On the matching extendability of graphs in surfaces

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Abstract

A graph $G$ with at least $2n + 2$ vertices is said to be $n$-extendable if every matching of size $n$ in $G$ extends to a perfect matching. It is shown that (1) if a graph is embedded on a surface of Euler characteristic $\chi$, and the number of vertices in $G$ is large enough, the graph is not 4-extendable; (2) given $g > 0$, there are infinitely many graphs of orientable genus $g$ which are 3-extendable, and given $g \geq 2$, there are infinitely many graphs of non-orientable genus $\bar{g}$ which are 3-extendable; and (3) if $G$ is a 5-connected triangulation with an even number of vertices which has genus $g > 0$ and sufficiently large representativity, then it is 2-extendable.

Keywords: Embedded graph; Genus; Matching; Extendability; Surface; Klein bottle; Projective plane; Torus

1. Introduction

A set $M$ of edges in a graph $G$ is a matching if no two members of $M$ share a vertex. A matching $M$ is perfect if every vertex of $G$ is covered by an edge of $M$. Let $n \geq 1$ be an integer. A graph $G$ having at least $2n + 2$ vertices is said to be $n$-extendable if every matching $M \subseteq E(G)$ with $|M| = n$, extends to (i.e., is a subset of) a perfect matching in $G$.

We shall have need of the following results. (See [7, Theorems 3.2 and 2.2].)

Lemma 1.1. Every $n$-extendable graph is $(n + 1)$-connected.
Lemma 1.2. Every $n$-extendable graph is $(n - 1)$-extendable.

A surface is a connected compact Hausdorff space which is locally homeomorphic to an open disc in the plane. If surface $\Sigma$ is obtained from the sphere by adding some number $g \geq 0$ of handles (respectively, some number $\bar{g} > 0$ of crosscaps), $\Sigma$ is said to be orientable of genus $g = g(\Sigma)$ (respectively non-orientable of genus $\bar{g} = \bar{g}(\Sigma)$). We shall follow the usual convention of denoting the surface of orientable genus $g$ (respectively non-orientable genus $\bar{g}$) by $S_g$ (respectively by $N_{\bar{g}}$).

An embedding of a graph $G$ on the orientable surface (respectively non-orientable surface) $\Sigma$ is minimal if $G$ cannot be embedded on any orientable (respectively non-orientable) surface $\Sigma'$ where $g(\Sigma') < g(\Sigma)$ (respectively $\bar{g}(\Sigma') < \bar{g}(\Sigma)$). Graph $G$ is said to have orientable genus $g$ (respectively, non-orientable genus $\bar{g}$) if $G$ minimally embeds on a surface of orientable genus $g$ (respectively, non-orientable genus $\bar{g}$).

An embedding of a graph $G$ on surface $\Sigma$ is said to be 2-cell if every face of the embedding is homeomorphic to a disc. For 2-cell embeddings, we have the important classical result of Euler.

Theorem 1.3. If $G$ is 2-cell embedded on surface $\Sigma$ having genus $g$ (respectively non-orientable genus $\bar{g}$) and if the embedded $G$ has $|V(G)| = p$ vertices, $|E(G)| = q$ edges and $|F(G)| = f$ faces, then $p - q + f = 2 - 2g$ (respectively $p - q + f = 2 - \bar{g}$).

The following two results are of paramount importance when working with minimal embeddings. The first is due to J.W.T. Youngs [14] and the second to Parsons, Pica, Pisanski and Ventre [6].

Theorem 1.4. Every minimal orientable embedding of a graph $G$ is 2-cell.

Theorem 1.5. Every graph $G$ has a minimal non-orientable embedding which is 2-cell.

The representativity (or face-width) of a graph embedded on a surface $\Sigma$ is the smallest number $k$ such that $\Sigma$ contains a noncontractible closed curve that intersects the graph in $k$ vertices. We shall also make use of the concept of “Euler Contribution.” Let $v$ be any vertex of a graph $G$ minimally embedded on an orientable surface of genus $g$ (respectively embedded on a non-orientable surface of genus $\bar{g}$). Define the Euler contribution of vertex $v$ to be

$$\phi(v) = 1 - \frac{\deg(v)}{2} + \sum_{i=1}^{\deg(v)} \frac{1}{f_i},$$

where the sum runs over the face angles at vertex $v$ and $f_i$ denotes the size of the $i$th face at $v$. (One should keep in mind here that a face may contribute more than one face angle at a vertex $v$. Think of $K_5$ embedded on the torus, for example.) The next result is essentially due to Lebesgue [4].

Lemma 1.6. If a connected graph $G$ is 2-cell embedded on a surface of orientable (respectively non-orientable) genus $g$ (respectively non-orientable genus $\bar{g}$), then $\sum_v \phi(v) = 2 - 2g$ (respectively $2 - \bar{g}$).

Given a surface $\Sigma$, orientable or non-orientable, let $\mu(\Sigma)$ denote the least integer $n$ such that no graph $G$ embeddable on the surface $\Sigma$ is $n$-extendable. We call $\mu(\Sigma)$ the extendability of
the surface $\Sigma$. In 1989, the third author showed [9] that no planar graph is 3-extendable. Hence, since there are planar graphs which are 1-extendable and 2-extendable, it follows that $\mu(S_0) = 3$. Later, Dean [1] showed for $N_1 =$ the projective plane $N_2 =$ the Klein bottle and $S_1 =$ the torus that $\mu(N_1) = 3$, $\mu(S_1) = \mu(S_2) = 4$ and, more generally, $\mu(\Sigma) = 2 + \lfloor \sqrt{2g(\Sigma)} \rfloor$, where $\chi(\Sigma)$ denotes the Euler characteristic of surface $\Sigma$. Thus for an orientable surface $\Sigma$ of genus $g$, we have $\mu(\Sigma) = 2 + \lceil \sqrt{2g} \rceil$, while if $\Sigma$ is a non-orientable surface of genus $\tilde{g}$, then $\mu(\Sigma) = 2 + \lfloor \sqrt{2\tilde{g}} \rfloor$. So the extendability function $\mu(\Sigma)$ is an increasing function of $g$ (and of $\tilde{g}$).

In the present paper, we derive three main results. In Section 2, we show that if $G$ is a connected graph of genus $g$ (non-orientable genus $\tilde{g}$), then if $G$ has a sufficiently large number of vertices, $G$ is not 4-extendable. In Section 3, we first show that for every (orientable) genus $g > 0$, there are infinitely many graphs with genus $g$ which are 3-extendable and then show that for every (non-orientable) genus $\tilde{g} \geq 2$, there are infinitely many graphs with non-orientable genus $\tilde{g}$ which are 3-extendable. Finally, in Section 4, we show that given $g > 0$, if $G$ is a 5-connected triangulation on an even number of vertices embedded on $S_g$ with representativity at least $f(g)$, then $G$ is 2-extendable.

2. Large graphs on any surface are not 4-extendable

We now present our first main theorem.

**Theorem 2.1.** Let $G$ be any connected graph of genus $g$ (respectively non-orientable genus $\tilde{g}$). Then if $|V(G)| \geq 8g - 7$ (respectively $|V(G)| \geq 4\tilde{g} - 7$), $G$ is not 4-extendable.

**Proof.** Suppose $G$ is 4-extendable. Then, by definition, $|V(G)| \geq 10$. Moreover, by Lemma 1.1, $G$ is 5-connected and hence mindeg$(G) \geq 5$. We shall prove that $\phi(v) \leq -1/4$ for every vertex $v \in V(G)$.

Let $v \in V(G)$ be an arbitrary vertex and let $R = V(G) - (\{v\} \cup N(v))$.

1. Suppose $v \in V(G)$ has degree 5 and let $x_1, x_2, x_3, x_4, x_5$ be the neighbors of $v$. If there are no triangular faces at $v$, then $\phi(v) \leq 1 - (5/2) + (5/4) = -1/4$. Thus we may assume that $vx_4x_5v$ is a triangular face. Since $G$ is 5-connected, we can find a matching $\{x_1y_1, x_2y_2, x_3y_3\}$ with $\{y_1, y_2, y_3\} \subseteq R$. If there is a subset $S \subseteq \{x_1, x_2, x_3\}$ such that $T = N(S) \cap R$ satisfies $|T| < |S|$, then $T \cup (N(v) - S)$ would be a cutset of order less than 5.) Thus the matching $\{x_1y_1, x_2y_2, x_3y_3, x_4x_5\}$ isolates vertex $v$ and hence cannot be extended to a perfect matching of $G$, a contradiction.

2. Suppose next that deg $v = 6$. Since $G$ is 3-extendable by Lemma 1.2, we may assume that the subgraph $G[N(v)]$ induced by $N(v)$ does not contain a perfect matching. Suppose that $G[N(v)]$ contains a path of length four, say $x_1x_2x_3x_4x_5$. Let $x_0$ be the remaining vertex in $N(v)$. Since $G[N(v)]$ has no perfect matching, $x_0$ is not adjacent to any of $x_1, x_3$ and $x_5$. Since deg $x_0 \geq 5$, $x_0$ is adjacent to at least two vertices in $R$. Also, since $G$ is 5-connected, at least one of $x_1, x_3$ or $x_5$, say $x_i$, has a neighbor $y$ in $R$. Taking $y_0 \in N(x_0) \cap (R - \{y\})$, we have a matching within $\{x_0, y_0, x_1, x_2, x_3, x_4, x_5, y\}$ which isolates $v$, a contradiction. Thus, $G[N(v)]$ does not contain a path of length four.

Now suppose there are at most three triangular faces at $v$. Then, $\phi(v) \leq 1 - (6/2) + (3/3) + (3/4) = -1/4$. So we may assume that there are at least four triangular faces at $v$. Since $G[N(v)]$ does not contain a perfect matching or a path of length four, we may assume that $vx_1x_2v, vx_2x_3v, vx_4x_5v, vx_5x_6v$ are the triangular faces. If $x_1$ is adjacent to one of $x_4, x_5$ or $x_6$, then $G[N(v)]$ would contain a perfect matching or a path of length at least four. Hence $x_1$ is
adjacent to none of \( x_4, x_5 \) or \( x_6 \). Since \( \deg x_1 \geq 5 \), vertex \( x_1 \) has at least two neighbors in \( R \). Also by symmetry, \( x_2 \) has at least two neighbors in \( R \). Thus we can find two distinct vertices \( y, y' \in R \) such that \( x_1y, x_4y' \in E(G) \). But then the matching \( \{x_1y, x_2x_3, x_4y', x_5x_6\} \) isolates the vertex \( v \), a contradiction.

(3) Suppose now that \( \deg v = 7 \). If there are at most five triangular faces at \( v \), then \( \phi(v) \leq 1 - (7/2) + (5/3) + (2/4) = -1/3 < -1/4 \). Hence we may assume that there are six triangular faces at \( v \) which we may denote by \( vx_i, x_{i+1}v \), \( 1 \leq i \leq 6 \). Since \( G \) is 5-connected, at least one of \( x_1, x_3, x_5 \) and \( x_7 \) is adjacent to a vertex \( y \in R \). Then we can easily find a matching within \( \{x_1, \ldots, x_7, y\} \) which isolates vertex \( v \), a contradiction.

(4) Finally, suppose that \( \deg v = k \geq 8 \). Then \( \phi(v) \leq 1 - (k/2) + (k/3) = 1 - k/6 \leq 1 - 8/6 = -1/3 < -1/4 \).

Thus for all \( v \in V(G) \) we have proved that \( \phi(v) \leq -1/4 \). Now suppose \( G \) is embedded on the orientable surface of genus \( g \). Then, by Lemma 1.6, we have

\[
2 - 2g = \sum_v \phi(v) \leq -|V(G)|/4,
\]

and hence \( |V(G)| \leq 8g - 8 \), contradicting our hypothesis.

A similar argument can be derived in the non-orientable case and hence the proof of the theorem is complete. \( \square \)

3. 3-extendable minimally embedded graphs

In this section, we will present examples of 3-extendable graphs which minimally embed on each surface, other than the plane and the projective plane. (It is known that no planar graph is 3-extendable \([9]\) and that no projective planar graph is 3-extendable \([1]\).)

We will have need of the following four results.

**Theorem 3.1.** (See \([13, \text{Corollary 6-16}]\).) If \( G \) is a connected bipartite graph having a quadrilateral embedding on some orientable surface and if \( G \) has \( q \) edges and \( p \) vertices, then \( g(G) = q/4 - p/2 + 1 \).

**Theorem 3.2.** (See \([13, \text{Corollary 11-8}]\).) If \( G \) is a connected graph with at least \( p \geq 3 \) vertices, \( q \) edges and no triangles, then \( \bar{g}(G) \geq q/2 - p + 2 \); equality holds if and only if a non-orientable quadrilateral embedding can be found for \( G \).

**Theorem 3.3.** (See \([8, \text{Corollary 2.6}]\).) Suppose \( n \geq 2 \) and \( 0 < r < n \). Suppose that \( G \) is an \( n \)-extendable bipartite graph with vertex partition \( V(G) = U \cup W \) and that \( u_1, \ldots, u_r \in U \) and \( w_1, \ldots, w_r \in W \). Then graph \( G - u_1 - \cdots - u_r - w_1 - \cdots - w_r \) is \((n - r)\)-extendable and hence \((n - r + 1)\)-connected.

**Theorem 3.4.** (See \([2, \text{Theorem 1}]\).) If \( G_1 \) and \( G_2 \) are \( k \)-extendable and \( \ell \)-extendable graphs, respectively, then their Cartesian product \( G_1 \times G_2 \) is \((k + \ell + 1)\)-extendable.

We begin with the orientable case and graph \( B_1 = C_4 \times C_4 = Q_4 \) which is sometimes called the 4-cube. Let us view \( B_1 \) embedded on the torus as shown in Fig. 3.1. Next, let \( B_2 \) denote the graph obtained from two copies of \( B_1 \) by deleting one edge in each copy and identifying the resulting hexagons. Embed \( B_2 \) on the double torus as shown in Fig. 3.2 where the six vertices of
the identified hexagon are $x_1, \ldots, x_6$ in cyclic order as shown. (Note that edge $x_2x_5$ was deleted from both copies of $B_1$ before the identification was made.)

We then construct graph $B_n$ inductively from $B_{n-1}$ and a copy of $B_1$ by deleting an edge from each which joins two degree four vertices and identifying the resulting hexagons. Then embed $B_n$ on $T^n$, the $n$-hole torus as illustrated in Fig. 3.3.

**Theorem 3.5.** For all $n \geq 1$, $g(B_n) = n$.

**Proof.** The proof is by induction on $n$. First suppose that $n = 1$. An easy vertex, edge and facial count and Euler’s formula show that $g = 1$.

So suppose that $g(B_{n-1}) = n - 1$, for $n \geq 2$ and consider $B_n$. By Theorem 3.1, since our embedding is quadrangular, $g(B_n) = \frac{|E(B_n)|}{4} - \frac{|V(B_n)|}{2} + 1 = \left(\frac{|E(B_{n-1})|}{4} + 24\right) - \frac{|V(B_{n-1})| + 10}{2} + 1 = \frac{|E(B_{n-1})|}{4} - \frac{|V(B_{n-1})|}{2} + 2 = g(B_{n-1}) + 1 = (n-1) + 1 = n$. □

Suppose $G_1$ and $G_2$ are two bipartite graphs and for $i = 1$ and 2, $G_i$ contains the induced subgraph $H_i$ shown below in Fig. 3.4.
Theorem 3.6. If \( G_1 \) and \( G_2 \) are each 3-extendable, so is \( G \).

Proof. Let \( M = \{ e_1, e_2, e_3 \} \) be a 3-matching in \( G \).

(1) Suppose that \(|M \cap E(H)| = 2\) or \(3\). In either case, without loss of generality, we may assume \( M \subseteq E(G_1) \). Let \( F_1 \) be a perfect matching in \( G_1 \) extending \( M \) and \( F_2 \), a perfect matching in \( G_2 \) extending \( s_2t_2, u_2v_2, x_2y_2 \) in \( G_2 \). Then, if \(|M \cap E(H)| = 3\) or, in the case when \(|M \cap E(H)| = 2\), if matching \( F_1 \) does not use the edge \( u_1v_1 \), perfect matching \( F = F_1 \cup (F_2 - s_2t_2 - u_2v_2 - x_2y_2) \) extends \( M \) in \( G \). If \(|M \cap E(H)| = 2\) and \( F_1 \) does use the edge \( u_1v_1 \), then let \( u_2' \) be a neighbor of \( u_2 \) in \( G_2 - V(H_2) \). Then extend \((M \cap E(H)) \cup \{u_2u_2'\}\) to a perfect matching \( F_2 \) in \( G_2 \) and \( F = (F_1 - u_1v_1) \cup F_2 \) is a perfect matching of \( G \) of the type sought.

(2) Suppose \(|M \cap E(H)| = 1\). Without loss of generality, we have two possibilities: either \(|M \cap E(G_1)| = 3\) or \(|M \cap E(G_1)| = 2\).

In the former case, take \( F_1 \) to be a perfect matching extending \( M \) in \( G_1 \). If \( F_1 \) uses \( u_1v_1 \), extend \( \{x_2y_2, s_2t_2, u_2u_2'\} \) (where \( u_2' \) is chosen as in Case 1) to a perfect matching \( F_2 \) of \( G_2 \) and let \( F = (F_1 - u_1v_1) \cup (F_2 - \{x_2y_2, s_2t_2\}) \). Otherwise, extend \( \{x_2y_2, s_2t_2, u_2v_2\} \) to a perfect matching \( F_2 \) of \( G_2 \) and then let \( F = F_1 \cup (F_2 - \{x_2y_2, s_2t_2, u_2v_2\}) \).

In the latter case, let us think of \( x_i, v_i, s_i \) as being colored black and \( y_i, u_i, t_i \) as being colored white, for \( i = 1, 2 \). Suppose, without loss of generality, that \( e_1 \in E(G_1) - E(H_1), e_2 \in E(G_2) - E(H_2) \) and \( e_3 \in E(H) \). By Theorem 3.3, there is a perfect matching \( F_1 \) extending \( \{e_1, e_3\} \) to a perfect matching of \( G_1 - b_1 - w_1 \), where \( b_1 \) is chosen to be a black vertex in \( H_1 \) and \( w_1 \) is chosen to be a white vertex in \( H_1 \), neither of which is an end vertex of \( e_1 \) or \( e_3 \) and...
one of which corresponds to an end vertex of $e_2$ if $e_2$ is incident with $H_2$ in $G_2$. Similarly, by Theorem 3.3, there is a perfect matching $F_2$ in $G_2 - b' - w'$ extending $\{e_2, e_3\}$ where $b'$ and $w'$ are the vertices in $H_2$ remaining once those corresponding to ends of $e_3$ and $b_1$ and $w_1$ are fixed. Then $F = F_1 \cup F_2$ is the desired perfect matching in $G$.

(3) Suppose next that $|M \cap E(H)| = 0$. Without loss of generality, there are two main cases to consider: $|M \cap E(G_1)| = 3$ and $|M \cap E(G_1)| = 2$.

We treat the first largely as before in Case 1. Let $F_1$ be a perfect matching in $G_1$ extending $M$. If $F_1$ uses $u_1v_1$, extend $\{x_2y_2, s_2f_2, u_2u'_2\}$ (where $u'_2$ is as in Case 1) to a perfect matching $F_2$ of $G_2$ and then let $F = (F_1 - \{u_1v_1\}) \cup (F_2 - \{x_2y_2, s_2f_2\})$. Otherwise, extend $\{x_2y_2, s_2f_2, u_2v_2\}$ to a perfect matching $F_2$ of $G_2$ and let $F = F_1 \cup (F_2 - \{x_2y_2, s_2f_2, u_2v_2\})$.

In the second case, we may suppose that $M = \{e_1, e_3\} \subseteq E(G_1)$ and $M = \{e_2\} \subseteq E(G_2)$. Now by Theorem 3.3, there is a perfect matching $F_1$ in $G_1 - b_1 - w_1$ extending $M_1$, where $b_1, w_1$ are chosen from $V(H_1)$ so that, if $e_2$ is incident with $H$, then one of $b_1$ and $w_1$ corresponds to an end vertex of $e_2$, the other is chosen to be one of $u_1, v_1$ if neither of these is already covered by $e_1, e_2$ or $e_3$. Similarly, by Theorem 3.3, there is a 1-factor $F_2$ in $G_2 - b' - w' - b'' - w''$ extending $M_2$, where $\{b', b'', w', w''\} = V(H_2) - b_2 - w_2$, where $b_2$ and $w_2$ are the vertices in $H_2$ corresponding to $b_1$ and $w_1$ respectively in $H_1$. This yields a perfect matching $F = F_1 \cup F_2$ in $G$ extending $M$. $\Box$

**Corollary 3.7.** The graphs $B_n$, $n \geq 1$, are 3-extendable.

**Proof.** Graph $B_1$ is 3-extendable by Theorem 3.4. Now $B_n$ is just a hex join of $B_{n-1}$ and $B_1$, so the result follows by the preceding theorem and induction on $n$. $\Box$

Thus for every orientable surface other than the sphere, we have constructed a 3-extendable graph which minimally embeds on that surface. To produce infinitely many 3-extendable graphs which minimally embed on each orientable surface, simply replace the initial $C_4 \times C_4$ with $C_{4k} \times C_{4k}$. The proofs of minimal embeddedness and 3-extendability are very similar to the proofs for $C_4 \times C_4$.

Now we turn to the non-orientable surfaces. We begin with the Klein bottle. Let $L_{2m}$ denote the Möbius ladder, namely the graph consisting of a cycle $x_1x_2 \cdots x_{2m}$ of length $2m$ together with the $m$ diagonals $x_1x_{m+1}, x_2x_{m+2}, \ldots, x_{m-1}x_{2m}$. Each of the graphs $G_n = L_{4n+2} \times K_2$ quadrangulates the Klein bottle and hence is minimally embeddable there. Moreover, each is 3-extendable by Theorem 3.4.

Next, let us consider the cases when $\tilde{g} \geq 3$. Suppose that $k \geq 2$ and that $G_1 = C_{2k} \times C_{2k}$ is embedded on the torus. Let $R$ denote the hexagonal region consisting of two adjacent quadrilateral faces $swut$ and $uxyv$ (as used in the hex join; see Fig. 3.4). Insert a crosscap inside region $R$ and pass edge $uv$ through the crosscap and add two new edges $sy$ and $xt$ also passing through the crosscap. Then the resulting graph $G_2 = G_1 \cup \{sy, xt\}$ is a quadrangulation of the surface $N_3$ and hence this embedding of $G_2$ is minimal by Theorem 3.2. Moreover, if one takes $m$ disjoint hexagonal regions in $G_1$ and applies the above operation to each of them, one obtains a quadrangulation $G_m$ of the non-orientable surface $N_{m+2}$.

Graph $G_1 = C_{2k} \times C_{2k}$ is 3-extendable by Theorem 3.4. Moreover, since $G_m$ is obtained from $G_1$ by adding edges which preserve the bipartite property, $G_m$ is also 3-extendable by Theorem 3.3.

Thus for each non-orientable surface of genus $\geq 2$, we have exhibited infinitely many 3-extendable graphs which genus embed there.
4. 2-extendable orientable triangulations

Suppose that $G$ is a 5-connected triangulation with an even number of vertices embedded on an orientable surface $\Sigma$. In this section we will show that if the representativity of the embedding is large enough, then $G$ is 2-extendable.

We make use of the concept of planarizing cycles introduced by Thomassen [12] for triangulations and later extended to general embedded graphs by Yu [15]. We follow closely the treatment of Yu as well as the terminology contained therein.

Let $G$ be a connected graph which is (2-cell) embedded on the (orientable) surface $\Sigma$ of genus $g$ and let $C$ be a non-contractible cycle of $G$. Cut the graph $G$ and the surface $\Sigma$ by cutting along cycle $C$ so as to produce a new graph $G'$ embedded on a new surface $\Sigma'$. When performing this cut, we duplicate the cycle $C$ to produce a cycle $C'$ on the “left side” of the cut and a cycle $C''$ on the “right side” of the cut. If a sequence of such cuts along non-contractible cycles $C_1, \ldots, C_m$ results in a planar graph, we say that $\{C_1, \ldots, C_m\}$ is a set of planarizing cycles. Yu obtained the following beautiful result about planarizing cycles. We hasten to point out that we are interested only in orientable embeddings in this section and will state only the orientable version of Yu’s result, although his full result applies to the non-orientable case as well.

We now adopt some of Yu’s notation. In particular, suppose a graph $G$ is embedded in surface $\Sigma$. Let $d_{G,\Sigma}(x, y)$ denote $\min\{|\Gamma \cap V(G)| \mid \Gamma$ is a simple curve in $\Sigma$ from $x$ to $y$ and $\Gamma \cap G \subset V(G)\}$. For two disjoint vertex sets $C$ and $D$ in $G$, let $d_{G,\Sigma}(C, D) = \min\{d_{G,\Sigma}(x, y) \mid x \in C$ and $y \in D\}.$

**Theorem 4.1.** (See [15, Theorem 4.3].) Let $G$ be a connected graph 2-cell embedded on a surface $\Sigma$ having orientable genus $g > 0$ and suppose the representativity $\rho(G, \Sigma)$ of this embedding satisfies $\rho(G, \Sigma) \geq 8(d + 1)(2^g - 1)$. Then $G$ can be reduced to a graph $H$ embedded on a disjoint union $S$ of spheres by cutting along a set of planarizing cycles $\{C_1, \ldots, C_m\}$ (in this order) such that

(i) each $C_i$ is induced,

(ii) for every integer $k$ with $0 \leq k \leq d/2$ there are induced cycles $D_i^{1k}$ and $D_i^k$ each bounding a closed disc in $S$ containing $C_i'$ and $C_i$, respectively, such that for every vertex $z \in D_i^{1k}$ (and $z \in D_i^k$) there is a simple curve $P$ in $S$ from $z$ to $C_i'$ (and $C_i$) with length equal to $d_{HS}(z, C_i') = k + 1$ (and $d_{HS}(z, C_i) = k + 1$) and $P \cap D_i^{1k} = \{z\}$ (and $P \cap D_i^k = \{z\}$), and

(iii) all $D_i^k$ and $D_i^{1k}$ are disjoint, and for each integer $k$ with $0 \leq k \leq d/2$, the closed disc bounded by $D_i^{1k}$ containing $C_i'$ is disjoint from the closed disc bounded by $D_i^k$ containing $C_i$, and both do not contain the disc bounded by $D_j^{1k}$ or $D_j^k$ containing $C_j'$ or $C_j$ for any $j > i$.

Yu then uses this result to prove the following.

**Theorem 4.2.** (See [15, Theorem 5.1].) Let $G$ be a 5-connected triangulation of a surface $\Sigma$ with genus $g$ and suppose that $\rho(G, \Sigma) \geq 96(2^g - 1)$. Then $G$ has a Hamilton cycle.

We now proceed to extend the techniques of Theorem 4.1 to the problem of extending matchings. To begin with, if $G$ is a 5-connected planar graph of even order, the 2-extendability of $G$ was proved by Lou [5] and (independently) by the third author [10]. So henceforth, we shall as-
sume that (orientable) genus $g$ of $G$ is positive (and hence that the representativity (face-width) is defined).

We now present the main theorem of this section.

**Theorem 4.3.** Let $G$ be a 5-connected triangulation with an even number of vertices which is 2-cell embedded on a surface of orientable genus $g > 0$. If $\rho(G, \Sigma) \geq 288(2^d - 1)$ then

(a) $G$ has a Hamilton cycle containing any two independent edges, and
(b) $G$ is 2-extendable.

**Proof.** Suppose $e_1 = u_1v_1$ and $e_2 = u_2v_2$ are two independent edges in $G$. We follow the construction and notation of Theorem 4.1 so as to produce a set of induced “planarizing cycles” $C_1, \ldots, C_m$ such that when the surface is cut along each $C_i$ and the cycles $C_i$ duplicated to form pairs $\{C_i, C'_i\}$, one can find induced cycles $D_1^1, \ldots, D_1^{17}$ (and $D_1^{1}, \ldots, D_1^{17}$) satisfying the conditions stated in Theorems 4.1 and 4.2. (Note that here we use $d = 35$, rather than $d = 11$ as used by Yu.) Let $H_i, 1 \leq i \leq m$, be the “cylindrical” graphs defined in the proof of Theorem 4.2, where $H_i$ is bounded by the cycles $D_i^5$ and $D_i^5$.

**Claim 1.** The cycles $C_i, D_i^5, D_i^5$ can be chosen so that the vertices of the edges $e_1$ and $e_2$ belong to no $H_i, 1 \leq i \leq m$.

To see this, think of $D_1^{17}, D_1^{16}, \ldots, D_1^1, C_i, D_i^1, D_i^2, \ldots, D_i^{17}$ as a set of thirty-five rings arranged in this order left to right along a cylinder. The edges $e_1$ and $e_2$ intersect at most four of these cycles. So when these four cycles are deleted, there remain at least thirty-one cycles in at most three clusters of consecutive cycles. Hence at least one of these three clusters contains at least eleven consecutive cycles. Then relabel so that these eleven cycles are $D_i^5, \ldots, D_i^1, C_i, D_i^1, \ldots, D_i^5$. (A similar procedure to eliminate edges from the cylinders $H_i$ was used in [3].) This proves the claim.

Thus we have produced $m$ cylindrical graphs $H_1, \ldots, H_m$ and a planar graph (the components of which are sometimes called “annulus graphs”) with some number of “holes.” Moreover, by our claim above, the edges $e_1$ and $e_2$ both lie in the annulus graph in such a way that the vertices of these two edges do not lie on the boundary of any hole. Now insert one new vertex inside each hole of the annulus graph and join it to the boundary of the hole via the prescription described in Theorem 4.2 and denote the resulting planar graph by $H'$. We note that graph $H'$ need not be connected, but, as proved in Theorem 4.2, each component of $H'$ is, in fact, 4-connected. Let these components of $H'$ be $H'_1, \ldots, H'_k$.

Without loss of generality, suppose $e_1 \in E(H'_1)$.

To prove (a) we proceed as follows. If $e_1$ and $e_2$ both lie in the same component of $H'$, say without loss of generality that both lie in $E(H'_1)$, then by Corollary 2 of [11], there is a Hamilton cycle in $H'_1$ containing both $e_1$ and $e_2$ and Hamilton cycles in each of $H'_2, \ldots, H'_k$. We may then proceed as in the proof of Theorem 4.2 to obtain a Hamilton cycle $C_0$ in $G$ containing both $e_1$ and $e_2$. Corollary 2 of [11] may also be used when the two edges lie in different components of $H'$ to give Hamilton cycles in each of the two components containing these two edges as well as arbitrary Hamilton cycles in each of the remaining components of $H'$ and again as in the proof of Theorem 4.2 a Hamilton cycle $C_0$ for $G$ containing $e_1$ and $e_2$ may then be constructed.

Now we turn to the proof of part (b). First, note that the Hamilton cycle $C_0$ guaranteed in part (a) may, or may not, contain a perfect matching which in turn contains edges $e_1$ and $e_2$. In
the case when $C_0$ does not contain such a perfect matching, that is, when edges $e_1$ and $e_2$ lie at an even distance from each other on $C_0$, we must proceed somewhat differently.

Let $J'_1 = H'_1 - u_1 - v_1$. Then $J'_1$ is 2-connected. Let $X$ be the cycle bounding the face created by the deletion of $u_1$ and $v_1$ which we shall consider to be the “infinite” face. Choose an edge $f$ in $H'_1$ to be $e_2$, if $e_2$ lies in $J'_1$, otherwise choose $f$ to be any arbitrary edge in $J'_1$. Then by Sanders’ theorem [11], there is a Tutte path $P_1$ in $J'_1$ joining the endvertices of edge $f$ such that if $B$ is a bridge of $P_1$ in $J'_1$, then (i) if $B$ contains an edge of $X$, $B$ has at most two vertices of attachment in $J'_1$ and otherwise, (ii) $B$ has at most three vertices of attachment in $J'_1$.

In case (i), let the vertices of attachment be $w_1$ and $w_2$ (if there are two such; otherwise, let $w_1$ be the single vertex of attachment). Then the vertex set $\{w_1, w_2, u_1, v_1\}$ is a 4-cut in graph $G$ (or $\{w_1, u_1, v_1\}$ is a 3-cut in $G$), in either case contradicting the assumption that $G$ is 5-connected. So no bridge containing an edge of $X$ can exist.

But now if $B$ is a bridge containing at most three vertices of attachment, but no edge of $X$, then these vertices of attachment form a cut of size at most three in graph $H'_1$, contradicting the fact that $H'_1$ is 4-connected.

So $P_1$ has no bridges in $J'_1$ and hence, when $e_2 \in E(J'_1)$, $P_1 \cup \{e_2\}$ is a Hamilton cycle in $J'_1$ containing edge $e_2$. If $e_2 \notin E(J'_1)$, we may suppose, without loss of generality, that $e_2 \in E(H'_2)$. But then by Corollary 2 of [11], there is a Hamilton cycle in $H'_2$ containing $e_2$ and Hamilton cycles in each of $H'_3, \ldots, H'_k$.

Now once again as in the proof of Theorem 4.2, these Hamilton cycles found in $H'_1, H'_2, \ldots, H'_k$ can be modified and combined with certain paths in the $H'_j$’s, $1 \leq j \leq m$, so as to produce a Hamilton cycle $C_0$ in $G - u_1 - v_1$ which contains edge $e_2$. Since $|V(G - u_1 - v_1)|$ is even, we can form a perfect matching $F$ in $G - u_1 - v_1$ by taking edge $e_2$ and every second edge around $C_0$. But then $F \cup \{e_1\}$ is a perfect matching in $G$ containing $e_1$ and $e_2$. □

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References