s*-ball around $f(\mathbf{x})$. This guarantees that $f|_\Omega$ attains any such value $\mathbf{y} \in B(f(\mathbf{x}), s)$, that is, $f[\Omega]$ contains $B(f(\mathbf{x}), s)$. For $\varepsilon < 2^{-k}$, this implies $\mathrm{Moo}_f(\mathbf{x}, k) \geqslant s/2 > 0$ and by Lemma 3(b) yields openness of *f*. $\square$

Regarding (c), *f* has no chance of being locally injective whenever $n > m$ so that (d) is not applicable in that case. Instead, exploiting differentiability, recall the Inverse Function Theorem from Real Analysis:

Fact 8. *Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ continuously differentiable, and $\mathbf{x}_0 \in U$ such that* $\operatorname{rank} f'(\mathbf{x}_0) = m$. *Then there exists a continuously differentiable local right inverse to f at $\mathbf{x}_0$,* *that is,* $\delta > 0$ *and a* C^1 *function*

$$g : B\,(f(\mathbf{x}_0), \delta) \subseteq \mathbb{R}^m \to U \quad \text{such that}$$
$$g\,(f(\mathbf{x}_0)) = \mathbf{x}_0, \quad f\,(g(\mathbf{y})) = \mathbf{y} \quad \forall \mathbf{y} \in B\,(f(\mathbf{x}_0), \delta) \tag{5}$$

If $n = m$, *then g is unique and locally left inverse to f, i.e.,* $g\,(f(\mathbf{x})) = \mathbf{x}$ *on* $B(\mathbf{x}_0, \varepsilon)$ *for some* $\varepsilon > 0$.

In particular, $f[U]$ covers the open ball $B\,(f(\mathbf{x}_0), \delta) \subseteq \mathbb{R}^m$. By taking $U := B(\mathbf{x}_0, 2^{-k}) \subseteq X$, we obtain $\mathrm{Moo}_f(\mathbf{x}, k) \geqslant \delta/2 > 0$ and Fact 6(c) finally follows with Lemma 3(b).

3. Effective continuity, effective openness

The present section is about an effectivization of Lemma 3. While positivity of $\mathrm{Moc}_f/\mathrm{Moo}_f$ is trivially equivalent to the existence of an according moc/moo, respectively, similar equivalences are by no means obvious with respect to computability. In fact for this purpose, both moc and moo have to be allowed to become *multi-valued* in the sense [Wei00, Definition 3.1.3.4] that the integer ℓ returned by a Type-2 Machine computing $\mathrm{moc}(\mathbf{x}, k)$ or $\mathrm{moo}(\mathbf{x}, k)$ may depend, rather than on the value of the argument $\mathbf{x}$ itself, also on the particular choice of rational approximations for $\mathbf{x}$. Such effects are well known in Recursive Analysis, see for instance [Wei00, Example 4.1.10 or Theorem 6.3.7].

Also recall, e.g. from [Wei00], that $\varrho_<$ is a representation for $\mathbb{R}$ connected to lower (also called *left*) computability in that it encodes rational approximations to the real number under consideration *from below*. Furthermore, ν denotes the standard notation of $\mathbb{N}$.

Theorem 9. *Let* $X \subseteq \mathbb{R}^n$ *be r.e. open. Parallel to* (*the numbering in*) *Lemma* 3, *we have*:

(a ii) *Fix some effective* (*i.e.,* $(\nu \to \varrho^n)$*-computable*) *enumeration* $(\mathbf{x}_j)_j$ *of a dense subset of X* (*like for instance* $X \cap \mathbb{Q}$).
A function $f : X \to \mathbb{R}^m$ *is computable iff the real sequence* $\big(f(\mathbf{x}_j)\big)_j$ *is computable and f admits a* $(\varrho^n \times \nu \rightrightarrows \nu)$*-computable multi-valued function* $\mathrm{moc} : X \times \mathbb{N} \rightrightarrows \mathbb{N}$ *such that Eq.* (2) *holds for all* $\ell \in \mathrm{moc}(\mathbf{x}, k)$, $\mathbf{x} \in X$, $k \in \mathbb{N}$.

(b i) *A computable* $f : X \to \mathbb{R}^m$ *is effectively open iff* $\mathrm{Moo}_f : X \times \mathbb{N} \to \mathbb{R}$ *is strictly positive and* $(\varrho^n \times \nu \to \varrho_<)$*-computable*;

(b ii) *equivalently*: *f admits a* $(\varrho^n \times \nu \rightrightarrows \nu)$*-computable multi-valued function* $\mathrm{moo} : X \times \mathbb{N} \rightrightarrows \mathbb{N}$ *such that Eq.* (4) *holds for all* $\ell \in \mathrm{moo}(\mathbf{x}, k)$, $\mathbf{x} \in X$, $k \in \mathbb{N}$.

Claim (a ii) is closely related to Theorem 6 in [Grz57]. Extending Definition 4, multi-valued functions moc/moo in the sense of Claims (a ii) and (b ii) will in the sequel also be called *moduli of continuity/openness*, respectively. An effective counterpart to Claim (a i) fails; cf. Remark 12 below. Before turning to the Proof of Theorem 9 in Section 3.1, we provide in Lemma 11 some tools on multi-valued computability which turn out to be useful.

By the main theorem of computable analysis, any computable real function *f* on a compact domain is continuous and thus bounded. However, the present work also considers *multi*-valued functions like moduli of continuity; and such functions can in general be unbounded even on compact domains.

Example 10. For a rational sequence $(x_j)_j$ with $|x_j - x| < 2^{-j}$ for all $j \in \mathbb{N}$, let

$$F\left((x_j)_j\right) \quad := \quad \left\lfloor \frac{1}{x_0+1} \right\rfloor.$$

Then, F is a computable realization of a multi-valued, unbounded function $f : [0,1] \rightrightarrows \mathbb{N}$.

Item (b) below basically says that such unpleasant cases can always be avoided by passing to another computable multi-valued function. To this end, we call $\tilde{f} : X \rightrightarrows Y$ a *sub-function* of $f : X \rightrightarrows Y$ if $\tilde{f}(\mathbf{x}) \subseteq f(\mathbf{x})$ for all $\mathbf{x} \in X$ and remark that, according to [Wei00, Definition 3.1.3.4], if $\tilde{f}$ is computable then so are *all* its super-functions *f*.

Lemma 11. *Let $X \subseteq \mathbb{R}^n$ be r.e. open and let ϱ_{sd} denote the* signed digit *representation.*

(a) *The partial function $G :\subseteq \mathfrak{O}^n \times \mathbb{R}^n \to \mathbb{R}$, $(U, \mathbf{x}) \mapsto \sup\{s \geqslant 0 : \overline{B}(\mathbf{x}, s) \subseteq U\}$, is $(\theta^n_< \times \varrho^n \to \varrho_<)$-computable; the multi-valued partial mapping $g :\subseteq \mathfrak{O}^n \times \mathbb{R}^n \rightrightarrows \mathbb{N}$ with* $\operatorname{Graph}(g) := \{(U, \mathbf{x}, k) : \overline{B}(\mathbf{x}, 2^{-k}) \subseteq U\}$ *is $(\theta^n_< \times \varrho^n \rightrightarrows \nu)$-computable.*

(b) *To every $(\varrho^n \rightrightarrows \varrho^m)$-computable multi-valued $f : X \rightrightarrows \mathbb{R}^m$, there exists a multi-valued $(\varrho^n \rightrightarrows \varrho^m)$-computable sub-function $\tilde{f}$ for which the image $\tilde{f}[K] := \bigcup_{\mathbf{x} \in K} \tilde{f}(\mathbf{x}) \subseteq \mathbb{R}^m$ of any compact subset K of X is bounded.*

(c) *For $m = 1$ and the function $\tilde{f}$ from* (b), *an upper bound $N \in \mathbb{N}$ on $\tilde{f}[K] \subseteq \mathbb{R}$ can be found effectively; formally: the multi-valued mapping $K \mapsto N$ with $\tilde{f}[K] \subseteq [-N, +N]$ is $(\kappa^n_> \rightrightarrows \nu)$-computable.*

Claims (b) *and* (c) *also hold uniformly in p for parametrized computable functions $p \mapsto f(p, \cdot) : X \rightrightarrows \mathbb{R}^m$.*

Proof of Lemma 11.

(a) By [Zie04, Lemma 4.1(b)], the property "$\overline{B}(\mathbf{x}, s) \subseteq U$" is $(\theta^n_<, \varrho^n, \varrho)$-r.e. open ('semi-decidable') in $(U, \mathbf{x}, s)$. So whenever $\mathbf{x} \in U$, dove-tailed search w.r.t. s gives lower approximations to $G(U, \mathbf{x})$; and restricting s to values 2^{-k} yields an admissible value $k \in g(U, \mathbf{x})$.

(b) Let $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ denote some computable (single-valued) realization of *f*. Exploiting $\varrho \equiv \varrho_{\text{sd}}$ according to [Wei00, Theorem 7.2.5.1], we pre-compose F with a computable function H converting ϱ^n_{sd}-names to ϱ^n-names. $F \circ H$ therefore realizes a $(\varrho^n_{\text{sd}} \rightrightarrows \varrho^m)$-computable sub-function $\tilde{f}$ of *f*, defined by $\tilde{f}(\mathbf{x}) = \left\{\varrho^m\left(F \circ H(\bar{\sigma})\right) : \varrho^n_{\text{sd}}(\bar{\sigma}) = \mathbf{x}\right\}$. By [Wei00, Exercise 7.2.9], the collection $\tilde{K} \subseteq (\Sigma^\omega)^n$ of all ϱ^n_{sd}-names $\bar{\sigma}$ of all $\mathbf{x} \in K$ is in particular compact. Being Cantor-continuous, $F \circ H$ maps $\tilde{K}$ to a compact set $(F \circ H)[\tilde{K}]$ whose image under ϱ^m, namely the set $\tilde{f}[K]$, is again compact by admissibility of ϱ^m.

(c) Rather than carefully adapting the proof of, e.g., [Wei00, Theorems 7.1.5], we slightly modify the Type-2 Machine *M* computing $F \circ H$ in (b) to operate as follows: upon input of a ϱ^n_{sd}-name for $\mathbf{x} \in X$ and while calculating rational approximations y_j to $y = \tilde{f}(\mathbf{x})$ with $|y_j - y| < 2^{-j}$, idly loop $\lceil |y_0| + 1 \rceil$ times before actually outputting the first symbol of that ϱ-name for *y* and then proceeding like *M*.

This new machine $\tilde{M}$ will thus satisfy $\operatorname{dom}(M) = \operatorname{dom}(\tilde{M})$ and $\operatorname{Time}_{\tilde{M}}(\bar{\sigma})(1) \geqslant \tilde{f}(\mathbf{x})$ for any ϱ^n_{sd}-name $\bar{\sigma}$ of $\mathbf{x} \in X$. In particular, $\operatorname{Time}^{\tilde{K}}_{\tilde{M}}(1) \in \mathbb{N}$ is an upper bound on $\tilde{f}[K]$ where $\tilde{K} \subseteq \Sigma^\omega$ denotes the collection of all ϱ^n_{sd}-names $\tilde{f}$ for all $\mathbf{x} \in K$.

According to [Wei00, Exercise 7.2.9], $K \mapsto \tilde{K}$ is $(\kappa^n_> \to \kappa^Y_>)$-computable; and [Wei00, Exercise 7.1.4(a)] implies that, from a $\kappa^Y_>$-name of $\tilde{K}$, one can effectively obtain an upper bound N on $\mathrm{Time}^{\tilde{K}}_{\tilde{M}}(1)$. □

Remark 12. An effective counterpart to Lemma 3(a i) unfortunately fails:

(a) Moc_f is in general not $(\varrho^n \times \nu \to \varrho_>)$-computable:
Take $X = \mathbb{R}$ and let $r > 1$ left but not right computable. Define closed co-r.e. [Wei00, Example 5.1.17.2(a)] $A := \mathbb{R} \setminus B(0, r)$ not containing 1. Consider a computable function $f : \mathbb{R} \to [0, 1]$ with $f(1) = 1$, $f|_A \equiv 0$, and $0 < f(x) < 1$ for all $x \notin A \uplus \{1\}$—such f can be obtained for instance from the Effective Urysohn Lemma [Wei00, Theorem 6.2.10.2]. Then, for $x := 1$, $f^{-1}[B(f(x), 1)] = f^{-1}[(0, 2)] = \mathbb{R} \setminus A = B(0, r)$ and so the value $\mathrm{Moc}_f(1, 0) = \sup\{s : \overline{B}(1, s) \subseteq B(0, r)\} = r - 1$ lacks $\varrho_>$-computability.

(b) Moc_f need not be $(\varrho^n \times \nu \to \varrho_<)$-computable either:
Take $X = (0, 2)$ and $f = \mathrm{id}$. Then, for all $0 < x < 1$, $\overline{B}(x, s) \cap X \subseteq B(x, 1)$ iff $s < 1$; hence $\mathrm{Moc}_f(x, 0) = 1$ in this case. Whereas for $x = 1$, all s satisfy $\overline{B}(x, s) \cap X \subseteq X = B(x, 1)$; so $\mathrm{Moc}_f(1, 0) = \infty$. This reveals Moc_f to lack the lower semi-continuity in x necessary for $(\varrho^n \times \nu \to \varrho_<)$-computability.

With the "sup" in its definition (1), upper computability of Moc_f should not be expected anyway; whereas the lack of lower computability—specifically the annoying influence of X on its values in (b)—has caused the author to ponder using, instead of Moc_f and Moo_f, the functions

$$\widetilde{\mathrm{Moc}}_f : (\mathbf{x}, k) \mapsto \sup\left\{s \geqslant 0 : f\left[\overline{B}(\mathbf{x}, s)\right] \subseteq B\left(f(\mathbf{x}), 2^{-k}\right) \wedge \overline{B}(\mathbf{x}, s) \subseteq X\right\}$$
$$\widetilde{\mathrm{Moo}}_f : (\mathbf{x}, k) \mapsto \sup\left\{s \geqslant 0 : \overline{B}\left(f(\mathbf{x}), s\right) \subseteq f\left[B(\mathbf{x}, 2^{-k})\right]\right\},$$
$$\mathrm{dom}\left(\widetilde{\mathrm{Moc}}_f\right) = X \times \mathbb{N}, \quad \mathrm{dom}\left(\widetilde{\mathrm{Moo}}_f\right) = \left\{(\mathbf{x}, k) : B(\mathbf{x}, 2^{-k}) \subseteq X\right\}$$

but he finally dismissed them because of the asymmetry between the continuous and the open case.

3.1. Proof of Theorem 9

This section collects the proofs of the several claims made in Theorem 9.

Claim 13. *Let $X \subseteq \mathbb{R}^n$ be r.e. open, $(\mathbf{x}_j)_j$ a computable sequence dense in X, and $f : X \to \mathbb{R}^m$ computable. Then the sequence $\left(f(\mathbf{x}_j)\right)$ is computable, and f admits a computable multi-valued modulus of continuity.*

Proof. The first sub-claim is immediate. For the second one, let $\mathbf{x} \in X$ and $k \in \mathbb{N}$ be given. From these, $\theta^d_<$-compute $U := f^{-1}\left[B\left(f(\mathbf{x}), 2^{-k}\right)\right] \cap X$ by virtue of [Wei00, Theorem 6.2.4.1 and Corollary 5.1.18.1]. Then invoke Lemma 11(a) to obtain some $\ell \in G(U, \mathbf{x})$. This satisfies Eq. (2) because $f[V] \subseteq U$ is equivalent to $V \subseteq f^{-1}[U]$. □

Claim 14. *Let $X \subseteq \mathbb{R}^n$ be r.e. open, $(\mathbf{z}_j)_j$ a computable sequence dense in X, $f : X \to \mathbb{R}^m$ such that $\left(f(\mathbf{z}_j)\right)$ is computable, and* $\mathrm{moc} : X \times \mathbb{N} \rightrightarrows \mathbb{N}$ *a computable multi-valued modulus of continuity. Then, f is computable.*

Proof. First note that *f* is continuous by Lemma 3(a). We show that it furthermore admits effective evaluation: given a sequence $\mathbf{x}_\ell \in X$ of rational vectors with $|\mathbf{x} - \mathbf{x}_\ell| < 2^{-\ell}$ for some $\mathbf{x} \in X$, one can computably obtain a sequence $\mathbf{y}_k$ such that $|f(\mathbf{x}) - \mathbf{y}_k| < 2^{-k}$.

Indeed, calculate by prerequisite $\ell \in \operatorname{moc}(\mathbf{x}, k)$; then search (dove-tailing) for some *j* with $\mathbf{z}_j \in X$ and $|\mathbf{z}_j - \mathbf{x}_{\ell+1}| < 2^{-\ell-1}$; finally let $\mathbf{y}_k := f(\mathbf{z}_j)$. It follows $|\mathbf{z}_j - \mathbf{x}| < 2^{-\ell}$ and thus, by Eq. (2), $|f(\mathbf{x}) - \mathbf{y}_k| < 2^{-k}$. $\square$

Claim 15. *Let $X \subseteq \mathbb{R}^n$ be r.e. open, $f : X \to \mathbb{R}^m$ computable and effectively open. Then,* $\mathrm{Moo}_f : X \times \mathbb{N} \to \mathbb{R}$ *is $(\varrho^n \times \nu \to \varrho_<)$-computable.*

Proof. The mapping $(\mathbf{x}, k) \mapsto B(\mathbf{x}, 2^{-k}) \cap X =: U$ is $(\varrho^n \times \nu \to \theta^n_<)$-computable since *X* is r.e. open [Wei00, Corollary 5.1.18.1]. By assumption on effective openness of *f*, one can therefore obtain a $\theta^m_<$-name for the open set $V := f[U] \ni \mathbf{y} := f(\mathbf{x})$. Then searching all rational $s \geqslant 0$ satisfying $\overline{B}(f(\mathbf{x}), s) \subseteq V$ is possible due to [Zie04, Lemma 4.1(b)] and yields lower approximations to (i.e., a $\varrho_<$-name for) the value $\mathrm{Moo}_f(\mathbf{x}, k)$. $\square$

Claim 16. *Let* $\mathrm{Moo}_f : X \times \mathbb{N} \to \mathbb{R}$ *be strictly positive and* $(\varrho^n \times \nu \to \varrho_<)$*-computable*; *then there is a computable multi-valued* moo.

Proof. From a $\varrho_<$-name of $s := \mathrm{Moo}_f(\mathbf{x}, k) > 0$, obtain some $\ell \in \mathbb{N}$ with $2^{-\ell} < s$; compare [Wei00, Example 4.1.10]. $\square$

For the converse claims in Theorem 9(b), the prerequisite of a computable *f* can actually be relaxed to continuity with computable values on a computable dense subset. This resembles conditions (9a) and (9b) in [Grz57] and is, without (9c) therein, more general than requiring computability of *f*.

Claim 17. *Let $X \subseteq \mathbb{R}^n$ be r.e. open, $(\mathbf{x}_j)_j$ a dense computable sequence in X, $f : X \to \mathbb{R}^m$ continuous, the sequence $\big(f(\mathbf{x}_j)\big)_j$ computable*, *and* moo *a computable multi-valued modulus of openness. Then f is effectively open.*

Proof. From Lemma 3(b) we already know that *f* is open. The goal is thus to $\theta^m_<$-compute $f[U]$, given a $\theta^n_<$-name of some open $U \subseteq X$. The proof of Lemma 3(b ii) has revealed that

$$f[U] \overset{\checkmark}{=} \bigcup_{\mathbf{x}\in U} B\left(f(\mathbf{x}), 2^{-\ell_{\mathbf{x}}}\right) \supseteq \bigcup_{j:\mathbf{x}_j\in U} B\left(f(\mathbf{x}_j), 2^{-\ell_{\mathbf{x}_j}}\right) =: V \tag{6}$$

for arbitrary $\ell_{\mathbf{x}} \in \operatorname{moo}(\mathbf{x}, G(U, \mathbf{x}))$ with *G* from Lemma 11(a). *V* is indeed contained in $f[U]$ as the union to the right ranges only over certain $\mathbf{x} \in U$ compared to all in the left one. Being only a countable union, *V* can be $\theta^m_<$-computed according to Example 5.1.19.1 in [Wei00]. More precisely, the $\theta^n_<$-name of *U* permits enumeration of all *j* such that $\mathbf{x}_j \in U$ by virtue of Lemma 11(a); the multi-valued mapping $h : (U, \mathbf{x}) \mapsto \operatorname{moo}(\mathbf{x}, G(U, \mathbf{x}))$ is $(\theta^n_< \times \varrho^n \rightrightarrows \nu)$-computable; and the multi-valued mapping $j \mapsto B\left(f(\mathbf{x}_j), 2^{-h(U,\mathbf{x}_j)}\right)$ is $(\nu \rightrightarrows \theta^m_<)$-computable since $j \mapsto \mathbf{x}_j$, $j \mapsto f(\mathbf{x}_j)$ both are by assumption.

To complete the proof of Claim 17, we shall show that in fact the reverse inclusion "$f[U] \subseteq V$" holds as well for any $\ell_{\mathbf{x}_j} \in \tilde{h}(U, \mathbf{x}_j)$ with $\tilde{h}$ denoting the computable sub-function according to Lemma 11(b). So take arbitrary $\mathbf{y} \in f[U]$, $\mathbf{y} = f(\mathbf{x})$ with $\mathbf{x} \in U$. If $\mathbf{x} = \mathbf{x}_j$ for some *j*, then $\mathbf{y} \in V$

by definition anyway. If $\mathbf{x}$ does not occur within the sequence $(\mathbf{x}_j)_j$, consider some compact ball $\overline{B} := \overline{B(\mathbf{x}, r)}$ sufficiently small to be contained in U. By the parametrized version of Lemma 11(b), there exists[3] an upper bound $L \in \mathbb{N}$ for $\tilde{h}(U, \cdot)$ on $\overline{B}$. Exploiting continuity, $|f(\mathbf{x}) - f(\mathbf{z})| < 2^{-L}$ for all $\mathbf{z}$ sufficiently close to $\mathbf{x}$. In particular for an appropriate $\mathbf{z} = \mathbf{x}_j$ and any $\ell \in \tilde{h}(U, \mathbf{x}_j) \leqslant L$ by choice of L, it holds that $f(\mathbf{x}) \in B\left(f(\mathbf{x}_j), 2^{-L}\right) \subseteq B\left(f(\mathbf{x}_j), 2^{-\ell}\right)$. The latter term occurs in the right-hand side union of (6); we have thus proven an arbitrary $\mathbf{y}=f(\mathbf{x})\in f[U]$ to lie in V. $\square$

4. Effectivized open mapping theorems

Here come the already announced effectivizations of the classical claims from Fact 6.

Theorem 18. *Let $X \subseteq \mathbb{R}^n$ be r.e. open.*

(a) *Every computable open $f : X \to \mathbb{R}$ (i.e., with one-dimensional range) is effectively open. More generally whenever a computable open $f : X \to \mathbb{R}^m$ maps open balls $B \subseteq X$ to convex sets $f[B] \subseteq \mathbb{R}^m$, then it is effectively open.*
(b) *Any surjective computable affinely linear mapping is effectively open.*
(c) *If computable $f : X \to \mathbb{R}^m$ is C^1, f' is computable and has rank m everywhere, then f is effectively open.*
(d) *Let both $f : X \to \mathbb{R}^n$ and $h : X \rightrightarrows \mathbb{N}$ be computable such that, for any $\mathbf{x} \in X$ and $\ell \in h(\mathbf{x})$, the restriction $f|_{B(\mathbf{x},2^{-\ell})\cap X}$ is injective. Then f is effectively open.*
(e) *Suppose X is bounded, $f : \overline{X} \to \mathbb{R}^n$ computable and locally injective; then $f|_X$ is effectively open. The same holds for unbounded X if $f : X \to \mathbb{R}^n$ is computable and globally injective.*

Proof. Claims (a) and (d) will be proven in Sections 4.1 and 4.2, respectively. Claim (b) is included in (c) just like in the classical case; and (c) in turn, once again similarly to the classical case, is a consequence of the *effectivized* Inverse Function Theorem 19; see the comment following it.

Claim (d) implies the second part of (e) as, $h(\mathbf{x}) :\equiv 0$ will do. For the first part[4] of (e), consider the "modulus of local injectivity" $\varepsilon : \overline{X} \to \mathbb{R}_{>0}$ from Fact 6(d) and observe that, f being locally injective on a *compact* domain, finitely many out of the balls $B(\mathbf{x}, \varepsilon(\mathbf{x}))$, $\mathbf{x} \in \overline{X}$, suffice to cover $\overline{X}$—say those with centers $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \overline{X}$. Therefore, $f|_{B(\mathbf{x}_i,\varepsilon_0)\cap X}$ is injective for all $i = 1, \ldots, N$ where $0 < \varepsilon_0 \leqslant \min_{i \leqslant N} \varepsilon(\mathbf{x}_i)$. By the Lebesgue Number Lemma—see e.g. [Haz00]—there exists some $\varepsilon_1 > 0$ such that every ball $B(\mathbf{x}, \varepsilon_1)$, $\mathbf{x} \in \overline{X}$, is contained in some ball $B(\mathbf{x}_i, \varepsilon_0)$, $1 \leqslant i \leqslant N$; w.l.o.g. $\varepsilon_1 = 2^{-\ell_1}$, $\ell_1 \in \mathbb{N}$. Therefore, $f|_{B(\mathbf{x},2^{-\ell_1})\cap X}$ is injective for all $\mathbf{x} \in \overline{X}$. Now $h(\mathbf{x}) :\equiv \ell_0$ defines a computable function, so Item (d) applies. $\square$

The following is a computable counterpart to Fact 8:

Theorem 19 (*Effectivized inverse function theorem*). *Let $U \subseteq \mathbb{R}^n$ be r.e. open, $f : U \to \mathbb{R}^m$ computable with computable derivative, and $\mathbf{x}_0 \in U$ computable such that* $\operatorname{rank} f'(\mathbf{x}_0) = m$. *Then there exists a computable local right inverse to f, that is, a computable function g with computable derivative satisfying* (5) *and* $\operatorname{dom}(g) = B(f(\mathbf{x}_0), \delta) \subseteq \mathbb{R}^m$ *for some rational $\delta > 0$.*

[3] Here we do not need to find this bound effectively.

[4] The author is indebted to an anonymous referee for pointing out a gap in an earlier version of this proof and for immediately filling that gap by pointing out Lebesgue's Number Lemma.

Moreover, such a $\delta = 2^{-\ell} > 0$ *is uniformly computable from* $\mathbf{x}_0$; *formally: for r.e. open* $X \subseteq \mathbb{R}^n$ *and computable, continuously differentiable* $f : X \to \mathbb{R}^m$ *with computable derivative* $f' : X \to \mathbb{R}^{m\times n}$, *the multi-valued mapping* $I : \mathfrak{O}^n \times X \rightrightarrows \mathbb{N}$ *with*

$$\mathrm{Graph}(I) := \Big\{ (U, \mathbf{x}_0, \ell) \Big| \, \mathbf{x}_0 \in U \subseteq X, \ \mathrm{rank}\left(f'(\mathbf{x}_0)\right) = m, \\ \exists g : B\left(f(\mathbf{x}_0), 2^{-\ell}\right) \to U \textit{ satisfying } (5) \Big\}$$

is $(\theta^n_< \times \varrho^n \rightrightarrows \nu)$*-computable.*

Similarly to the classical case, $f[U]$ in particular covers the open ball $B(f(\mathbf{x}_0), \delta) \subseteq \mathbb{R}^m$. Setting $\mathrm{moo}(\mathbf{x}, k) := I\left(B(\mathbf{x}, 2^{-k}), \mathbf{x}\right)$ therefore proves Theorem 18(c) by virtue of Theorem 9(b).

We emphasize that Theorem 19 can be generalized to hold even uniformly in (f, f'). Furthermore, the multi-valued computation is extendable to yield not only δ but also g and g'. As the domain of these partial functions varies, an according formalization however requires an appropriate representation such as δ_1 from Exercise 6.1.11 in [Wei00] and is beyond our present interest.

Let us also point out that, although the proofs to Theorem 19 (in Section 4.3) as well as the one to Theorem 18(c+d) proceed by presenting according algorithms, they are not necessarily constructive in the intuitionistic sense since the correctness of these algorithms relies on Brouwer's Fixed-Point Theorem.

4.1. Proof of Theorem 18(a)

Claim 20. *Let* $X \subseteq \mathbb{R}^n$ *be r.e. open. If computable open* $f : X \to \mathbb{R}^m$ *maps open Euclidean balls to convex sets, then it is effectively open.*

Proof. Recall that a $\theta^n_<$-name for $U \subseteq X$ is (equivalent to) a list of all closed rational Euclidean balls $\overline{B}_i = \overline{B(\mathbf{z}_i, r_i)}$ contained in U [Wei00, Definition 5.1.15.1 and Exercise 5.1.7].

Since it is easy to obtain a $\psi^n_<$-name for each such $\overline{B}_i$, one can $\psi^m_<$-compute $\overline{f[\overline{B}_i]}$ by virtue of Theorem 6.2.4.3 in [Wei00]. In fact $\overline{f[\overline{B}_i]} = f[\overline{B}_i] = \overline{f[B_i]}$ since f is continuous and $\overline{B}$ compact; cf. [Zie04, Lemma 4.4(d)]. The prerequisite asserts $f[B_i]$ to be convex, and its closure is thus convex and even regular by Proposition 1.1(f) in [Zie04]. By virtue of [Zie02, Theorem 4.12(a)], the $\psi^m_<$-name for $\overline{f[B_i]}$ can hence be converted into a matching $\theta^m_<$-name, that is, a $\theta^m_<$-name for $\overset{\circ}{\overline{f[B_i]}}$ which is a subset of $f[U]$ as $\overline{B}_i \subseteq U$ and $\overline{f[B_i]} = f[\overline{B}_i]$.

Doing so for all $\overline{B}_i$ listed in the $\theta^n_<$-name of U and taking their countable union according to [Wei00, Exercise 5.1.19], constitutes an algorithm $\mathcal{A}$ which produces a $\theta^m_<$-name for some open subset V of $f[U]$. To see that V in fact coincides with $f[U]$, consider some $\mathbf{y} \in f[U]$, $\mathbf{y} = f(\mathbf{x})$ with $\mathbf{x} \in U$. Then some entire ball $B(\mathbf{x}, s)$ is contained inside of U. By density, there exist $\mathbf{z} \in \mathbb{Q}^n$ and $0 < r \in \mathbb{Q}$ such that $\mathbf{x} \in B(\mathbf{z}, r) \subseteq \overline{B}(\mathbf{z}, r) \subseteq B(\mathbf{x}, s)$. This $\overline{B} := \overline{B}(\mathbf{z}, r)$ will thus occur in the list fed into $\mathcal{A}$ as $\theta^n_<$-encoding of U; and will in turn cause $\mathcal{A}$'s output list $\theta^m_<$-encoding V to contain an entry $\overset{\circ}{\overline{f[B]}} \supseteq f[B] \ni f(\mathbf{x}) = \mathbf{y}$, cf. [Zie04, Lemma 4.2(i)]. As $\mathbf{y} \in f[U]$ was arbitrary, this proves $V \supseteq f[U]$. $\square$

4.2. Proof of Theorem 18(d)

By combination with Theorem 9(b), the claim follows uniformly in *f*.

Lemma 21. *Fix r.e. open $X \subseteq \mathbb{R}^n$. The multi-valued mapping $H : C(X, \mathbb{R}^n) \times X \times \mathbb{N} \rightrightarrows \mathbb{N}$ with*

$$\mathrm{Graph}(H) := \Big\{ (f, \mathbf{x}, k, \ell) \Big| \, f : X \to \mathbb{R}^n \text{ injective on } B(\mathbf{x}, 2^{-k}) \subseteq X, \\ B\left(f(\mathbf{x}), 2^{-\ell}\right) \subseteq f[B(\mathbf{x}, 2^{-k})] \Big\}$$

is $([\varrho^n \to \varrho^n] \times \varrho^n \times \nu \rightrightarrows \nu)$-computable.

Proof. For *f* injective on $B(\mathbf{x}, 2^{-k})$, $f[B(\mathbf{x}, 2^{-k})]$ is indeed an open set because of Fact 6(d). Recall its proof based on Theorem 4.3 in [Dei85] together with Theorem 3.1(d4+d5) therein. The latter reveal that, for each $\Omega := B(\mathbf{x}, 2^{-k-1})$—observe $\overline{\Omega} \subseteq X$—$f[\Omega]$ covers $B(f(\mathbf{x}), r)$ where $r > 0$ denotes the distance of $f(\mathbf{x})$ to the set $K := f\left[\partial\Omega\right]$. The sphere boundary $\partial\Omega$ being obviously κ^n-computable from $(\mathbf{x}, k)$, *K*'s distance function is uniformly computable by virtue of Theorem 6.2.4.4 in [Wei00]. In particular, one can effectively evaluate this function at $f(\mathbf{x})$ and thus obtain the aforementioned *r*. From this it is easy to get some $\ell \in \mathbb{N}$ with $2^{-\ell} < r$. $\square$

4.3. Proof of effectivized inverse function theorem

An important part in the proof of Theorem 19 relies on the following result on computability of unique zeros of real functions. It generalizes Corollary 6.3.5 in [Wei00] from one to higher dimensions.

Lemma 22. *Consider the class of continuous real functions f in n variables on the closed unit ball $\overline{B}(0, 1) \subseteq \mathbb{R}^n$ attaining the value zero in* exactly *one point. Hereon, the $\overline{B}$-valued function Z_u, defined by*

$$Z_u(f) = \mathbf{x} \quad :\Longleftrightarrow \quad \mathbf{x} \text{ is the (unique) zero of } f,$$

is $([\varrho^n \to \varrho] \to \varrho^n)$-computable.

Proof. By Theorems 6.2.4.2 and 5.1.13.2 in [Wei00] one can, given a $[\varrho^n \to \varrho]$-name of *f*, $\psi^n_>$-compute the set $f^{-1}[\{0\}] \subseteq \overline{B}$. This computation actually yields a $\kappa^n_>$-name of this set which, by prerequisite, consists of exactly one point. Now apply Exercise 5.2.3 in [Wei00]. $\square$

Recall the second claim from Theorem 19 which shall be proven first:

Claim 23. *Let $X \subseteq \mathbb{R}^n$ be r.e. open, $f : X \to \mathbb{R}^m$ computable with computable derivative $f' : X \to \mathbb{R}^{m \times n}$. Then the multi-valued mapping $I : \mathfrak{O}^n \times X \rightrightarrows \mathbb{N}$ with*

$$\mathrm{Graph}(I) := \Big\{ (U, \mathbf{x}_0, \ell) \mid \mathbf{x}_0 \in U \subseteq X, \ \mathrm{rank}\left(f'(\mathbf{x}_0)\right) = m, \\ \exists g : B\left(f(\mathbf{x}_0), 2^{-\ell}\right) \to U \text{ satisfying } (5) \Big\}$$

is $(\theta^n_< \times \varrho^n \rightrightarrows \nu)$-computable.

Proof. Given a $\theta^n_<$-name of $U \subseteq X$ and $\mathbf{x}_0 \in U$, determine according to Lemma 11(a) some $k_0 \in \mathbb{N}$ such that $B(\mathbf{x}_0, 2^{-k_0}) \subseteq U$. Exploit differentiability of f to write

$$f(\mathbf{x}) \quad = \quad f(\mathbf{x}_0) \; + \; f'(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \; + \; r(\mathbf{x}) \tag{7}$$

with computable and computably differentiable r satisfying $r(\mathbf{x})/|\mathbf{x}| \to 0$ as $\mathbf{x} \to \mathbf{x}_0$.

- Since the computable matrix-valued function $\mathbf{x} \mapsto f'(\mathbf{x})$ was required to have rank m in $\mathbf{x}_0$, certain m of its columns are linearly independent. In fact, one can effectively find a regular $m \times m$ submatrix $A = A(\mathbf{x}_0)$ of $f'(\mathbf{x}_0)$: by dove-tailing w.r.t. all (finitely many) possible candidates and looking for one with non-zero determinant.
  For ease of notation, suppose that $f'(\mathbf{x}_0)$ is of the form $(A|B)$ with $B \in \mathbb{R}^{(n-m)\times m}$. Continuity of the function $\mathbf{x} \mapsto \det A(\mathbf{x})$ with non-zero value at $\mathbf{x}_0$ yields that $A(\mathbf{x})$ is regular on a whole ball around $\mathbf{x}_0$; $\mathbf{x} \mapsto \det A(\mathbf{x})$ even being computable, a corresponding radius $2^{-k_1} \leqslant 2^{-k_0}$ can in fact be found effectively.
- By (computable) translation, it suffices to prove the claim for the computable function on only m variables

$$\tilde{f} \;:\; \mathbb{R}^m \supseteq B(\mathbf{0}, 2^{-k_0}) \quad \ni \quad \mathbf{x} \quad \mapsto \quad f(\mathbf{x}_0 + \mathbf{x}) \quad \in \quad \mathbb{R}^m.$$

  Indeed, any local right inverse $\tilde{g} : B\left(\tilde{f}(\mathbf{x}_0), \tilde{\delta}\right) \subseteq \mathbb{R}^m \to \tilde{U} := B(\mathbf{0}, 2^{-k_0}) \subseteq \mathbb{R}^m$ for this restriction can straight-forwardly (and computably) be extended to one for f by letting $g(\mathbf{y}) := (\tilde{g}(\mathbf{y}), \mathbf{0}) + \mathbf{x}_0 \in \mathbb{R}^n$.
- $A = \tilde{f}'(\mathbf{0})$ being regular, $c := \min_{|\mathbf{x}|=1} |A \cdot \mathbf{x}|$ is non-zero and, according to Corollary 6.2.5 in [Wei00], can be effectively calculated from the given data.
- Effective continuity of $r'(\,\cdot\,)$ together with $r'(\mathbf{0}) = (0)_{ij}$ implies that one can computably find an integer $k_2 \geqslant k_1$ satisfying $\|r'(\mathbf{z})\| \leqslant c/2$ for all $|\mathbf{z}| \leqslant 2^{-k_2}$. Here, $\|B\| := \sqrt{\sum_i \sum_j |b_{ij}|^2}$ denotes the square sum norm on matrices which is known to be submultiplicative: $|B \cdot \mathbf{x}| \leqslant \|B\| \cdot |\mathbf{x}|$. Consequently, by taking the norm on both sides of the Mean Value Theorem

$$r(\mathbf{y} + \mathbf{h}) - r(\mathbf{y}) \quad = \quad \left(\int_0^1 \left(r'(\mathbf{y} + t\mathbf{h})\right) dt \right) \cdot \mathbf{h},$$

  it follows with $\mathbf{h} := \mathbf{x} - \mathbf{y}$ that for all $\mathbf{x}, \mathbf{y} \in \overline{B}(\mathbf{0}, 2^{-k_2})$ we have

$$|r(\mathbf{x}) - r(\mathbf{y})| \leqslant \left(\int_0^1 \|r'(\underbrace{\mathbf{y} + t\mathbf{h}}_{\in \overline{B}(\mathbf{0}, 2^{-k_2}) \text{ convex}})\| \, dt \right) \cdot |\mathbf{h}| \leqslant \tfrac{c}{2} \cdot |\mathbf{x} - \mathbf{y}|. \tag{8}$$

- This asserts injectivity of $\tilde{f}|_{\overline{B}(\mathbf{0}, 2^{-k_2})}$. Indeed, $\tilde{f}(\mathbf{x}) = \tilde{f}(\mathbf{y})$ implies with Eq. (7) that $A \cdot \mathbf{x} + r(\mathbf{x}) = A \cdot \mathbf{y} + r(\mathbf{y})$ and thus

$$c \cdot |\mathbf{x} - \mathbf{y}| \leqslant |A \cdot (\mathbf{y} - \mathbf{x})| \; = \; |r(\mathbf{x}) - r(\mathbf{y})| \overset{(8)}{\leqslant} \tfrac{c}{2} \cdot |\mathbf{x} - \mathbf{y}| :$$

  a contradiction for $\mathbf{x} \neq \mathbf{y}$.

We may thus apply Lemma 21 to obtain some $\ell \in \mathbb{N}$ such that any $\mathbf{y} \in B\left(\tilde{f}(\mathbf{0}), 2^{-\ell}\right)$ is the image of one and exactly one $\mathbf{x} \in B(\mathbf{0}, 2^{-k_2}) \subseteq \tilde{U}$. Finally, setting $\tilde{g}(\mathbf{y}) := \mathbf{x}$ shows that $\tilde{f}$ does have a local right inverse. $\square$

The first part of Theorem 19 claims the right inverse we have just constructed to be computable and differentiable with computable derivative:

Claim 24. *Let $U \subseteq \mathbb{R}^n$ be r.e. open, $f : U \to \mathbb{R}^m$ computable with computable derivative, and $\mathbf{x}_0 \in U$ computable such that* $\operatorname{rank} f'(\mathbf{x}_0) = m$. *Then there exists a computable C^1 function g with computable derivative on some open ball $B\left(f(\mathbf{x}_0), \delta\right) \subseteq \mathbb{R}^m$ satisfying* (5).

Proof. Recall from the proof of Claim 23 the reduction from the case $n \geqslant m$ to the case $n = m$ leading to a function $\tilde{f}$ instead of f which turned out to be injective on some $2^{-k'}$-ball around $\mathbf{x}_0$. Let $\tilde{U} := B(\mathbf{x}_0, 2^{-k'-1})$ and apply to $\tilde{f}$ the classical Inverse Mapping Theorem, in particular the last line of Fact 8: it asserts $\tilde{f}$ to have, on some (possibly smaller) open ball $B(\mathbf{x}_0, \tilde{\varepsilon}) \subseteq B(\mathbf{x}_0, 2^{-k'-1})$ around $\mathbf{x}_0$, a unique and continuously differentiable local inverse $\tilde{g}$. For any $\mathbf{y}$ from its domain $B\left(\tilde{f}(\mathbf{x}_0), \tilde{\delta}\right)$, the value $\tilde{g}(\mathbf{y})$ is according to Eq. (5) the unique $\mathbf{x} \in B(\mathbf{x}_0, \tilde{\varepsilon})$ with $\tilde{f}(\mathbf{x}) = \mathbf{y}$. Since $\tilde{f}$ is injective on $B(\mathbf{x}_0, 2^{-k'}) \supseteq \overline{\tilde{U}} \supseteq B(\mathbf{x}_0, \tilde{\varepsilon})$, $\tilde{g}(\mathbf{y})$ is the unique zero of $\mathbf{x} \mapsto f(\mathbf{x}) - \mathbf{y}$ on $\overline{\tilde{U}}$. Computing $\mathbf{y} \mapsto \tilde{g}(\mathbf{y})$ can thus be performed by finding this zero by virtue of Lemma 22; to actually apply it, straight-forward scaling and translation effectively reduces $\overline{\tilde{U}} = \overline{B}(\mathbf{x}_0, 2^{-k'-1})$ to $\overline{B}(\mathbf{0}, 1)$.

Differentiability of $\tilde{g}$ is asserted already classically. Moreover, the Chain Rule of Differentiation yields the formula $\tilde{g}'(\mathbf{y}) = \tilde{f}'\left(\tilde{g}(\mathbf{y})\right)^{-1}$ which (Cramer's Rule and computability of determinants) reveals that g' is computable as well. □

5. Computable open semi-algebraic functions are effectively open

Open mapping theorems give conditions for continuous functions to be open. However being only sufficient, they miss many continuous open functions.

Example 25. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x^3 + z^2, y^3 + z^2)$. Then f is open although no item from Fact 6 is applicable: (a) fails due to the 2D range, (b) fails due to nonlinearity, (c) fails because $f'(\mathbf{0}) = \mathbf{0}$, and (d) fails as f lacks injectivity everywhere.

Section 4 of the present work provided effectivizations of those classical results where the prerequisites were strengthened from continuity to computability in order to assert, in addition to openness, *effective* openness. They therefore cannot be applied to cases such as Example 25 where the classical theorems fail already. The main result of this section is of a different kind in that it requires openness in order to conclude effective openness. It is concerned with semi-algebraic functions in the sense of, e.g., [BPR03, Section 2.4.2].

Definition 26. Let $F \subseteq \mathbb{R}$ denote a field. A set $S \subseteq \mathbb{R}^n$ is *basic semi-algebraic over F* if

$$S = \left\{\mathbf{x} \in \mathbb{R}^n \;:\; p_1(\mathbf{x}) \geqslant 0 \;\wedge\; \cdots \wedge\; p_k(\mathbf{x}) \geqslant 0 \;\;\wedge\;\; q_1(\mathbf{x}) > 0 \;\wedge \cdots \wedge\; q_\ell(\mathbf{x}) > 0\right\}$$

for certain $k, \ell \in \mathbb{N}$, $p_1, \ldots, q_\ell \in F[X_1, \ldots, X_n]$, that is, if S is the set of solutions to some finite system of polynomial inequalities both strict and non-strict with coefficients from F. S is *semi-algebraic over F* if it is a finite boolean combination (intersection and union) of basic semi-algebraic sets over F. A partial function $f :\subseteq \mathbb{R}^n \to \mathbb{R}^m$ is *semi-algebraic over F* if $\operatorname{Graph}(f) = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in \operatorname{dom}(f) \wedge \mathbf{y} = f(\mathbf{x})\} \subseteq \mathbb{R}^{n+m}$ is semi-algebraic over F. In the case $F = \mathbb{R}$, the indication "*over F*" may be omitted.

The class of semi-algebraic functions is very rich:

Example 27. (a) Any rational function $f \in \mathbb{R}(X_1, \ldots, X_n)$ is semi-algebraic.
(b) The roots of a univariate polynomial $p = \sum_{i=0}^{n-1} p_i \cdot x^i \in \mathbb{R}[X]$, considered as a partial function of its coefficients $(p_0, \ldots, p_{n-1})$, are semi-algebraic.
(c) For semi-algebraic f and g, both composition $g \circ f$ and juxtaposition (f, g) are again semi-algebraic. Projection $\mathbb{R}^{n+m} \to \mathbb{R}^n$, $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}$ is also semi-algebraic.

Proof. (a) Let $f = p/q$ with co-prime $p, q \in \mathbb{R}[X_1, \ldots, X_n]$. Observe that

$$(\mathbf{x}, y) \in \mathrm{Graph}(f) \quad \Leftrightarrow \quad q(\mathbf{x}) \neq 0 \ \wedge \ p(\mathbf{x}) = y \cdot q(\mathbf{x})$$

which is a boolean combination of polynomial inequalities. For (b) and (c) as well as for further examples of semi-algebraic functions, refer to [BPR03, Section 2.4.2]. □

The main result of the present section thus covers many more in addition to Example 25.

Theorem 28. *Let $f :\subseteq \mathbb{R}^n \to \mathbb{R}^m$ be computable, open, and semi-algebraic over $\mathbb{R}_c$ with open* $\mathrm{dom}(f) =: X$. *Then X is r.e. and f is effectively open.*

The proof requires some tools and is therefore deferred to Section 5.2.

5.1. Applications of quantifier elimination to recursive analysis

Quantifier elimination is an important tool in the algebraic framework of computability and complexity [BCSS98,BPR03]. Its reliance on (in)equality as a decidable primitive seemingly renders it useless for the framework of Recursive Analysis. It does however have interesting consequences to *non*-uniform computability as revealed in this section. Specifically, it is employed in (the proofs of) Lemma 29(b), Proposition 30(c), and Theorem 28.

The following lemma will be applied to $E := \mathbb{R}$ and $F := \mathbb{R}_c$ the set of computable real numbers, a real closed field [Wei00, Corollary 6.3.10], but might be of independent interest and is therefore formulated a bit more generally.

Lemma 29. (a) *Let F denote a real closed field with field extension E and $f \in F[X_1, \ldots, X_n]$. If $g \in E[X_1, \ldots, X_n]$ divides f considered as polynomial over E, then $\lambda g \in F[X_1, \ldots, X_n]$ for some non-zero $\lambda \in E$.*
(b) *Let F denote a field with extension E. If $f, g \in F[X_1, \ldots, X_n]$ and $h \in E[X_1, \ldots, X_n]$ is a* gcd *of f and g considered as polynomials over E, then $\lambda h \in F[X_1, \ldots, X_n]$ for some non-zero $\lambda \in E$.*
(c) *Let $\emptyset \neq X \subseteq \mathbb{R}^n$ be r.e. open. Suppose $p, q \in \mathbb{R}[X_1, \ldots, X_n]$ are coprime with $q(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and such that $p/q : X \to \mathbb{R}$ is computable. Then $\lambda p, \lambda q \in \mathbb{R}_c[X_1, \ldots, X_n]$ for some non-zero $\lambda \in \mathbb{R}$; that is, the coefficients of the rational function p/q may w.l.o.g. be presumed computable.*

Proof. We start with the easier arguments.

(b) In the uni-variate case $n = 1$, this follows from the Euclidean Algorithm since its calculation of the gcd uses only arithmetic operations $+, -, \times, \div$ and thus remains within the coefficient field of the input polynomials f and g. In the multi-variate case, the gcd is still well-defined (up to multiples $\lambda \in E$) based on unique factorization in $E[X_1, \ldots, X_n]$ [CLOS97, Exercise 4.§2.9]. Moreover, it can be calculated via Gröbner Bases [CLOS97, Propositions 4.§3.13+14], again using only arithmetic operations and thus remaining within the field F.

(a) By prerequisite, the formula

$$\exists \hat{g}, \hat{h} \in E[X_1, \ldots, X_n]: \quad \hat{g} \neq 0 \ \wedge \ f = \hat{g} \cdot \hat{h} \tag{9}$$

admits a solution (namely g and f/g). This equation of n-variate polynomials over E translates to a finite bilinear system of equalities for the $\mathcal{O}(d^n)$ sought coefficients ($\hat{g}_i, \hat{h}_i$, say) of $\hat{g}$ and $\hat{h}$, given those (f_i) of f where $d := \deg(g)$. The absolute terms for instance must satisfy $g_0 \cdot h_0 = f_0$ and the leading term of g must be non-zero. Observe that, although the solution (g_i, h_i) may live in E, the system is posed using only numbers f_i, that is, in the smaller field F. This is our ticket to the Tarski-Seidenberg Transfer Principle [BPR03, Theorem 2.78] asserting that, in addition to the solution $g \in E[X_1, \ldots, X_n]$, (9) will also admit a solution $\tilde{g} \in F[X_1, \ldots, X_n]$. Due to the aforementioned condition on the leading term, we have $\deg(g) = \deg(\tilde{g})$; and, by uniqueness of factorization as in (b) above, it follows that $\tilde{g} = \lambda g$ for some non-zero $\lambda \in E$.

(c) Let $d > \deg(p) + \deg(q)$ and consider a $(d \times d \times \cdots \times d)$-grid of computable vectors $\mathbf{x} \in X$, that is, n sets $X_1, \ldots, X_n \subseteq \mathbb{R}_c$ of cardinality $|X_i| = d$ such that $X_1 \times X_2 \times \cdots \times X_n \subseteq X$; such exist because $\mathbb{R}_c$ is dense and X is non-empty and open. By prerequisite, $y_j := p(\mathbf{x}_j)/q(\mathbf{x}_j) \in \mathbb{R}_c$ for each $\mathbf{x}_j \in \prod_i X_i$, $j = 1, \ldots, d^n$. Expanding the equations $p(\mathbf{x}_j) - y_j \cdot q(\mathbf{x}_j) = 0$ in the multinomial standard basis yields a homogeneous system of linear equations with respect to the coefficients of both p and q to be solved for.

On the other hand the system itself is composed from (products of components of) computable reals $\mathbf{x}_j$ and y_j. It follows from [ZB04, Corollary 15] that this system also admits a *computable* non-zero solution $\tilde{p}, \tilde{q} \in \mathbb{R}_c[X_1, \ldots, X_n]$. In particular, $\tilde{p}/\tilde{q}$ is defined and coincides with p/q almost everywhere on X.

For $h := \gcd(\tilde{p}, \tilde{q})$, $\hat{p} := \tilde{p}/h$ and $\hat{q} := \tilde{q}/h$ are coprime and, based on Items (a) and (b), still belong to $\mathbb{R}_c[X_1, \ldots, X_n]$. Moreover, induction on n reveals an n-variate polynomial of maximum degree $< d$ to be uniquely specified by its values on an $(d \times \cdots \times d)$-grid like $X_1 \times \cdots \times X_n$; in particular, $\hat{p} \cdot q = \hat{q} \cdot p$. As q divides $\hat{p} \cdot q = \hat{q} \cdot p$, coprimality with p requires it to divide $\hat{q}$. Similarly $\hat{q}$ divides q. Thus $\hat{q} = \lambda q$ for some non-zero $\lambda \in E$ and consequently $\hat{p} = \lambda p$. $\square$

It is well-known in Recursive Analysis that equality of reals lacks even semi-decidability. Surprisingly it does become decidable for rational arguments to real polynomial equations:

Proposition 30. (a) *Let $p \in \mathbb{R}[X_1, \ldots, X_n]$ denote an n-variate polynomial.*[5] *Then $\{\mathbf{x} \in \mathbb{Q}^n : p(\mathbf{x}) = 0\}$ is decidable in the classical* (*i.e.*, *Type*-1) *sense.*

(b) *Let $\Psi(X_1, \ldots, X_n)$ denote a finite Boolean combination of polynomial equalities and inequalities in variables $X_1, \ldots, X_n$ with computable real coefficients. Then $\{\mathbf{x} \in \mathbb{Q}^n : \Psi(\mathbf{x})\}$ is* (*classically*) *semi-decidable.*

(c) *Let $X \subseteq \mathbb{R}^n$ be open and semi-algebraic over $\mathbb{R}_c$. Then X is r.e, that is, $\theta^n_<$-computable.*

[5] Its coefficients do not even need to be computable!

Proof. Again, we first take care of the easy parts:

(b) Without loss of generality, Ψ consists—apart from equalities—of *strict* inequalities only; otherwise replace any "$p(\mathbf{x}) \leqslant 0$" with "$p(\mathbf{x}) < 0 \ \vee \ p(\mathbf{x}) = 0$". Since $p \in \mathbb{R}_c[X_1, \ldots, X_n]$ is computable by assumption, strict inequalities are obviously semi-decidable. This yields a reduction from Claim (b) to Claim (a) proven next.

(a) Let

$$p(\mathbf{x}) = \sum_{k_1=0}^{d_1} \cdots \sum_{k_n=0}^{d_n} a_{(k_1,\ldots,k_n)} \cdot x_1^{k_1} \cdots x_n^{k_n}$$

with $a_{(k_1,\ldots,k_n)} \in \mathbb{R}$. Have among these $a_\mathbf{k}$ a basis $\{b_0 = 1, b_1, \ldots, b_m\}$ chosen[6] for the finite-dimensional $\mathbb{Q}$-vector space $V := \{q_0 + q_\mathbf{k} a_\mathbf{k} + \cdots + q_{(d_1,\ldots,d_n)} a_{(d_1,\ldots,d_n)} : q_\mathbf{k} \in \mathbb{Q}\}$. Consequently, each coefficient of p is of the form $a_\mathbf{k} = \sum_{i=0}^{m} A_{i,\mathbf{k}} b_i$ with fixed $A_{i,\mathbf{k}} \in \mathbb{Q}$. Now for given $\mathbf{x} \in \mathbb{Q}^n$,

$$0 \quad = \quad p(\mathbf{x}) \quad = \quad \sum_{i=0}^{m} b_i \cdot \underbrace{\sum_{k_1=0}^{d_1} \cdots \sum_{k_n=0}^{d_n} A_{i,\mathbf{k}} \cdot x_1^{k_1} \cdots x_n^{k_n}}_{=:R_i(\mathbf{x}) \in \mathbb{Q}}$$

holds if and only if $R_i(\mathbf{x}) = 0$ for all $i = 0, \ldots, m$ because the b_i are linearly independent over $\mathbb{Q}$. The equalities $R_i(\mathbf{x}) = 0$ in turn are of course decidable by means of exact rational arithmetic.

(c) Let $\mathbf{x} \in \mathbb{Q}^n$ and $0 < r \in \mathbb{Q}$. Then "$B(\mathbf{x}, r) \subseteq X$" is equivalent to

$$\forall \mathbf{y} \in \mathbb{R}^n : \quad \left(\sum_{i=1}^{n} (y_i - x_i)^2 < r^2 \ \Rightarrow \ \mathbf{y} \in X \right)$$

a first-order formula $\Phi(\mathbf{x}, r)$ in the language of ordered fields with coefficients by assumption from the real closed field $\mathbb{R}_c$. By Tarski's Quantifier Elimination,[7] there exists an equivalent quantifier-free formula $\Psi(\mathbf{x}, r)$ over $\mathbb{R}_c$ [BPR03, Theorem 2.74]; but for rational $(\mathbf{x}, r)$, $\Psi(\mathbf{x}, r)$ is semi-decidable according to (b) and $X = \bigcup \left\{ B(\mathbf{x}, r) : \mathbf{x} \in \mathbb{Q}^n, 0 < r \in \mathbb{Q}, B(\mathbf{x}, r) \subseteq X \right\}$ is therefore $\theta_<^n$-computable. □

5.2. Proof of Theorem 28 and consequences

Proof of Theorem 28. The domain of f is semi-algebraic over $\mathbb{R}_c$ according to [BPR03, Proposition 2.81] and thus r.e. due to Proposition 30(c). Similarly to the proof there,

[6] Observe the strong non-uniformity inherent in this step; for example a still open problem of number theory asks whether $e \cdot \pi$ or $e + \pi$ is rational [EHH*91, p. 153].

[7] This proof bears some similarity to [BV99]; there however the sets under consideration are BSS-semi-decidable (i.e., roughly speaking, countable unions of semi-algebraic ones) and therefore $\theta_<$-computable (recursively enumerable) only *relative* to the Halting problem.

we observe:

$$\overline{B}\,(f(\mathbf{x}), s) \subseteq f\left[B(\mathbf{x}, 2^{-k}) \;\cap\; X\right] \qquad \Longleftrightarrow$$

$$\exists\, \mathbf{v} \in \mathbb{R}^m\ \forall \mathbf{y} \in \mathbb{R}^m\ \exists \mathbf{u} \in \mathbb{R}^n \;:\quad (\mathbf{x}, \mathbf{v}) \in \mathrm{Graph}(f) \;\wedge\; (\mathbf{u}, \mathbf{y}) \in \mathrm{Graph}(f) \;\wedge\; \left(\sum_{j=1}^{m} (y_j - v_j)^2 \leqslant s^2 \;\Rightarrow\; \sum_{i=1}^{n} (x_i - u_i)^2 < 2^{-2k}\right)$$

Since the latter is a first-order formula $\Phi(\mathbf{x}, s)$, by assumption with coefficients from $\mathbb{R}_c$, there exists by [BPR03, Theorem 2.74] an equivalent quantifier-free formula $\Psi(\mathbf{x}, s)$ again over $\mathbb{R}_c$. This in turn is semi-decidable for rational $(\mathbf{x}, s)$ by virtue of Proposition 30(b) so that Moo_f can be approximated from below on $\mathbb{Q}^n$. Now apply the lemma below (which we could have included into this proof but found it might be of independent interest). $\square$

Example 5 illustrated that in Lemma 3(b) as well as in Theorem 9(b), it does not suffice to consider Moo_f only on a dense subset of X. On the other hand if f is already asserted as open, then computability of Moo_f on rationals already does guarantee effective openness:

Lemma 31. *Let $X \subseteq \mathbb{R}^n$ be r.e. open; furthermore let $f : X \to \mathbb{R}^m$ be computable and open. If* $\mathrm{Moo}_f : (X \cap \mathbb{Q}^n) \times \mathbb{N} \to \mathbb{R}$ *is* $(\nu_{\mathbb{Q}}^n \times \nu \to \varrho_<)$*-computable, then f is effectively open.*

Here, $\nu_{\mathbb{Q}}$ denotes a canonical encoding of rational numbers as in [Wei00, Definition 3.1.2.4].

Proof. The goal is to $\theta^m_<$-compute $f[U]$, given a $\theta^n_<$-name of some open $U \subseteq X$. To this end observe that, similarly to the proof of Claim 17,

$$f[U] \overset{\checkmark}{=} \bigcup_{\mathbf{x} \in U} B\left(f(\mathbf{x}),\quad \mathrm{Moo}_f\,(\mathbf{x}, G(U, \mathbf{x}))\,/2\right)$$

$$\supseteq \quad \bigcup_{\mathbf{x}' \in U \cap \mathbb{Q}^n} B\left(f(\mathbf{x}'), \mathrm{Moo}_f\left(\mathbf{x}', G(U, \mathbf{x}')\right)/2\right) \;=:\; V$$

with G from Lemma 11(a). And, again, the countable union V can be $\theta^m_<$-computed because $\mathbb{Q}^n \times \mathbb{R} \ni (\mathbf{z}, r) \mapsto B\,(f(\mathbf{z}), r)$ is $(\nu^n \times \varrho_< \to \theta^m_<)$-computable and the multi-valued mapping $h : \mathbb{Q}^n \ni \mathbf{z} \mapsto \mathrm{Moo}_f\,(\mathbf{z}, G(U, \mathbf{z}))\,/2$ is $(\nu^n \rightrightarrows \varrho_<)$-computable by assumption. It remains to show that, again, the reverse inclusion "$f[U] \subseteq V$" holds as well for a suitable computable subfunction. More precisely w.l.o.g. replace G from Lemma 11(a) by $\tilde{G}$ according to Lemma 11(b) such that $G(U, \cdot)$ is bounded on compact subsets of $\mathbb{R}^n$. Now consider some $\mathbf{x} \in U \setminus \mathbb{Q}^n$. We show that then $f(\mathbf{x}) \in V$:

Let some compact ball $\overline{B} := \overline{B}(\mathbf{x}, r)$ be contained in U and take an upper bound $L \in \mathbb{N}$ for $\tilde{G}(U, \cdot)$ on $\overline{B}$. By assumption, $\delta := \mathrm{Moo}_f(\mathbf{x}, L + 1)$ is strictly positive. The computable f is continuous so that, for some $0 < r' \leqslant \min\{r, 2^{-L-1}\}$, $f(\mathbf{x}') \in B\,(f(\mathbf{x}), \delta/2)$ whenever $\mathbf{x}' \in \overline{B}' := \overline{B}(\mathbf{x}, r')$. $\mathbb{Q}^n$ being dense in U, there exists some rational $\mathbf{x}' \in \overline{B}'$. Now observe that

(i) $\overline{B}(\mathbf{x}, 2^{-(L+1)}) \subseteq \overline{B}(\mathbf{x}', 2^{-L})$ by choice of $\mathbf{x}'$, thus $f\left[\overline{B}(\mathbf{x}, 2^{-(L+1)})\right] \subseteq f\left[\overline{B}(\mathbf{x}', 2^{-L})\right]$;

(ii) from continuity of f it follows $\overline{B}\left(f(\mathbf{x}'), s - \frac{\delta}{2}\right) \subseteq \overline{B}\,(f(\mathbf{x}), s)$ for any $s \geqslant \frac{\delta}{2}$.

(iii) Combining (i) and (ii) yields $\mathrm{Moo}_f(\mathbf{x}', L) + \frac{\delta}{2} \geqslant \mathrm{Moo}_f(\mathbf{x}, L + 1) = \delta$ because, by Eq. (3), $\mathrm{Moo}_f(\mathbf{x}', L)$ is the *supremum* of feasible radii s.

(iv) Any $\ell' \in \tilde{G}(U, \mathbf{x}')$ has $\ell' \leqslant L$ by choice of L; therefore
(v) $\mathrm{Moo}_f(\mathbf{x}', \ell') \geqslant \mathrm{Moo}_f(\mathbf{x}', L)$ as $\mathrm{Moo}_f(\mathbf{x}', \cdot)$ is monotonic according to Eq. (3).

We conclude that $\delta' := \mathrm{Moo}_f(\mathbf{x}', \ell') \geqslant \frac{\delta}{2}$ and $f(\mathbf{x}) \in B\left(f(\mathbf{x}'), \delta'\right) \subseteq V$. $\square$

Corollary 32. *If the rational functions* $f_i \in \mathbb{R}(X_1, \ldots, X_n)$ *are computable for* $i = 1, \ldots, m$ *and the function* $(f_1, \ldots, f_m) :\subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is open, then it is effectively open.*

Proof. According to Example 27(a), f_i as well as its domain is semi-algebraic; in fact semi-algebraic over $\mathbb{R}_c$ by virtue of Lemma 29(c). Now apply Theorem 28. $\square$

In Theorem 28, f was explicitly required to be semi-algebraic over $\mathbb{R}_c$; yet it seems reasonable, similarly to Lemma 29(c), to

Conjecture 33. *Let* $F \subseteq \mathbb{R}$ *be a real closed subfield. Furthermore let* $f :\subseteq \mathbb{R}^n \to \mathbb{R}$ *be continuous and semi-algebraic (over* $\mathbb{R}$*!) with* $\mathrm{dom}(f)$ *semi-algebraic over* F *and such that* $f(\mathbf{x}) \in F$ *whenever* $\mathbf{x} \in F^n \cap \mathrm{dom}(f)$. *Then* f *is semi-algebraic already over* F.

6. Effective openness and computability

The preceding sections presented sufficient conditions for a computable function f to be effectively open. The present one aims more generally at the logical relation between openness, continuity, effective openness, and computability of real functions.

The two classical properties for instance are well-known mutually independent: continuity does not imply openness; nor does openness require continuity. (Counter-)Examples (c) and (d) below reveal that the same still holds under effectivized prerequisites.

Example 34.
(a) There exists a function $h : \mathbb{R} \to \mathbb{R}$ such that $h[(a, b)] = (0, 1)$ for any $a < b$.
(b) There exists an open but not effectively open real function.
(c) There exists a computable but not open real function; e.g. $f : \mathbb{R} \to \mathbb{R}, x \mapsto 0$.
(d) There exists an effectively open but uncomputable real function.
(e) There exist open functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ such that $f_1 \neq f_2$ but $f_1[U] = f_2[U]$ for any open $U \subseteq \mathbb{R}$.

Proof.
(a) Cf., e.g., item no.100 in the Guide at the beginning of [GO90].
(b) Let u be right-uncomputable and $v > u$ be left-uncomputable. Let $g : \mathbb{R} \to \mathbb{R}$, $g(x) = u + (v - u)x$. Then, with h from (a), $g \circ h : \mathbb{R} \to \mathbb{R}$ has image (u, v) for any non-empty open U and is thus open; but under $\theta_<$-computable $U := (0, 1)$, this image lacking $\theta_<$-computability [Wei00, Example 5.1.17.2(a)] reveals that $g \circ h$ is not effectively open.
(d) The function h from (a) is open but maps the compact interval $[0, 1] \subseteq \mathbb{R}$ to the non-compact interval $(0, 1)$

$$(0, 1) = h\left[\left(\tfrac{1}{3}, \tfrac{2}{3}\right)\right] \subseteq h\,[[0, 1]] \subseteq h\,[(-1, 2)] = (0, 1)$$

and thus cannot be continuous nor computable. For $(\theta_< \to \theta_<)$-computing $U \mapsto h[U]$, it suffices to output a $\theta_<$-name of $(0, 1)$ [Wei00, Example 5.1.17.2(c)] independent of the input $U \neq \emptyset$. The test "$U \neq \emptyset$" is obviously semi-decidable, formally: $\mathfrak{O} \setminus \{\emptyset\}$ is $\theta_<$-r.e. [Wei00, Definition 3.1.3.2].

(e) Let $f_1 := h$ from (a) and $f_2 := g \circ h$ with $g : \mathbb{R} \to \mathbb{R}$, $g(y) = y^3$, open as composition of two open functions. As $h[(0, 1)] = (0, 1)$, there is some $x \in (0, 1)$ such that $h(x) = y := \frac{1}{2}$. Then $f_2(x) = \frac{1}{8}$ reveals that $f_1 \neq f_2$. $\square$

Attempts to strengthen Examples 34(c) and (d) immediately raise the following

Question 35. (a) *Is there a computable, open but not effectively open real function?*
(b) *Is there a continuous, effectively open but uncomputable real function?*

Regarding Theorem 18(a), a putative example for Question 35(a) must have domain and range both of dimension at least two, that is, a graph living in $\mathbb{R}^d$ for some $d \geqslant 4$. Moreover, it cannot be semi-algebraic because of Theorem 28. Concerning candidates to 35(b), the following result allows to restrict research to functions with one-dimensional range on domains of dimension at least two.

Theorem 36. (a) *On r.e. open $X \subseteq \mathbb{R}$, any continuous and effectively open $f : X \to \mathbb{R}$ is also computable.*
(b) *Let $X \subseteq \mathbb{R}^n$ be r.e. open, $f = (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ continuous and effectively open but not computable. Then some $f_i : X \to \mathbb{R}$, too, is continuous and effectively open, but not computable.*

Proof. As usual, the easy part is taken care of first.

(b) Recall that the projections $\mathrm{pr}_i : \mathbb{R}^n \to \mathbb{R}$, $(x_1, \ldots, x_n) \mapsto x_i$ are computable (hence continuous) and open; even effectively open: Theorem 18(a) or (b). By closure under composition, the component functions $f_i = \mathrm{pr}_i \circ f : X \to \mathbb{R}$ are therefore continuous and effectively open themselves. Regarding that a vector-valued f is computable iff its components are [Wei00, Lemma 4.1.19.5], it follows that at least some f_i cannot be computable.

(a) To evaluate f at a given $x \in X$, we are given two monotonic sequences $(u_j)_j$ and $(v_j)_j$ of rational numbers converging to x from below and above, respectively. As $x \in X$ is open, the entire interval $[u_J, v_J]$ belongs to X for some $J \in \mathbb{N}$; and, since (u_j) and (v_j) are, respectively, increasing and decreasing to x, also $x \in [u_j, v_j] \subseteq X$ for all $j \geqslant J$. In fact, such J can be found effectively because the property "$[u_J, v_J] \subseteq X$" is semi-decidable by virtue of [Zie04, Lemma 4.1(b)].
Now, for each $j \geqslant J$, $\theta_<$-compute the open intervals $U_j := (u_j, u_{j+1})$ and $V_j := (v_{j+1}, v_j)$ as well as (by prerequisite) their images $f[U_j]$ and $f[V_j]$ and choose rational numbers $a_j \in f[U_j]$ and $b_j \in f[V_j]$. According to Lemma 37 below, both sequences (a_j) and (b_j) converge to $f(x)$ monotonically from different sides; and comparing a_J to b_J immediately reveals which one constitutes the lower and which one the upper approximations. $\square$

The following lemma can be regarded as a one-dimensional converse to Fact 6(d) because it implies that, for arbitrary open $X \subseteq \mathbb{R}$, a continuous open function $f : X \to \mathbb{R}$ is injective on any connected component of X.

Lemma 37. *Let $X \subseteq \mathbb{R}$ be open and connected, $f : X \to \mathbb{R}$ continuous and open. Then f is either strictly increasing or strictly decreasing.*

Proof. Let $a, b, c \in X$, $a < b < c$. W.l.o.g. presuming $f(a) < f(c)$—otherwise consider $-f$ instead of f—we show $f(a) < f(b) < f(c)$. Now suppose for instance that $f(b) > f(c)$. As f is continuous on $[a, c]$, it attains its maximum therein at some $x \in [a, c]$ with a value $f(x) \geqslant f(b) > \max\{f(a), f(c)\}$; in particular, $x \in (a, c)$. Therefore, the interval $f[(a, c)]$ is closed on its upper end contradicting that f is open. By considering the minimum of f on $[a, c]$, the case $f(b) < f(a)$ similarly raises a contradiction. $\square$

7. Application

Section 1 has already mentioned that effectively open functions arise in computations on regular sets such as in solid modeling. For instance when encoding bounded regular $R \subseteq \mathbb{R}^d$ as a list of open rational balls with union dense in R (representation $\overline{\theta}^d_<$), this will render not only union and intersection computable but also pre-image and image $R \mapsto g[R]$ under computable effectively open functions g [Zie04, Theorem 3.9]. According to Theorem 18(c), that requirement on g is satisfied by any computably differentiable function with regular derivative everywhere. However, some g might be computably differentiable and open with $g'(\mathbf{x})$ occasionally singular. The following result based on the Morse–Sard Theorem establishes that even then, $R \mapsto g[R]$ is $\overline{\theta}^d_<$-computable:

Theorem 38. *Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be computable, open, and C^1 with computable derivative g'. Then its image mapping on bounded regular sets $R \mapsto g[R]$ is $(\overline{\theta}^d_< \to \overline{\theta}^d_<)$-computable.*

Proof. Let $U_0 := \left\{\mathbf{x} \in \mathbb{R}^d : \operatorname{rank}\left(g'(\mathbf{x})\right) = d\right\}$ denote the set of regular points of g.

Consider the function $G := \operatorname{rank} \circ g' : \mathbb{R}^d \to \mathbb{N}$, $\mathbf{x} \mapsto \operatorname{rank}\left(g'(\mathbf{x})\right)$. Because of its discrete range, the pre-image $G^{-1}\left[(d - \frac{1}{2}, \infty)\right]$ obviously coincides with U_0. Moreover, being the composition of the lower semi-computable rank-function—see [ZB00, Proposition 6] or [ZB04, Theorem 7(i)]—with computable g', G is in particular lower semi-continuous and $U_0 \subseteq \mathbb{R}^d$ therefore open [Rud74, Definition 2.8]; in fact r.e. open, see Lemma 39 below.

By Theorem 18(c), at least the restriction $g|_{U_0}$ is thus effectively open. So given as $\overline{\theta}^d_<$-name for $R \subseteq \mathbb{R}^d$ a $\theta^d_<$-name for open $U \subseteq \mathbb{R}^d$ with $\overline{U} = R$, $\theta^d_<$-compute $U \cap U_0$ according to [Wei00, Corollary 5.1.18.1]; then exploit effective openness of $g|_{U_0}$ to $\theta^d_<$-compute $V := g[U \cap U_0]$.

We claim that this yields a valid $\overline{\theta}^d_<$-name for the regular set $g[R]$, i.e., it holds that $\overline{V} = g[R]$. To this end, observe that $\overline{U_0} = \mathbb{R}^d$; for if $A_0 := \mathbb{R}^d \setminus U_0$ had non-empty interior, then the set $V_0 := g\left[\overset{\circ}{A}_0\right]$ of critical values would be open (since g is open by prerequisite) and non-empty rather than having measure zero according to Fact 7. U_0 thus being dense, [Zie04, Lemma 4.3(c) and Lemma 4.4(d)] imply $\overline{U \cap U_0} = R$ and $\overline{V} = g[R]$. $\square$

Lemma 39. *Let $h : \mathbb{R}^d \to \mathbb{R}$ be lower semi-computable. Then the mapping $\mathbb{R} \ni \alpha \mapsto h^{-1}[(\alpha, \infty)] \subseteq \mathbb{R}^d$ is $(\varrho_> \to \theta^d_<)$-computable.*

Proof. Recall that lower semi-computability of h means that evaluation of h at some $\mathbf{x} \in \mathbb{R}^d$, given open rational balls $B_j \ni \mathbf{x}$ of radius $r_j \to 0$, yields rational numbers β_j tending from below to $h(\mathbf{x})$.

So feed into this h-oracle *all* open rational balls $B_j \subseteq \mathbb{R}^d$ and, whenever the answer β_j is strictly greater than α (semi-decidable, given a $\varrho_>$-name for α), report this B_j. The resulting sequence obviously covers exactly $h^{-1}[(\alpha, \infty)]$ and consequently is a $\theta^d_<$-name for this set. $\square$

7.1. Conclusion

The present work investigated conditions for an open function $f : \mathbb{R}^n \to \mathbb{R}^m$ to be *effectively* open in the sense that the image mapping $U \mapsto f[U]$ is $(\theta^n_< \to \theta^m_<)$-computable. This property is so as to speak dual to function *computability* because the latter holds for $f : \mathbb{R}^n \to \mathbb{R}^m$ iff the *pre*-image mapping $V \mapsto f^{-1}[V]$ is $(\theta^m_< \to \theta^n_<)$-computable.

Remark 40. This characterization of computable real functions gave in Definition 6.1.6 of [Wei00] rise to a natural representation—equivalent to many other ones [Wei00, Lemmas 6.1.7 and 6.1.10]—for the space $C(X, \mathbb{R}^m)$ of *all* (not necessarily computable) continuous functions $f : X \to \mathbb{R}^m$, namely by $\theta^n_<$-encoding, for each open rational ball $B \subseteq \mathbb{R}^m$, the open set $f^{-1}[B]$.

Analogy might suggest to represent the family of *all* (not necessarily computable) open functions $f : X \to \mathbb{R}^m$ by $\theta^m_<$-encoding, for each open rational ball $B \subseteq \mathbb{R}^n$, the open set $f[B]$. However Example 34(e) reveals that such a representation would not be well-defined.

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