On distorted probabilities and $m$-separable fuzzy measures

Yasuo Narukawa \textsuperscript{a,b,*}, Vicenç Torra \textsuperscript{c}

\textsuperscript{a} Toho Gakuen, 3-1-10 Naka, Kunitachi, Tokyo 186-0004, Japan
\textsuperscript{b} Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology, 4259 Nagatuta, Midori-ku, Yokohama 226-8502, Japan
\textsuperscript{c} IIIA-CSIC, Institut d’Investigació en Intel·ligència Artificial, Campus de Bellaterra, 08193 Bellaterra, Catalonia, Spain

ARTICLE INFO

Article history:
Available online 22 January 2011

Keywords:
Fuzzy measures
Distorted probabilities
$m$-Dimensional distorted probabilities
$m$-Symmetric fuzzy measures

ABSTRACT

Fuzzy measures are used in conjunction with fuzzy integrals for aggregation. Their role in the aggregation is to permit the user to express the importance of the information sources (either criteria or experts). Due to the fact that fuzzy measures are set functions, the definition of such measures requires the definition of $2^n$ parameters, where $n$ is the number of information sources. To make the definition easier, several families of fuzzy measures have been defined in the literature.

In this paper $m$-separable fuzzy measures are introduced. We present some results on this type of measures and we relate them to some of the previous existing ones. We study generating functions for $m$-separable fuzzy measures and some properties related to these generating functions.

1. Introduction

Fuzzy integrals (e.g. Choquet and Sugeno) can be classified as a type of aggregation operators [37] to be used with numerical information. Some of them, as the Sugeno integral, are also useful when data is qualitative. The use of integrals in applications [18, 17, 14] requires not only the input data but also the definition of a fuzzy measure. This fuzzy measure is used to express our prior knowledge on the information source that supply the data.

Informally, fuzzy measures can be said to have the same role played by weighting vectors in the weighted mean. That is, they are used to measure the importance, or reliability of the sources.

The main difficulty for defining the measures is that they are set functions on the set of information sources. Due to this, when the number of sources is large, the number of parameters required by these measures becomes very large.

Different families of fuzzy measures have been defined to reduce the number of parameters. The first one was the Sugeno $\lambda$-measure [31]. Others have been proposed more recently as the $k$-order additive fuzzy measure [11, 12], the $m$-symmetric [19, 20], and $m$-dimensional decomposable fuzzy measures [24].

In this paper we establish some connections between some families of fuzzy measures. We focus on $m$-separable fuzzy measures, and introduce $m$-sequence separable fuzzy measures. Then, we establish some connections between these measures and other existing in the literature as, e.g., distorted probabilities [24, 27, 8, 9], the $m$-symmetric fuzzy measures [19, 20], and hierarchically decomposable ones [35].

The structure of the paper is as follows: In Section 2, we review some concepts that are needed later on. In Section 3, we focus on symmetric fuzzy measures. In Section 4, we define $m$-separable fuzzy measures and present some results that establish connections among these measures and other existing ones. The generating functions for $m$-separable fuzzy measures are also studied in detail. The paper finishes with some conclusions.

* Corresponding author at: Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology, 4259 Nagatuta, Midori-ku, Yokohama 226-8502, Japan.
E-mail addresses: narukawa@d4.dion.ne.jp, nrkwy@ybb.ne.jp (Y. Narukawa), vtorra@iiia.csic.es (V. Torra).

0888-613X/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.ijar.2011.01.004
2. Preliminaries

This section reviews some previous results in the literature that are needed in the rest of the paper. We start by defining fuzzy measures, and some of their families. Among them, we review m-dimensional distorted probabilities and a few results concerning these measures. The section finishes with a review of a few aggregation operators that are relevant for the purpose of this paper.

2.1. Fuzzy measures

In this paper we will consider fuzzy measures on a finite universal set \( X = \{x_1, \ldots, x_n\} \). For the sake of simplicity, when possible, we will consider \( X := \{1, \ldots, n\} \). Now, we review the definition of fuzzy measure.

**Definition 2.1.** A set function \( \mu : 2^X \rightarrow [0, 1] \) is a fuzzy measure if it satisfies the following axioms:

(i) \( \mu(\emptyset) = 0 \) (boundary conditions).

(ii) \( A \subseteq B \) implies \( \mu(A) \leq \mu(B) \) (monotonicity).

In order to distinguish measures satisfying (i) and (ii) with others that also satisfy some additional constraints (e.g., additivity \( \mu(A \cup B) = \mu(A) + \mu(B) \) when \( A \cap B = \emptyset \)), we use the terms *unconstrained* fuzzy measures for the former ones and *constrained* fuzzy measures for the others.

Given a fuzzy measure \( \mu \) on \( X \), we consider the concept of \( \oplus \)-interadditivity where \( \oplus \) is a pseudo-addition. This concept is a generalization of the concept of interadditivity defined in [22].

**Definition 2.2.** Let \( P := \{X_1, X_2, \ldots, X_m\} \) be a partition of \( X \). Then, we say that \( P \) is an \( \oplus \)-interadditive partition of \( X \) with respect to \( \mu \) if

\[
\mu(A) = \oplus_{i=1}^{n} \mu(A \cap X_i).
\]

When \( \oplus \) corresponds to the addition, we say that \( P \) is an interadditive partition of \( X \) with respect to \( \mu \).

The definition of \( m \)-symmetric fuzzy measures is based on the concept of set of indifference. Roughly speaking, a set of indifference is defined by elements that do not affect the value of the measure. That is, the elements of a set are indistinguishable with respect to the fuzzy measure.

**Definition 2.3** [19,20]. Given a subset \( A \) of \( X \), we say that \( A \) is a set of indifference if and only if:

\[
\forall B_1, B_2 \subseteq A, |B_1| = |B_2|, \forall C \subseteq X \setminus A \mu(B_1 \cup C) = \mu(B_2 \cup C).
\]

In this definition \( |\cdot| \) corresponds to the cardinality of a set. We now consider \( m \)-symmetric fuzzy measures for the particular case of \( m = 2 \) and, then, we give the general definition.

**Definition 2.4** [19,20]. Given a fuzzy measure \( \mu \), we say that \( \mu \) is an at most 2-symmetric fuzzy measure if and only if there exists a partition of the universal set \( \{X_1, X_2\} \), with \( X_1, X_2 \neq \emptyset \) such that both \( X_1 \) and \( X_2 \) are sets of indifference. An at most 2-symmetric fuzzy measure is 2-symmetric if \( X \) is not a set of indifference.

**Definition 2.5** [19,20]. Given a fuzzy measure \( \mu \), we say that \( \mu \) is an at most \( m \)-symmetric fuzzy measure if and only if there exists a partition of the universal set \( \{X_1, \ldots, X_m\} \), with \( X_1, \ldots, X_m \neq \emptyset \) such that \( X_1, \ldots, X_m \) are sets of indifference.

Every fuzzy measure \( \mu \) is an at most \( n \)-symmetric fuzzy measure for \( n = |X| \). So, all fuzzy measures can be considered as \( m \)-symmetric for a value of \( m \) large enough.

**Example 2.6.** Let \( X = \{x_1, x_2, x_3, x_4, x_5\} \) and let \( X_1 = \{x_1, x_2, x_3\} \) and \( X_2 = \{x_4, x_5\} \). Then, the measure defined as follows is a 2-symmetric fuzzy measure.

\[
\begin{align*}
\mu(\emptyset) &= 0; \\
\mu(\{x_1\}) &= \mu(\{x_2\}) = \mu(\{x_3\}) = 0.3; \\
\mu(\{x_1, x_2\}) &= \mu(\{x_2, x_3\}) = \mu(\{x_1, x_3\}) = 0.5; \\
\mu(\{x_1, x_2, x_3\}) &= 0.8; \\
\mu(\{x_4\}) &= \mu(\{x_5\}) = 0.4; \\
\mu(\{x_1, x_4\}) &= \mu(\{x_2, x_4\}) = \mu(\{x_3, x_4\}) = 0.6; \\
\mu(\{x_1, x_5\}) &= \mu(\{x_2, x_5\}) = \mu(\{x_3, x_5\}) = 0.6;
\end{align*}
\]
• \(\mu(\{x_1, x_2, x_4\}) = \mu(\{x_2, x_3, x_4\}) = \mu(\{x_1, x_3, x_4\}) = 0.7;\)
• \(\mu(\{x_1, x_2, x_5\}) = \mu(\{x_2, x_3, x_5\}) = \mu(\{x_1, x_3, x_5\}) = 0.7;\)
• \(\mu(\{x_1, x_2, x_3, x_4\}) = 0.8;\)
• \(\mu(\{x_1, x_2, x_3, x_5\}) = 0.8;\)
• \(\mu(\{x_4, x_5\}) = 0.9;\)
• \(\mu(\{x_1, x_4, x_5\}) = \mu(\{x_2, x_4, x_5\}) = \mu(\{x_3, x_4, x_5\}) = 0.9;\)
• \(\mu(\{x_1, x_2, x_4, x_5\}) = \mu(\{x_2, x_3, x_4, x_5\}) = \mu(\{x_1, x_3, x_4, x_5\}) = 1.0;\)
• \(\mu(\{x_1, x_2, x_3, x_4, x_5\}) = 1.0.\)

**Definition 2.7** [19,20]. Given two partitions \(\{X_1, \ldots, X_p\}\) and \(\{Y_1, \ldots, Y_q\}\) on the finite universal set \(X\), we say that \(\{X_1, \ldots, X_p\}\) is coarser than \(\{Y_1, \ldots, Y_q\}\) if the following holds:

\[\forall X \exists Y_i\text{ such that } Y_i \subseteq X.\]

**Definition 2.8.** Given a fuzzy measure \(\mu\), we say that \(\mu\) is \(m\)-symmetric if and only if the coarsest partition of the universal set in sets of indifference contains \(m\) non empty sets. That is, the coarsest partition is of the form: \(\{X_1, \ldots, X_m\}\), with \(X_i \neq \emptyset\) for all \(i \in \{1, \ldots, m\}\).

**Proposition 2.9** [19,20]. Let \(\mu\) be an \(m\)-symmetric measure with respect to the partition \(\{X_1, \ldots, X_m\}\). Then, the number of values that are needed in order to determine \(\mu\) is:

\[\left(|X_1| + 1\right) \cdots \left(|X_m| + 1\right) - 2.\]

An \(m\)-symmetric fuzzy measure can be represented in a \((|X_1| + 1) \cdots (|X_m| + 1)\) matrix \(M\).

To illustrate this proposition, we consider the following example.

**Example 2.10.** Let us consider again the 2-symmetric fuzzy measure in Example 2.6. Then, as before, \(X = \{x_1, x_2, x_3, x_4, x_5\},\) \(X_1 = \{x_1, x_2, x_3\}, X_2 = \{x_4, x_5\}\).

This measure can be represented by a matrix of dimension \(|X_1| \cdots |X_2|\), that is, \(|X_1| \cdots |X_2| = 4 \cdot 3 = 12.\) Nevertheless, two of these values correspond to the measure of the empty set, i.e., zero, and the measure of \(X\), i.e., one. Thus, only \(|X_1| \cdots |X_2| - 2 = 2\) are required.

More specifically, this fuzzy measure can be represented by the following matrix:

\[
\begin{bmatrix}
|A \cap X_1| = 3 & 0.8 & 0.8 & 1 \\
|A \cap X_1| = 2 & 0.5 & 0.7 & 1 \\
|A \cap X_1| = 1 & 0.3 & 0.6 & 0.9 \\
|A \cap X_1| = 0 & 0 & 0.4 & 0.9 \\
\end{bmatrix}
\]

(1)

2.2. Hierarchically S-decomposable fuzzy measures

Torra [35] introduced Hierarchically S-decomposable fuzzy measures. These measures (HDFM for short) can be seen as a generalization of S-decomposable measures. An important characteristic of S-decomposable fuzzy measures, is that the measure for any subset of \(X\) can be built from the measures on the singletons and a t-conorm \(S\). When interactions among information sources are considered, such construction means that the interactions among pairs (or subsets) of sources can be expressed in a single and unique way. In particular, all interactions are modeled using the t-conorm \(S\).

The so-called hierarchically S-decomposable fuzzy measures define a more general family of fuzzy measures as they permit us to express different kind of interactions between different subsets. This is achieved permitting us the use of different t-conorms for combining the measures of different singletons (and of different subsets).

This is obtained as follows: (i) the elements in \(X\) are structured in a hierarchy that gathers together elements that are similar (from the interactions point of view); (ii) each node of the hierarchy has associated a t-conorm to be used to combine these interactions. In this way, a richer variety of interactions can be expressed.

For example, if we have a fuzzy measure \(\mu\) with \(X = \{x_1, x_2, x_3, x_4, \ldots, x_n\}\) such that \(\mu(\{x_1\}) = 0.2, \mu(\{x_2\}) = 0.4, \mu(\{x_3\}) = 0.3\) and \(\mu(\{x_4\}) = 0.3.\) Then, we have a negative interaction between \(x_1\) and \(x_2\) defining \(\mu(\{x_1, x_2\}) = \max(\mu(\{x_1\}), \mu(\{x_2\})).\) Instead, for a positive interaction between \(x_3\) and \(x_4\) we define \(\mu(\{x_3, x_4\}) = \min(1, \mu(\{x_3\}) + \mu(\{x_4\})).\) Both situations can be modeled with the t-conorms \(S_1(x, y) = \max(x, y)\) and \(S_2(x, y) = \min(1, x + y).\) Then, a hierarchically decomposable fuzzy measure including nodes \(\{x_1, x_2\}\) and \(\{x_3, x_4\}\) with t-conorms \(S_1\) and \(S_2\) can represent these situations.
We give below the definition for the particular case of 2-level HDFM. That is, a measure where the hierarchy has only two levels.

**Definition 2.11** [35]. Given a fuzzy measure \( \mu \), we say that \( \mu \) is a 2-level Hierarchically Decomposable Fuzzy Measure (2-level HDFM) if there is a partition \( \{X_1, \ldots, X_m\} \). On \( X \) (we denote the elements in \( X_i \) by \( X_i = \{x_{i,1}, \ldots, x_{i,m_i}\} \)) and t-conorms \( S, S_1, \ldots, S_m \) such that:

\[
\mu(A) = S(r_1(A), \ldots, r_m(A)),
\]

where

\[
r_i(A) = S_i(\mu(\{x_{i,1}\} \cap A), \ldots, \mu(\{x_{i,m_i}\} \cap A)).
\]

In the general case of HDFM, not presented here, a complete hierarchy is permitted and, then, the measure is defined recursively for each node using the t-conorm attached to the node, and the partition associated to the node.

### 2.3. Distorted probabilities and m-dimensional distorted probabilities

As briefly described in the introduction, distorted probabilities correspond to fuzzy measures that can be represented by a probability distribution and a distortion function. We formalize these measures as well as the required concepts below:

**Definition 2.12.** Let \( P : 2^X \rightarrow [0, 1] \) be a probability measure. Then, we say that a function \( f \) is strictly increasing with respect to \( P \) if and only if

\[
P(A) > P(B) \text{ implies } f(P(A)) > f(P(B)).
\]

**Remark.** Since we suppose that \( X \) is a finite set, when there is no restriction on the function \( f \), a strictly increasing function \( f \) with respect to \( P \) can be regarded as a strictly increasing function on \([0, 1]\). Note that with respect to increasingness only the points in \( \{P(A) | A \in 2^X\} \) are essential, the others are not considered by \( f(P(A)) \).

**Definition 2.13** [2, 3]. Let \( \mu \) be a fuzzy measure. We say that \( \mu \) is a distorted probability if there exists a probability distribution \( P \) and a strictly increasing function \( f \) with respect to \( P \) such that \( \mu = f \circ P \).

The next theorem gives the necessary and sufficient condition for a fuzzy measure \( \mu \) to be a distorted probability. The theorem is based on Scott’s condition [28]:

**Definition 2.14** [24]. Let \( \mu \) be a fuzzy measure, \( \mu \) satisfies Scott’s condition when for all \( A_i, B_i \in 2^X \) such that \( \sum_{i=1}^n 1_{A_i} = \sum_{i=1}^n 1_{B_i} \) the condition below holds:

\[
\mu(A_i) \leq \mu(B_i) \text{ for } i = 2, 3, \ldots, n \implies \mu(A_1) \geq \mu(B_1).
\]

Here \( 1_A \) represents the characteristic function of the set \( A \). That is \( 1_A(x) = 1 \) if and only if \( x \in A \).

Using this condition, we can characterize distorted probabilities as follows:

**Theorem 2.15** [24]. Let \( \mu \) be a fuzzy measure; then, \( \mu \) is a distorted probability if and only if Scott’s condition holds.

\( m \)-Dimensional distorted probabilities were presented in [24] to overcome the limited expressiveness of distorted probabilities. They are defined as follows:

**Definition 2.16** [24]. Let \( \{X_1, X_2, \ldots, X_m\} \) be a partition of \( X \); then, we say that \( \mu \) is an at most \( m \) dimensional distorted probability if there exists a function \( f \) on \( \mathbb{R}^m \) and probabilities \( P_i \) on \( (X_i, 2^{X_i}) \) such that:

\[
\mu(A) = f(P_1(A \cap X_1), P_2(A \cap X_2), \ldots, P_m(A \cap X_m)),
\]

where \( f \) on \( \mathbb{R}^m \) is strictly increasing with respect to each variable.

We say that an at most \( m \) dimensional distorted probability \( \mu \) is an \( m \) dimensional distorted probability if \( \mu \) is not an at most \( m - 1 \) dimensional.

### 2.4. Aggregation operators

Now we define the OWA and the WOWA operators. They will be of relevance in this work. As explained in detail in [34], the OWA operator permits to give importance to the data (with respect to their position) while the WOWA permits to give
Proposition 3.1. Let $\mu$ be a fuzzy measure, then the Choquet integral of a function $f : X \rightarrow \mathbb{R}^+$ with respect to the fuzzy measure $\mu$ is defined by:

$$(C) \int f \, d\mu(= C_\mu(f)) = \sum_{i=1}^{n} [f(x_{s(i)}) - f(x_{s(i-1)})] \mu(A_{s(i)}),$$

where $x_i \in X$ and where $f(x_{s(i)})$ indicates that the indices have been permuted so that

$0 \leq f(x_{s(1)}) \leq \cdots \leq f(x_{s(n)})$,

$A_{s(i)} = \{x_{s(i)}, \cdots, x_{s(n)}\}$ and $f(x_{s(0)}) = 0$.

3. Symmetric fuzzy measures

We start showing that a 1-Symmetric fuzzy measure is a special case of distorted probabilities.

**Proposition 3.1.** Let $\mu = f \circ P$ be a distorted probability. Then, $\mu$ is a 1-symmetric fuzzy measure if and only if $P(A) = |A|/|X|$.

**Proof.** Suppose that $\mu$ is 1-symmetric fuzzy measure and let $x_i, x_j \in X$ for $i \neq j$. Then, since $f$ is strictly increasing, we have that $P(x_i) = P(x_j)$ for every $i, j$. Therefore, we have $P(\{x_i\}) = 1/n$. So, $P(A) = |A|/|X|$.

Conversely, suppose that $P(A) = |A|/|X|$ for $A \subset X$. Then, if $|A| = |B|$ for $A, B \subset X$, we have that $P(A) = P(B)$. Therefore, $\mu(A) = f \circ P(A) = f \circ P(B) = \mu(B)$. \(\square\)

Now we show that all $m$-symmetric fuzzy measures are $m$-dimensional distorted probabilities. This implies that 1-symmetric fuzzy measures are distorted probabilities.
Proposition 3.2. Let \( \mu \) be an \( m \)-symmetric fuzzy measure with respect to the partition \( \{X_1, \ldots, X_m\} \). Then, \( \mu \) is an \( m \)-dimensional distorted probability.

Proof. Let \( \mu \) be an \( m \)-symmetric fuzzy measure with respect to the partition \( \{X_1, \ldots, X_m\} \). Then, according to Proposition 2.9, \( \mu \) can be defined in terms of a matrix with \( ([|X_1| + 1] \times \cdots \times [|X_m| + 1]) \) values. Let \( T \) be such matrix:

\[
T = \{t_{i_1, \ldots, i_m} \mid i_1 \in \mathbb{N}_{|X_1|}, \ldots, i_m \in \mathbb{N}_{|X_m|}, \}
\]

where \( \mathbb{N}_m \) is \( \{0, 1, \ldots, m\} \).

Proposition 3.2 states that a fuzzy measure defined in terms of a matrix can be represented in terms of a distorted probability.

Example 3.3. The fuzzy measure defined in Example 2.10 can be represented in terms of a distorted probability as follows:

- \( p_1(x_1) = 1/|X_1| = 1/3 \) for all \( x_1 \in X_1 = \{x_1, x_2, x_3\} \);
- \( p_2(x_1) = 1/|X_2| = 1/2 \) for all \( x_1 \in X_2 = \{x_4, x_5\} \)

and \( f \) defined in terms of the matrix of the example following the last proposition. That is, \( f(i_1/|X_1|, i_2/|X_2|) = t_{i_1, i_2} \) for all \( i_r \in \{0, \ldots, |X_r|\} \) for \( r \) in \( \{1, \ldots, m\} \).

\[
\begin{bmatrix}
i_1 = 3 & 0.8 & 0.8 & 1 \\
i_1 = 2 & 0.5 & 0.7 & 1 \\
i_1 = 1 & 0.3 & 0.6 & 0.9 \\
i_1 = 0 & 0 & 0.4 & 0.9 \\
i_2 = 0 & i_2 = 1 & i_2 = 2 \\
\end{bmatrix}
\] (3)

Although the converse of the last proposition is not true, the next proposition characterizes one case in which \( m \)-dimensional distorted probabilities are \( m \)-symmetric fuzzy measures.

Proposition 3.4. Let \( \mu \) be an \( m \)-dimensional distorted probability. If \( p_i(x_j) = p_i(x_k) \) for all \( x_j, x_k \in X_i \) and for all \( i = 1, \ldots, m \), then \( \mu \) is an \( m \)-symmetric fuzzy measure.

Proof. If \( p_i(x_j) = p_i(x_k) \) for all \( x_j, x_k \in X_i \), then \( X_i \) is a set of indifference. This is so because for the set \( X_i \) the equality \( p_i(A \cap X_i) = p_i(A \cap X_j) \) holds for any \( x_j \in X_i \). Therefore, for such \( X_i \), \( P_i(B_1) = \mu(B_2) \) if \( |B_1| = |B_2| \subseteq X_i \). Now, from this, and as

\[
\mu(A) = f(P_i(A \cap X_1), \ldots, P_i(A \cap X_m))
\]

it is clear that \( \forall B_1, B_2 \subseteq X_i \) with \( |B_1| = |B_2| \) and \( \forall C \subseteq X \setminus X_i \) it holds that \( \mu(B_1 \cup C) = \mu(B_2 \cup C) \).

As this condition corresponds to the condition of \( X_i \) being a set of indifference, and as this applies to all \( X_i \), the proposition is proven. □

It is known that OWA operators are equivalent to Choquet integrals with respect to symmetric fuzzy measures. Therefore, \( m \)-symmetric fuzzy measures permit us to define a generalization of OWA operators. The \( m \)-dimensional OWA is defined below:

Definition 3.5. The \( m \)-dimensional OWA is defined as the Choquet integral with respect to an \( m \)-symmetric fuzzy measure.

Example 3.6. Using the table in Example 3.3, we can define a 2-dimensional OWA. That is, the OWA is defined as the Choquet integral with respect to the 2-dimensional fuzzy measure defined in the matrix of Example 3.3.

As proven in [33], a Weighted OWA (WOWA) operator is equivalent to a Choquet integral with respect to a distorted probability. Therefore, a Choquet integral with an \( m \)-dimensional probability can be seen as a generalization of the WOWA operator. We define an \( m \)-dimensional WOWA as follows:
Definition 3.7. The \( m \)-dimensional WOWA is defined as the Choquet integral with respect to an \( m \)-dimensional distorted probability.

Example 3.8. Let \( X = \{x_1, x_2, x_3, x_4, x_5\} \), and \( X_1 \) and \( X_2 \) define a partition with \( X_1 = \{x_1, x_2, x_3\} \) and \( X_2 = \{x_4, x_5\} \), let \( p_1(x_1) = 0.4 \), \( p_1(x_2) = 0.35 \), \( p_1(x_3) = 0.25 \), let \( p_2(x_4) = 0.6 \) and \( p_2(x_5) = 0.4 \), and let \( f(x, y) = (x + y + y^2x)/3 \). The partition, the two probability distributions and the function represent a 2-dimensional distorted probability. The WOWA of \( f \), \( p_1 \) and \( p_2 \) is defined as the Choquet integral with respect to this 2-dimensional distorted probability.

Then, considering Definitions 3.5 and 3.7 above, we have the following corollary from Proposition 3.2:

**Corollary 3.9.** An \( m \)-dimensional OWA is a particular case of an \( m \)-dimensional WOWA. In other words, a Choquet integral with respect to an \( m \)-symmetric fuzzy measure is a particular case of a Choquet integral with respect to an \( m \)-dimensional distorted probability.

4. \( m \)-Separable fuzzy measure

In this section we introduce \( m \)-separable fuzzy measures, giving some examples and then establishing some connections between these fuzzy measures and some of the already existing ones in the literature. Some results about the Choquet integral are also reported.

4.1. Definition and basic properties

As stated above fuzzy measures are set functions on a reference set \( X \). It is often the case that the elements of this set can be divided into some classes, and there are essential interactions among these classes. The following example illustrates this case.

Example 4.1. Let \( X := \{x_i, i = 1, 2, 3, 4, 5\} \) be the set of subjects taught in a certain school. We will use for illustration the following subjects: \( x_1 \) : Algebra, \( x_2 \) : Analysis, \( x_3 \) : Geometry, \( x_4 \) : English reading, \( x_5 \) : English writing. To evaluate the students of the school, we need to aggregate their score on each subject.

Let \( M := \{x_1, x_2, x_3\} \) be the set on mathematics, and \( E := \{x_4, x_5\} \) be the set on English. The measure \( \mu(A) \) for \( A \subset X \) can be divided into some classes, and there are essential interactions among these classes. The following example illustrates this case.

**Definition 4.2.** Let \( \mu \) be a fuzzy measure. Then, we say that \( \mu \) is an \( m \)-separable fuzzy measure if there exists a function \( g \) and a partition \( \{X_1, \ldots, X_m\} \) of \( X \) such that

\[
\mu(A) = g(\mu(A \cap X_1), \ldots, \mu(A \cap X_m)), \tag{4}
\]

where \( g \) is an \( m \)-dimensional function on \( \mathbb{R}^m \). \( g \) is required to be symmetric. We say that \( g \) is a generating function for \( \mu \).

We say that a generating function \( g \) is induced by \( h \) on \( R \times R \) if \( g(x_1, \ldots, x_m) = h(h(\ldots h(x_1, x_2), \ldots, x_{m-1}), x_m) \)

\[
g(x_1, x_2, 0, \ldots, 0) = h(x_1, x_2).
\]

Note that \( g \) is required to be symmetric because \( \{X_1, \ldots, X_m\} \) is a partition, and, thus, the order of the elements in the partition is irrelevant. To make the order relevant, we need to consider a sequence \( \langle X_1, \ldots, X_m \rangle \) defining a partition. In this case, \( g \) does not need to be symmetric. The next definition illustrates this case.

**Definition 4.3.** Let \( \mu \) be a fuzzy measure. Then, we say that \( \mu \) is an \( m \)-sequence separable fuzzy measure if there exists a function \( g \) and a sequence \( \langle X_1, \ldots, X_m \rangle \) that defines a partition of \( X \) such that

\[
\mu(A) = g(\mu(A \cap X_1), \ldots, \mu(A \cap X_m)), \tag{5}
\]

where \( g \) is an \( m \)-dimensional function on \( \mathbb{R}^m \). We say that \( g \) is a generating function for \( \mu \).

We say that a generating function \( g \) is induced by \( h \) on \( R \times R \) if \( g(x_1, \ldots, x_m) = h(h(\ldots h(x_1, x_2), \ldots, x_{m-1}), x_m) \)

\[
g(x_1, x_2, 0, \ldots, 0) = h(x_1, x_2).
\]

**Proposition 4.4.** The following holds:

- \( m \)-Separable fuzzy measures are \( m \)-sequence separable fuzzy measures by any permutation of \( \{X_1, \ldots, X_m\} \).
- \( m \)-Sequence separable fuzzy measures with \( g \) generated by associative and strict \( h \) are \( m \)-separable fuzzy measures.
The second condition is true because, associativity and strict monotonicity of \( h \) lead to symmetric \( h \) and also to symmetric \( g \).

Let us consider some examples of \( m \)-separable fuzzy measures and \( m \)-sequence separable fuzzy measures induced by particular functions \( h \). The examples illustrate the possible cases of symmetric and nonsymmetric measures generated from a sequence that defines a partition.

**Example 4.5.** Let \( \langle X_1, \ldots, X_m \rangle \) be a sequence that defines a partition of \( X \).

1. Suppose \( g(x_1, \ldots, x_m) = x_1 + \cdots + x_m \), so \( g \) is induced by \( h(x, y) = x + y \). Then we have
   \[
   \mu(A) = \mu(A \cap X_1) + \cdots + \mu(A \cap X_m).
   \]
   This is an interadditivity.

2. Suppose \( g(x_1, \ldots, x_m) = x_1 \lor \cdots \lor x_m \), so \( g \) is induced by \( h(x, y) = x \lor y \). Then we have
   \[
   \mu(A) = \mu(A \cap X_1) \lor \cdots \lor \mu(A \cap X_m).
   \]

3. Suppose \( g(x_1, \ldots, x_m) = (x_1^2 + \cdots + x_m^2)^{1/2} \), so \( g \) is induced by \( h(x, y) = (x^2 + y^2)^{1/2} \). Then we have
   \[
   \mu(A) = (\mu(A \cap X_1)^2 + \cdots + \mu(A \cap X_m)^2)^{1/2}.
   \]

4. Suppose \( g(x_1, \ldots, x_m) = x_1/(2^{m-1}) + \sum_{i=2}^{m} x_i/(2^{m-(i-1)}) \), so \( g \) is induced by \( h(x, y) = (x + y)/2 \). Then we have
   \[
   \mu(A) = \frac{\mu(A \cap X_1)}{2^{m-1}} + \sum_{i=2}^{m} \frac{\mu(A \cap X_i)}{2^{m-(i-1)}}.
   \]

5. Suppose \( g(x_1, \ldots, x_m) = \sum_{i=1}^{m} x_i/(3^{m-i}) \), so \( g \) is induced by \( h(x, y) = x/3 + y \). Then we have
   \[
   \mu(A) = \sum_{i=1}^{m} \frac{\mu(A \cap X_i)}{3^{m-i}}.
   \]

In this example, the first three measures use symmetric and associative functions \( h \), the fourth case uses a symmetric \( h \) but the measure is not symmetric (because \( h \) is not associative), and, finally, the last measure uses a non associative and non symmetric \( h \) which results also into a non symmetric measure. Note that the first three measures are \( m \)-separable fuzzy measures while the last two ones are only \( m \)-sequence separable fuzzy measures.

From an application point of view, the last two measures, which are not symmetric, permits us to model situations in which the relevance of one \( X_i \) is twice (one third) the relevance of \( X_{i-1} \).

An example of separable fuzzy can be found in Walley’s paper [38]. In the context of belief functions, Walley introduces a possibility measure inside partition elements and \( g = + \). This is just the opposite of example (2) above.

Now, we consider the relationship between the \( m \)-separable fuzzy measures and some other families of measures. The next propositions are obvious from the definition.

**Proposition 4.6.** 2-Level HDFMs with \( S_i \) Archimedean t-conorms are \( m \)-separable fuzzy measures.

**Proof.** We prove this proposition by construction. Let us consider a 2-level HDFM. If \( \mu \) is such a measure, then \( \mu \) is represented in terms of a top node \( X \) with t-conorm \( S_X \), subnodes \( X_i \) for \( i = 1, \ldots, m \) with (Archimedean) t-conorms \( S_i \) and the leaves with the elements \( x \) in \( X_i \). Let the elements of \( X_i \) be denoted by \( X_i = \{ x_{i,1}, \ldots, x_{i,m_i} \} \).

Now, we define first the function \( g \) as equivalent to \( S_i \). That is, \( g = S \). Note that the t-conorm \( S \) is a symmetric function. Now, we show that

\[
r_i(A) = S_i(\mu(\{x_{i,1}\} \cap A), \ldots, \mu(\{x_{i,m_i}\} \cap A))
\]

can be rewritten in the form of \( f_i \circ P(A) \). Defining the function \( v(x_i,j) \) as follows:

\[
\begin{align*}
  v(x) &= 0 \text{ if } x \notin A; \\
  v(x) &= \mu(x) \text{ if } x \in A.
\end{align*}
\]

We have that \( r_i(A) \) is equivalent to:

\[
r_i(A) = S_i(v(x_{i,1}), \ldots, v(x_{i,m_i})).
\]
As, \( S_i \) is an archimedean t-conorm, we have that:

\[ S_i(a, b) = h_i^{-1}(h_i(a) + h_i(b)). \]

In this case,

\[ r_i(A) = h_i^{-1}\left(\sum_{x_{ij}} h_i(\nu(x_{ij}))\right). \]

Thus, defining \( f_i = h_i^{-1}(R \cdot z) \) and \( p'(x_{ij}) = h_i(\nu(x_{ij}))/R \) with \( R = \sum_{x_{ik}} \nu(x_{ij}) \), we have that \( p' \) is a probability distribution and \( f_i(p'(A)) = r_i(A) \).

Therefore, the proposition is proven. \( \square \)

It is obvious from the definition that \( m \)-dimensional distorted probabilities are \( m \)-separable fuzzy measures. Since \( m \)-symmetric fuzzy measures are \( m \)-dimensional distorted probabilities [23], \( m \)-symmetric fuzzy measures are \( m \)-separable fuzzy measures.

Since the Choquet integral is additive with respect to the fuzzy measures, we have the next theorem.

**Theorem 4.7.** Let \( \{X_1, \ldots, X_m\} \) be a partition of \( X \) and \( \mu_i \) \( i = 1, \ldots, m \) be distorted probabilities represented by \( f_i \) and \( P_i \) (i.e., \( \mu_i = f_i \circ P_i \)). Then, there exists an \( m \)-separable fuzzy measure \( \mu \) such that

\[
\sum_{i=1}^{m}((C) \int f \circ P_i) = (C) \int f d\mu
\]

for all measurable function \( f \).

**Proof** Define \( g : \mathbb{R}^m \rightarrow \mathbb{R} \) by

\[ g(x_1, \ldots, x_m) = x_1 + \cdots + x_m. \]

Then we can define a type 2 fuzzy measure by \( \mu = g(f_1(P_1), \ldots, f_m(P_m)) \). Since \( \{X_1, \ldots, X_m\} \) is a partition, if \( A \subset X_i \) for \( i = 1, \ldots, m \) then \( \mu(A \cap X_i) = f_i(P_i(A)) \) and \( \mu(A \cap X_j) = 0 \) if \( i \neq j \). Therefore we have

\[
\mu(A) = g(f_1(P_1(A)), \ldots, f_m(P_m(A)))
\]

\[
= f_1(P_1(A)) + \cdots + f_m(P_m(A))
\]

\[
= \mu(A \cap X_1) + \cdots + \mu(A \cap X_m).
\]

Therefore \( \{X_1, \ldots, X_m\} \) is an interadditive partition of \( X \). Then it follows from a theorem in [22] that

\[
\sum_{i=1}^{m}((C) \int_{X_i} f d\mu) = (C) \int f d\mu.
\]

Since \( f_i(P_i(A)) = \mu(A \cap X_i) \), we have

\[
(C) \int_{X_i} f d\mu = (C) \int_{X_i} f d f_i \circ P_i.
\]

Therefore

\[
\sum_{i=1}^{m}(C) \int_{X_i} f d f_i \circ P_i = (C) \int f d\mu. \quad \square
\]

As a corollary of this theorem, we have that the Choquet integral with respect to a \( m \)-separable fuzzy measure \( \mu \) with \( g(x_1, \ldots, x_m) = x_1 + \cdots + x_m \) can be represented as a two step Choquet integral.

**Corollary 4.8.** Let \( \mu \) be a \( m \)-separable fuzzy measure \( \mu \) with \( g(x_1, \ldots, x_m) = x_1 + \cdots + x_m \). Then the Choquet integral with respect to \( \mu \) is represented as a two step Choquet integral of a 1st step integral with respect to a probability on \( (1, \ldots, m) \). That is,

\[
(C) \int f d\mu = \int ((C) \int f d f_i \circ P_i) dP(i).
\]
4.2. Generating function

The properties of \( m \)-separable fuzzy measures depend on their generating functions. We have some results concerning the generating functions.

Definition 4.9. Let \( \{X_1, \ldots, X_m\} \) be a partition of \( X \). We say that an \( m \)-separable fuzzy measure is distorted interadditive if there exists a function \( \varphi : \mathbb{R}^+ \to \mathbb{R} \) such that \( \mu(A) = \varphi(\mu(A \cap X_1) + \mu(A \cap X_2) + \cdots + \mu(A \cap X_m)) \).

Theorem 4.10. Suppose that a generating function \( g \) of an \( m \)-separable fuzzy measure \( \mu \) is differentiable on \( \mathbb{R}^m \). \( \mu \) is distorted interadditive if and only if \( \frac{\partial g}{\partial x_i} = \frac{\partial g}{\partial x_j} \) for \( i \neq j \) and \( i, j = 1, 2, \ldots, m \).

Proof. Suppose that a generating function \( g \) is represented as \( g(x_1, x_2, \ldots, x_n) = \varphi(x_1 + x_2 + \ldots + x_n) \). It is obvious that \( \frac{\partial g}{\partial x_i} = \frac{\partial g}{\partial x_j} \) for \( i \neq j \).

Conversely, let \( s_1 := x_1 + x_2 + \cdots + x_n \), \( s_k := x_k \) for \( k = 2, 3, \ldots, m \) and

\[
f(s_1, s_2, \ldots, s_m) = g(x_1, x_2, \ldots, x_n).
\]

Then we have \( \frac{\partial f}{\partial s_i} = -1 \), \( \frac{\partial f}{\partial s_j} = 1 \) for \( i = 2, 3, \ldots, m \) since \( x_1 = s_1 - s_2 - \cdots - s_m \) and \( x_i = s_i \). If \( i \neq j \), we have \( \frac{\partial f}{\partial s_j} = 0 \).

Therefore we have

\[
\frac{\partial f}{\partial s_i} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial s_i} + \sum_{k=2}^m \frac{\partial f}{\partial x_k} \frac{\partial x_k}{\partial s_i}
\]

\[
= -\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_i}
\]

\[
= -\frac{\partial g}{\partial x_1} + \frac{\partial g}{\partial x_i} = 0
\]

for \( i = 2, 3, \ldots, m \). Therefore \( f \) is a univariate function of \( s_1 \). We can put \( f(s_1, s_2, \ldots, s_m) := \varphi(s_1) \). That is \( g(x_1, x_2, \ldots, x_m) = \varphi(x_1 + x_2 + \cdots + x_m) \). \( \square \)

To find the dimension \( m \) of the generator \( g \) of a separable fuzzy measure is an important problem. We have some results for this problem.

Definition 4.11. Let \( \varphi \) be a strictly monotone function on \( \mathbb{R} \). \( \varphi \)-Möbius inverse \( m^\varphi \) of \( \mu \) is defined by

\[
m^\varphi(A) := \sum_{B \subset A} (-1)^{|A \setminus B|} \varphi(\mu(B))
\]

for \( A \in 2^X \).

Let \( M := \{A \mid m^\varphi(A) \neq 0\} \). Then, we define \( A_1 := \{A \mid A \in M, x_1 \in A\} \) for \( X = \{x_1, \ldots, x_l, \ldots, x_n\} \) and

\[
M_l := \max \{|A| \mid A \in A_l\}.
\]

\( M \) is the class of sets in which each element has essentially some interaction. \( A_l \) is the class of sets which have interaction with an element \( x_l \). The proposition below shows the relation between an \( m \)-separable fuzzy measure and the structure of the sets.

Proposition 4.12. Let \( \mu \) be a \( m \)-separable fuzzy measure generated by \( g \), and \( g \) be induced by a strictly monotone and associative \( h \), that is \( h(h(x, y), z) = h(x, h(y, z)) \). Then we have

\[
m \times \min_{l \in \{1, \ldots, n\}} M_l \leq n \leq m \times \max_{l \in \{1, \ldots, n\}} M_l.
\]

Proof. Suppose that \( \mu \) is a \( m \)-separable fuzzy measure generated by \( g \) and that \( g \) is induced by \( h \). Since \( g \) is symmetric, then \( h \) is symmetric, that is \( h(x, y) = h(y, x) \). Since \( h \) is strictly monotone, symmetric and associative, there exists a strictly
monotone function $\varphi$ such that $h(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y))$. Define $\varphi$-Möbius inverse $m^\varphi$ by

$$m^\varphi(A) := \sum_{B \subseteq A} (-1)^{|A|\setminus|B|} \varphi(\mu(B)).$$

Let $P := X_1, X_2, \ldots, X_m$ be a partition of $X$ for an $m$-separable fuzzy measure. Applying Theorem 5.2 in [10], $A \not\subset C, C \in P \Rightarrow m^\varphi(A) = 0$.

Since $A \subset C, C \in P$ for $A \in A_l$, we have $\min_{i \in [1,\ldots,n]} M_l \leq |X_k| \leq \max_{i \in [1,\ldots,n]} M_l$ for all $k = 1, 2, \ldots, m$. Therefore

$$m \times \min_{i \in [1,\ldots,n]} M_l \leq \sum_{k=1}^m |X_k| \leq m \times \max_{i \in [1,\ldots,n]} M_l.$$

Since $\sum_{k=1}^m |X_k| = n$, the proposition is proven. \(\square\)

We say that a fuzzy measure $\mu$ is a $(\varphi, k)$-order additive if $\max\{|A|\setminus A \in M| = k$. If $\varphi(x) = x$, a $(\varphi, k)$-order additive fuzzy measure is a $k$-order additive fuzzy measure [11, 12].

**Corollary 4.13.** Let $\mu$ be a $m$-separable fuzzy measure generated by $g$, and $g$ be induced by strict monotone and associative $h$. If $\mu$ is $k$-additive, then we have $m \times k \geq n$.

**Example 4.14.** Let $X := \{x_1, x_2, x_3\}, P := \{\{x_1, x_2\}, \{x_3\}\}$ and a $2$-separable fuzzy measure $\mu$ generated by $g(x, y) := (x^2 + y^2)^{1/2}$.

Since $\frac{\partial g}{\partial x} \neq \frac{\partial g}{\partial y}$, $\mu$ is not distorted interadditive.

Let $\varphi(x) = x^2$ and consider $\varphi$-Möbius inverse $m^\varphi$ of $\mu$.

Since $\mu(\{x_i, x_3\}) = (\mu(\{x_i\})^2 + \mu(\{x_3\}))^{1/2}$ for $i = 1, 2$, we have

$$m^\varphi(\{x_i, x_3\}) = \mu(\{x_i, x_3\})^2 - \mu(\{x_i\})^2 - \mu(\{x_3\})^2 = \mu(\{x_i\})^2 + \mu(\{x_3\})^2 - (\mu(\{x_i\})^2 - (\mu(\{x_3\})^2 = 0.$$

for $i = 1, 2$. Since $\mu(\{x_1, x_2, x_3\}) = (\mu(\{x_1, x_2\})^2 + \mu(\{x_3\}))^{1/2}$, we have

$$m^\varphi(\{x_1, x_2, x_3\}) = (\mu(\{x_1, x_2\})^2 - \sum_{i \neq j}(\mu(\{x_i, j\})^2 + \sum_i(\mu(\{x_i\})^2$$

$$= \mu(\{x_1, x_2\})^2 + \mu(\{x_3\})^2 - \sum_{i \neq j}(\mu(\{x_i, x_j\})^2 + \sum_i(\mu(\{x_i\})^2$$

$$= \mu(\{x_3\})^2 - (\mu(\{x_1, x_3\})^2 - (\mu(\{x_2, x_3\})^2 + \sum_i(\mu(\{x_i\})^2$$

$$= 0.$$

Then we have $M = \{\{x_1, x_2\}, \{x_1\}, \{x_2\}, \{x_3\\}, A_l = \{\{x_1, x_2\}, \{x_i\}) i = 1, 2 \text{ and } A_3 = \{\{x_3\}\}$. Therefore $M_1 = M_2 = 2, M_3 = 1$.

5. Conclusions

In this paper we have established some connections between existing measures, and introduced a new family of fuzzy measures, $m$-sequence separable fuzzy measures. This new family of measures permits us to represent the information in a fuzzy measure in a compact way, with a function, and requiring less parameters than unconstrained fuzzy measures.

Further work will be done to establish a tight connection between the $m$-sequence separable fuzzy measures and the hierarchically decomposable ones. In addition, we will consider the properties described in [7, 6, 5] and their relationship with $m$-sequence separable fuzzy measures.

Acknowledgements

Partial support by the Spanish MEC (projects ARES – CONSOLIDER INGENIO 2010 CSD2007-00004 – and eAEGIS – TSI2007-65406-C03-02) is acknowledged.

The authors are grateful to Guest Editors and to the anonymous referees for their comments and suggestions.
References