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A Combinatorial View of Andrews' Proof of the L–M–W Conjectures

A. M. GARSIA AND J. REMMEL

0. INTRODUCTION

In a remarkable paper [1] G. Andrews gives an analytic proof of two important [4] and difficult conjectures of Lusztig, Macdonald and Wall. The proof is essentially elementary and not difficult to follow. However, some of the steps appear rather surprising and unmotivated. Indeed, there is a certain 'out of the hat' aspect to them which can only be explained as pure analytical wizardry.

The L–M–W conjectures can be stated as follows. Let $\chi_n = \chi_n(a, b; q)$ be the sequence of polynomials in q satisfying the recursions

$$\begin{aligned} &\text{(a)} \quad \chi_{2n+1} = \chi_{2n} + q^{2n+1}\chi_{2n-1} \\ &\text{(b)} \quad \chi_{2n+2} = \chi_{2n+1} + (q^{n+1} + q^{2n+2})(\chi_{2n-1} + \chi_{2n}) \end{aligned} \tag{0.1}$$

and the initial conditions

$$\chi_{-1} = a, \qquad \chi_0 = b. \tag{0.2}$$

Set

$$\chi(a, b; q) = \lim_{n\to\infty} \chi_n(a, b; q).$$

Then the following identities hold true

$$\begin{aligned} &\text{(a)} \quad \chi(1, 1; q) = \sum_{k\geq 0} q^{k^2} \Big/ \prod_{n\geq 1} (1 - q^n), \\ &\text{(b)} \quad \chi(0, 1; q) = \sum_{k\geq 0} q^{k(k+1)} \Big/ \prod_{n\geq 1} (1 - q^n). \end{aligned} \tag{0.3}$$

In this paper we give a 'lattice path' setting to the recursions in (0.1) which reveals that some of the puzzling steps referred to above have a rather natural combinatorial explanation. Indeed, our methods allow us to put together a completely combinatorial proof of the identities in (0.3).

Q-series identities are most often interpreted and proved combinatorially in the setting of the theory of partitions. The present paper should help bring to light the fact that for some q-series the lattice path setting may be more natural and easier to work with.

1. OUTLINE OF ANDREWS' PROOF

Before proceeding with our proof, it may be good to go over the essential points of Andrews' proof. There are essentially four separate steps which may be described as follows.

Step 1. A further variable z is introduced and the given recursions in I.1 are replaced by the new recursions

$$\begin{aligned} &\text{(a)} \quad \chi_{2n+1} = \chi_{2n} + z^2q^{2n+1}\chi_{2n-1}, \\ &\text{(b)} \quad \chi_{2n+2} = \chi_{2n+1} + (zq^{n+1} + z^2q^{2n+2})(\chi_{2n-1} + \chi_{2n}). \end{aligned} \tag{1.1}$$

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Define $\chi_n(a, b; q, z)$ to be the solution of (1.1) which satisfies the initial conditions in (0.2). This given it is not difficult to show that the limit

$$\chi(a, b; q, z) = \lim_{n\to\infty} \chi_n(a, b; q, z)$$

is well defined as a formal power series in the two variables z, q. We can also see that the recursions in (1.1) reduce to trivialities as $n\to\infty$, and thus they deliver no direct information concerning the limiting series $\chi(a, b; q, z)$.

Step 2. To get around this difficulty Andrews obtains a new pair of recursions involving the polynomials $\chi_n(a, b; q, z)$.

$$\begin{aligned}
&\text{(a)}\quad \chi_{2n+2}(0,1;q,z) = \chi_{2n}(1,1;q,zq) + (zq+z^2q^2)\chi_{2n}(0,1;q,zq),\\
&\text{(b)}\quad \chi_{2n+1}(1,1;q,z) = \chi_{2n+1}(0,1;q,z) + z^2q\chi_{2n-1}(1,1;q,zq)\\
&\qquad\qquad + (zq+z^2q^2)\chi_{2n-1}(0,1;q,zq)
\end{aligned}\tag{1.2}$$

Crudely speaking Andrews obtains these recursions from those in (1.1) by expressing the polynomials as determinants and noticing that last columns expansions of these determinants yields (1.1) and first row expansion yields (1.2). The reader is referred to [1] for details on this aspect of Andrews' proof.

In contrast with (1.1) the recursions in (1.2) are not devoid of content at infinity. Indeed, passing to the limit and setting

$$P(z) = \chi(0, 1; q, z), \qquad Q(z) = \chi(1, 1; z, q),$$

we obtain the 'functional equations'

$$\begin{aligned}
&\text{(a)}\quad P(z) = Q(zq) + (zq+z^2q^2)P(zq),\\
&\text{(b)}\quad Q(z) = P(z) + z^2qQ(zq) + (zq+z^2q^2)P(zq).
\end{aligned}\tag{1.3}$$

Step 3. Andrews 'guesses' the power series expansions

$$\begin{aligned}
&\text{(a)}\quad P(z) = \sum_{n,m\geq 0} q^{n^2+m^2-nm} z^{n+m} q^n/(q)_n(q)_m,\\
&\text{(b)}\quad Q(z) = \sum_{n,m\geq 0} q^{n^2+m^2-nm} z^{n+m}/(q)_n(q)_m,
\end{aligned}\tag{1.4}$$

where $(q)_n = (1-q)(1-q)\cdots(1-q^n)$. The guess is then confirmed by verifying that the formal series in (1.4) are indeed solutions of the system in (1.3).

Step 4. Finally, the identities in (0.2) are then established by setting $z=1$ in (1.4) and manipulating the resulting expressions by means of the standard q-series identities.

Our program here is to show that Step 1 and Step 2 have very natural combinatorial explanations. This is carried out in Sections 2 and 3. In Section 4, we replace Step 3 of Andrews' proof by giving a bijective proof of a pair of identities that result from (1.4) by means of the 'roofing' operation introduced in [3]. Our bijection of Section 4 is a rather remarkable path factorization which essentially results by taking the relatively simple proof of the identities of (1.4) which uses the roofing methods of [3] and replacing each analytic step of that proof by its bijective counterpart. Finally, in Section 5 we show that this factorization leads to a purely combinatorial proof of the identities in (0.3a, b).

One fact we shall have to leave unexplained here is Andrews' guess that $P(z)$ and $Q(z)$ have such simple expressions as those given in (1.4). While we can prove these identities combinatorially, we have been unable to come up with a model in which one would be naturally led to such identities. Thus it appears that the deepest step in Andrews' proof is precisely his venturing this particular guess.

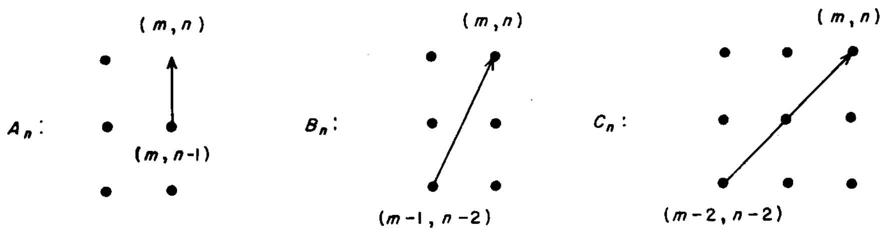
2. A Lattice Path Setting for Recursions

In this section we present a procedure for giving a lattice path interpretation to families of polynomials defined by recursions. This procedure essentially stems from the formal language methods of the Schutzenberger school and most particularly from the recent work of Flajolet [2] and Viennot [3]. We shall illustrate this procedure in a very special case and in the next section we shall apply it to the L-M-W recursions.

Let us consider the family $\mathbb{F}$ of lattice paths whose steps are of the following types

$$A_n: \quad (m, n-1) \Rightarrow (m, n),$$
$$B_n: \quad (m-1, n-2) \Rightarrow (m, n),$$
$$C_n: \quad (m-2, n-2) \Rightarrow (m, n).$$

Pictorially these steps may be represented by the following 'arrows'



Let us associate weights a_n, b_n, c_n to the arrows of types A_n, B_n, C_n respectively. The weight $w(P)$ of a path P is then defined as the product of the weights of the arrows of P. For instance for the path P given in Figure 1 we have $w(P) = a_2 b_4 c_6$.

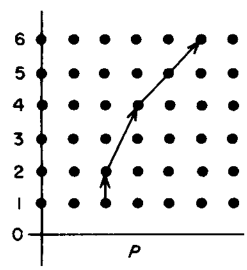


Figure 1

Finally, the weight of a family of paths is taken to be the sum of the weights of its individual members. This given let $\chi_n(P_0)$ denote the weight of the family $\mathbb{F}_n(P_0)$ consisting of all paths in $\mathbb{F}$ starting at the point P_0 and ending at some point of height n. In symbols

$$\chi_n(P_0) = \sum_{\mathscr{P} \in \mathbb{F}_n(P_0)} w(\mathscr{P}).$$

Our construction immediately yields that this sequence of polynomials satisfies the recursions

$$\chi_n(P_0) = \chi_{n-1}(P_0) a_n + \chi_{n-2}(P_0) b_n + \chi_{n-2}(P_0) c_n. \tag{2.1}$$

Indeed, this is simply obtained by grouping together the paths of $\mathbb{F}_n(P_0)$ according to the type of edge they end up with.

We shall refer to the equations in (2.1) as the *fan-in* equations, since they are produced by the different arrows that may fan into a given lattice point. In Figure 2 we have depicted the fan-in of the lattice point (m, n). Accordingly, there is one term in (2.1) corresponding to each arrow in this figure.

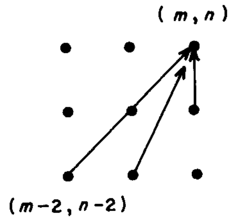


FIGURE 2

However, the same paths can be regrouped according to the arrows they start with. This leads to what we shall refer to as the *fan-out* equations. For instance, taking P to be the origin we get

$$\chi_n(0,0) = a_1\chi_n(0,1) + b_2\chi_n(1,2) + c_2\chi_n(1,2). \tag{2.2}$$

Each term in this equation corresponds to one arrow in the fan-out of the point $(0, 0)$, which we have depicted in Figure 3.

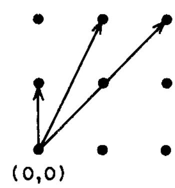


FIGURE 3

We can add another feature to our procedure by observing that, under the present circumstances, the paths starting at any point whatever are simply translates of paths starting at the origin. Thus all our equations can be rewritten in terms of this special family of paths. This can be easily carried out by means of a 'shift' operator. More precisely, let S be the operator which acts linearly on polynomials and replaces in each monomial the variables a_n, b_n, c_n respectively by a_{n+1}, b_{n+1}, c_{n+1}. Thus, for instance we have

$$S(a_3b_5c_7 + c_4a_5b_7) = a_4b_6c_8 + c_5a_6b_8.$$

This given, the fan-out equations in (2.2) can be rewritten in the form

$$\chi_n(0,0) = a_1S\chi_{n-1}(0,0) + b_2S^2\chi_{n-2}(0,0) + c_2S^2\chi_{n-2}(0,0). \tag{2.3}$$

To see this, we need only observe, for instance, that every path from $(0, 1)$ to a point of height n can be obtained by translating upwards a path from $(0, 0)$ to a point of height $n-1$. Thus every term in $\chi_n(0, 1)$ has a corresponding term in $\chi_{n-1}(0, 0)$. However, by the very manner which our weights are put together, we see that each monomial in $\chi_{n-1}(0, 0)$ has to be 'shifted' by S to yield the corresponding monomial in $\chi_n(0, 1)$. This gives that

$$\chi_n(0,1) = S\chi_{n-1}(0,0).$$

The remaining terms in Equation (2.3) can be justified in a similar way.

There are some special choices of weights for which the shift operator takes a very simple form. To see how this comes about let us look again at our example. For simplicity let us suppose that there are no arrows of type C_n. In this case the paths starting at the origin can be identified with partitions with parts differing by at least two and smallest part greater than or equal to two.

For instance, the path of Figure 4 can be identified with the partition 2, 6, 9, 11. More generally, the parts of the partition corresponding to a given path are simply the columns of lattice points lying below the tips of the arrows (see Figure 4).

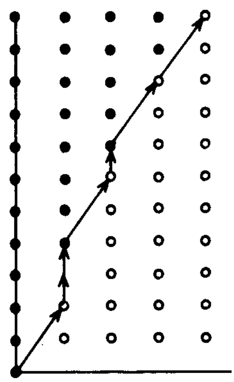


FIGURE 4

Suppose now that we specialize our weights by setting

$$a_n = 1, \qquad b_n = zq^n, \qquad c_n = 0. \tag{2.4}$$

Then, a given path $\mathscr{P}$ will have weight $z^k q^m$ where k can be viewed as the *width* of the path (the length of the projection of the path on the x-axis) and m represents the *area* (the number of lattice points) below the path. In terms of the partition corresponding to $\mathscr{P}$, k is the number of parts and m is the sum of the parts.

Let us denote then by $\chi_n(P; q, z)$ the polynomial resulting from $\chi_n(P)$ upon this choice of weights. This given, the recursions in (2.1) reduce to

$$\chi_n(P; q, z) = \chi_{n-1}(P; q, z) + \chi_{n-2}(P; q, z)zq^n. \tag{2.5}$$

On the other hand, the fan-out equations can now be written in the form

$$\chi_n(0, 0; q, z) = \chi_{n-1}(0, 0; q, zq) + zq^2\chi_{n-2}(0, 0; q, zq^2). \tag{2.6}$$

Indeed, the operator S in this situation simply consists of replacing z by zq. The easiest way to see this is to note that in lifting a path upwards one unit the number of lattice points below the path increases by the width of the path. Thus a path of weight $z^k q_n$, after the lift, will have weight

$$z^k q^{n+k} = (zq)^k q^n.$$

Note that if we allow our paths to have arrows of type C_n and set

$$c_n = z^2 q^{2n},$$

then the fan-in and fan-out equations, respectively, become

$$\chi_n(P; q, z) = \chi_{n-1}(P; q, z) + (zq^n + z^2q^{2n})\chi_{n-2}(P; q, z), \tag{2.7}$$

$$\chi_n(0, 0; q, z) = \chi_{n-1}(0, 0; q, zq) + (zq^2 + z^2q^2)\chi_{n-2}(0, 0; q, zq^2). \tag{2.8}$$

More generally, we can easily see that if every arrow of width k and tip of height n is given weight $(zq^{an})^k$ (for some fixed constant a) then the action of the shift operator on one of our polynomials $P(z)$ is simply of the form

$$SP(z) = P(zq^a). \tag{2.9}$$

We terminate the section by observing that we can also give a lattice path interpretation to the formal series

$$\chi(P; q, z) = \lim_{n\to\infty} \chi_n(P; q, z). \tag{2.10}$$

Clearly, $\chi(P; q, z)$ should be the enumerator of infinite paths out of P. However we can see from (2.7) that a path which has an infinite number of arrows of types B_n or C_n, can leave no contribution to the limit formal series. Thus we should interpret $(P; q, z)$ as the enumerator of paths out of P which terminate with an infinite string of vertical arrows.

Q-series are most often interpreted combinatorially as enumerators of classes of partitions by number of parts and by sum of the parts. Clearly we can do the same by enumerating classes of lattice paths by width and area. It develops that this model has wider applicability since there are q-series whose terms can be so represented by means of lattice paths which at the same time cannot be unambiguously represented by means of partitions.

3. The L–M–W Paths

We are now in a position to construct the lattice paths underlying the L–M–W recursions. We shall reverse the process illustrated in the previous section and determine the fan-in from the recursions.

Clearly Equations (0.1) indicate that points of odd and even heights must have fan-in consisting of two and five arrows respectively. Furthermore, the example of Section 2, most particularly Equation (2.5), suggests that the first term in both Equations (0.1a, b) should correspond to a vertical arrow of weight one. However, most revealing are the four terms

$$q^{n+1}\chi_{2n}, \qquad q^{n+1}\chi_{2n-1}, \qquad q^{2n+2}\chi_{2n}, \qquad q^{2n+2}\chi_{2n-1} \tag{3.1}$$

of Equation (0.1b).

Assume, to be consistent with our previous example, that each arrow of width k and tip of height n is assigned weight $(q^{na})^k$, where a is to be fixed once and for all. This given, the simplest possible way to justify simultaneously the four factors in (3.1) is to take $a = \frac{1}{2}$ with $k = 1$ for the first two factors and $k = 2$ for the remaining two. Putting all this information together we deduce that our paths should have the five types of arrows with tip of even height shown in Figure 5.

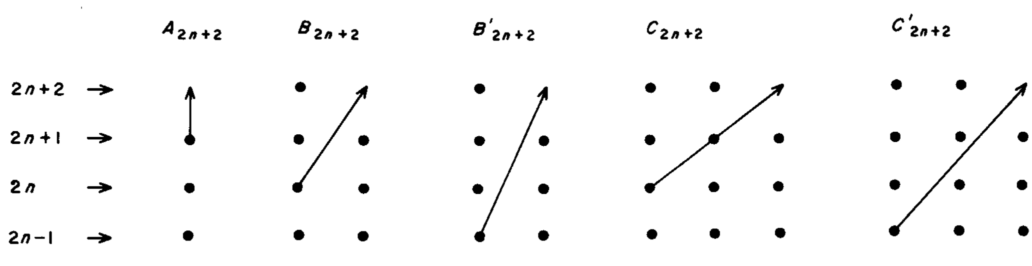


Figure 5

To be consistent with our choice of weights, the second term in (0.1b) must then correspond to an arrow of width 2. Thus the arrows with tip of odd height are of the following two types

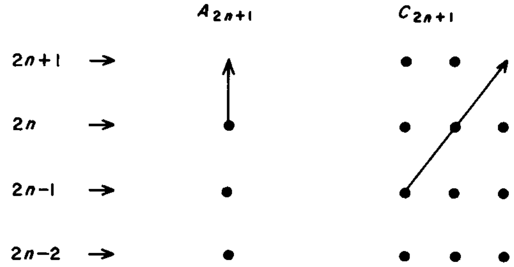


FIGURE 6

Let us again denote by $\chi_n(P)$ the polynomial giving the weight of the family consisting of all paths starting at P and ending at a point of height n. Assume for a moment that the arrows of types A_n, B'_n, B_n, C_n, C'_n are given generic weights a_n, b_n, b'_n, c_n, c'_n. The fan-in equations are then

$$\begin{aligned}&\text{(a)}\quad \chi_{2n+2}(P)=\chi_{2n+1}(P)a_{2n+2}+\chi_{2n}(P)b_{2n+2}+\chi_{2n-1}(P)b'_{2n+2}\\&\qquad\qquad +\chi_{2n}(P)c_{2n+2}+\chi_{2n-1}(P)c'_{2n+2},\\&\text{(b)}\quad \chi_{2n+1}(P)=\chi_{2n}(P)a_{2n+1}+\chi_{2n-1}(P)c_{2n+1}.\end{aligned}\tag{3.2}$$

Let us now derive the fan-out equations. To this end observe first that under the present situation a path which starts at a point of even height is a translate of a path starting at (0, 0), while a path which starts at a point of odd height may be thought of as a translate of a path starting at (0, −1).

Note then that the fan-out at (0, 0) consists of three arrows of types A, B and C (see Figure 7), while the fan-out at (0, −1) consists of four arrows of types A, B', C' and C, respectively (see Figure 8). Using this information we derive the fan-out equations

$$\begin{aligned}&\text{(a)}\quad \chi_n(0,0)=a_1S^2\chi_{n-2}(0,-1)+b_2S^2\chi_{n-2}(0,0)+c_2S^2\chi_{n-2}(0,0),\\&\text{(b)}\quad \chi_n(0,-1)=a_0\chi_n(0,0)+c_1S^2\chi_{n-2}(0,-1)+b'_2S^2\chi_{n-2}(0,0)+c'_2S^2\chi_{n-2}(0,0).\end{aligned}\tag{3.3}$$

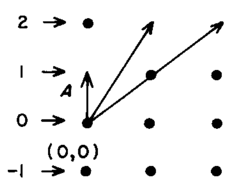


FIGURE 7

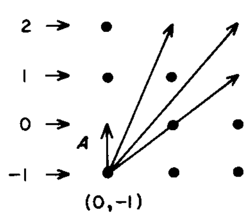


FIGURE 8

Choosing $a_n=1$, $b_n=b'_n=q^{n/2}$, $c_n=c'_n=q^n$ reduces the equations in (3.2) to those in (0.1). However these weights are not much help in the fan-out equations. On the other hand, if we take

$$a_n=1,\qquad b_n=b'_n=zq^{n/2},\qquad c_n=c'_n=z^2q^n,\tag{3.4}$$

we can express S by means of (2.9).

For convenience let us denote by $P_n(z)$ and $Q_n(z)$ the polynomials resulting from $\chi_n(0,0)$ and $\chi_n(0,-1)$, respectively, upon choosing the weights according to (3.4). Then, using (2.9) with $a=\frac{1}{2}$, we can rewrite the equations in (3.3) in the explicit form

$$\begin{aligned}&\text{(a)}\quad P_n(z)=Q_{n-2}(zq)+zqP_{n-2}(zq)+z^2q^2P_{n-2}(zq),\\ &\text{(b)}\quad Q_n(z)=P_n(z)+z^2qQ_{n-2}(zq)+zqP_{n-2}(zq)+z^2q^2P_{n-2}(zq).\end{aligned}\tag{3.5}$$

The polynomials $P_n(z)$ and $Q_n(z)$ must of course satisfy the recursions in (1.1), since the latter are precisely what the equations in (3.2) reduce to when the weights are as in (3.4). Moreover, it is easy to see that we do have

$$P_{-1}(z)=0,\qquad P_0(z)=1,\qquad Q_{-1}(z)=1,\qquad Q_0(z)=1,$$

and therefore we must conclude that

$$P_n(z)=\chi_n(0,1;q,z),\qquad Q_n(z)=\chi_n(1,1;q,z).$$

We have thus shown that the equations in (1.2) are none other than fan-out equations. Moreover, we see that the additional variable z is simply a device for obtaining an algebraically simple form for the fan-out equations. Finally, since we must have

$$\lim_{n\to\infty}P_n(z)=\lim_{n\to\infty}\chi_n(0,1;q,z)=P(z),\qquad \lim_{n\to\infty}Q_n(z)=\lim_{n\to\infty}\chi_n(1,1;q,z)=Q(z)$$

we also obtain that the formal series $P(z)$ and $Q(z)$ are none other than the weights of the families of paths which start respectively at $(0,0)$ and $(0,-1)$ and end with an infinite string of vertical arrows.

4. Path Factorization

In this section, we shall obtain a factorization of a L-M-W path into a pair of simpler paths. In the next section we shall show that this factorization leads to a purely combinatorial proof of the L-M-W identities (0.3a, b).

Step 3 of Andrews' proof consists in verifying the expansion of (1.4). The formal manipulation can be simplified somewhat by means of the roofing operator introduced in [3]. We shall see that our path factorization can essentially be read off from the roofing method proof of (1.4) and provide a bijective proof of the identities in (1.4). Since, the procedure developed here can be used to construct bijective proofs in a variety of situations we shall spend some time detailing how one is led to our path factorization.

We recall that a q-series is a formal power series

$$A(z)=\sum_{n\geq 0}A_n(q)z^n$$

whose coefficients $A_n(q)$ are rational functions of q. In [3] we have defined the "roof" of a q-series $A(z)$ by the formula

$$\hat{A}(z)=\sum_{n\geq 0}A_n(q)q^{-\binom{n}{2}}z^n.$$

Similarly the 'unroof' is defined by setting

$$\check{A}(z)=\sum_{n\geq 0}A_n(q)q^{\binom{n}{2}}z^n.$$

The usefulness of the roofing operator is due to the fact that it behaves like a 'Fourier transform' with respect to 'tangled' products. More precisely, if $A(z)$ and $B(z)$ are q-series then

$$\Big(\sum_{n\geq 0} A_n(q)z^nB(zq^n)\Big)^{\wedge} = \hat{A}(z)\hat{B}(z). \tag{4.1}$$

Similarly we have

$$\Big(\sum_{n\geq 0} A_n(q)z^nB(z/q^n)\Big)^{\vee} = \check{A}(z)\check{B}(z). \tag{4.2}$$

In particular we also have

$$(z^nB(z))^{\wedge} = z^nq^{-\binom{n}{2}}\hat{B}(z/q^n) \quad \text{and} \quad (z^nB(z))^{\vee} = z^nq^{\binom{n}{2}}\check{B}(zq^n). \tag{4.3}$$

Now suppose for the moment we let $\mathcal{P}(z)$ and $\mathcal{Q}(z)$ denote the right-hand side of the identities in (1.4). That is,

$$\begin{aligned} &\text{(a)} \quad \mathcal{P}(z) = \sum_{n,m\geq 0} q^{n^2+m^2-nm}z^{n+m}q^m/(q)_n(q)_m \\ &\text{(b)} \quad \mathcal{Q}(z) = \sum_{n,m} q^{n^2+m^2-nm}z^{n+m}/(q)_n(q)_m. \end{aligned} \tag{4.4}$$

Note now that the exponent of q in (4.4a, b) can be written in the form

$$n^2+m^2-nm = 3\binom{n}{2}+3\binom{m}{2}+n+m-\binom{n+m}{2}.$$

Thus unroofing both expressions in (4.4) we obtain

$$\begin{aligned} &\text{(a)} \quad \check{\mathcal{P}}(z) = \sum_{n\geq 0} q^{3\binom{n}{2}+n}z^n/(q)_n \sum_{m\geq 0} q^{3\binom{m}{2}+2m}z^m/(q)_m = \theta(z)\theta(zq) \\ &\text{(b)} \quad \check{\mathcal{Q}}(z) = \sum_{n\geq 0} q^{3\binom{n}{2}+n}z^n/(q)_n \sum_{m\geq 0} q^{3\binom{m}{2}+m}z^m/(q)_m = \theta(z)\theta(z) \end{aligned} \tag{4.5}$$

where for convenience we have set

$$\theta(z) = \sum_{n\geq 0} q^{3\binom{n}{2}+n}z^n/(q)_n.$$

Note then that

$$\hat{\hat{\theta}}(z) = \sum_{n\geq 0} q^{\binom{n}{2}+n}z^n/(q)_n = \prod_{m\geq 1}(1+zq^m).$$

This immediately gives

$$\hat{\hat{\theta}}(z) = (1+zq)\hat{\hat{\theta}}(zq) = \hat{\hat{\theta}}(zq)+zq\hat{\hat{\theta}}(zq).$$

Unroofing twice, we obtain

$$\theta(z) = \theta(zq)+zq\theta(zq^3). \tag{4.6}$$

The methods developed in Section 2 allow us to easily read off from (4.6) a lattice path interpretation for $\theta(z)$. Indeed, let us consider the family of paths whose fan out at any point (m, n) consists of the two types of arrows shown in Figure 9. Using the

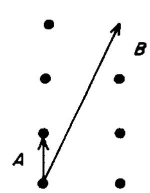


FIGURE 9

notation of Section 2 and generic weight a_n, b_n we obtain the fan out equation

$$\chi_n(0,0)=a_1 S\chi_{n-1}(0,0)+b_3 S^3\chi_{n-3}(0,0). \tag{4.7}$$

Comparing (4.7) with (4.6) suggests that we should make the following choice of weights:

$$a_n=1, \qquad b_n=zq^{n-2}.$$

Indeed, let $\mathscr{G}$ denote the set of paths which use only the steps pictured in Figure 9 and which end in an infinite number of vertical arrows. We shall call such paths 'Gauss' paths. Given $p\in\mathscr{G}$, let $\omega(p)$ denote the weight of p given by the choices $a_n=1$ and $b_n=zq^{n-2}$. We can now give a simple bijection to prove that

$$\theta(z)=\sum_{n\geq 0} q^{3\binom{n}{2}+n}z^n/(q)_n=\sum_{p\in\mathscr{G}}\omega(p). \tag{4.8}$$

The bijection is best illustrated by an example. The coefficient of z^n in the L.H.S. of 4.8 is clearly just the sum of the weights of Gauss paths of width n. The weight of such a path P is then $q^{l(P)}$ where $l(P)$ equals the number of lattice points which lie directly under and at least two levels below the head of an arrow of type B in P. For the path pictured in Figure 10, the lattice points counted by $l(P)$ are circled.

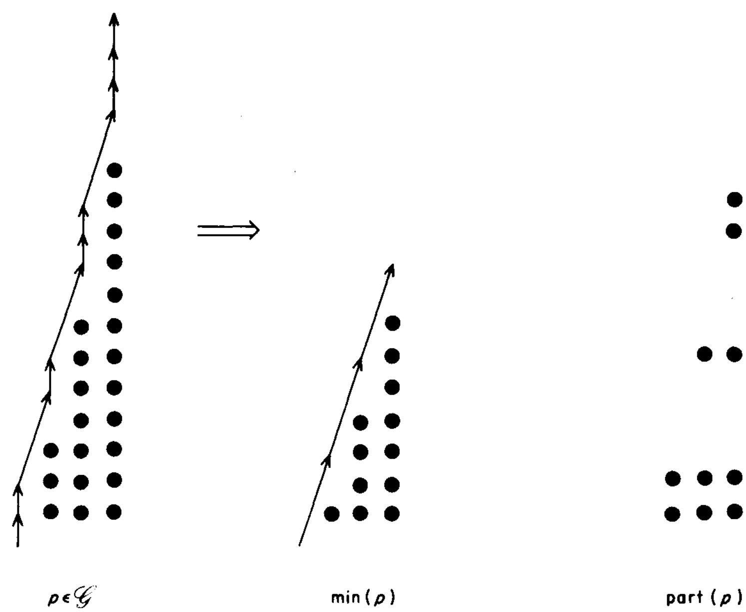


FIGURE 10

It is then easy to see that as in Figure 10, we can decompose the lattice points counted by $l(p)$ into two parts by separating out those lattice points which are to the right of a head of some vertical arrow in P. Those lattice points which are separated out are easily seen to correspond to a partition whose parts are no larger than n and hence such points account for the factor $1/(q)_n$ in the R.H.S. of (4.8). The remaining lattice points correspond to the path of minimum weight among those paths of width n and are counted by $1+4+\cdots+(3n+1)=3\binom{n}{2}+n$.

Let $\mathscr{LMW}(0,0)$ and $\mathscr{LMW}(0,-1)$ denote the collection of all L-M-W paths out of $(0,0)$ and $(0,-1)$, respectively. Also let $S(\mathscr{G})$ denote the set of all 'shifted' Gauss paths, i.e. those Gauss paths that start with a vertical arrow. In light of 4.5 and our interpretations of $P(z)$, $Q(z)$, and $\theta(z)$, it follows that we can prove (1.4a, b) if we can establish the following.

THEOREM 4.1. *There are bijections Γ_1: $\mathscr{LMW}(0,0) \to \mathscr{G} \times S(\mathscr{G})$ and Γ_2: $\mathscr{LMW}(0,-1) \to \mathscr{G} \times \mathscr{G}$ such that if $\Gamma_i(P) = (LG(P), RG(P))$, then*

$$\omega(P) = \omega(LG(P))\omega(RG(P))q^{-\binom{n(P)+m(P)}{2}}, \tag{4.9}$$

where $n(P)$ and $m(P)$ denote the widths of $LG(P)$ and $RG(P)$, respectively.

PROOF. As pointed out in the introduction, the bijections Γ_1 and Γ_2 result from taking the roofing proofs of (1.4a, b) and replacing analytic steps by bijective ones. To see how this comes about let us give the roofing proofs of (1.4a, b) which result from repeated use of the recursion in (4.6). Indeed starting from (4.5a), we get

$$\begin{aligned}\check{\mathscr{P}}(z) &= \theta(z)\theta(zq) = (\theta(zq) + zq\theta(zq^3))\theta(zq)\\ &= \check{\mathscr{Q}}(zq) + zq\theta(zq^3)(\theta(zq^2) + zq^2\theta(zq^4))\\ &= \check{\mathscr{Q}}(zq) + zq\check{\mathscr{P}}(zq^2) + z^2q^3\check{\mathscr{P}}(zq^3).\end{aligned} \tag{4.10}$$

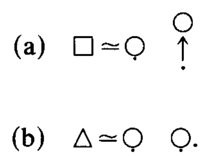
Similarly, from (4.5b) we get

$$\begin{aligned}\check{\mathscr{Q}}(z) &= \theta(z)\theta(z) = \theta(z)(\theta(zq) + zq\theta(zq^3))\\ &= \check{\mathscr{P}}(z) + zq(\theta(zq) + zq\theta(zq^3))\theta(zq^3)\\ &= \check{\mathscr{P}}(z) + zq(\theta(zq^2) + zq^2\theta(zq^4))\theta(zq^3) + z^2q^2\check{\mathscr{Q}}(zq^3)\\ &= \check{\mathscr{P}}(z) + zq\check{\mathscr{P}}(zq^2) + z^2q^3\check{\mathscr{P}}(zq^3) + z^2q^2\check{\mathscr{Q}}(zq^3).\end{aligned}$$

Roofing and using 4.3 gives that $\mathscr{P}(z)$ and $\mathscr{Q}(z)$ satisfy (1.3a) and (1.3b) from which it easily follows that $\mathscr{P}(z) = P(z)$ and $\mathscr{Q}(z) = Q(z)$.

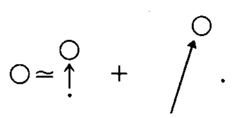
Next let us denote by the symbols $\square$ and $\triangle$ the formal sums of all paths in $\mathscr{LMW}(0,0)$ and $\mathscr{LMW}(0,-1)$, respectively, and let the symbol $\bigcirc$ denote the formal sum of all the Gauss paths out of $(0,0)$.

This given, the equations in (4.5a, b) can be rewritten symbolically in the form:



(4.12)

Similarly, 4.5 can be expressed in the form



(4.13)

Using this symbolism we can translate each of the 'analytic' steps we carried out in (4.10) and (4.11) into 'combinatorial' steps. We are thus led to the symbolic identities shown in Figures 11 and 12. In this figure, for convenience, we have placed under each combinatorial construct the corresponding analytic expression.

The desired correspondence between L-M-W arrows and pairs of Gauss arrows can now be obtained by matching corresponding terms in Equations (a) and (b) of Figure

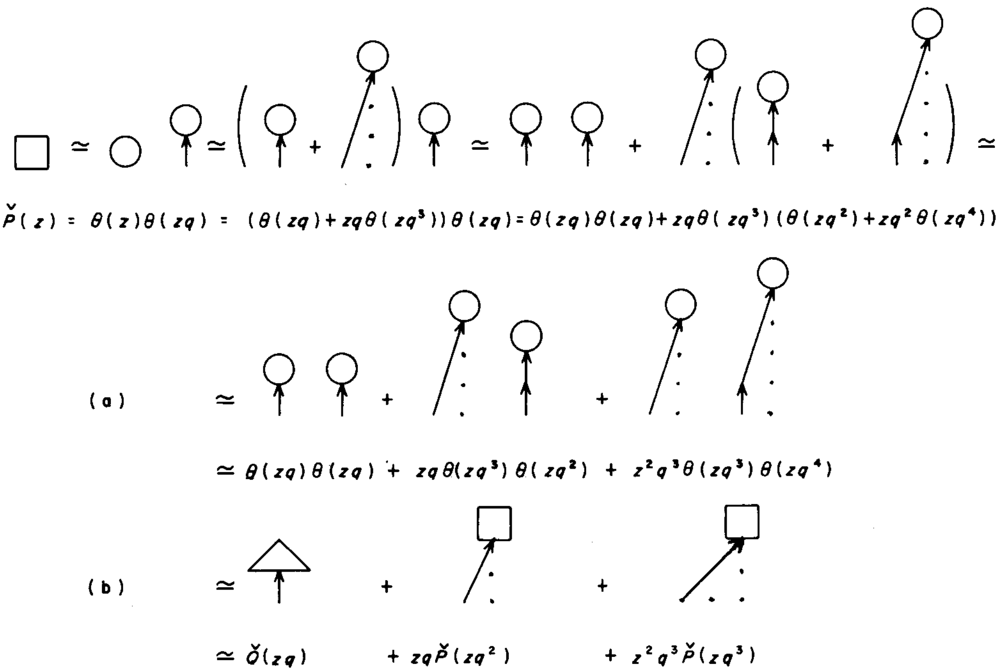


FIGURE 11

11 and of Figure 12, respectively. However, some care should be exerted in expressing the non-commutativity which is present in the combinatorial setting. In particular the term corresponding to $\theta(zq^3)\theta(za^2)$ should be distinguished from the term corresponding to $\theta(zq^2)\theta(zq^3)$.

Taking all these facts into account we are thus led to the conversion table of Figure 13. For the use of this table, we imagine that the image pair $(LP(G), RG(P))$ of a given path $\mathbb{P}$ is to be constructed by means of a two state device and that the table gives the transition mechanism of the device.

More precisely, the input of the device consists of a state and an L-M-W arrow, the output is a pair consisting of one (or more) G arrows and a new state as indicated by the table. There are two cases according as the given L-M-W arrow starts at a point of even or odd height. In each case the table gives the image arrows for each arrow in the fan-out.

Given any L-M-W path P we start with the device in state $\boxed{+}$. The arrows of P are then processed one by one by the device and the left and right arrows thus produced are successively attached upon $LG(P)$ and $RG(P)$, respectively.

This construction is best understood by working on an example. For instance, if we process the L-M-W path of Figure 14 then we should obtain the pair of Gauss paths shown in Figure 15. Note that $RG(P)$ is a path which starts at $(0, 1)$. We should expect this whenever P starts at $(0, 0)$. Note further that the weight of $LG(P)$ is z^3q^{25}, where 25 represents the number of circled points below $LG(P)$ in the figure. We do not count the first two rows of lattice points here due to the fact that $b_n = zq^{n-2}$. Similarly, the weight of $RG(P)$ is z^4q^{27}. To calculate the weight of P itself we proceed as follows. First of all, we circle the lattice points 'below' P. That is for each arrow tip we circle one or

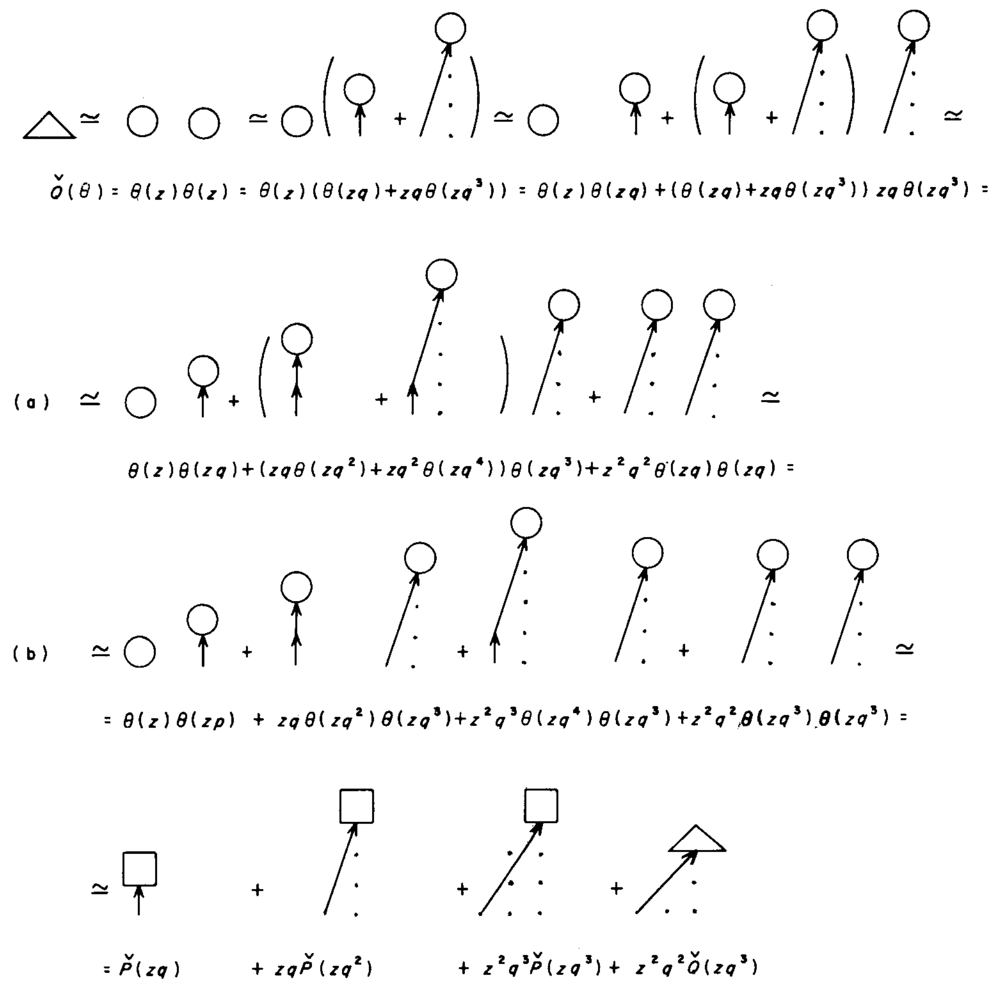


FIGURE 12

two columns of points up to the level of the tip. This given, we count the number of circled points (which is 62 in this case) and divide by 2 (since here $a=\frac{1}{2}$). We thus obtain that the weight of P is z^7q^{31}, (7 is the width). Thus $z^7q^{31}=z^3q^{25}z^4q^{27}q^{-\binom{7}{2}}$ as required by (4.9).

It is now routine to verify in case by case manner that the path factorization algorithm which results from our conversion table is completely reversible and has the required weight preserving properties in (4.9). Thus this factorization algorithm gives us the bijections Γ_1 and Γ_2 required in Theorem 4.1.

5. Combinatorial Proof of the M-L-W Identities

Our factorization yields an elementary proof of the identities in (0.3). We give our arguments here in the case of (0.3a), the other identity can be treated in an entirely analogous manner.

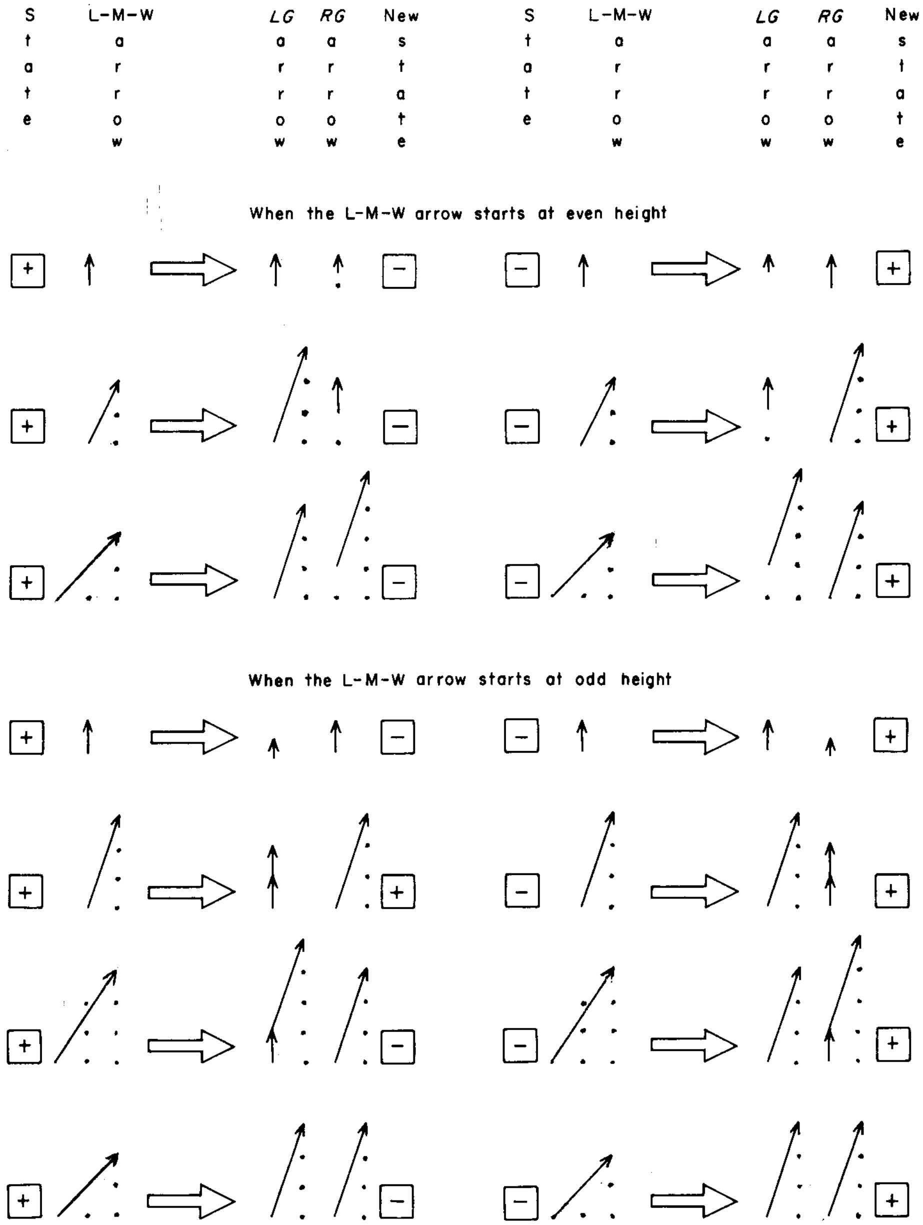


FIGURE 13. The conversion table.

We start by giving a combinatorial setting to the right hand side of (0.3a). To this end let $\mathbb{D}$ denote the family of all Ferrers diagrams. For a given $D \in \mathbb{D}$ we let the weight D be

$$w(D) = q^{\# D} \tag{5.1}$$

where $\# D$ denotes the number of squares in D.

It is well known and it is not difficult to see that the infinite product on the right hand side of (0.3a) is the weight of $\mathbb{D}$. That is

$$\sum_{D \in \mathbb{D}} w(D) = \prod_{m \geq 1} \frac{1}{1-q^m}.$$

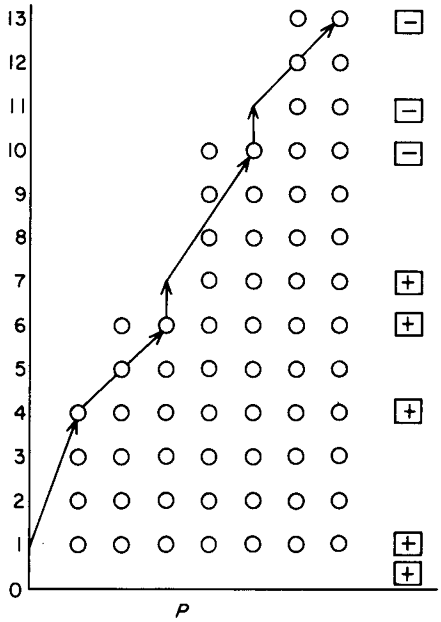


FIGURE 14

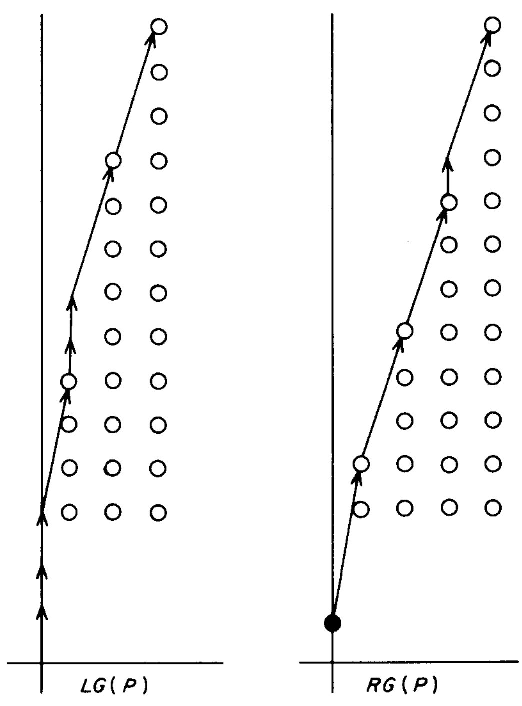


FIGURE 15

To take care of the other factor in (0.3a) we shall resort to a family $\mathbb{S}$ of signed, square Ferrers diagrams. More precisely, we let

$$\mathbb{S}=\{S_0, \pm S_1, \pm S_2, \ldots, \pm S_k, \ldots\}$$

where S_k for $k>0$ denotes a square diagram of side k, and S_0 denotes the empty diagram. Using the weight defined in (5.1) we may express the sum factor in (0.3a) in the form

$$\sum_{k=-\infty}^{+\infty} q^{k^2}=\sum_{S\in\mathbb{S}} w(S).$$

Thus the product

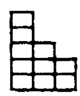
$$\sum_{k=-\infty}^{+\infty} q^{k^2} \prod_{m\geq 1} \frac{1}{1-q^m}$$

may be interpreted as the weight of the family $\mathbb{S}\times\mathbb{D}$ consisting of all pairs $(\pm S_k, D)$. This is about the simplest possible combinatorial interpretation we can come up with for the right hand side of (0.3a).

On the other hand, we know that the left hand side of (0.3a) is the weight of a path P here it is taken to be q to the power the number of lattice points 'below' P.

This given, to prove (0.3a) combinatorially we need only give a weight preserving bijection between $\mathbb{S}\times\mathbb{D}$ and the family of L–M–W paths out of $(-1,0)$.

Now it develops that our factorization yields a very simple recipe for such a bijection. Recall from our bijective proof of (4.8) that each Gauss path P of width n can be decomposed into a pair (min(P), part(P)) consisting of the path min(p) which has the minimum weight $=3\binom{n}{2}+n=n^2+\binom{n}{2}$ for paths of length n and a partition on Ferrers diagram with rows no longer than n. For example, the Ferrers diagram for the path in Figure 10 is



Conversely, by reversing the process, given n and a Ferrers diagram D with rows of length less than or equal to n we can uniquely reconstruct a Gauss path G such that

$$w(G)=q^{n^2+\binom{n}{2}}q^{\# D}. \tag{5.2}$$

In particular this implies that the weight of the family of Gauss paths of width n is

$$\frac{q^{n^2+\binom{n}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)}. \tag{5.3}$$

Let now P be a given L–M–W path out of $(-1,0)$. Construct, the Ferrers diagrams D_L and D_R corresponding to $LG(P)$ and $RG(P)$ and let $m(P)$, $n(P)$ be the widths of these two Gauss paths. Take a rectangular Ferrers diagram with $m(P)$ columns and $n(P)$ rows and place upon it the diagram D_L and on the right of it the transpose of the diagram D_R (see Figure 17). Since D_L has no more than $m(P)$ columns and the transpose of D_R has no more than $n(P)$ rows, the resulting figure is again a Ferrers diagram, let us denote it by $D(P)$. Set

$$k=|m(P)-n(P)|.$$

This given, if $k>0$, we shall take as image of P the pair

$$(+S_k, D(P)) \text{ or } (-S_k, D(P))$$

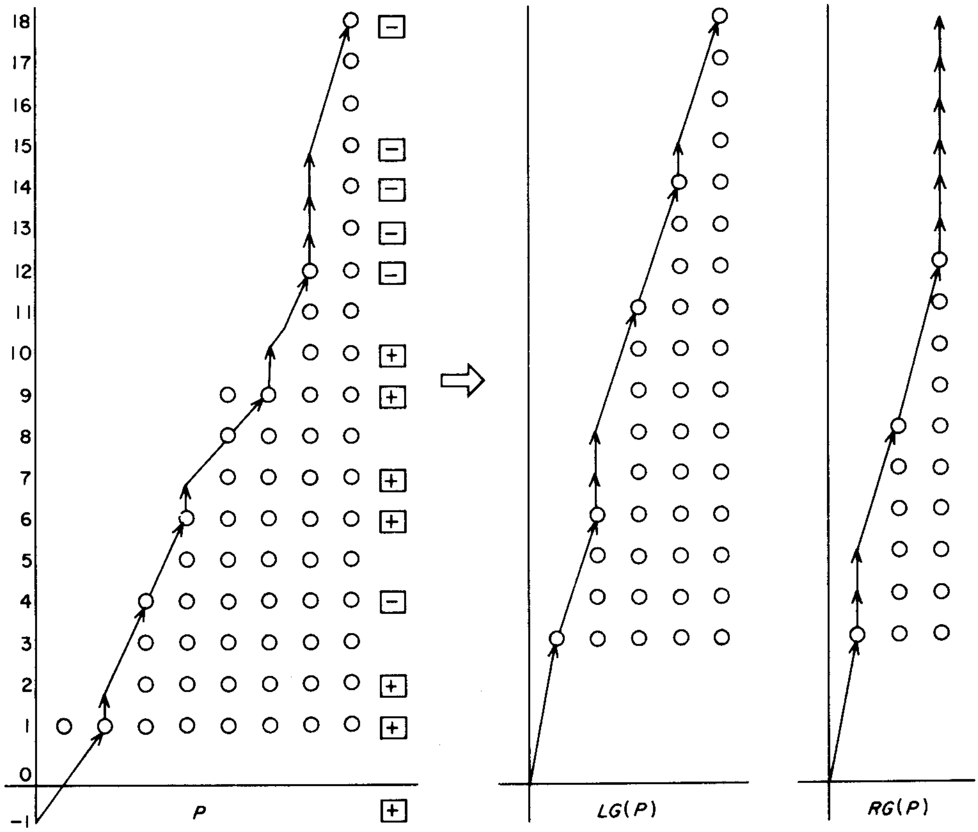


FIGURE 16

according as

$$m(p)-n(P)>0 \quad \text{or} \quad <0.$$

If $k=0$ we let the image of P be

$$(S_0, D(P)).$$

This construction is clearly invertible. Thus to complete our argument we need only verify that it is weight preserving. However, this follows immediately from (4.13) and (5.2). Indeed, using (5.2) we get

$$w(LG(P))=q^{m^2(P)+\binom{m(P)}{2}+\#D_L},$$

$$w(RG(P))=q^{n^2(P)+\binom{n(P)}{2}+\#D_R},$$

substituting in 4.13, with a little arithmetic, we derive

$$w(P)=q^{(m(P)-n(P))^2+n(P)m(P)+\#D_L+\#D_R}.$$

Our construction of $D(P)$ implies that

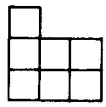
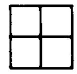
$$m(P)n(P)+\#D_L+\#D_R=\#D(P).$$

Thus

$$w(P)=q^{k^2}q^{\#D(P)},$$

and this is precisely the weight of the image pair $(\pm S_k, D(P))$.

We terminate by illustrating the whole construction with an example in Figure 16. Here we have shaded rows of squares yielding D_L and D_R, thus

$D_L =$  , $D_R =$ 

Since here $m(P)=5$, $n(P)=3$, $k=2$, we finally obtain the image pair of Figure 17.

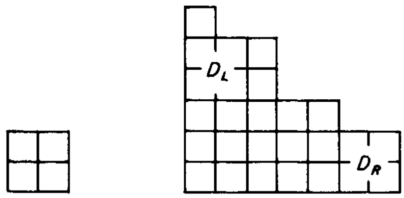


FIGURE 17

Clearly the methods we have introduced in this paper can be used in a variety of similar situations in the theory of partitions. A most tempting undertaking in this respect is to obtain a combinatorial interpretation of the proof of the Rogers–Ramanujan identities that was given in [3]. Indeed, in [3] the roofing operator yields a factorization of the sum side of the Rogers–Ramanujan identities that is quite similar to that we have obtained here for $P(z)$ and $Q(z)$. This strongly suggests that there may be a 'path factorization' result underlying also these remarkable identities.

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A. M. GARSIA AND J. REMMEL
Department of Mathematics, University of California at San Diego,
La Jolla, California 92093, U.S.A.