Algebraic degeneracy of non-Archimedean analytic maps [*]

by Ta Thi Hoai An [a], William Cherry [b] and Julie Tzu-Yueh Wang [c]

[a] *Institute of Mathematics, 18 Hoang Quoc Viet, Cau Giay, Hanoi, Vietnam*
[b] *Department of Mathematics, University of North Texas, 1155 Union Circle #311430, Denton, TX 76203, USA*
[c] *Institute of Mathematics, Academia Sinica, Nankang, Taipei 11529, Taiwan*

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ABSTRACT

We prove non-Archimedean analogs of results of Noguchi and Winkelmann showing algebraic degeneracy of rigid analytic maps to projective varieties omitting an effective divisor with sufficiently many irreducible components relative to the rank of the group they generate in the Néron–Severi group of the variety.

1. INTRODUCTION

Dufresnoy [7] (or see [9, §3.10]) proved that a holomorphic map from the complex plane $\mathbf{C}$ that omits $n + k$ hyperplanes in general position in projective space $\mathbf{P}^n$ must be contained in a linear subspace of dimension at most n/k. Noguchi and Winkelmann [16] generalized this result to show that a holomorphic curve in an arbitrary projective manifold (or more generally a compact Kähler manifold) omitting sufficiently many irreducible hypersurfaces relative to the rank of the group generated by their cohomology classes must be algebraically degenerate. This

puts the Dufresnoy theorem in context and clarifies the roll played by the rank of the Néron–Severi group. Noguchi and Winkelmann's precise result is the following theorem.

Theorem (Noguchi/Winkelmann). *Let M be a compact Kähler manifold of dimension m. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible hypersurfaces in general position. Let r be the rank of the group generated by $\{c_1(D_i)\}_{i=1}^{\ell}$. Let W be a closed subvariety of M of dimension n and irregularity q. Suppose there exists an algebraically non-degenerate holomorphic map from the complex plane $\mathbf{C}$ to W that omits each of the D_i that does not contain all of W. Then*

(i) $\#\{W \cap D_i \neq W\} + q \leqslant n + r$;

(ii) *If $\ell > m$ and in addition each of the D_i are ample, then*

$$n \leqslant \frac{m}{\ell - m} \max\{0, r - q\}.$$

Here, c_1 denotes the first Chern class, and the irregularity q is the dimension of the space of holomorphic 1-forms on a desingularization of W, which is the dimension of the Albanese variety. When the irregularity exceeds the dimension, Bloch [3] (or see [17] for a rigorous proof) proved all holomorphic curves are algebraically degenerate by showing the image in the Albanese variety is degenerate. This was extended by Noguchi (and by Noguchi and Winkelmann in the non-algebraic Kähler case) – see the references in [16] – to conclude that a holomorphic map from $\mathbf{C}$ omitting an effective divisor D such that the logarithmic irregularity with respect to D, i.e., the dimension of the space of logarithmic 1-forms with poles along D, exceeds the dimension. This was again done by composing with the quasi-Albanese morphism, but this time since the quasi-Albanese variety need not be compact, there are additional difficulties.

Our purpose here is to explain the rigid analytic analog of the Noguchi–Winkelmann theorem for non-Archimedean analytic maps, at least in the case of projective algebraic varieties. The non-Archimedean analog of Bloch's theorem was proven by Cherry in [5]. Modulo the standard constructions of Albanese and Picard varieties as in [10], it is then a routine matter to conclude analogous algebraic degeneracy results.

One could perhaps argue that the natural category for us to work in is that of complete rigid analytic spaces, rather than projective varieties. However, the Albanese map is crucial for our arguments, and as far as we know, this has not been worked out for non-algebraic rigid analytic spaces. Since we restrict ourselves to projective varieties, we do not hesitate to appeal to algebraic results when convenient, even when there are alternative approaches that work for more general rigid analytic spaces.

Let $\mathbf{F}$ be an algebraically closed field complete with respect to a non-Archimedean absolute value and of arbitrary characteristic. Throughout, a variety will mean an algebraic variety defined over $\mathbf{F}$, and a morphism will mean an algebraic morphism defined over $\mathbf{F}$. We will use $\mathbf{A}^1$ to denote the affine line over $\mathbf{F}$ and $\mathbf{A}^{1\times}$

to denote $\mathbf{A}^1 \setminus \{0\}$. We will use $\mathbf{G}_m$ to denote the multiplicative group, which as an analytic space or a variety is of course the same thing as $\mathbf{A}^{1\times}$. Analytic will mean rigid analytic over $\mathbf{F}$. One can think concretely by thinking of an analytic map from $\mathbf{A}^1$ (resp. $\mathbf{A}^{1\times}$) to an algebraic variety X as given by a solution to the defining equations of X in formal power series (resp. formal Laurent series) with coefficients in $\mathbf{F}$ and converging for arbitrary positive radii.

2. NON-ARCHIMEDEAN ANALYTIC MAPS TO SEMI-ABELIAN VARIETIES

We begin by recalling the main results of [5] and extending them to semi-Abelian varieties.

Theorem 2.1. *Any analytic map from $\mathbf{A}^1$ to a semi-Abelian variety must be constant.*

Proof. The proof is essentially the argument on p. 401 of [5]. What was actually shown in [5] was that an analytic map into the extension of an Abelian variety with good reduction by a torus must be constant. This was then used with the semi-Abelian uniformization theorem to show that an analytic map to an arbitrary Abelian variety must be constant. Once one has this, the same argument repeated then gives that an analytic map from $\mathbf{A}^1$ to an arbitrary semi-Abelian variety must be constant. □

In characteristic zero, Theorem 2.1 can also be proven by the method of [6].

Corollary 2.2. *If X is a variety admitting a non-constant morphism to a semi-Abelian variety, then any analytic map from $\mathbf{A}^1$ to X is algebraically degenerate.*

Corollary 2.3. *If X is a non-singular projective variety in characteristic zero with positive irregularity, then any analytic map from $\mathbf{A}^1$ to X must be algebraically degenerate.*

For the case of surfaces, Corollary 2.3 was pointed out in [4].

Proof of Corollary 2.3. Since we assume X to be non-singular, the Albanese map is a morphism, see e.g., [10, Ch. 2], and since we have assumed characteristic zero, the dimension of the Albanese variety is the same as the irregularity. □

Corollary 2.4. *If X is a projective variety admitting a non-constant rational map to an Abelian variety, then any analytic map from $\mathbf{A}^1$ to X is algebraically degenerate.*

Proof. Let $\phi : X \to A$ be a non-constant rational map to an Abelian variety and let f be an analytic map from $\mathbf{A}^1$ to X. The idea is that $\phi \circ f$ is a meromorphic mapping from $\mathbf{A}^1$ to A and hence analytic, unless f is contained in the indeterminacy locus of ϕ, whence degenerate. The corollary then follows from the theorem.

Lacking a convenient reference for the general fact, we give an ad-hoc proof here. Embed X and A in projective spaces. Then ϕ can be represented as $[\phi_0, \ldots, \phi_N]$ where the ϕ_i are homogeneous polynomials and N is the dimension of the projective space in which we have embedded A. Then, $\phi_i \circ f$ are analytic functions on $\mathbf{A}^1$. If they are all identically zero, then the image of f is contained in the indeterminacy locus of ϕ and is algebraically degenerate. Otherwise, factoring out the greatest common divisor (well defined up to a non-zero constant) from the $\phi_i \circ f$ if necessary, we see $\phi \circ f$ can be made to be well defined on all of $\mathbf{A}^1$. Hence, $\phi \circ f$ is an analytic map to A and hence constant by the theorem. Therefore, f is algebraically degenerate. $\square$

Proposition 2.5. *Let T be a multiplicative torus and let f be an analytic map (not assumed to be a group homomorphism) from T to $\mathbf{G}_m$. Then f is the translation of a group homomorphism.*

Proof. Embed T in affine n-space $\mathbf{A}^n$ in the natural way with affine coordinates $z = (z_1, \ldots, z_n)$ on $\mathbf{A}^n$. Then, f can be written as a Laurent series in multi-index notation as

$$\sum_{\gamma \in \mathbf{Z}^n} a_\gamma z^\gamma .$$

An easy argument involving valuation polygons shows that there exists precisely one multi-index γ such that $a_\gamma \neq 0$, from which the proposition follows.

Indeed, suppose there exist two multi-indices

$$\mu = (\mu_1, \ldots, \mu_n) \neq (v_1, \ldots, v_n) = v$$

with $a_\mu \neq 0$ and $a_v \neq 0$. Then, there must be some k such that $\mu_k \neq v_k$. Without loss of generality by reordering the coordinates if necessary, assume $\mu_n \neq v_n$. Let $u = (u_1, \ldots, u_{n-1})$ be such that $|u_j| = 1$. Let $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_{n-1})$ be the reduction of u in $\mathbf{A}^{n-1}(\widetilde{\mathbf{F}})$, where $\widetilde{\mathbf{F}}$ denotes the residue class field of $\mathbf{F}$. We want to make a substitution of the form $z_n = z$ and $z_j = u_j$ for $1 \leqslant j \leqslant n-1$ to get a Laurent series in one variable z with at least two non-zero coefficients, but we need to choose the u_j so as not to have any accidental cancellation. But clearly if we choose u with $\tilde{u}$ generic, meaning that there is a non-zero polynomial with coefficients in $\widetilde{\mathbf{F}}$ such that if $\tilde{u}$ is not in the zero locus of that polynomial, then

$$\left| \sum_{\gamma \text{ s.t. } \gamma_n = \mu_n} a_\gamma u_1^{\gamma_1} \cdots u_{n-1}^{\gamma_{n-1}} \right| = \sup\{|a_\gamma| : \gamma_n = \mu_n\} \geqslant |a_\mu| \neq 0$$

and

$$\left| \sum_{\gamma \text{ s.t } \gamma_n = v_n} a_\gamma u_1^{\gamma_1} \cdots u_{n-1}^{\gamma_{n-1}} \right| = \sup\{|a_\gamma| : \gamma_n = v_n\} \geqslant |a_v| \neq 0,$$

and so we will indeed get a one-variable Laurent series with at least two non-zero coefficients. This contradicts the assumption that f is zero-free by the theory of valuation polygons, and so there could have only been one multi-index γ with $a_\gamma \neq 0$. $\square$

Theorem 2.6. *Let $f : \mathbf{A}^{1\times} \to S$ be an analytic map to a semi-Abelian variety S. Then f is the translate of a group homomorphism from $\mathbf{G}_m$ to S.*

Proof. By composing with a translation of S, we may assume without loss of generality that $f(1) = 1$. Let $\phi : S \to A$ be the homomorphism defining S as an extension (by a multiplicative torus) of the Abelian variety A. By the argument on p. 401 of [5], $\phi \circ f$ is a group homomorphism from $\mathbf{G}_m$ to A. Now let $(z_1, z_2) \in \mathbf{G}_m \times \mathbf{G}_m$ and consider the analytic map ψ from $\mathbf{G}_m \times \mathbf{G}_m$ to S defined by

$$\psi(z_1, z_2) = f(z_1 z_2)[f(z_1)]^{-1}[f(z_2)]^{-1}.$$

Because ϕ and $\phi \circ f$ are group homomorphisms, $\phi \circ \psi$ is the constant map from $\mathbf{G}_m \times \mathbf{G}_m$ to the identity element of A. Hence ψ can be thought of as an analytic map to a multiplicative torus T. It then follows from Proposition 2.5 by projecting from T onto each of its factors that ψ is the translation of a group homomorphism. But, since $\psi(1, 1) = 1$, we have that ψ is in fact a group homomorphism. Clearly, $\psi(z_1, 1) = \psi(1, z_2) = 1$, and hence ψ is the constant map to the identity. In other words, f is a group homomorphism. $\square$

Theorem 2.7. *Let $X \subset S$ be a closed subvariety of a semi-Abelian variety S. Let $f : \mathbf{A}^1 \to X$ be a non-constant analytic map. Then, the image of f is contained in the translate of a non-trivial semi-Abelian subvariety of S contained in X.*

Proof. By Theorem 2.6, the map f is the translate of a group homomorphism. Thus, by [12, p. 84], the Zariski closure of the image of f is the translate of a subgroup of S. $\square$

Corollary 2.8 (Non-Archimedean Bloch theorem [5]). *If X is a non-singular projective variety whose Albanese variety has dimension larger than $\dim X$, then any analytic map from $\mathbf{A}^{1\times}$ to X is algebraically degenerate.*

Remark. In [5], Corollary 2.8 was incorrectly stated in terms of the irregularity $q = \dim H^1(X, \mathcal{O}_X)$ rather than the dimension of the Albanese variety. In characteristic zero, both numbers are equal, but an example of Igusa [8] shows that in positive characteristic the dimension of the Albanese variety can be smaller than the dimension of the space of regular 1-forms.

In this section we will develop the non-Archimedean analogs of the work of Noguchi and Winkelmann [16].

Let X be a projective variety non-singular in codimension one. Recall that the space $\mathrm{Pic}(X)$ classifies Cartier divisors on X up to linear equivalence. Those divisor classes in $\mathrm{Pic}(X)$ which are algebraically equivalent to zero are denoted by $\mathrm{Pic}^0(X)$, and $\mathrm{Pic}^0(X)$ is an Abelian variety, known as the Picard variety which is dual to the Albanese variety, see e.g., [10]. The quotient $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is a finitely generated group called the Néron–Severi group of X and denoted $\mathrm{NS}(X)$. Finite generation of $\mathrm{NS}(X)$ is a theorem of Severi in characteristic zero and Néron in positive characteristic. It also follows from the Lang–Néron theorem, see e.g., [11], or by étale cohomology, see e.g., [15]. We will refer to the canonical image of a divisor D in $\mathrm{NS}(X)$ as the *Chern class* of the divisor and denote it by $c_1(D)$. In characteristic zero when X is non-singular, this agrees with the classical notion of Chern class in $H^2(X, \mathbf{Z})$ coming from the exponential sheaf sequence. In positive characteristic, one can embed $\mathrm{NS}(X)$ in an étale cohomology group, see e.g., [15], and think of c_1 that way. For us, the cohomological interpretation will not be important, so we prefer to simply think of c_1 as a homomorphism from divisors to $\mathrm{NS}(X)$.

If $\iota: Y \to X$ is a morphism (or more generally a rational map) from a projective variety Y, non-singular in codimension one, to a non-singular projective variety X, then if D is in $\mathrm{Pic}^0(X)$, then $\iota^* D$ is in $\mathrm{Pic}^0(Y)$ by [10, Ch. V, Prop. 1]. Hence, the pull-back map on divisor classes $\iota^*: \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ induces a homomorphism $\iota^*: \mathrm{NS}(X) \to \mathrm{NS}(Y)$.

We begin by discussing analytic maps from $\mathbf{A}^1$ omitting sufficiently many divisors relative to the size of the group generated by their Chern classes.

Theorem 3.1. *Let Y be a possibly singular projective variety and let $\iota: Y \to X$ be a morphism to a smooth projective variety X. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible effective divisors on X such that $\{\iota^* D_i\}_{i=1}^{\ell}$ form ℓ distinct effective Cartier divisors on Y. Assume the number of irreducible components ℓ is larger than the rank of the subgroup generated by the $c_1(D_i)$ in $\mathrm{NS}(X)$. Then, any analytic map from $\mathbf{A}^1$ to Y is either algebraically degenerate or intersects the support of at least one of the $\iota^* D_i$.*

Any r algebraically independent entire functions form an algebraically non-degenerate analytic map from $\mathbf{A}^1$ to $Y = X = (\mathbf{P}^1)^r$ that omits the r divisors defined by taking the point at ∞ on one of the $\mathbf{P}^1$ factors and thus show the theorem is optimal in its dependence on the rank of the group generated by the $c_1(D_i)$.

Typically what we have in mind for Y is a closed subvariety of X. In that case the map ι is the inclusion in X, and $\iota^* D_i$ is set-theoretically $D_i \cap Y$.

Notice that unlike [16], we do not need to make any kind of general position assumption on the D_i, other than that the $\iota^* D_i$ are distinct.

In the case that $Y = X = \mathbf{P}^n$, we recover the well-known trivial fact that a non-Archimedean analytic map from $\mathbf{A}^1$ to $\mathbf{P}^n$ that omits two distinct hypersurfaces is algebraically degenerate.

Proof of Theorem 3.1. Let $f : \mathbf{A}^1 \to Y$ be an algebraically non-degenerate analytic map.

Let $\widetilde{Y}$ be the normalization of Y, which of course is non-singular in codimension one. Let $\widetilde{\iota} : \widetilde{Y} \to X$ denote the composition of the natural map from $\widetilde{Y}$ to Y with $\iota : Y \to X$.

If f is not algebraically degenerate, then Y is not contained in the indeterminacy locus of the rational map from Y to $\widetilde{Y}$, and hence lifts (as in the proof of Corollary 2.4) to an analytic map $\widetilde{f}$ from $\mathbf{A}^1$ to $\widetilde{Y}$.

By our assumption that there are more components D_i than the rank of the group the $c_1(D_i)$ generate in $\mathrm{NS}(X)$, we can find integers a_i not all zero so that $\sum a_i c_1(D_i) = 0$. Thus, $\sum a_i c_1(\widetilde{\iota}^* D_i) = \widetilde{\iota}^*(\sum a_i c_1(D_i)) = 0$ in $\mathrm{NS}(\widetilde{Y})$, and thus $\sum a_i \widetilde{\iota}^* D_i$ is algebraically equivalent to zero on $\widetilde{Y}$. Because $\widetilde{Y}$ maps onto Y, by our assumption that the $\iota^* D_i$ are distinct, we also have that the $\widetilde{\iota}^* D_i$ are distinct. Also, because not all the a_i are zero, we conclude that $\sum a_i \widetilde{\iota}^* D_i$ is not the zero divisor on $\widetilde{Y}$.

If there is a non-constant rational map from Y to an Abelian variety, then f is already algebraically degenerate by Corollary 2.4. Thus, without loss of generality, we may assume there are no non-constant rational maps from $\widetilde{Y}$ to Abelian varieties, or in other words that the Albanese variety of $\widetilde{Y}$ is trivial. Because the Picard variety $\mathrm{Pic}^0(\widetilde{Y})$ is Cartier dual to the Albanese variety, $\mathrm{Pic}^0(\widetilde{Y})$ is also trivial. But $\mathrm{Pic}^0(\widetilde{Y})$ is precisely the set of divisors algebraically equivalent to zero modulo those divisors linearly equivalent to zero. Hence, every divisor algebraically equivalent to zero on $\widetilde{Y}$ is also linearly equivalent to zero. Thus, we can find a non-constant rational function h on $\widetilde{Y}$ such that

$$\mathrm{div}(h) = \sum a_i \widetilde{\iota}^* D_i.$$

If f omits the supports of all the $\iota^* D_i$, then its lift $\widetilde{f} : \mathbf{A}^1 \to \widetilde{Y}$ is an analytic map omitting the supports of all the $\widetilde{\iota}^* D_i$. Then, $h \circ \widetilde{f}$ is an analytic map from $\mathbf{A}^1$ to $\mathbf{A}^{1\times}$, and hence constant. Thus, $\widetilde{f}$ is algebraically degenerate and so is f. $\square$

Next, we recall that a collection of irreducible effective ample divisors D_i in a non-singular projective variety X of dimension m are said to be *in general position* if for each $1 \leqslant k \leqslant m+1$ and each choice of indices $i_1 < \cdots < i_k$, each irreducible component of

$$D_{i_1} \cap \cdots \cap D_{i_k}$$

has codimension k in X, so in particular is empty when $k = m+1$.

Corollary 3.2. *Let Y be a closed positive dimensional subvariety of a non-singular projective variety X. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible, effective, ample divisors in*

general position on X. Let r be the rank of the subgroup of NS(X) *generated by* $\{c_1(D_i)\}_{i=1}^{\ell}$. *If there exists an algebraically non-degenerate analytic map from* $\mathbf{A}^1$ *to Y omitting each of the D_i that does not contain all of Y, then*

$$\ell \leqslant \max\left\{ r + \operatorname{codim} Y, r \cdot \frac{\dim X}{\dim Y} \right\}.$$

When $r = 1$, the term $r + \operatorname{codim} Y = 1 + \operatorname{codim} Y$ is largest, and the example of Y a linear subspace of $X = \mathbf{P}^n$ shows the inequality is optimal. We do not have examples to show optimality when $r > 1$, and we suspect the inequality may not be optimal in that case. When $Y \subset X = \mathbf{P}^n$, Corollary 3.2 was proven by An, Wang and Wong [2].

Corollary 3.3. *Let X be a non-singular projective variety. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible, effective, ample divisors in general position on X. Let r be the rank of the subgroup of* NS(X) *generated by* $\{c_1(D_i)\}_{i=1}^{\ell}$. *Let f be an analytic map from $\mathbf{A}^1$ to X omitting each of the D_i. Then the image of f is contained in an algebraic subvariety Y of X such that*

$$\dim Y \leqslant \max\left\{ r + \dim X - \ell, \frac{r \cdot \dim X}{\ell} \right\}.$$

In particular, if

$$\ell \geqslant \max\{r + \dim X, r \cdot \dim X + 1\},$$

then f is constant.

Note that when $Y = X = \mathbf{P}^n$, the fact that an analytic map from $\mathbf{A}^1$ omitting $n + 1$ hypersurfaces in general position must be constant also follows from Ru's defect inequality [18]. The fact that an analytic map from $\mathbf{A}^1$ to a projective variety $X \subset \mathbf{P}^N$ omitting $\dim X + 1$ hypersurfaces of $\mathbf{P}^N$ in general position with X is a consequence of An's defect inequality [1].

Proof of Corollary 3.2. Suppose f is an algebraically non-degenerate analytic map from $\mathbf{A}^1$ to Y omitting the D_i. Let ℓ_0 be the cardinality of the set

$$\{D_i \cap Y \colon D_i \not\supset Y\},$$

and note that $D_i \cap Y \neq \emptyset$ for all i because the D_i are assumed ample. By the theorem,

(1) $\qquad \ell_0 \leqslant r.$

We now estimate ℓ_0 as in [16]. Let $n = \dim Y$. Without loss of generality we may assume that $D_1 \cap Y, D_2 \cap Y, \ldots, D_{\ell_0} \cap Y$ are distinct. For $1 \leqslant j \leqslant \ell_0$, let s_j be the

number of divisors D_i with $D_\iota \cap Y = D_J \cap Y$. Rearranging the indices, we may assume that

(2) $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_{\ell_0}$.

We first consider the case when $\ell_0 \leqslant n$. Because the D_ι are ample and by the definition of ℓ_0, we have

$$\emptyset \neq Y \cap \left(\bigcap_{j=1}^{\ell_0} D_J \right) = Y \cap \left(\bigcap_{j=1}^{\ell} D_J \right).$$

Because the D_i are in general position, this implies

$$\dim Y - \ell_0 \leqslant \dim X - \ell,$$

and hence

$$\ell \leqslant \ell_0 + \operatorname{codim} Y \leqslant r + \operatorname{codim} Y$$

by (1).

The remaining case is $\ell_0 > n$. Again, because the D_i are ample,

$$\emptyset \neq Y \cap \left(\bigcap_{j=1}^{n} D_J \right) = Y \cap \left(\bigcap_{\iota \in I} D_i \right),$$

where $I = \{i : D_i \supset Y \text{ or } D_i \cap Y = D_J \cap Y \text{ for some } 1 \leqslant j \leqslant n\}$. Let s_0 denote the number of divisors D_i such that $D_i \supset Y$. Since the divisors are in general position, this implies

(3) $$\sum_{\iota=0}^{n} s_\iota = \#I \leqslant \dim X.$$

On the other hand, it follows from (2) that

$$\frac{1}{\ell_0} \sum_{\iota=1}^{\ell_0} s_i \leqslant \frac{1}{n} \sum_{\iota=1}^{n} s_i.$$

Therefore,

(4) $$\sum_{i=1}^{\ell_0} s_i \leqslant \frac{\ell_0}{n} \sum_{\iota=1}^{n} s_i.$$

As $\ell_0 > n$, we have $s_0 \leqslant \frac{\ell_0}{n} s_0$. Combining this with (4), (3), and (1), we have

$$\ell = \sum_{i=0}^{\ell_0} s_i \leqslant \frac{\ell_0}{n} \sum_{\iota=0}^{n} s_i \leqslant \frac{\ell_0}{n} \dim X \leqslant \frac{r}{n} \dim X. \qquad \square$$

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We conclude with the analog of the Noguchi–Winkelmann theorem for non-Archimedean analytic maps from $\mathbf{A}^{1\times}$. We remind the reader that in the complex case, the exponential map provides a non-constant complex analytic map from $\mathbf{A}^1(\mathbf{C})$ to $\mathbf{A}^{1\times}(\mathbf{C})$ and of course $\mathbf{A}^{1\times}(\mathbf{C}) \subset \mathbf{A}^1(\mathbf{C})$, so that in complex analytic geometry there is no difference in algebraic degeneracy theorems from maps from $\mathbf{A}^1(\mathbf{C})$ or $\mathbf{A}^{1\times}(\mathbf{C})$. However, in the non-Archimedean case the only analytic maps from $\mathbf{A}^1$ to $\mathbf{A}^{1\times}$ are the constants, and thus it is more difficult to have an algebraically non-degenerate analytic map from $\mathbf{A}^1$ than it is to have an algebraically non-degenerate analytic map from $\mathbf{A}^{1\times}$.

Theorem 3.4. *Let Y be a possibly singular projective variety that admits a desingularization $\widetilde{Y} \to Y$, and let $\iota : Y \to X$ be a morphism to a smooth projective variety X. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible effective divisors on X such that $\{\iota^* D_i\}_{i=1}^{\ell}$ form ℓ distinct effective Cartier divisors on Y. Let a denote the dimension of the Albanese variety of Y. Let r be the rank of the subgroup generated by the $c_1(D_i)$ in $\mathrm{NS}(X)$. If $\ell > r + \dim Y - a$, then any analytic map from $\mathbf{A}^{1\times}$ to Y is either algebraically degenerate or intersects the support of at least one of the $\iota^* D_i$.*

Remark. In characteristic zero by Hironaka's Theorem, any Y admits a desingularization $\widetilde{Y}$. Because resolution of singularities is not yet known in positive characteristic, we make the existence of a desingularization an explicit hypothesis. Unlike in Theorem 3.1, here we will need a morphism rather than a rational map to the Albanese variety, so working with a normalization is not sufficient.

Proof of Theorem 3.4. Let $\tilde{\iota}$ be the natural map from $\widetilde{Y}$ to X induced by ι. Let Y' be the variety obtained by deleting the supports of $\tilde{\iota}^* D_i$ from $\widetilde{Y}$. By [19], there is a morphism α from Y' to a semi-Abelian variety S such that S is generated by the differences of points in the image of Y' and such that S is the extension of the Albanese variety A of $\widetilde{Y}$ by a multiplicative torus. As in [20], let I denote the free Abelian group generated by the $\tilde{\iota}^* D_i$ and let J be the kernel of the mapping from I to $\mathrm{NS}(\widetilde{Y})$. Then, it follows from the discussion in [20] that the dimension of S is the dimension of A plus the rank of J.

Let K be the subgroup of the free Abelian group generated by the D_i that maps to zero in $\mathrm{NS}(X)$. By hypothesis, K has rank $\ell - r$. Consider the map $\tilde{\iota}^* : K \to J$. By our assumption that the $\iota^* D_i$ are distinct and the fact that $\widetilde{Y}$ maps onto Y, we have that the $\tilde{\iota}^* D_i$ are distinct. Hence, $\tilde{\iota}^*$ injects K into J, and thus J has rank at least $\ell - r$. So, if $\ell > r + \dim Y - a$, then $\dim S \geq \ell - r + a > \dim Y$. If f is an analytic map to Y not contained in the singular locus and not intersecting the supports of $\iota^* D_i$, then f lifts to an analytic map $\tilde{f}$ to Y'. By Theorem 2.7, $\alpha \circ \tilde{f}$ is contained in the translate of a semi-Abelian subvariety of S contained in the proper subvariety $\alpha(Y')$. Because differences of points in $\alpha(Y')$ generate S, the variety $\alpha(Y')$ cannot be a translate of a proper semi-Abelian subvariety of S, and hence $\tilde{f}$ and f are algebraically degenerate. $\square$

Applying the argument on pp. 606–607 of [16] i.e., replacing $\ell_0 \leqslant r$ with $\ell_0 \leqslant r + \dim Y - a$ in (1) in the proof of our Corollary 3.2, then yields the following corollaries.

Corollary 3.5. *Let Y be a closed positive dimensional subvariety of a non-singular projective variety X admitting a desingularization $\widetilde{Y} \to Y$. Let a be the dimension of the Albanese variety of Y. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible, effective, ample divisors in general position on X. Let r be the rank of the subgroup of $\mathrm{NS}(X)$ generated by $\{c_1(D_i)\}_{i=1}^{\ell}$. If there exists an algebraically non-degenerate analytic map from $\mathbf{A}^{1\times}$ to Y omitting each of the D_i that does not contain all of Y, then*

$$(\ell - \dim X) \cdot \dim Y \leqslant \dim X \cdot \max\{0, r - a\}.$$

Corollary 3.6. *Let X be a non-singular projective variety in characteristic zero. Let $\{D_i\}_{i=1}^{\ell}$ be $\ell > \dim X$ irreducible, effective, ample divisors in general position on X, and let r be the rank in $\mathrm{NS}(X)$ of the subgroup generated by $\{c_1(D_i)\}_{i=1}^{\ell}$. Let f be an analytic map from $\mathbf{A}^{1\times}$ to X omitting each of the D_i. Then the image of f is contained in an algebraic subvariety Y of X such that*

$$\dim Y \leqslant \frac{r \cdot \dim X}{\ell - \dim X}.$$

In particular, if

$$\ell \geqslant (r + 1) \cdot \dim X + 1,$$

then f is constant.

ADDENDUM ADDED AUGUST 21, 2008

We remarked that the inequality in Corollary 3.2 may not be optimal when the rank r of $\mathrm{NS}(X)$ is greater than one. While this article was under review, Lin and Wang [14] proved that any analytic map from $\mathbf{A}^1$ to a normal projective variety X omitting $\ell > \dim X$ irreducible, effective, ample divisors in general position must be constant. This is evidence that under the hypotheses of Corollary 3.2, the number of components ℓ may be able to be bounded above independent of r, i.e., bounded only in terms of the dimension of X and the dimension of the image of the analytic map. Lin and Wang made a precise conjecture to that effect. We would also like to refer the reader to related conjectures about complex holomorphic curves and integral points in the work of Levin [13].

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