ELSEVIER Journal of Computational and Applied Mathematics 57 (1995) 163–170

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Perturbation of the coefficients in the recurrence relation of a class of polynomials

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Received 23 October 1992; revised 12 April 1993

Abstract

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a system of polynomials satisfying the recurrence relation

$$P_{-1}(x)=0,\quad P_0(x)=1,\quad P_{n+1}(x)+h_nP_{n-1}(x)+c_nP_n(x)=xP_n(x),$$

where h_n, c_n are real sequences and $h_n>0$, $n=0,1,2,\ldots$. The co-recursive polynomials $\{P_n^*(x)\}_{n=0}^{\infty}$ satisfy the same recurrence relation except for $n=1$, where $P_1^*(x)=\gamma x-c_0-\beta$, $\gamma\neq 0$. It is well known that the problem of determining the zeros of $P_n(x)$ is equivalent to the problem of determining the eigenvalues of a generalized eigenvalue problem $Tf=\lambda Af$, where T and A are symmetric matrices. In this paper the problem of determining the zeros of the co-recursive polynomials is reduced to a perturbation problem of the operators T and A perturbed by perturbations of rank one. A function $\varphi(\lambda)=\varphi(\lambda,\lambda_1,\lambda_2,\ldots,\lambda_k)$ is found, $k=1,2,\ldots,n$, whose zeros are the zeros of $P_n^*(x)$, and λ_k are the zeros of the polynomial $P_n(x)$ of degree n, for $\gamma\neq 0$. This function unifies many results concerning interlacing between the zeros of $P_n(x)$ and $P_n^*(x)$ for $\gamma\neq 0$. Moreover we obtain from this function similar results in the unstudied case $\gamma=0$.

Keywords: Co-recursive polynomials; Perturbations of rank one

1. Introduction

Consider the polynomials $R_n(x)$ of degree n which are defined by

$$R_{n+1}(x)+R_{n-1}(x)=2x(1-\alpha\delta_{n,0})R_n(x),\quad n=0,1,\ldots,\tag{1.1}$$
$$R_{-1}(x)=0,\quad R_0(x)=1,$$

where $0\leqslant\alpha<1$, $\delta_{n,0}=1$ for $n=0$ and $\delta_{n,0}=0$ for $n\neq 0$. For $\alpha=\frac{1}{2}$ and $\alpha=0$ these polynomials are the Tchebichef polynomials of the first and second kind, respectively. More precisely, the Tchebichef polynomials $C_n(x)$ are obtained from (1.1) by setting $C_n(x)=R_n(x)$, $n=0,1,2,\ldots$, and

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SSDI 0377-0427(93)E0242-E

$a = \frac{1}{2}$ or $\alpha = 0$. The polynomials (1.1) may also be defined by

$$R_n(x) = 2\alpha \cos n\varphi + (1 - 2\alpha)\frac{\sin(n+1)\varphi}{\sin\varphi}, \quad \cos\varphi = x, \quad n = 0, 1, 2, \ldots . \tag{1.2}$$

This is a variational connection between the two kinds of Tchebichef polynomials and was for us the first motivation to study perturbations of the coefficients of the general form of orthogonal polynomials:

$$P_{n+1}(x) + h_n P_{n-1}(x) + c_n P_n(x) = x P_n(x), \tag{1.3}$$

$$P_{-1}(x) = 0, \quad P_0(x) = 1, \quad h_n > 0 \text{ and } c_n \text{ real sequence.}$$

Later it was brought to our attention that such perturbations were studied by other authors in the past [1,2,8,9] because of their applications in several problems of physics and harmonic analysis. In 1957 Chihara [2] studied the following perturbed polynomials,

$$P^*_{n+1}(x) + h_n P^*_{n-1}(x) + (c_n + \beta\delta_{n,0})P^*_n(x) = x P^*_n(x), \quad \beta \neq 0, \tag{1.4}$$

which he called co-recursive orthogonal polynomials. Among others he proved that the zeros x_j, $j = 1, 2, \ldots, n$, of $P_n(x)$ and x^*_j of $P^*_n(x)$ are mutually separated,

$$x_{j-1} < x^*_{j-1} < x_j < x^*_j, \quad j = 2, 3, \ldots, n, \quad \beta > 0, \tag{1.5}$$

with the roles of x_j and x^*_j reversed for $\beta < 0$.

Recently Slim [9] has studied the more general case

$$F^*_{n+1}(x) + h_n F^*_{n-1}(x) + (c_n + \beta\delta_{n,0})F^*_n(x) = x(1 + (\gamma - 1)\delta_{n,0})F^*_n(x), \tag{1.6}$$

$$F^*_{-1}(x) = 0, \quad F^*_0(x) = 1, \quad h_n > 0, \quad \beta \neq 0, \quad \gamma \neq 0. \tag{1.7}$$

He has proved that all the zeros of (1.6) are real and simple for $\gamma \neq 0$ and he found a series of sufficient conditions in order that relation (1.5) be satisfied.

There is a case where the polynomials $F_n(x)$ and $F^*_n(x)$ have the same zeros. This is the case where $\beta \neq 0$, $\gamma \neq 1$ and $\beta = (\gamma - 1)\lambda_k$, where λ_k is any zero of $F_n(x)$. We assume here that $\beta \neq (\gamma - 1)\lambda_k$, $k = 1, 2, \ldots, n$, and prove that the number λ is a zero of the polynomial $F^*_n(x)$ defined recursively by (1.6) and (1.7) if and only if $\lambda_k \neq \lambda$ and λ is a zero of the function

$$\varphi(\lambda) = 1 - (\lambda(\gamma - 1) - \beta)\sum_{k=1}^{n}\frac{\sigma_k^2}{\lambda_k - \lambda}, \tag{1.8}$$

where σ_k are real numbers such that $\sum_{k=1}^{n}\sigma_k^2 = 1$. This result unifies many results concerning interlacing between the zeros of $F_n(x)$ and $F^*_n(x)$ for $\gamma \neq 0$. Moreover we obtain from (1.8) similar results for the case $\gamma = 0$, which has not been studied until now. In that case the degree of $F^*_n(x)$ is unpredictable. We find conditions in order that $F^*_n(x)$ is of degree $n - 1$ and has real and simple zeros which lie between the n zeros of $F_n(x)$.

2. Reduction of the problem of zeros of $F_n^*(x)$ to the problem of the zeros of (1.8)

The relation (1.6), by setting $\alpha_n = \sqrt{h_{n+1}}$ and $F_n^*(x) = \sqrt{h_1 h_1 \cdots h_n}\, Q_n(x)$, $F_0^*(x) = Q_0(x)$, can be reduced to

$$\alpha_n Q_{n+1}(x) + \alpha_{n-1} Q_{n-1}(x) + (c_n + \beta\delta_{n,0}) Q_n(x) = x(1 + (\gamma - 1)\delta_{n,0}) Q_n(x), \tag{2.1}$$

where the polynomials $Q_n(x)$ and $F_n^*(x)$ have the same zeros.

According to an abstract setting [6, 7] λ is a zero of the polynomial $Q_n(x)$ if and only if it is an eigenvalue of the problem

$$(AV^* + VA + C + \beta P_0)x = \lambda(1 + (\gamma - 1)P_0)x \tag{2.1α}$$

in the space H_n.

In (2.1) H_n is a finite-dimensional Hilbert space with the orthonormal basis e_k, $k = 0, 1, \ldots, n-1$, A and C are the diagonal operators $Ae_k = \alpha_k e_k$, $Ce_k = c_k e_k$, $k = 0, 1, \ldots, n-1$, V is the truncated shift ($Ve_k = e_{k+1}$, $k = 0, 1, \ldots, n-2$, $Ve_{n-1} = 0$), V^* the adjoint of $V(V^* e_k = e_{k-1}$, $V^* e_0 = 0)$ and P_0 is the orthogonal projection of the subspace spanned by the element e_0, i.e. $P_0 x = (x, e_0)e_0$, $x \in H_n$. For completeness we give below the proof of the above statement.

Let λ be an eigenvalue of the problem (2.1α). Since $\alpha_k \neq 0$, $k = 0, 1, 2, \ldots, n-1$, we have $(x, e_0) \neq 0$, because otherwise $(x, e_1) = (x, e_2) = \cdots = (x, e_{n-1}) = 0$, i.e. $x = 0$. So we normalize x by setting $(x, e_0) = 1$. Then from (2.1α) we find $(x, e_1) = Q_1(\lambda)$, $(x, e_2) = Q_2(\lambda)$, $\ldots$, $(x, e_{n-1}) = Q_{n-1}(\lambda)$. Since $Ve_{n-1} = 0$, scalar product multiplication of (2.1α) by e_{n-1} leads to

$$\alpha_{n-2} Q_{n-2}(\lambda) + c_{n-1} Q_{n-1}(\lambda) = \lambda Q_{n-1}(\lambda),$$

which together with (2.1α) gives $Q_n(\lambda) = 0$.

Conversely if $Q_n(\lambda) = 0$, then it is easy to see that the vector $x = \sum_{k=0}^{n-1} Q_k(\lambda) e_k$, $Q_0(\lambda) = 1$, satisfies (2.1α). Note that $x \neq 0$ because $Q_0(\lambda) = 1$.

We write the problem (2.1α) in the form

$$(T_0 + \beta P_0)x = \lambda(1 + (\gamma - 1)P_0)x$$

or

$$T_0 x - \lambda x = [\lambda(\gamma - 1) - \beta] P_0 x, \tag{2.2}$$

where

$$T_0 = AV^* + VA + C.$$

In (2.2) T_0 is a self-adjoint operator, whose eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \tag{2.3}$$

are the zeros of the unperturbed polynomial $P_n(x)$ of degree n. For $\gamma = 1$ the eigenvalue problem (2.2) is the problem

$$(T_0 + \beta P_0)x = \lambda x, \tag{2.4}$$

where βP_0 is a perturbation of rank one. In the case n tends to infinity the operator V is the unilateral shift operator on an abstract separable Hilbert space H with the orthonormal basis e_n, $n = 0, 1, 2, \ldots$. For more details of the truncated shift V see [5]. It is known that if T_0 is self-adjoint (not necessarily in a finite-dimensional Hilbert space) with a discrete spectrum then between every distinct pair of eigenvalues $(\lambda_i, \lambda_{i+1})$ of T_0 there is precisely one eigenvalue of $T_0 + \beta P_0$ in one of the intervals $[\lambda_i, \lambda_{i+1})$ or $(\lambda_i, \lambda_{i+1}]$ or $(\lambda_i, \lambda_{i+1})$ [4]. Here the possible case for the operators T_0 and $T_0 + \beta P_0$ to have a common eigenvalue is excluded because of a peculiarity of the perturbation P_0. This peculiarity is expressed in the following lemma.

Lemma 2.1. *Let $\beta \neq \lambda(\gamma - 1)$, and assume that x satisfies* (2.2) *with some real $\lambda \neq 0$. Then λ is a regular point of the operator T_0.*

Proof. Let λ be an eigenvalue of T_0, i.e.

$$T_0 x_0 = \lambda x_0, \quad x_0 \neq 0. \tag{2.5}$$

Then scalar product multiplication of (2.2) by x_0 gives

$$(\lambda(\gamma - 1) - \beta)(P_0 x, x_0) = 0$$

or

$$(P_0 x, x_0) = (x, e_0)(e_0, x_0) = 0.$$

This is impossible because $(x, e_0) \neq 0$ and $(x_0, e_0) \neq 0$. □

Now with a slightly modified version of the method used in [4] we prove the following theorem.

Theorem 2.2. *Let $\beta \neq \lambda(\gamma - 1)$. Then $\lambda \neq 0$ satisfies* (2.2) *with some $x \neq 0$, if and only if λ is a zero of the function*

$$\varphi(\lambda) = 1 - [\lambda(\gamma - 1) - \beta]((T_0 - \lambda I)^{-1} e_0, e_0), \tag{2.6}$$

from which (1.8) *follows.*

Proof. Let λ be a zero of (2.6), i.e.

$$1 - [\lambda(\gamma - 1) - \beta]((T_0 - \lambda I)^{-1} e_0, e_0) = 0$$

or

$$(e_0, e_0) - [\lambda(\gamma - 1) - \beta]((T_0 - \lambda I)^{-1} e_0, e_0) = 0$$

or

$$(e_0, e_0 - [\lambda(\gamma - 1) - \beta](T_0 - \lambda I)^{-1} e_0) = 0.$$

The last means that the element

$$y = e_0 - [\lambda(\gamma - 1) - \beta](T_0 - \lambda I)^{-1} e_0 \tag{2.7}$$

is orthogonal to e_0, i.e. $(y, e_0) = 0$. Thus from (2.7)

$$-e_0 + y = [\lambda(\gamma - 1) - \beta](T_0 - \lambda I)^{-1} P_0(-e_0 + y) \tag{2.8}$$

because $P_0(-e_0 + y) = -e_0$. From (2.8) we see that $x = -e_0 + y$ is different from the zero element and satisfies (2.2). Conversely let $\lambda \neq 0$ satisfy (2.2). Then because of Lemma 2.1, λ is a regular point of T_0, i.e. $(T_0 - \lambda I)^{-1}$ exists as an operator on H_n and we have

$$x = [\lambda(\gamma - 1) - \beta](T_0 - \lambda I)^{-1} P_0 x \tag{2.9a}$$

or

$$x = [\lambda(\gamma - 1) - \beta](T_0 - \lambda I)^{-1}(x, e_0) e_0 \tag{2.9b}$$

or

$$x = (x, e_0)[\lambda(\gamma - 1) - \beta](T_0 - \lambda I)^{-1} e_0. \tag{2.9c}$$

From (2.2) $(x, e_0) \neq 0$ because otherwise $(x, e_1) = 0$, $(x, e_2) = 0$, ... , $(x, e_{n-1}) = 0$ and $x = 0$. Thus from (2.9) we see that

$$(x, e_0) = (x, e_0)[\lambda(\gamma - 1) - \beta]((T_0 - \lambda I)^{-1} e_0, e_0)$$

and λ is a zero of the function (2.6). Expanding the element $(T_0 - \lambda I)^{-1} e_0$ in terms of the complete orthonormal system y_k, $k = 1, 2, \ldots, n$, of T_0 and the eigenvalues λ_k, i.e.

$$\begin{aligned}(T_0 - \lambda I)^{-1} e_0 &= \sum_{k=1}^{n} ((T_0 - \lambda I)^{-1} e_0, y_k) y_k = \sum_{k=1}^{n} (e_0, (T_0 - \lambda I)^{-1} y_k) y_k \\ &= \sum_{k=1}^{n} \left(e_0, \frac{1}{\lambda_k - \lambda} y_k \right) y_k = \sum_{k=1}^{n} \frac{1}{\lambda_k - \lambda} (e_0, y_k) y_k,\end{aligned}$$

we find easily that (2.6) can be taken in the form (1.8), where $\sigma_k^2 = |(y_k, e_0)|^2$ and

$$1 = \|e_0\|^2 = \sum_{k=1}^{n} |(e_0, y_k)|^2 = \sum_{k=1}^{n} \sigma_k^2. \qquad \square$$

Remark 2.3. Adding and subtracting the term $(\gamma - 1) \sum_{k=1}^{n} \lambda_k \sigma_k^2$ in (1.8) and using the relation $\sum_{k=1}^{n} \sigma_k^2 = 1$, the relation (1.8) for $\gamma \neq 1$ can be written in the form

$$\varphi(\lambda) = \gamma + (\gamma - 1) \sum_{k=1}^{n} \frac{\sigma_k^2 (\beta/(\gamma - 1) - \lambda_k)}{\lambda_k - \lambda}. \tag{2.10}$$

3. Interlacing of zeros

Theorem 3.1. *For $\beta \neq 0$ and $\gamma = 1$ the zeros λ_k^* of $F_n^*(x)$ are real and simple and interlaced with the zeros λ_k of $F_n(x)$ as*

$$\lambda_1 < \lambda_1^* < \lambda_2 < \cdots < \lambda_n < \lambda_n^* \tag{3.1}$$

for $\beta > 0$ and

$$\lambda_1^* < \lambda_1 < \lambda_2^* < \cdots < \lambda_n^* < \lambda_n \tag{3.2}$$

for $\beta < 0$.

Proof. For $\beta > 0$ and $\gamma = 1$ we observe from the function

$$\varphi(\lambda) = 1 + \beta \sum_{k=1}^{n} \frac{\sigma_k^2}{\lambda_k - \lambda}, \tag{3.3}$$

that in the interval $(\lambda_n, +\infty)$ there exists at least one zero of $\varphi(\lambda)$. In fact we have $\varphi(+\infty) = \lim_{\lambda \to +\infty} \varphi(\lambda) = 1$ and $\lim_{\lambda \to \lambda_n - 0} \varphi(\lambda) = -\infty$ because $\lambda_n - \lambda < 0$. So by the intermediate theorem there exists a zero of $\varphi(\lambda)$ in $(\lambda_n, +\infty)$. Also from (3.3) by the intermediate theorem it follows that between two successive zeros of $F_n(x)$, λ_i and λ_{i+1}, there exists at least one zero of $F_n^*(x)$. Thus we prove the existence of n real and different zeros of $F_n^*(x)$, $\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*$, such that relation (3.1) holds. For $\beta < 0$ we also have the existence of n different zeros of $F_n^*(x)$. The $n-1$ zeros lie between the zeros λ_k of the polynomial $F_n(x)$. The first zero λ_1^* lies in the interval $(-\infty, \lambda_1)$ because $\varphi(-\infty) = \lim_{\lambda \to -\infty} \varphi(\lambda) = 1$ and $\lim_{\lambda \to \lambda_1 - 0} \varphi(\lambda) = -\infty$. Thus we obtain relation (3.2). □

Remark 3.2. Theorem 3.1 has been proved by Chihara in [2] by a different method.

Theorem 3.3. *Let $\beta = 0, \gamma \neq 0$ and let two successive parts in the sum* (2.10) *have the same sign. Then between two successive zeros of $F_n(x)$ there exists at least one zero of $F_n^*(x)$.*

Proof. This follows from (2.10) by the intermediate theorem. □

Theorem 3.4. *Let $\beta = 0, \gamma > 1$ and let all zeros λ_k of $F_n(x)$ be positive. Then the zeros of $F_n(x)$ and the zeros λ_k^* of $F_n^*(x)$ are interlaced as*

$$\lambda_1^* < \lambda_1 < \lambda_2^* < \cdots < \lambda_n^* < \lambda_n. \tag{3.4}$$

Proof. The existence of $n-1$ zeros follows from Theorem 3.3, and the existence of λ_1^*, in the interval $(-\infty, \lambda_1)$, follows from (2.10) because for $\gamma > 1$, $\varphi(-\infty) = \lim_{\lambda \to -\infty} \varphi(\lambda) = \gamma > 0$ and $\lim_{\lambda \to \lambda_1 - 0} \varphi(\lambda) = \infty$. □

Theorem 3.5. *Let one of the three sets of conditions,*
(a) $\beta/(\gamma-1) < \lambda_1, \gamma > 1$,
(b) $\beta/(\gamma-1) < \lambda_1, \gamma < 0$,
(c) $\beta/(\gamma-1) > \lambda_n, 0 < \gamma < 1$,
be satisfied. Then the zeros λ_k^* of $F_n^*(x)$ and the zeros λ_k of $F_n(x)$ are interlaced as

$$\lambda_1^* < \lambda_1 < \lambda_2^* < \cdots < \lambda_n^* < \lambda_n.$$

Proof. The proof of the existence of $n-1$ different zeros follows from Theorem 3.3 because $(\beta/(\gamma-1)) - \lambda_k$ have the same sign. The existence of λ_1^* in the interval $(-\infty, \lambda_1)$ follows because
(a) for $\gamma > 1$, $\varphi(-\infty) = \gamma > 0$ and $\lim_{\lambda \to \lambda_1 - 0} \varphi(\lambda) = -\infty$,
(b) for $\gamma < 0$, $\varphi(-\infty) = \gamma < 0$ and $\lim_{\lambda \to \lambda_1 - 0} \varphi(\lambda) = +\infty$ and
(c) for $0 < \gamma < 1$, $\varphi(-\infty) = \gamma > 0$ and $\lim_{\lambda \to \lambda_n - 0} \varphi(\lambda) = -\infty$,
since $\gamma - 1 < 0, (\beta/(\gamma-1)) - \lambda_k > 0\ \forall k = 1, 2, \ldots, n$ and $\lambda_1 - \lambda > 0$. □

In the same way we can easily prove the following.

Theorem 3.6. *Let one of the three conditions,*
(a) $\beta/(\gamma-1) < \lambda_1, 0 < \gamma < 1$,
(b) $\beta/(\gamma-1) > \lambda_n, \gamma < 0$,
(c) $\beta/(\gamma-1) > \lambda_n, \gamma > 1$,
be satisfied. Then the zeros λ_k of $F_n(x)$ and the zeros λ_k^* of $F_n^*(x)$ are interlaced as

$$\lambda_1 < \lambda_1^* < \lambda_2 < \cdots < \lambda_n < \lambda_n^*.$$

Remark 3.7. In [9] it was proved that the conclusions of Theorems 3.5 and 3.6 hold true if in the conditions (a), (b), (c) the numbers λ_1 and λ_n are replaced by ζ_1 and n_1, where $[\zeta_1, n_1]$ is the true interval of orthogonality of $F_n(x)$. Moreover in [9] it was assumed that $\zeta_1 > -\infty$ and $n_1 < +\infty$, which restrict the class of the perturbed polynomials $F_n(x)$.

Remark 3.8. During the conference Prof. Galliano Valent informed us that the results of Slim [9] were also proved by Allaway [1] in his Ph.D. thesis in 1972, which was never published.

4. The special case $\gamma = 0$

This case has not been studied previously by other authors because in that case one of the terms $1 + (\gamma - 1)\delta_{n,0}$ of relation (1.6) vanishes, and the degree of the polynomial $F_n^*(x)$ defined by the recurrence relation (1.6) is unpredictable. However from function (2.10), which in this case takes the form

$$\varphi(\lambda) = \sum_{k=1}^{n} \frac{(\beta + \lambda_k)\sigma_k^2}{\lambda_k - \lambda}, \tag{4.1}$$

we obtain the following result.

Theorem 4.1. *Suppose that* $\gamma = 0$ *and* $\beta > -\lambda_k$, $k = 1, 2, \dots, n$, *or* $\beta < -\lambda_k$, $k = 1, 2, \dots, n$. *Then the degree of the polynomial* $F_n^*(x)$ *is* $n - 1$ *and has real and simple zeros, which lie between the* n *zeros of* $F_n(x)$.

Proof. From (4.1), using the intermediate theorem, we establish the existence of $n - 1$ different real zeros of the polynomial $F_n^*(x)$. On the other hand, from the recurrence relation we see easily that the degree of $F_n^*(x)$ cannot be greater than $n - 1$. $\square$

Acknowledgements

The authors wish to express their thanks to the referees for their helpful remarks and suggestions.

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