Discrete Applied Mathematics 3 (1981) 257–262
North-Holland Publishing Company

PLANAR KERNEL AND GRUNDY WITH $d \leq 3$, $d_{\text{out}} \leq 2$, $d_{\text{in}} \leq 2$ ARE NP-COMPLETE

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Received 12 June 1980
Revised 17 February 1981

It is proved that the questions whether a finite digraph G has a kernel K or a Sprague–Grundy function g are NP-complete even if G is a cyclic planar digraph with degree constraints $d_{\text{out}}(u) \leq 2$, $d_{\text{in}}(u) \leq 2$ and $d(u) \leq 3$. These results are best possible (if P ≠ NP) in the sense that if any of the constraints is tightened, there are polynomial algorithms which either compute K and g or show that they do not exist. The proof uses a single reduction from planar 3-satisfiability for both problems.

Throughout $G=(V,E)$ denotes a finite digraph, and Z^0 the set of non-negative integers. Define the set of *followers* of $u \in V$ by

$$F(u)=\{v \in V: (u,v) \in E\}$$

and the set of *predecessors* of $w \in V$ by

$$P(w)=\{v \in V: (v,w) \in E\}.$$

Then the *out-degree* of $u \in V$ is $d_{\text{out}}(u)=|F(u)|$, the *in-degree* $d_{\text{in}}(u)=|P(u)|$ and $d(u)=d_{\text{out}}(u)+d_{\text{in}}(u)$. If $F(u)=\emptyset$, u is called a *sink*.

A *kernel* of G is a subset $K \subseteq V$ such that for every $u \in V$,

$$u \in K \Leftrightarrow F(u) \cap K = \emptyset,$$

that is, K is independent ($\Rightarrow$) and dominating ($\Leftarrow$). See e.g., Berge [1, Ch. 14] or Garey and Johnson [4].

If S is a finite subset of Z^0, define the "minimum excluded value" of S by $\text{mex}\, S = \min \bar{S}$ = least nonnegative integer not in S. A (classical) *Sprague–Grundy* function $g: V \to Z^0$ (also called *Grundy* function) is defined by $g(u)=\text{mex}\, g(F(u))$, where for any set T and any function h on T, $h(T)=\{h(t): t \in T\}$. See Berge [1] or Garey and Johnson [4]. The kernel and Sprague–Grundy function have applications in combinatorial game theory and elsewhere. See Berge [1]. There is a "one-way" connection between the two:

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Lemma 1. *If $G=(V,E)$ is a digraph and g a Sprague–Grundy function on G, then $K=\{u\in V: g(u)=0\}$ is a kernel of G.*

The proof is immediate. The converse is false. For both see [1, Ch. 14, § 3].

Since $\text{mex}\,\emptyset=0$, $g(u)=0$ for a sink u. For "simple" digraphs, K and g exist uniquely [1, Theorem 7], and can be computed easily:

Lemma 2. *If $G=(V,E)$ is a finite acyclic digraph, then K and g exist uniquely on G and they can be computed in polynomial time.*

Proof. We give an algorithm for computing g and K in "endorder", that is, labeling a vertex only after all its followers have been labeled: Label all sinks by 0; if u is any unlabeled vertex such that $F(u)$ has already been labeled, then label u with $\text{mex}\, l(F(u))$, where $l(u)$ denotes the label of u.

Since G is finite and acyclic, at least one sink always exists, and so the algorithm can always be started. Since G is acyclic, no predecessor of an unlabeled vertex u is labeled when the set $F(u)$ has been labeled. Hence the labeling exists and satisfies $l(u)=\text{mex}\, l(F(u))$ for every $u\in V$. The labels are unique, since if we assume that all labels in $l(F(u))$ are unique, then also $l(u)$ is unique. The algorithm clearly requires only $O(|V|\,|E|)$ steps.

The result now follows by letting $g(u)=l(u)$ for all $u\in V$ and $K=\{u\in V: g(u)=0\}$. □

Define the following two decision problems:

KERNEL (KE). *Given a finite digraph $G=(U,V)$, does G have a kernel?*

SPRAGUE–GRUNDY (SG). *Given a finite digraph $G=(U,V)$, does G have a Sprague–Grundy function?*

Both of these problems are known to be NP-complete. See Chvátal [2], and van Leeuwen [7]. The problem SG is NP-complete even when restricted to planar digraphs with $d_{\text{out}}(u)\le 5$ and $d_{\text{in}}(u)\le 5$ for every $u\in V$. See [3]. For another reference for both problems as well as for the general concepts of NP-completeness and polynomial reduction used below, see Garey and Johnson [4].

The decision problems RESTRICTED PLANAR KERNEL (PKE3) and RESTRICTED PLANAR SPRAGUE–GRUNDY (PSG3) are defined as above, except that $G=(V,E)$ is a finite cyclic planar digraph with $d_{\text{out}}(u)\le 2$, $d_{\text{in}}(u)\le 2$ and $d(u)\le 3$ for every $u\in V$. Since $g(u)$ is clearly bounded by the maximum out-degree of the given digraph, $\text{SG}\in\text{NP}$. Also $\text{KE}\in\text{NP}$. Thus a fortiori $\text{PKE3}, \text{PSG3}\in\text{NP}$. Our main result is that even these restricted problems seem difficult:

Theorem. PKE3 *and* PSG3 *are* NP-*complete.*

For the proof we define an additional decision problem: Let 3CNF be the set of Boolean formulas $B = C_1 \wedge \cdots \wedge C_m$ in the variables $x_1, \ldots, x_n$, where each clause C_i is the disjunction of three literals, and the set of literals is $\{x_1, \ldots, x_n\} \cup \{\bar{x}_1, \ldots, \bar{x}_n\}$. A Boolean formula B is *satisfiable* if there is a truth assignment $(t_1, \ldots, t_n) \in \{0,1\}^n$ to the variables such that $B = 1$, where $0 \vee 0 = 0 \wedge 0 = 0 \wedge 1 = 1 \wedge 0 = 0$, $0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1 \wedge 1 = 1$. The *multigraph* $G(B) = (V, E)$ *of* B is defined by

$$V = \bigcup_{i=1}^{m} c_i \bigcup_{j=1}^{n} x_j, \qquad E = E^1 \cup E^2,$$

where

$$E^1 = \bigcup_{i=1}^{m} \{(c_i, x_j) : x_j \in C_i\}, \qquad E^2 = \bigcup_{i=1}^{m} \{(c_i, x_j) : \bar{x}_j \in C_i\}.$$

See Fig. 1. Note that different formulas B may map into the same multigraph $G(B)$ (such as $\bar{x}_1 \vee \bar{x}_2 \vee x_3$ and $x_1 \vee \bar{x}_2 \vee \bar{x}_3$).

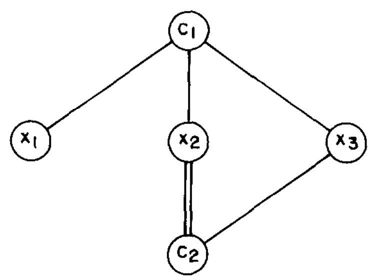

Fig. 1. The multi-graph $G(B)$ for the case $B = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee \bar{x}_2 \vee x_3)$.

PLANAR 3-SATISFIABILITY (P3SAT). *Given a Boolean formula* $B \in$ 3CNF *such that* $G(B)$ *is planar, is* B *satisfiable*?

Lemma 3. (Lichtenstein [6]). P3SAT *is* NP-*complete.*

Proof of Theorem. We use a single reduction from P3SAT for both problems. Let $B = C_1 \wedge \cdots \wedge C_m$ in $x_1, \ldots, x_n$ be an instance of P3SAT. Construct the planar multigraph $G(B)$ of B as defined above. Now "blow up" the vertices of $G(B)$ as follows: Replace each c_i by a directed "triangle with tree" $T_i = (V_i, E_i)$, where

$$V_i = \{c_i, d_i, e_i, p_i, q_i, r_i, s_i, t_i, u_i\},$$

$$E_i = \{(c_i, d_i), (d_i, e_i), (e_i, c_i), (c_i, p_i), (p_i, q_i), (q_i, r_i), (r_i, t_i), (q_i, s_i), (s_i, u_i)\}.$$

As in Garey, Johnson and Stockmeyer [5], let $m(j)$ denote the total number of times x_j and $\bar{x}_j$ appear in B. Replace each vertex x_j in $G(B)$ by a directed cycle $S_j = (V_j', E_j')$, where

$$V_j' = \{x_{1j}, \bar{x}_{1j}, \ldots, x_{m(j),j}, \bar{x}_{m(j),j}\},$$

$$E_j' = \{(x_{1j}, \bar{x}_{1j}), (\bar{x}_{1j}, x_{2j}), \ldots, (x_{m(j),j}, \bar{x}_{m(j),j}), (\bar{x}_{m(j),j}, x_{1j})\}.$$

For fixed i $(1 \le i \le m)$, each edge $e_i = (c_i, x_j) \in E(B)$ is replaced by a directed edge (t_i, x_{hj}) or (u_i, x_{hj}) (if $e_i \in E^1$) or by a directed edge $(t_i, \bar{x}_{hj})$ or $(u_i, \bar{x}_{hj})$ (if $e_i \in E^2$), with distinct h for each edge, such that $d_{\text{out}}(t_i) = 2$, $d_{\text{out}}(u_i) = 1$ (see Fig. 2). This gives a set E_i'' of three edges. All of this can be done so that planarity is preserved. Thus the resulting digraph $G(V, E)$ is planar, with $d_{\text{out}}(u) \le 2$, $d_{\text{in}}(u) \le 2$, $d(u) \le 3$ for all $u \in V$, where

$$V = \bigcup_{i=1}^{m} V_i \bigcup_{j=1}^{n} V_j', \qquad E = \bigcup_{i=1}^{m} \{E_i, E_i''\} \bigcup_{j=1}^{n} E_j'.$$

It is also clear that the construction is polynomial.

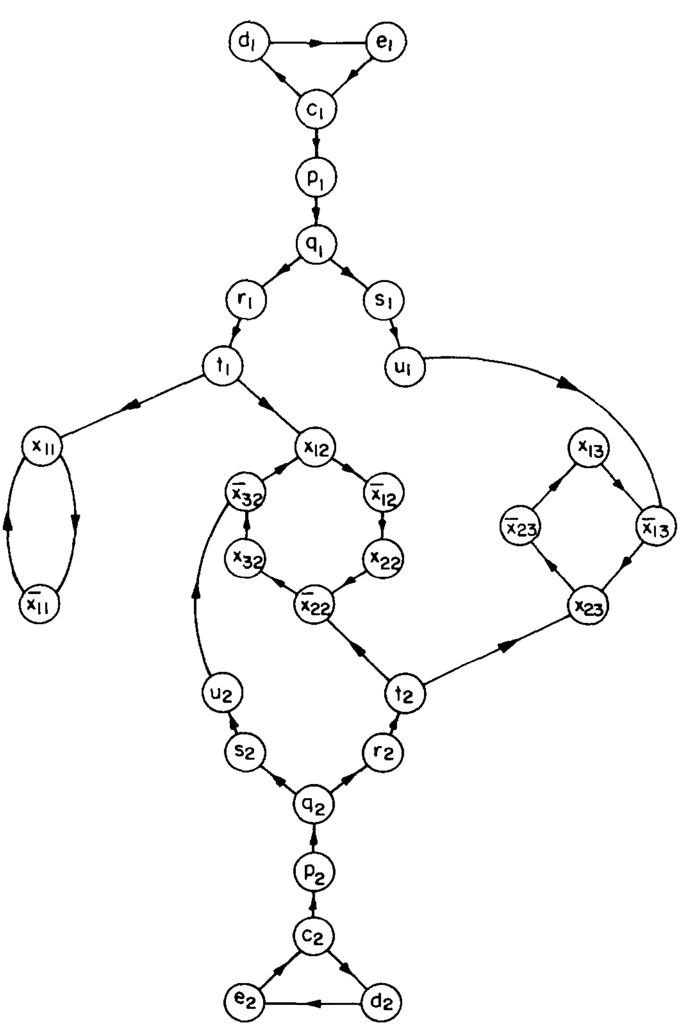

Fig. 2. The construction for $B = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee \bar{x}_2 \vee x_3)$.

Regarding the subgraph G_i consisting of the vertices q_i, r_i, s_i, t_i, u_i and the edges (q_i, r_i), (r_i, t_i), (q_i, s_i), (s_i, u_i), we note that q_i is in a kernel K if and only if $(F(t_i) \cup F(u_i)) \cap K = \emptyset$; and $g(q_i) = 0$ if and only if $0 \notin g(F(t_i) \cup F(u_i))$ $(1 \le i \le m)$.

Now suppose that we are given a satisfying truth assignment for B. For each j $(1 \le j \le m)$, label all $m(j)$ vertices $x_{1j}, \ldots, x_{m(j),j}$ by 0 and all vertices $\bar{x}_{1j}, \ldots, \bar{x}_{m(j),j}$ by 1 (if $x_j = 1$) or the reverse (if $x_j = 0$). Since $C_i = 1$, at least one vertex in $F(t_i) \cup F(u_i)$ is labeled 0 for every i $(1 \le i \le m)$. Label each of the vertices of the subgraph G_i by the mex of the labels of their followers. This results in a positive label (1 or 2) at q_i. Also label e_i and p_i by 0, d_i by 1 and c_i by 2 $(1 \le i \le m)$. It is immediate that the labels are the value of a Sprague–Grundy function on G, and that $K = \{u \in V: g(u) = 0\}$ is a kernel (Lemma 1).

Conversely, suppose that G has a kernel K. For each i $(1 \le i \le m)$, exactly one vertex of each triple $\{c_i, d_i, e_i\}$ has to be in K, and it is clear that this must be e_i. This implies $p_i \in K$, $q_i \notin K$. Hence $(F(t_i) \cup F(u_i)) \cap K \ne \emptyset$. So there is either some edge $(t_i, x_{hj}) \in E''$ or $(u_i, x_{hj}) \in E''$ with $x_{hj} \in K$ in which case we put $x_j = 1$, or there is an edge $(t_i, \bar{x}_{hj}) \in E''$ or $(u_i, \bar{x}_{hj}) \in E''$ with $\bar{x}_{hj} \in K$ in which case we put $x_j = 0$. This induces a consistent satisfying truth assignment on a subset of the variables since clearly each S_j has precisely $m(j)$ vertices in K, either all of the form x_{ij} or all of the form $\bar{x}_{ij}$.

Now suppose that G has a Sprague–Grundy function g. If $g(e_i) > 0$, then $g(d_i) = 0$, hence $g(c_i) > 0$ and so $g(e_i) = 0$, a contradiction. Hence $g(e_i) = 0$, $g(d_i) = 1$, $g(p_i) = 0$, $g(c_i) = 2$, $g(q_i) > 0$. Hence there must be some $x_{hj} \in F(t_i) \cup F(u_i)$ such that $g(x_{hj}) = 0$ or some $\bar{x}_{hj} \in F(t_i) \cup F(u_i)$ such that $g(\bar{x}_{hj}) = 0$. In the former case we have actually $g(x_{hj}) = 0$, $g(\bar{x}_{hj}) = 1$ $(1 \le h \le m(j))$ and we put $x_j = 1$, and in the latter case the equalities are reversed and we put $x_j = 0$. In any case $C_i = 1$ $(1 \le i \le m)$ and B is satisfiable. □

Note. The theorem is best possible (if $P \ne NP$). To see this, we may assume that $G = (V, E)$ is a connected cyclic digraph. Because if it is not connected, apply the polynomial procedures below to each connected component, and if it is acyclic then Lemma 2 applies and g can be computed polynomially.

(i) Suppose that $d(u) = d_{\text{out}}(u) + d_{\text{in}}(u) \le 2$ for all $u \in V$. This implies that G consists of a single simple cycle. Then G has a kernel K and a Sprague–Grundy function g if and only if $|V| \equiv 0 \pmod 2$. If the condition holds, the g-values 0 and 1 alternate along the cycle, and $K = \{u \in V: g(u) = 0\}$.

(ii) Suppose that $d_{\text{out}}(u) \le 1$ for all $u \in V$. Then G consists of a single cycle, possibly with several ingoing trees impinging on it. If V' is the set of vertices on the cycle, then G has a kernel and a Sprague–Grundy function if and only if $|V'| \equiv 0 \pmod 2$. If this holds, g along the cycle is determined as in (i), and these values clearly prescribe g-values on the trees. A kernel is determined as in (i).

(iii) Suppose that $d_{\text{in}}(u) \le 1$ for all $u \in V$. Then G consists of a single cycle, possibly with several outgoing trees cropping up from it. The g-values on the trees can be computed by the algorithm indicated in the proof of Lemma 2. Let u be any vertex on the cycle. If any of the labels 0, 1 or 2 for u leads to a consistent assignment of labels for all vertices on the cycle, then G has a Sprague–Grundy function, otherwise it does not have one. A slight variation of this algorithm determines a kernel if there is one or shows that there is none.

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