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Counting lattice paths taking steps in infinitely many directions under special access restrictions

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Abstract

We count lattice paths that are confined to the first quadrant by the nature of their step vectors. If no further restrictions apply, a path can go from any point to infinitely many others, but each point on the path has only finitely many predecessors. By " further restrictions" we mean a boundary line above which the paths may have to stay. Access privilege to the boundary line itself is granted from certain lattice points in the form of a special access step set, which itself may be infinite. We also count the number of paths that contact the weak boundary a given number of times. We approach explicit solutions of such enumeration problems via Sheffer polynomials and functionals, using results of the Umbral Calculus.

Keywords: Lattice path counting; Infinite step set; Umbral Calculus; Sheffer sequence; Privileged access

1. Introduction

Denote by $D(n,m)$ the number of lattice paths from the origin to the integer lattice point (n,m) and let $\mathfrak{S}$ be the set of steps the path can take. A step vector $\langle i,j\rangle \in \mathfrak{S}$ denotes a path step from (n,m) to $(n+i,m+j)$. The lattice paths stay in the first quadrant and above an *access restricted* (*half*) *line* $f(n)=a(n-\ell)+b$ where a,b, and ℓ are nonnegative integers. We allow paths to step onto the restricted line for $n\geqslant \ell$ if they arrive there by *privileged access* step vectors from a special set $\mathfrak{P}$ as introduced in [5,6].

For a simple example taken from [4], a lattice path has the finite step set $\mathfrak{S}=\{\uparrow, \rightarrow, \langle 3,1\rangle\}$, but only with step vectors from the *privileged access* set $\mathfrak{P} = \{\langle j,0\rangle \mid j>0,\ j\in\mathbb{Z}\}$ the path can reach the restricted line $2(n-3)$ for $n\geqslant 3$.

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m	1	6	21	63	165	339
4	1	5	15	41	97	**159**
3	1	4	10	25	52	■
2	1	3	6	14	**24**	
1	1	2	3	7	■	1
0	1	1	1	**3**		
−1	■	■	■	■		
−2						
−3						
	0	1	2	3		$4 \to n$

$D(n,m) = \#\{\text{paths to } (n,m)\}$

m	1	6	21	63	165	339
4	1	5	15	41	97	**159**
3	1	4	10	25	52	52
2	1	3	6	14	**24**	−6
1	2	3	7	8	−33	
0	1	1	1	**3**	0	−42
−1	1	0	0	1	−3	−42
−2	1	−1	0	0	−3	−39
−3	1	−2	1	−1	−1	−37
	0	1	2	3	$4 \to n$	

Polynomial extension $d_n(m)$

Fig. 1. $\{\uparrow, \rightarrow, \langle 3,1\rangle\}$-paths with access steps $\mathfrak{P} = \{\langle i,0\rangle | i \in \mathbb{N}_1\}$ to the restricted line $m = 2(n-3)$.

The path counts are in the first table of Fig. 1; numbers in bold are the path counts on the restricted line. Our goal is to find explicit expressions for the path counts $D(n,m)$ by constructing a polynomial sequence $(d_n(x))_{n \geqslant 0}$ where $d_n(m) = D(n, mt)$. We can view each column n in Fig. 1 as the values of a polynomial $d_n(x)$ evaluated at $x = m$, the height of the path. The second table in Fig. 1 shows the polynomial values extended outside the *support* of D, the region supp(D) where the path counts $D(n,m)$ are positive. For this example, $d_n(m) = D(n,m)$ everywhere in supp(D). We say that $D(n,m)$ can be *extended as a polynomial*. The polynomial sequence starts as $d_0(x) = 1$, $d_1(x) = x+1$, $d_2(x) = (x+2)(x+1)/2$, $d_3(x) = (x+2)(4x+x^2+9)/6$.

The number of paths are explicitly found from the closed form solution

$$D(n,m) = b_n(6+m) - 4b_{n-1}(4+m) + \sum_{k=0}^{n-1} 2^k b_{n-1-k}(6+m) - 4\sum_{k=0}^{n-2} 2^k b_{n-2-k}(4+m),$$

where $b_n(x) = ((x-2n)/x)\sum_{j=0}^{\lfloor n/3 \rfloor} \binom{x}{j}\binom{x+n-1-3j}{n-3j}$. The solution utilizes the Umbral Calculus as given in [7,11,10]. A sequence of polynomials like $(d_n(x))$ is a polynomial sequence iff $\deg d_n = n$. Our polynomial sequences are Sheffer sequences derived from a convenient polynomial basis (like $(b_n(x))$), not from a generating function. The polynomial basis is found from the recursion of the path counts as determined by the step vector set $\mathfrak{S}$. Formal power series are also involved in this approach, but at a different level; we need Lagrange inversion on a power series expansion of a linear operator on polynomials. Thus the polynomial approach cannot save us from Lagrange inversion; however, it allows us to express the initial conditions of the polynomial at the restricted line, as determined by the access set $\mathfrak{P}$, in an algebraic way as a functional on polynomials. This becomes an integral part of the expansion.

We will not attempt giving an overview of the vast literature on lattice path enumeration. For asymptotic enumeration of lattice paths the extraction of the generating function is the preferred method (for recent results see [1], applying the kernel method [2]).

In previous work [4], we showed how to find polynomial solutions to privileged access problems with a two piece boundary (an initial horizontal piece that intersects with a straight line) for standard lattice paths, $\mathfrak{S}=\{\rightarrow,\uparrow\}$, and when the lattice paths can choose among three step directions, $\mathfrak{S}=\{\rightarrow,\uparrow,\langle c,\gamma\rangle\}$. Central to that paper were the necessary conditions for obtaining $d_n(m)$ equal to $D(n,m)$ in $\mathrm{supp}(D)$. In this paper, we consider paths with infinite step sets (they may become finite when interpreted as weighted steps; see Remark 2). Only finitely many paths may arrive at any lattice point, but the path has infinitely many options how to continue. The privileged access step sets may also be infinite. In expanding the step sets, we can no longer demand that $d_n(m)=D(n,m)$ in all of the support of D. Some of the solutions are only *eventually polynomial*; for every n there is an m_n such that $d_n(m)=D(n,m)$ for $m\geqslant m_n$. Only very few points at each n may need special manipulation. The path counts may not be polynomial values at all, but may still be expressed in terms of a polynomial sequence. In problem 7,

$$D(n,m)=2^{m-1}d_n(m)\quad \text{for } m\geqslant 0,\ n>0.$$

In the last problem, we show how to use the Umbral Calculus approach to derive the number $D(n,m;c)$ of paths that contact the weak boundary c times.

Section 2 begins with a discussion of how the path count recursion determines the operator equation needed to find the solution. The section then introduces the three examples of infinite step sets $\mathfrak{S}$ we demonstrate in this paper. For each step set type, we set up problems based on some restricted line and access step set $\mathfrak{P}$. For each problem (labelled P-#) we explain how the polynomials will give the path counts. The functional that incorporates our initial conditions, in general, is derived from the notion that for points (n,m) on the restricted line holds $D(n,m)-\sum_{\langle p,q\rangle\in\mathfrak{P}} D(n-p,m-q)=0$. For each problem, we find more efficient expressions for the initial conditions (labelled IC-#) in an attempt to decrease the number of summations in our solution. For example, in problem 4, the polynomial values on the restricted line give the initial condition that $d_n(n)-\sum_{j\geqslant 2} d_{n-j}(n-j+1)=0$ if $n>0$ but this can be simplified to the condition $d_n(n)=d_{n-1}(n-1)+d_{n-2}(n-1)$ for all $n\geqslant 2$.

In Section 3, we introduce the necessary theory from Umbral Calculus for linear operators and functionals needed for our main theorem, the Functional Expansion Theorem.

Section 4 derives the solutions (labelled S-#) to the ten problems. Our goal is an explicit expression (in terms of *Sheffer polynomials*, as defined in Section 3) for the number $D(n,m)$ of paths. We start the section by deriving a general expression for a necessary component for all the solutions, the *basic sequence*, also defined in Section 3. Then for each infinite step set example we restate the parameters for the problems to follow and give the specific basic sequence. Unless we can solve for $d_n(x)$ directly from the simpler *Binomial Theorem for Sheffer Sequences*, we derive the functional from the initial condition, calculate the corresponding operator inverse which is needed in our theorem, and then find the explicit equation for $d_n(x)$.

We use the notation $\mathbb{N}_0$ for nonnegative integers and $\mathbb{N}_1$ for positive integers throughout our work.

2. Examples

Our examples have infinite step sets $\mathfrak{S}$, supports bounded by the restricted half line $a(n-\ell)+b$ for $n \geqslant \ell$, where $a,b,\ell \in \mathbb{N}_0$ are given parameters, and may have infinite privileged access step sets $\mathfrak{P}$ to the restricted line.

The path counts $D(n,m)$ from $(0,0)$ to (n,m) follow the recursion $D(n,m)=$

$$\begin{array}{ll} \sum_{\langle i,j\rangle \in \mathfrak{S}} D(n-i,m-j) & \text{if } (n,m) \in \operatorname{supp}(D) \backslash \{(n, a(n-\ell)+b) \mid n \geqslant \ell\}, \\ \sum_{\langle p,q\rangle \in \mathfrak{P}} D(n-p,m-q) & \text{if } n \geqslant \ell \text{ and } m = a(n-\ell)+b, \\ 0 & \text{if } (n,m) \notin \operatorname{supp}(D), \end{array}$$

with initial value $D(0,0)=1$. Recursion formulas for the numbers $D(n,m)$ always assume that $D(n,m)=0$ outside $\operatorname{supp}(D)$, without stating this as an initial condition. For polynomial sequences (d_n) the only such implicit assumption is $d_n(m)=0$ for $n<0$, the half plane inaccessible to our lattice paths independent of any imposed restrictions. Thus we define the following enlargement of the support.

Definition 1. The *allowed region* of a lattice path problem is the support, $\operatorname{supp}(D)$, extended to the left half plane $\{(n,m)\in\mathbb{Z}^2 : n<0\}$.

Denote the backwards difference operator by ∇. If the backwards difference of a function is a polynomial of degree $n-1$, then the function is a polynomial of degree n. Therefore, for example, the recursion for the step set $\mathfrak{S}=\{\uparrow, \langle j,\gamma(j-1)\rangle \mid j\in\mathbb{N}_1\}, \gamma\in\mathbb{N}_0$, where

$$\nabla D(n,m) = D(n,m) - D(n,m-1) = \sum_{j\geqslant 1} D(n-j, m-\gamma(j-1)),$$

shows that $D(n,m)$ can be *extended to a polynomial* $d_n(m)$ of degree n for almost all m, where $D(n-1,m)$ already has a polynomial extension. The polynomial extension (d_n) follows the same recursion

$$d_n(m) - d_n(m-1) = \sum_{j\geqslant 1} d_{n-j}(m-\gamma(j-1)).$$

To solve this recursion, we define W as the linear operator on polynomials which maps d_n to d_{n-1}, thus

$$\nabla = \sum_{j\geqslant 1} E^{-\gamma(j-1)} W^j = \frac{W}{1-E^{-\gamma}W}, \tag{1}$$

where E^a is the shift by a operator, $E^a f(x) := f(x+a)$. This operator equation can be written as

$$\nabla = W + \nabla E^{-\gamma} W.$$

All of the polynomial solutions in this paper follow the more general operator recursion

$$\nabla = W + \alpha \nabla E^{-\gamma} W + \beta \nabla E^{-\eta} W^2, \tag{2}$$

where $\alpha, \beta, \gamma, \eta$ are integers. Thus they have an equivalent six term recursion

$$d_n(m) = d_n(m-1) + d_{n-1}(m) + \alpha d_{n-1}(m-\gamma) - \alpha d_{n-1}(m-\gamma-1) \\ + \beta d_{n-2}(m-\eta) - \beta d_{n-2}(m-\eta-1).$$

Remark 2. The above recursion may be interpreted as originating from the enumeration of *weighted* paths taking six weighted steps. As long as the generating function (1) of $\nabla = \phi(W)$ is rational, there will always exist an equivalent weighted problem with only finitely many steps. Our approach remains valid whether ϕ is rational or not; however, for a general ϕ it is most likely impossible to extract the coefficient of W^n in ϕ^m, as required in (15) for an explicit enumeration result. For example, let $f : \mathbb{N}_1 \to \mathbb{N}_1$ be injective, $f(1) = 1$, and consider the step set $\{\uparrow, \langle f(j), 0\rangle : j \in \mathbb{N}_1\}$. The answer to enumeration problems based on this step set will be as explicit as we can describe the number of compositions of n into m terms $f(j_1), \ldots, f(j_m)$. On the other hand, the enumeration of weighted paths with an infinite step set can be easy. Give the weight 1 to $\uparrow$, and the weights $1/j!$ to $\langle j, 0\rangle$, $j \in \mathbb{N}_1$. Thus $\nabla = \mathrm{e}^W - 1$ is not rational, but it is easy to derive (with our or other approaches) for $0 < n \leqslant m$ that in terms of Stirling numbers (second kind) $\frac{m-n}{m!n!} \sum_{j=1}^{n} (j+m-1)! S(n,j)$ is the total weight of all paths to (n,m) starting at the origin and continuing strictly above the diagonal.

In the remainder of this section we set up the examples, show their polynomial extensions, and state the recursions the polynomials follow along with their initial conditions (labelled IC-#). The solution to path counts $D(n,m)$ in terms of the (Sheffer) polynomial $d_n(m)$ is given and labelled P-#. Explicit expressions for $d_n(x)$ are derived in Section 4 and are labelled S-#.

2.1. Almost all steps

We say that a path can proceed with *almost all steps* if $\mathfrak{S} = \{\rightarrow, \langle i,j\rangle \in \mathbb{N}_1^2\}$. In Problem 1 the boundary for the paths is $m = n - \ell$ where $\ell = 3$ and the set $\mathfrak{P}$ of access steps is empty (Table 1). The path counts $D(n,m)$ can be recursively calculated from initial conditions and

$$D(n,m) = D(n-1,m) + \sum_{j \geqslant 1} \sum_{i \geqslant 1} D(n-i, m-j).$$

If there are no restrictions from below, this recursion is equivalent to

$$D(n,m) = D(n-1,m) + D(n,m-1) + \sum_{i\geqslant 2} D(n-i,m-1)$$

for $n\geqslant 1$. Thus we have derived the corresponding operator equation

$$\nabla = W + E^{-1}\frac{W^2}{1-W} = W + \nabla W - \nabla W^2.$$

Let $(d_n(m))_{n\geqslant 0}$ be a polynomial sequence that follows this recursion. Selecting $\alpha=1$, $\beta=-1$ and $\gamma=\eta=0$ in Eq. (2) shows that

$$d_n(m) = d_n(m-1) + 2d_{n-1}(m) - d_{n-1}(m-1) - d_{n-2}(m) + d_{n-2}(m-1). \quad (3)$$

The number of paths to (n,m) equals

$$\begin{aligned} &D(0,m) = \delta_{0,m} \\ &D(n,0) = 1 \quad \text{for } n < \ell = 3 \\ &D(n+1,m) = d_n(m) \quad \text{for } n \geqslant 0 \quad \text{and} \quad m \geqslant \max\{1, n-1\}. \end{aligned} \quad \text{(P.1)}$$

Table 1 shows the path counts $D(n,m)$ and their polynomial extension $d_n(m)$. The numbers are italicized where $d_n(m)\neq D(n+1,m)$ in the supp(D); bold entries are values on the boundary $m=n-(\ell-1)$. At $(1,0)$ the polynomial picks up an additional 1, compensating for the missing $D(0,0)$.

The initial condition $D(n,n-\ell)=0$ for $n\geq\ell$ obviously does not extend to the polynomials. However, the initial condition implies that

$$\begin{aligned} &D(n,n-\ell+1) \\ &\quad = D(n-1,n-\ell+1) + \sum_{i\geqslant 1}\sum_{j\geqslant 1} D(n-i,n-\ell+1-j) \\ &\quad = 2D(n-1,n-\ell+1) + \sum_{j\geqslant 1} D(n-1,n-\ell+1-j) - D(n-2,n-\ell+1) \\ &\quad = 2D(n-1,n-\ell+1) + D(n-1,n-\ell) - D(n-2,n-\ell+1). \end{aligned}$$

Table 1
$\{\longrightarrow, \langle i,j\rangle \in \mathbb{N}_1\times\mathbb{N}_1\}$-paths with boundary line $m=n-3$

Problem 1

Path counts $D(n,m)$

m	0	1	6	24	76	198	419
3	0	1	5	17	46	99	■
2	0	1	4	11	24	■	
1	0	1	3	6	■		
0	1	1	1	■			
	0	1	2	3	4	5	$\rightarrow n$

Polynomial extension $d_n(m)$ (shifted scale)

m	0	1	6	24	76	198	416	**640**
3	0	1	5	17	46	99	**152**	53
2	0	1	4	11	24	**37**	13	−176
1	0	1	3	6	**9**	3	−43	−211
0	*0*	1	*2*	**2**	0	−11	−51	−162
−1	0	1	1	−1	−4	−12	−37	−96
−2	0	1	0	−3	−4	−6	−19	−47
	−1	0	1	2	3	4	5	$\rightarrow n$

The polynomials (translated by $n \to n-1$) therefore satisfy the initial condition

$$d_n(n-\ell+2) = 2d_{n-1}(n-\ell+2) + d_{n-1}(n-\ell+1) - d_{n-2}(n-\ell+2)$$

for $n \geqslant \ell$. Applying the five term recursion (3) further simplifies the above condition to the boundary condition

$$d_n(n-\ell) = d_{n-1}(n-\ell) - d_{n-2}(n-\ell) \quad \text{for } n \geqslant \ell. \tag{IC.1}$$

Remark 3. This example shows us the importance of the allowed region (Definition 1); we had to create a $\tilde{D}(n,m) := D(n+1,m)$, change $\tilde{D}(-1,0)$ to 0 and $\tilde{D}(1,0)$ to 2, for creating the proper allowed region that makes a polynomial extension possible.

2.2. *Steep paths*

In our second example, a *steep path* has the infinite step set

$$\mathfrak{S} = \{\uparrow, \langle j, \gamma(j-1)\rangle | j \in \mathbb{N}_1\}, \quad \gamma \in \mathbb{N}_0.$$

The path counts solve the recursion

$$D(n,m) = D(n,m-1) + \sum_{j\geqslant 1} D(n-j, m-\gamma(j-1)).$$

The polynomial extension $d_n(m)$ follows the same recursion and, as stated in the beginning of this section, leads to the operator equation

$$\nabla = W + \nabla E^{-\gamma} W.$$

2.2.1. *Above a line and natural privileged access*

For Problems 2 and 3 we choose $\gamma = 1$, and the boundary lines $m = n - \ell$ for $\ell = 1$ and 4. The set $\mathfrak{P}$ of access steps is empty. Tables 2 and 3 show some path counts $D(n,m)$ and values of the corresponding polynomial extension $d_n(m)$ inside and outside the support.

For $\ell = 1$,

$$D(n,m) = d_n(m). \tag{P.2}$$

In Problem 2, the recursive calculations of $d_n(m)$ at points $(n,m) \in \text{supp}(D)$ only refer back to points (i,j) in the allowed region, thus $d_n(m) = D(n,m)$. This is no longer true in Problem 3. Italicized entries are values of $d_n(m) \neq D(n,m)$ in supp(D). The path counts are only eventually values of polynomials, when $m \geqslant n-2$. Starting at $n = 2$, the recursion for $d_n(n-2)$ picks up the entry $d_0(-1) = 1 \neq 0 = D(0,-1)$. This additional entry must be compensated for by defining $d_n(n-3) := D(n, n-3) - 1$. Hence

$$\begin{aligned} D(n,m) &= d_n(m) \quad \text{when } m > n-3 \geqslant -1 \\ D(n,n-3) &= d_n(n-3) + 1 \quad \text{for } n \geqslant 2. \end{aligned} \tag{P.3}$$

Table 2
$\{\uparrow, \langle j, j-1\rangle \mid j \in \mathbb{N}_1\}$-path with boundary line $m = n - 1$

Problem 2

m	1	5	18	49	101
4	1	4	12	26	36
3	1	3	7	10	■
2	1	2	3	■	
1	1	1	■		
0	1	■			
	0	1	2	3	$\rightarrow n$

Path counts $D(n,m)$

m	1	5	18	49	101
4	1	4	12	26	36
3	1	3	7	10	**0**
2	1	2	3	**0**	−15
1	1	1	**0**	−5	−16
0	1	**0**	−2	−6	−9
−1	**1**	−1	−3	−4	1
−2	1	−2	−3	0	10
	0	1	2	3	$\rightarrow n$

Polynomial extension $d_n(m)$

Table 3
$\{\uparrow, \langle j, j-1\rangle \mid j \in \mathbb{N}_1\}$-path with boundary line $m = n - 4$

Problem 3

m	1	5	19	8	149	326	590
3	1	4	13	34	74	132	168
2	1	3	8	17	29	36	■
1	1	2	4	6	7	■	
0	1	1	1	1	■		
	0	1	2	3	4	5	$\rightarrow n$

Path counts $D(n,m)$

m	1	5	19	58	149	326	590
3	1	4	13	34	74	132	*167*
2	1	3	8	17	29	*35*	**−2**
1	1	2	4	6	*6*	**−2**	−42
0	1	1	1	*0*	**−2**	−7	−33
−1	1	0	−1	**−2**	−1	0	−20
−2	1	−1	**−2**	−1	4	6	−22
	0	1	2	3	4	5	$\rightarrow n$

Polynomial extension $d_n(m)$

The bold entries in Tables 2 and 3 are values on the boundary and can be used as initial values for the polynomial extension:

For $\ell = 1$,

$$d_n(n-1) = \delta_{n,0} \tag{IC.2}$$

For $\ell = 4$,

$$d_n(n-4) = -2 \quad \text{for } n > 0 \quad \text{and} \quad d_0(-\ell) = 1. \tag{IC.3}$$

(The -2 is the result of an additional compensation, because $d_0(-2) = 1$ and $d_0(-1) = 1$ enter the recursion for $d_n(n-4)$ if $n \geqslant 2$.)

In Problems 2 and 3, we notice that paths reaching the weak boundary $m = n - \ell + 1$ may use any step from $\mathfrak{S}$ but $\uparrow$. We call this *natural* privileged access (to the restricted line $m = n - \ell + 1$). The natural privileged access step set $\mathfrak{P}$ equals $\mathfrak{S} \setminus \{\uparrow\}$.

Table 4
$\{\uparrow, \langle j, j-1\rangle | j \in \mathbb{N}_1\}$-path with $\{\langle j+1, j\rangle | j \in \mathbb{N}_1\}$-access to $m = n$

Problem 4

m	1	3	8	13	**6**
3	1	2	4	**2**	
2	1	1	**1**		
1	1	**0**			
0	**1**				
	0	1	2	3	$\to n$

Path counts $D(n,m)$

m	1	3	8	13	**6**
3	1	2	4	**2**	−13
2	1	1	**1**	−4	−17
1	1	**0**	−1	−6	−12
0	**1**	−1	−2	−5	−3
−1	1	−2	−2	−2	6
−2	1	−3	−1	2	12
	0	1	2	3	$\to n$

Polynomial extension $d_n(m)$

Table 5
$\{\uparrow, \langle j, j-1\rangle | j \in \mathbb{N}_1\}$-path with $\{\langle j+1, j\rangle | j \in \mathbb{N}_1\}$-access to $m = n - 3$

Problem 5

m	1	5	19	57	140	277	392
3	1	4	13	33	66	93	**28**
2	1	3	8	16	22	**6**	
1	1	2	4	5	**1**		
0	1	1	1	**0**			
	0	1	2	3	4	5	$\to n$

Path counts $D(n,m)$

m	1	5	19	57	140	277	392
3	1	4	13	33	66	93	**27**
2	1	3	8	16	22	**5**	−95
1	1	2	4	5	**0**	−24	−98
0	1	1	1	−**1**	−7	−22	−61
−1	1	0	−**1**	−3	−5	−9	−28
−2	1	−**1**	−2	−2	1	2	−17
	0	1	2	3	4	5	$\to n$

Polynomial extension $d_n(m)$

The two problems above can be interpreted as lattice path problems with access set $\mathfrak{P} = \{\langle j+1, j\rangle | j \in \mathbb{N}_0\}$ to the restricted line $m = n - (\ell - 1)$.

2.2.2. *Privileged access from a diagonal*

Problems 2 and 3 illustrated natural privileged access, $\mathfrak{P} = \mathfrak{S} \backslash \{\uparrow\}$. For Problems 4 and 5, we further reduce the privileged access step set, considering $\mathfrak{P} = \mathfrak{S} \backslash \{\uparrow, \to\}$. Thus we allow privileged access to the restricted line $m = n - \ell$ for all $n \geqslant \ell$, with access steps $\mathfrak{P} = \{\langle j+1, j\rangle | j \in \mathbb{N}_1\}$. We let $\ell = 0$ or $\ell = 3$; see Tables 4 and 5.

The polynomial extension must satisfy the access condition on the boundary

$$d_n(n - \ell) = \sum_{(p,q) \in \mathfrak{P}} d_{n-p}(n - \ell - q) = \sum_{j=2}^{n} d_{n-j}(n - \ell - j + 1) \tag{4}$$

for all $n \geqslant \max\{\ell, 1\}$. If $\ell = 0$, the access recursion

$$d_n(n) = \sum_{j \geqslant 2} d_{n-j}(n - j + 1) \quad \text{if } n > 0$$

only refers to points in the allowed region and therefore

$$D(n,m) = d_n(m) \quad \text{for all } m \geqslant n \geqslant 0. \tag{P.4}$$

We could use this access recursion as the initial condition on the boundary, but prefer the simpler equivalent to (4),

$$d_n(n-\ell) = d_{n-1}(n-1-\ell) + d_{n-2}(n-1-\ell), \tag{5}$$

which holds for large enough n. If $\ell = 0$, then

$$d_n(n) = d_{n-1}(n-1) + d_{n-2}(n-1) \quad \text{for all } n \geqslant 2. \tag{IC.4}$$

In Problem 5 the columns in the table of path counts are only eventually polynomial because the recursion

$$d_n(m) = d_n(m-1) + \sum_{j\geqslant 1} d_{n-j}(m-j+1)$$

stays in the allowed region only if $m \geqslant n-1$. For $m = n-2$ the recursion includes the term $d_0(-1)$, outside the support of D, when $D(0,-1)=0$ but $d_0(-1)=1$. To compensate we make $d_n(n-3) = D(n,n-3)-1$. Thus,

$$\begin{aligned} &D(n,n-3) = d_n(n-3)+1 \quad \text{for } n > 0,\\ &D(n,m) = d_n(m) \quad \text{for } m \geqslant 0 \quad \text{and } n = 0,1,2,\\ &D(n,m) = d_n(m) \quad \text{for } m > n-3 \geqslant 0. \end{aligned} \tag{P.5}$$

The access boundary condition

$$D(n,n-3) = \sum_{j\geqslant 2} D(n-j, n-2-j)$$

does not carry over to the polynomials, but its two term equivalent

$$D(n,n-3) = D(n-1,n-4) + D(n-2,n-4)$$

stays in the support for all $n \geqslant 4$, and at $n=3$ the recursion (5) gives $d_3(0) = -1 = d_2(-1) + d_1(-1)$ as desired. Therefore, the polynomials satisfy the initial condition

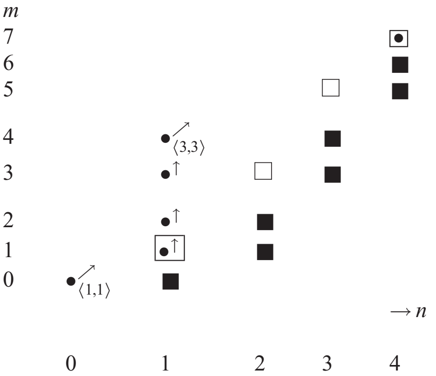
$$d_n(n-3) = d_{n-1}(n-4) + d_{n-2}(n-4) \quad \text{for all } n \geqslant 3. \tag{IC.5}$$

2.2.3. Steep boundary and diagonal access

In Problem 6 on steep paths, we let $a=2$, $\ell = b = 1$ for a boundary line of $m = 2(n-1)+1$ for $n \geqslant 1$, and $\gamma = 2$, creating the step set $\mathfrak{S} = \{\uparrow, \langle j, 2j-2\rangle | j \in \mathbb{N}_1\}$. We let the access step set to the " steep" restricted line $m = 2n-1$ be positive steps of slope one, $\mathfrak{P} = \{\langle j,j\rangle | j \in \mathbb{N}_1\}$. Table 6 shows a sample path that accesses the point

Table 6
$\{\uparrow, \langle j, 2j-2\rangle | j \in \mathbb{N}_1\}$-path with $\{\langle j, j\rangle | j \in \mathbb{N}_1\}$-access to $m = 2n-1$

Problem 6



Sample path to (4,7) in 5 steps

m						
7	1	7	29	73	**57**	*−450*
6	1	6	21	38	*−34*	*−524*
5	1	5	14	**12**	*−83*	*−484*
4	1	4	8	*−6*	*−100*	⋮
3	1	3	**3**	*−17*	*−94*	
2	1	2	*−1*	*−22*	⋮	
1	1	**1**	*−4*	*−22*		
0	1	*0*	*−6*	*−18*		
−1	*1*	*−1*	*−7*	*−1*		
	0	1	2	3	4	$\to n$

$D(n,m)$ and the extension $d_n(m)$

(4, 7) on the boundary with a $\langle 3,3\rangle$ step, and the boundary point (1, 1) with a $\langle 1,1\rangle$ step.

The path counts and polynomials solve the recursion

$$d_n(m) = d_n(m-1) + \sum_{j \geqslant 1} d_{n-j}(m - 2(j-1)),$$

and the boundary condition implies that

$$d_n(2n-1) = \sum_{j \geqslant 1} d_{n-j}(2n-1-j) \quad \text{for all } n > 0. \tag{IC.6}$$

The path and privileged access recursions both stay in the allowed region of D, thus

$$D(n,m) = d_n(m) \quad \text{for all } m \geqslant 2n-1 \quad \text{and} \quad n > 0. \tag{P.6}$$

2.3. All steps

An *all-steps-path* takes steps from the set

$$\mathfrak{S} = \{\langle i,j\rangle \in \mathbb{N}_0 \times \mathbb{N}_0 \backslash \langle 0,0\rangle\}.$$

This example was inspired by Exercise 6.16 in Stanley [12] and by Sulanke [13]. The path counts follow the recursion

$$D(n,m) = \sum_{\langle i,j\rangle \in \mathfrak{S}} D(n-i, m-j). \tag{6}$$

Table 7
Unrestricted all-steps-path, $\mathfrak{S} = \{\langle i,j\rangle\} \in \mathbb{N}_0 \times \{\mathbb{N}_0 \backslash \langle 0,0\rangle\}$

Problem 7

m	32	256	...	...	...	...	...
5	16	112	544	...	...	...	...
4	8	48	208	768	...	...	...
3	4	20	76	252	768	...	...
2	2	8	26	76	208	544	...
1	1	3	8	20	48	112	256
0	1	1	2	4	8	16	32
−1							
	0	1	2	3	4	5	$\to n$

Path counts $D(n,m)$

m	1	8	43	190	743	...	...
5	1	7	34	138	501	...	...
4	1	6	26	96	321	...	...
3	1	5	19	63	192	552	...
2	1	4	13	38	104	272	...
1	1	3	8	20	48	112	576
0	2	2	4	8	16	32	64
−1		*1*	*1*	*1*	*1*	*1*	*1*
	0	1	2	3	4	5	$\to n$

$2^{1-m}D(n,m)$, *with extension*

Table 8
$\{\langle i,j\rangle \in \mathbb{N}_0 \times \mathbb{N}_0 \backslash \langle 0,0\rangle\}$-path above $m = 2(n-2)+1$

Problem 8

m	32	256	1216	4032	8064
5	16	112	464	1264	■
4	8	48	168	336	■
3	4	20	56	■	
2	2	8	16	■	
1	1	3	■		
0	1	1	■		
−1					
−2					
−3					
	0	1	2	3	$\to n$

Path counts $D(n,m)$

m	1	8	38	126	252
5	1	7	29	79	**79**
4	1	6	21	42	−37
3	1	5	14	**14**	−107
2	1	4	8	−6	−141
1	1	3	**3**	−19	−148
0	*1*	2	−1	−26	−136
−1	1	**1**	−4	−28	−112
−2	1	0	−6	−26	−82
−3	**1**	−1	−7	−21	−51
	0	1	2	3	$\to n$

$d_n(m)$

The recursion can be simplified to the three term recursion

$$D(n,m) = 2D(n,m-1) + 2D(n-1,m) - 2D(n-1,m-1) \tag{7}$$

for all $(n,m) \in \operatorname{supp}(D)$ with the exception of those points (n,m) where $D(n,m-1)$, $D(n-1,m)$, or $D(n-1,m-1)$ are initial values. There are no nontrivial polynomial solutions to this recursion (see Table 7); however, if we let $d_n(m) = 2^{1-m}D(n,m)$ for large enough m, it follows that $2^{1-m}D(n,m)$ is eventually polynomial, because

$$d_n(m) = d_n(m-1) + 2d_{n-1}(m) - d_{n-1}(m-1). \tag{8}$$

The corresponding operator equation

$$\nabla = W + \nabla W$$

can be seen as the case $\alpha = 1$, and $\beta = \gamma = \eta = 0$ in Eq. (2).

2.3.1. Unrestricted

In Problem 7 we show how the unrestricted all-steps enumeration fits into the polynomial approach. It can be proven inductively from (6) that

$$D(0,n) = D(n,0) = 2^{n-1}. \tag{P.7a}$$

Using the above and the three term recursion (8), it can be shown that the initial conditions

$$d_n(-1) = 1 \tag{IC.7}$$

will result in the polynomial sequence $(d_n(m))$ with the property

$$\begin{aligned} d_n(m) &= 2^{1-m}D(n,m) \quad \text{for } m \geqslant 0,\ n > 0 \\ d_0(m) &= 1 = 2^{1-m}D(0,m) \quad \text{for } m > 0. \end{aligned} \tag{P.7}$$

Problems involving boundary lines, privileged access and contacts to boundary lines are all solved by polynomials that obey the three term recursion (8) but have different initial conditions.

2.3.2. Bounded by a line

Problem 8 (see Table 8) discusses the number of all-steps-paths that stay above the line $m = 2n-3 = 2(n-2)+1$ for $n \geqslant \ell = 2$.

For $n \geqslant 2$ the path counts at points just above the boundary must be

$$D(n, 2n-2) = 2D(n-1, 2n-2),$$

since no path comes up from the boundary or below. This gives an initial condition for the polynomials:

$$d_n(2n-2) = 2d_{n-1}(2n-2).$$

An even simpler initial condition on the boundary follows if we combine the above condition with the three term recursion (8)

$$d_n(2n-3) = d_{n-1}(2n-2) \quad \text{for all } n \geqslant 1. \tag{IC.8}$$

The polynomials represent the path counts as

$$D(n,m) = 2^{m-1}d_n(m) \text{ for } m \geqslant \max\{1, 2n-3\} \quad \text{and } n > 0. \tag{P.8}$$

2.3.3. Privileged access from the left

In Problem 9 we allow privileged access by steps $\mathfrak{P} = \{\langle i, 0\rangle | i \in \mathbb{N}_1\}$ to the restricted line $m = 2(n-2)+1$ for $n \geqslant \ell = 2$. Table 9 shows the path counts with the privileged path counts to the restricted line in bold. The italicized numbers in the polynomial extension table are the values $2^{1-m}D(n,m) \neq d_n(m)$; the initial values $d_n(2n-2)$ are underlined.

Table 9
$\{\langle i,j\rangle \in \mathbb{N}_0 \times \mathbb{N}_0 \setminus \langle 0,0\rangle\}$-path with $\mathfrak{P} = \{\langle i,0\rangle | i \in \mathbb{N}_1\}$-access to $m = 2(n-2)+1$

Problem 9

m	32	256	1280	4640	11440
5	16	112	496	1536	**2160**
4	8	48	184	456	
3	4	20	64	**88**	
2	2	8	20		
1	1	3	**4**		
0	1	1			
−1					
−2					
−3					
	0	1	2	3	$\to n$
Path counts $D(n,m)$					

m	1	8	40	145	357.5
5	1	7	31	96	*135/163.5*
4	1	6	23	57	28.5
3	1	5	16	*22/27*	−58.5
2	1	4	10	5	−107.5
1	1	3	*4/5*	−10	−127.5
0	*1*	2	1	−19	−126.5
−1	1	1	−2	−23	−111.5
−2	1	0	−4	−23	...
−3	1	−1	−5	...	
	0	1	2	3	$\to n$
$2^{1-m}D(n,m)$ and $d_n(m)$					

Steps that reach the point $(n,2n-2)$ immediately above the boundary either come from the boundary, $(n,2n-3)$ or from some point (i,j) with $i<n$, $j \leqslant 2n-2$. Hence

$$\begin{aligned} D(n,2n-2) &= D(n,2n-3) + \sum_{i\geqslant 1}\sum_{j\geqslant 0} D(n-i,2n-2-j) \\ &= D(n,2n-3) + 2D(n-1,2n-2). \end{aligned}$$

We substitute the access recursion for $D(n,2n-3)$ and describe $D(n,m)$ with the help of (7) as

$$D(n,m) = 2D(n,m-1) + 2D(n-1,m) - 2D(n-1,m-1) \quad \text{for } m \geqslant \max\{2,2n-1\} \tag{9}$$

$$D(n,2n-2) = 2D(n-1,2n-2) + \sum_{i\geqslant 1} D(n-i,2n-3) \quad \text{for } n > 1$$

$$D(0,0) = D(0,1) = D(1,0) = 1 \quad \text{and} \quad D(1,1) = 3.$$

This description has the advantage that the recursions refer to points inside the allowed region of D; therefore there exists a polynomial extension $d_n(m)$ of $2^{1-m}D(n,m)$ for $m \geqslant \max\{1,2n-2\}$. The extension solves the recursion (8), $d_n(m) = d_n(m-1) + 2d_{n-1}(m) - d_{n-1}(m-1)$, for $m \geqslant \max\{2,2n-1\}$ and has at $(n,2n-2)$ the initial values

$$2d_n(2n-2) = 4d_{n-1}(2n-2) + \sum_{i\geqslant 1} d_{n-i}(2n-3) \text{ for } n > 1. \tag{10}$$

In order to agree with $2^{1-m}D(n,m)$ for all $m \geqslant \max\{1,2n-2t\}$, $n \geqslant 0$, we must ascertain that $d_0(1) = D(0,1) = 1$ and $d_1(1) = D(1,1) = 3$ when recursion (8) is applied to $d_n(m)$. This implies $d_1(0) = 2$, because $d_0 \equiv 1$ (see Table 9). And although the points on the restricted line are not included in the polynomial extension they are easily recovered as $D(n,2n-3) = D(n,2n-2) - 2D(n-1,2n-2)$. We find the path counts in terms of $d_n(m)$,

$$D(n,m) = 2^{m-1}d_n(m) \quad \text{for } m \geqslant \max\{1,2n-2\}, n \geqslant 0,$$
$$D(n,2n-3) = 2^{2n-3}(d_n(2n-2) - 2d_{n-1}(2n-2)) \quad \text{for } n \geqslant 1. \tag{P.9}$$

For a simpler expression of the initial condition, we can substitute recursion (8) into the initial condition (10) for three iterations and derive

$$2d_n(2n-4) = d_{n-1}(2n-4) \quad \text{for all } n \geqslant 2. \tag{IC.9}$$

2.3.4. The number of paths making c contacts with a boundary

In Problem 10 we study the number of paths $D(n,m;c)$ to (n,m) making c contacts with the (weak) boundary $m = 2(n-1)$. Denote the generating function (weighted path count) of $D(n,m;c)$ by $D(n,m)$,

$$D(n,m) = \sum_{c \geqslant 0} D(n,m;c)s^c.$$

Above the boundary we recursively calculate $D(n,m) = \sum_{\langle i,j\rangle \in \mathfrak{S}} D(n-i,m-j)$ because of the all-steps recursion. The path “ picks up” a factor s every time it steps on the weak boundary, thus

$$D(n,2n-2) = s \sum_{\langle i,j\rangle \in \mathfrak{S}} D(n-i,2n-2-j)$$
$$= 2sD(n-1,2n-2) \quad \text{(as in Problem 8)}. \tag{11}$$

Table 10 shows the weighted path counts, $D(n,m)$, weakly above the boundary $m = 2n-2$, and the polynomials $d_n(m) = 2^{1-m}D(n,m)$ for $n \geqslant 1$, $d_0(m) \equiv 1$. Italicized entries denote where $2^{1-m}D(n,m) \neq d_n(m)$ in the supp(D); the values for $d_n(m)$ are in bold. Thus the path counts are

$$D(n,m) = 2^{m-1}d_n(m) \quad \text{when } m \geqslant \max\{1,2n-1\} \text{ and}$$
$$D(n,2n-2) = 2^{2n-2}sd_{n-1}(2n-2) \quad \text{for } n \geqslant 1. \tag{P.10}$$

Specific solutions for $D(n,m;c)$ will be given in (S.10b) and (S.10c).

At the point above, the weak boundary holds

$$D(n,2n-1) = 2D(n-1,2n-1) + D(n,2n-2)$$
$$= 2D(n-1,2n-1) + 2sD(n-1,2n-2),$$

referring only to points above the line $m = 2n-2$. Thus we obtain initial values for the polynomials,

$$d_n(2n-1) = 2d_{n-1}(2n-1) + sd_{n-1}(2n-2) \quad \text{for all } n \geqslant 1. \tag{IC.10}$$

Table 10
$\{\langle i,j\rangle \in \mathbb{N}_0^2 \backslash \langle 0,0\rangle\}$-path weighted by the number of contacts with $m=2(n-1)$

Problem 10

m	32	$224+32s$	$928+256s+32s^2$	$2528+1184s+288s^2+32s^3$
5	16	$96+16s$	$336+112s+16s^2$	$672+448s+128s^2+16s^3$
4	8	$40+8s$	$112+48s+8s^2$	$224s+96s^2+16s^3$
3	4	$16+4s$	$32+20s+4s^2$	■
2	2	$6+2s$	$12s+4s^2$	
1	1	$2+s$	■	
0	1	s		
	0	1	2	$3 \rightarrow n$

Path counts $D(n,m)$

m	1	$7+6$	$29+8s+s^2$	$79+37s+9s^2+s^3$
5	1	$6+s$	$21+7s+s^2$	$42+28s+8s^2+s^3$
4	1	$5+s$	$14+6s+s^2$	$28s+12s^2+2s^3/\mathbf{14+20s+7s^2+s^3}$
3	1	$4+s$	$8+5s+s^2$	$-6+13s+6s^2+s^3$
2	1	$3+s$	$6s+2s^2/\mathbf{3+4s+s^2}$	$-19+7s+5s^2+s^3$
1	1	$2+s$	$-1+3s+s^2$	$-26+2s+4s^2+s^3$
0	*2/1*	$2s/\mathbf{1+s}$	$-4+2s+s^2$	$-28-2s+3s^2+s^3$
-1	1	s	$-6+s+s^2$	$-26-5s+2s^2+s^3$
-2	1	$s-1$	...	...
-3	1	$s-2$	...	...
	0	1	2	$3 \rightarrow n$

$2^{1-m}D(n,m)$ and the polynomial extension $d_n(m)$

3. Theory

We need a few facts from Umbral Calculus (as developed in [7,10,11]) to explain our method of solving recursions. Remember that we are looking for polynomial solutions $p_0, p_1, \ldots$, where p_n is of degree n. We will use the word 'operator' as a synonym for 'linear operator on polynomials', and do the same for 'functionals'. The derivative operator $\mathscr{D} := \mathrm{d}/\mathrm{d}x$ is an important example, and it follows from Taylor's formula that $\mathrm{e}^{a\mathscr{D}}$ is the *shift operator* by a, hence

$$e^{a\mathscr{D}} = E^a : f(x) \longmapsto f(x+a).$$

An operator T is *shift invariant* iff T commutes with all shift operators, $E^aT = TE^a$ for all a. Examples are the derivative operator, and therefore all power series in $\mathscr{D}$. It has been shown [11] that the shift invariant operators are isomorphic to formal power series; the invertible shift invariant operators correspond to power series with nonzero constant term.

A power series $\phi(t)$ is a *delta series* iff $\phi(0) = 0$ and $\phi'(0) \neq 0$. The corresponding operator $\phi(\mathscr{D})$ is a *delta operator*. Examples are $\mathscr{D}$, of course, the (forward) difference operator $\Delta = \mathrm{e}^{\mathscr{D}} - 1 = E - 1$, and the backwards difference operator $\nabla = 1 - \mathrm{e}^{-\mathscr{D}} = 1 - E^{-1}$. Delta series have compositional inverses, and this makes it possible to express every delta operator as a delta series in any other delta operator; we have $\mathscr{D} = \ln(1+\Delta) = -\ln(1-\nabla)$, and therefore $\Delta = \nabla/(1-\nabla)$, etc.

Delta operators reduce the degree of polynomials by exactly 1. Suppose B is a delta operator. A *B-Sheffer sequence* (s_n) is a polynomial sequence such that $Bs_n(x) = s_{n-1}(x)$ for all $n \in \mathbb{N}_0$. In applications, the combinatorial recursion generates this system of operator equations. For example, the binomial coefficients $\binom{n+x}{n}$ are ∇-Sheffer polynomials because

$$\nabla \binom{n+x}{n} = \binom{n+x}{n} - \binom{n+x-1}{n} = \binom{n-1+x}{n-1}.$$

Different B-Sheffer sequences solve the same recursion, but satisfy different initial conditions. There exists a unique B-Sheffer sequence (b_n), the *B-basic sequence*, which has the initial values $b_n(0) = \delta_{n,0}$ for all $n \in \mathbb{N}_0$. The basic sequence serves as the basis for expanding Sheffer sequences, and carries the recursion information of the combinatorial problem. The ∇-basic sequence is $(\binom{n-1+x}{n})_{n \geqslant 0}$.

The second ingredient in the expansion formula is a functional L that describes the initial conditions. In this paper such a functional could be as simple as evaluation at 0, asking that $\langle L|s_n\rangle = \langle \mathrm{Eval}_0|s_n(x)\rangle = s_n(0)$ for all $n>0$, or as involved as

$$\langle L|s_n\rangle = s_n(2n-1) - \sum_{j \geqslant 1} s_{n-j}(2n-1-j)$$

in the privileged access condition (IC.6). With every functional L comes a unique shift invariant operator

$$\mu(L) := \sum_{n \geqslant 0} \langle L|b_n\rangle B^n. \tag{12}$$

It is shown in [10] that the operator $\mu(L)$ is invariant under the choice of the delta operator B and its basic sequence (b_n). For example, if $L = \mathrm{Eval}_a$ it is convenient to choose the pair $\mathscr{D}$ and $(x^n/n!)$ to show that

$$\mu(\mathrm{Eval}_a) = \sum_{n \geqslant 0} \frac{a^n}{n!} \mathscr{D}^n = \mathrm{e}^{a\mathscr{D}}, \tag{13}$$

the shift operator E^a. If $\langle L|1\rangle \neq 0$ then $\mu(L)$ is invertible, as in the case of shift operators. Now we are ready to state the Functional Expansion Theorem [9].

Theorem 4. *Suppose* $(s_n)_{n \in \mathbb{N}_0}$ *is a B-Sheffer sequence and L a functional such that* $\langle L|1\rangle \neq 0$. *The polynomials* $s_n(x)$ *can be expanded in terms of the B-basic sequence* $(b_n)_{n \in \mathbb{N}_0}$ *as*

$$s_n(x) = \sum_{k=0}^{n} \langle L|s_k\rangle \mu(L)^{-1} b_{n-k}(x).$$

The *Binomial Theorem for Sheffer Sequences* [11],

$$s_n(x+a) = \sum_{k=0}^{n} s_k(a) b_{n-k}(x) \tag{14}$$

is a special case of the Functional Expansion Theorem if we choose $L = \mathrm{Eval}_a$. After the recursion and initial values of a polynomial solution have been extracted from the combinatorial problem, there remain two obstacles to apply the theorem: finding $\mu(L)^{-1}$ and (b_n) explicitly. For $\mu(L)$ we can use expansion (12), which has the following easy to prove corollary.

Corollary 5. *If the functional L acts on polynomials p as $\langle L|p\rangle = \langle \mathrm{Eval}_a|B^k p\rangle$ for some delta operator B and $k \in \mathbb{N}_0$, then*

$$\mu(L) = E^a B^k.$$

Often the B-basic sequence (b_n) can be guessed; if not, it may be possible to connect (b_n) with a known Q-basic sequence (q_n), say. The following *transfer formula* (15) is shown in [8], based on results in [11]. Delta operators can also be expanded as delta series ϕ with coefficients in the ring of shift invariant operators. This expansion is no longer unique, and may generate identities. Suppose $Q = \phi(B)$. With the help of Lagrange–Bürmann inversion it can be shown that

$$b_n(x) = \sum_{i=0}^{n-1} x\alpha_{n,i} \frac{1}{x} q_{n-i}(x) \tag{15}$$

for positive n, where $\alpha_{n,i}$ is the coefficient of B^n in $\phi(B)^{n-i}$. For example, if $Q = E^a B$ then for all $n > 0$

$$b_n(x) = \sum_{i=0}^{n-1} x([B^n](E^a B)^{n-i})\frac{1}{x} q_{n-i}(x) = xE^{an}\frac{1}{x} q_n(x) = \frac{x}{x+an} q_n(x+an). \tag{16}$$

4. Solutions

With the help of the theory from the preceding section we find explicit expressions for the number $D(n,m)$ of paths from the origin to (n,m) in all ten problems. The discussion of these examples show how similar problems can be solved. In all examples, we defined W as the operator which maps d_n into d_{n-1}, and we found that the Sheffer polynomials solve a recursion of the type

$$\nabla = W + \alpha\nabla E^{-\gamma} W + \beta\nabla E^{-\eta} W^2,$$

where $\alpha, \beta, \gamma, \eta$ are integers, depending on the step set. Because ∇ is a delta series in W we know that W must be a delta operator, and we find

$$\begin{aligned}\alpha_{n,i} &= [W^n]W^{n-i}(1 + \alpha\nabla E^{-\gamma} + \beta\nabla E^{-\eta} W)^{n-i}\\ &= \sum_{k=0}^{n-i} \binom{n-i}{k}\binom{k}{i} (\alpha\nabla E^{-\gamma})^{k-i}(\beta\nabla E^{-\eta})^i.\end{aligned}$$

Using Lagrange–Bürmann expansion (15), the W-basic polynomials are $w_0 = 1$ and for $n > 0$ holds

$$\begin{aligned} w_n(x) &= \sum_{i=0}^{n-1} x\alpha_{n,i} \frac{1}{x} q_{n-i}(x), \\ &= \sum_{i=0}^{n-1} \frac{x}{n-i} \sum_{k=0}^{n-i} \binom{n-i}{k} \binom{k}{i} \alpha^{k-i} \beta^i \\ &\quad \times \binom{n+x+(\gamma-\eta-1)i-(1+\gamma)k-1}{n-i-k-1}. \end{aligned} \tag{17}$$

When our boundary condition acts along a line $a(n-\ell)+b$ it is helpful to introduce the $E^{-a}W$-Sheffer sequence $(\hat{d}_n)$, where

$$\hat{d}_n(x) = d_n(an+x), \tag{18}$$

$$E^{-a}W\hat{d}_n(x) = E^{-a}d_{n-1}(an+x) = d_{n-1}(an+x-a) = \hat{d}_{n-1}(x).$$

By observation (16), the $E^{-a}W$-basic sequence $(\hat{w}_n)$ becomes

$$\hat{w}_n(x) = \frac{x}{x+an} w_n(x+an).$$

For each example, we adjust the basic sequence $w_n(x)$ to the choice of parameters, and may use when needed $\hat{w}_n(x)$ as defined above. To solve a problem we either apply the Binomial Theorem for Sheffer Sequences (14), or we derive the functional $\langle L \mid \hat{d}_n \rangle$ from the initial condition, calculate the corresponding operator inverse $\mu(L)^{-1}$, and find the explicit equation for $d_n(x)$ using Theorem 4.

4.1. Almost all steps

Parameters: For the almost-all-steps step set we found $\nabla = W + \nabla W - \nabla W^2$, thus $\alpha = 1, \beta = -1$, and $\gamma = \eta = 0$

Basic Sequence $(w_n(x))$: It follows from (17) that $w_0(x) = 1$, and for $n \geqslant 1$

$$w_n(x) = \sum_{i=0}^{n-1} \frac{x}{n-i} \sum_{k=0}^{n-i} \binom{n-i}{k} \binom{k}{i} (-1)^i \binom{n+x-i-k-1}{n-1-i-k},$$

which simplifies for all $n \geqslant 0$ to

$$w_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{x}{i} \binom{n-i+x-1}{n-2i}.$$

Problem 1.

Functional L: The initial condition $d_n(n-\ell) = d_{n-1}(n-\ell) - d_{n-2}(n-\ell)$ for $n \geqslant \ell$ in (IC-1) can be written as

$$\hat{d}_n(-\ell) - \hat{d}_{n-1}(1-\ell) + \hat{d}_{n-2}(2-\ell) = 0 \quad \text{for } n \geqslant \ell$$

and is expressed as $\langle L|\hat{d}_n\rangle = 0$ for $n \geqslant \ell$ if we define the functional L as

$$\langle L|\hat{d}_n\rangle = \langle \mathrm{Eval}_{-\ell}|\hat{d}_n\rangle - \langle \mathrm{Eval}_{1-\ell}|E^{-1}W\hat{d}_n\rangle + \langle \mathrm{Eval}_{2-\ell}|(E^{-1}W)^2\hat{d}_n\rangle.$$

If $\ell = 3$, the initial values $\langle L|\hat{d}_2\rangle = -1$, $\langle L|\hat{d}_1\rangle = -1$ and $\langle L|\hat{d}_0\rangle = 1$ can be found with the help of Table 1.

Inverse operator: $\mu(L)^{-1} = E^{\ell}(1 - W + W^2)^{-1}$. The well-known identity [3]

$$\sum_{j=1}^{\infty} t^j \sin jx = (t \sin x)/(1 - 2t \cos x + t^2)$$

shows that $[t^j](1 - W + W^2)^{-1} = 2/\sqrt{3} \sin([\pi(j+1)]/3)$, which equals 0 if $j + 1 = 0 \bmod 3$, and $(-1)^{\lfloor (j+1)/3 \rfloor}$ otherwise. Hence

$$\mu(L)^{-1} = \frac{2}{\sqrt{3}} \sum_{j \geqslant 0} \sin\left(\frac{\pi(j+1)}{3}\right) E^{(\ell+j)}(E^{-1}W)^j.$$

Sheffer polynomial $d_n(x)$: Applying the Functional Expansion Theorem 4 we construct $d_n(x)$ for the case $\ell = 3$ as $d_n(x) = E^{-n}\hat{d}_n(x)$

$$\begin{aligned} &= \frac{2}{\sqrt{3}} E^{-n} \sum_{j \geqslant 0} \sin \frac{\pi(j+1)}{3} E^{(3+j)}(E^{-1}W)^j(\hat{w}_n(x) - \hat{w}_{n-1}(x) - \hat{w}_{n-2}(x)) \\ &= \frac{2}{\sqrt{3}} \sum_{j=0}^{n} \sin \frac{\pi(n-j+1)}{3} \\ &\quad \times (x+3-j)\left(\frac{w_j(x+3)}{x+3} - \frac{w_{j-1}(x+2)}{x+2} - \frac{w_{j-2}(x+1)}{x+1}\right), \qquad \text{(S.1)} \end{aligned}$$

with $w_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{x}{i}\binom{n-i+x-1}{n-2i}$.

4.2. Steep paths

Parameters: $\alpha = 1, \beta = -1$, and $\eta = 0$ describe the operator equation $\nabla = W + \nabla E^{-\gamma} W$ for steep paths.

Basic Sequence $(w_n(x))$: $w_0(x) = 1$ and for $n \geqslant 1$,

$$w_n(x) = \frac{x}{n} \sum_{i=0}^{n-1} \binom{n}{i} \binom{n + x - (1+\gamma)i - 1}{n - 1 - i}.$$

In Problems 2–5 we have chosen $\gamma = 1$, and for Problem 6, $\gamma = 2$.

Problems 2 and 3.

Functional L: The initial conditions on $d_n(n - \ell) = \hat{d}_n(-\ell)$ in (IC.2) and (IC.3) determine the functional L as

$$\langle L|\hat{d}_n\rangle = \langle \mathrm{Eval}_{-\ell}|\hat{d}_n\rangle.$$

Inverse operator: $\mu(L)^{-1} = E^{\ell}$.

Sheffer polynomial $d_n(x)$: In Problem 2, where $\ell = 1$, the initial condition (IC.2) can be expressed as $\langle L|\hat{d}_n\rangle = \delta_{n,0}$. By the Functional Expansion Theorem 4

$$\begin{aligned} d_n(x) &= E^{-n}\hat{d}_n(x) = E^{-n}\sum_{i=0}^{n}\langle \mathrm{Eval}_{-1}|\hat{d}_i\rangle E\hat{w}_{n-i}(x) = E^{-n}\hat{w}_n(x+1) \\ &= \frac{x+1-n}{x+1}\, w_n(x+1). \end{aligned} \tag{S.2}$$

In Problem 3, where $\ell = 4$, the initial condition (IC.3) can be expressed as $\langle L|\hat{d}_n\rangle = -2$ if $n > 0$, and $\langle L|\hat{d}_0\rangle = 1$. Thus

$$\begin{aligned} d_n(x) &= E^{-n}\hat{d}_n(x) = E^{-n}\hat{w}_n(x+4) - 2\sum_{k=1}^{n} E^{-n}\hat{w}_{n-k}(x+4) \\ &= (x+4-n)\left(\frac{w_n(x+4)}{x+4} - 2\sum_{k=1}^{n}\frac{w_{n-k}(x+4-k)}{x+4-k}\right). \end{aligned} \tag{S.3}$$

In both problems,

$$w_n(x) = \frac{x}{n}\sum_{i=0}^{n-1}\binom{n}{i}\binom{n+x-2i-1}{n-1-i} \quad \text{for } n > 0.$$

Problems 4 and 5.

Functional L: The privileged access initial condition on $\hat{d}_n(-\ell) - \hat{d}_{n-1}(-\ell) - \hat{d}_{n-2}(1-\ell)$ (see (5)) determines the functional L as

$$\langle L|\hat{d}_n\rangle = \langle \mathrm{Eval}_{-\ell}|\hat{d}_n\rangle - \langle \mathrm{Eval}_{-\ell}|E^{-1}W\hat{d}_n\rangle - \langle \mathrm{Eval}_{1-\ell}|(E^{-1}W)^2\hat{d}_n\rangle.$$

Inverse operator:

$$\begin{aligned} \mu(L)^{-1} &= (E^{-\ell} - E^{-\ell}E^{-1}W - E^{(1-\ell)}(E^{-1}W)^2)^{-1} \\ &= \sum_{m\geqslant 0}\sum_{k=0}^{m}\binom{m-k}{k}E^{(\ell+k)}(E^{-1}W)^m. \end{aligned}$$

Sheffer polynomial $d_n(x)$: For $\ell = 0$ (Problem 4) the initial condition (IC.4) shows that $\langle L|\hat{d}_0\rangle = 1$ and $\langle L|\hat{d}_1\rangle = -1$. By the Functional Expansion Theorem 4 holds

$$\begin{aligned} d_n(x) &= E^{-n}\hat{d}_n(x) = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{m-k}{k}(\hat{w}_{n-m}(x-n+k) - \hat{w}_{n-1-m}(x-n+k)) \\ &= \frac{x-n}{x}\, w_n(x) + \sum_{m=1}^{n-1}\sum_{k=0}^{m-1}\binom{m-1-k}{k} \\ &\quad \times \frac{x-n+1+k}{x-m+k}\, w_{n-1-m}(x-m+k). \end{aligned} \tag{S.4}$$

For Problem 5 with $\ell = 3$, we know from (IC.5) that $\langle L|\hat{d}_n\rangle = 0$ for all $n \geqslant 3$. With the help of Table 5 we determine $\langle L|\hat{d}_2\rangle = -1$, $\langle L|\hat{d}_1\rangle = -2$, $\langle L|\hat{d}_0\rangle = 1$.

Hence $d_n(x) = E^{-n}\hat{d}_n(x)$

$$\begin{aligned}
&= \sum_{m\geqslant 0}\sum_{k=0}^{m}\binom{m-k}{k}(\hat{w}_{n-m}(x+3+k-n) - 2\hat{w}_{n-1-m}(x+3+k-n)\\
&\quad -\hat{w}_{n-2-m}(x+3+k-n))\\
&= \frac{x+3-n}{x+3}\,w_n(x+3) - \frac{x+3-n}{x+2}\,w_{n-1}(x+2) + \frac{x+4-n}{x+2}\,w_{n-2}(x+2)\\
&\quad + \sum_{m=0}^{n-2}\sum_{k=0}^{m}\left(\binom{k}{m-k-2} - 2\binom{k}{m-k}\right)\frac{x+3+m-k-n}{x+1-k}\\
&\quad \times w_{n-2-m}(x+1-k). \qquad \text{(S.5)}
\end{aligned}$$

In both problems,

$$w_n(x) = \frac{x}{n}\sum_{i=0}^{n-1}\binom{n}{i}\binom{n+x-2i-1}{n-1-i} \quad \text{for } n > 0.$$

Problem 6.

Recall for Problem 6, that $\gamma = 2$ in the basic sequence. The boundary line has slope $a = 2$, thus we will calculate the $E^{-2}W$-Sheffer polynomials $\hat{d}_n(x) := d_n(2n + x)$.

Functional L: The privileged access initial condition on $\hat{d}_n(-1) - \sum_{j\geqslant 1}\hat{d}_{n-j}(j-1)$ (see (IC.6)) determines the functional L as

$$\langle L|\hat{d}_n\rangle = \langle \mathrm{Eval}_{-1}|\hat{d}_n\rangle - \sum_{j\geqslant 1}\langle \mathrm{Eval}_{j-1}|(E^{-2}W)^j\hat{d}_n\rangle,$$

which vanishes for all $n\geqslant 1$, just leaving us with $\langle L|\hat{d}_0\rangle = 1$.

Inverse operator: $\mu(L)^{-1} = E(1 - \sum_{j\geqslant 1}(E^{-1}W)^j)^{-1} = E(1 + \frac{E^{-1}W}{1-2E^{-1}W}) = E(1 + \sum_{k\geqslant 0} 2^k E^{(k+1)}(E^{-2}W)^{k+1})$.

Sheffer polynomial $d_n(x)$: By the Functional Expansion Theorem 4

$$\begin{aligned}
d_n(x) = E^{-2n}\hat{d}_n(x) &= \hat{w}_n(x-2n+1) + \sum_{k=0}^{n-1} 2^k\hat{w}_{n-1-k}(x-2n+2+k),\\
&= \frac{x-2n+1}{x+1}\,w_n(x+1) + \sum_{k=0}^{n-1} 2^k\,\frac{x-2n+2+k}{x-k}\,w_{n-1-k}(x-k), \qquad \text{(S.6)}
\end{aligned}$$

where

$$w_n(x) = \frac{x}{n}\sum_{i=0}^{n-1}\binom{n}{i}\binom{n+x-3i-1}{n-1-i} \quad \text{for } n > 0.$$

4.3. All steps

Parameters: For the all-steps step set we found $\nabla = W + \nabla W$, thus $\alpha = 1$, and $\beta = \gamma = \eta = 0$.

Basic Sequence $(w_n(x))$: It follows from (17) that for $n\geqslant 0$,

$$w_n(x)=\frac{x}{n}\sum_{i=0}^{n}\binom{n}{i}\binom{x+n-i-1}{n-i-1}=\sum_{i=0}^{n}\binom{n-1}{i}\binom{x+n-i-1}{n-i}.$$

Note that in Problems 8–10 the boundary line has slope $a=2$, thus we will calculate the $E^{-2}W$-Sheffer polynomials $\hat{d}_n(x):=d_n(2n+x)$.

Problem 7.

For Problem 7 we count the unrestricted paths. We can use the Binomial Theorem for Sheffer Sequences (14) with the initial condition $d_n(-1)=1$ (see (IC.7)) to expand $d_n(x)=\sum_{i=0}^{n} d_i(-1)w_{n-i}(x+1)$. It is easy to verify that $w_n(-1)=-1$ for $n>0$, thus

$$d_n(x)=2w_n(x+1)-\sum_{i=0}^{n} w_i(-1)w_{n-i}(x+1)=2w_n(x+1)-w_n(x).$$

Because $w_n(x)=w_n(x-1)+2w_{n-1}(x)-w_{n-1}(x-1)$ (recursion (8)) we obtain

$$\begin{aligned} d_n(x)&=w_{n+1}(x+1)-w_{n+1}(x)\\ &=\sum_{k=0}^{n}\binom{n}{k}\left(\binom{x+n-k+1}{n+1-k}-\binom{x+n-k}{n+1-k}\right)\\ &=\sum_{k=0}^{n}\binom{n}{k}\binom{x+k}{k}. \end{aligned} \tag{S.7}$$

Problem 8.

Functional L: The initial condition on $\hat{d}_n(-3)-\hat{d}_{n-1}(-1)$ (see (IC.8)) determines the functional L as

$$\langle L|\hat{d}_n\rangle=\langle \mathrm{Eval}_{-3}|\hat{d}_n\rangle-\langle \mathrm{Eval}_{-1}|E^{-2}W\hat{d}_n\rangle.$$

Inverse operator:

$$\mu(L)^{-1}=(E^{-3}-E^{-1}E^{-2}W)^{-1}=\sum_{k\geqslant 0}E^{(3+2k)}(E^{-2}W)^k.$$

Sheffer polynomial $d_n(x)$: From $\langle L|\hat{d}_n\rangle=\delta_{0,n}$ follows

$$\begin{aligned} d_n(x)&=E^{-2n}\hat{d}_n(x)\\ &=E^{-2n}\sum_{k=0}^{n}E^{(3+2k)}(E^{-2}W)^k\hat{w}_n(x)=\sum_{k=0}^{n}\hat{w}_{n-k}(x-2n+3+2k)\\ &=\sum_{k=0}^{n}\frac{x+3-2k}{x+3}\sum_{i=0}^{k}\binom{k-1}{i}\binom{x+k-i+2}{k-i}. \end{aligned} \tag{S.8}$$

Problem 9.

Functional L: The initial conditions on $2\hat{d}_n(-4) - \hat{d}_{n-1}(-2)$ (see (IC.9)) determine the functional L as

$$\langle L|\hat{d}_n\rangle = 2\langle \mathrm{Eval}_{-4}|\hat{d}_n\rangle - \langle \mathrm{Eval}_{-2}|E^{-2}W\hat{d}_n\rangle.$$

Inverse operator:

$$\mu(L)^{-1} = (2E^{-4} - E^{-2}E^{-2}W)^{-1} = \sum_{k\geqslant 0} 2^{-1-k}E^{(4+2k)}(E^{-2}W)^k.$$

Sheffer polynomial $d_n(x)$: Because of condition (IC.9), $\langle L|\hat{d}_n\rangle = 0$ for $n>1$, and with the help of Table 9 we find $\langle L|\hat{d}_0\rangle = 2$ and $\langle L|\hat{d}_1\rangle = -1$. By the Functional Expansion Theorem 4,

$$\begin{aligned} d_n(x) = E^{-2n}\hat{d}_n(x) &= E^{-2n}\sum_{k\geqslant 0} 2^{-1-k}E^{(4+2k)}(E^{-2}W)^k(2\hat{w}_n(x) - \hat{w}_{n-1}(x)),\\ &= \sum_{k=0}^{n} 2^{-k}\hat{w}_{n-k}(x+4+2k-2n) - 2^{-1-k}\hat{w}_{n-1-k}(x+4+2k-2n),\\ &= \sum_{k=0}^{n} 2^{k-1-n}(x+4-2k)\left(\frac{2w_k(x+4)}{x+4} - \frac{w_{k-1}(x+2)}{x+2}\right), \end{aligned} \tag{S.9}$$

where

$$w_n(x) = \sum_{i=0}^{n}\binom{n-1}{i}\binom{x+n-i-1}{n-i}.$$

Problem 10.

Functional L: The initial condition $\hat{d}_n(-1) - 2\hat{d}_{n-1}(1) - s\hat{d}_{n-1}(0) = \delta_{0,n}$ (see (IC.10)) determines the functional L as

$$\langle L|\hat{d}_n\rangle = \langle \mathrm{Eval}_{-1}|\hat{d}_n\rangle - 2\langle \mathrm{Eval}_1|E^{-2}W\hat{d}_n\rangle - s\langle \mathrm{Eval}_0|E^{-2}W\hat{d}_n\rangle.$$

Inverse operator: $\mu(L)^{-1} = (E^{-1} - 2E^{-1}W - sE^{-2}W)^{-1} = E(1 - W(2+sE^{-1}))^{-1} = E\sum_{k\geqslant 0} W^k \sum_{j=0}^{k}\binom{k}{j}2^{k-j}s^jE^{-j}$.

Sheffer polynomial $d_n(x)$: By Theorem 4,

$$\begin{aligned} d_n(x) &= E^{-2n}\hat{d}_n(x,s)\\ &= \sum_{k\geqslant 0}\sum_{j=0}^{k}\binom{k}{j}2^{k-j}s^jE^{(2(k-n)-j+1)}(E^{-2}W)^k\hat{w}_n(x),\\ &= \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}2^{k-j}s^j\hat{w}_{n-k}(x+2(k-n)-j+1). \end{aligned} \tag{S.10a}$$

To find $D(n,m;c)$ we must extract the coefficient of s^c from $D(n,m)$ as given in (P.10). Hence we find for $m \geqslant \max\{1, 2n-1\}$

$$D(n,m;c) = [s^c]2^{m-1}d_n(m) = 2^{m-1}\sum_{k=0}^{n}\binom{k}{c}2^{k-c}\hat{w}_{n-k}(m-2n+2k-c+1)$$
$$= 2^{m-c-1}\sum_{k=c}^{n}\binom{k}{c}2^{k}\frac{m-2n+2k-c+1}{m-c+1}w_{n-k}(m-c+1) \quad \text{(S.10b)}$$

and

$$D(n,2n-2;c) = [s^c]2^{2n-2}sd_{n-1}(2n-2),$$
$$= 2^{2n-c-1}\sum_{k=c-1}^{n-1}\binom{k}{c-1}2^{k}\frac{2(k+1)-c}{2n-c}w_{n-1-k}(2n-c), \quad \text{(S.10c)}$$

where

$$w_n(x) = \sum_{i=0}^{n}\binom{n-1}{i}\binom{x+n-i-1}{n-i}.$$

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