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Comparison Principle and Nonlinear Contractions in Abstract Spaces

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1. INTRODUCTION

The theory of differential and integral equations exploits comparison and iterative techniques which do not fall under the Contractive Mapping Principle. For they make use of partial orderings and maximal solutions; concepts which have no significance in a metric space. These methods take their proper place in the theory of cones [1, 2].

In this paper, Banach's Contraction Mapping Principle and comparison and iterative methods are brought together under a single roof which houses various results from the theory of differential equations [3], in addition to an interesting generalization of Banach's Theorem [4]. One is thus led, in a natural way to a generalized Bellman–Gronwall–Reid Inequality and a discussion of nonlinear contractions of a space whose open neighborhoods are conic segments.

The advantage of such a uniform principle is twofold. First, by means of it, one is able to present many different results in one stroke while focusing more clearly on the basic ideas involved; second, one is able to isolate concepts widely used in the theory of differential equations for further application to the general theory of nonlinear analysis.

2. DEFINITIONS

Let E be a real Banach space. A set $k \subset E$ is called a *cone* if: (i) k is closed; (ii) if $u, v \in k$ then $\alpha u + \beta v \in k$ for all $\alpha, \beta \geqslant 0$; (iii) of each pair of vectors u, $-u$ at least one does not belong to k, provided $u \neq \theta$, where θ is the zero of the space E. We say that $u \geqslant v$ if and only if $u - v \in k$. A cone k is called *normal* if a $\delta > 0$ exists such that $\| e_1 + e_2 \| \geqslant \delta$ for $e_1, e_2 \in k$ and $\| e_1 \| = \| e_2 \| = 1$. A cone is called *regular* if each *decreasing* monotonic sequence has a limit.

Let ψ be a mapping from a cone k into itself. The mapping ψ is said to be monotone if $\psi u \geqslant \psi v$ whenever $u \geqslant v$. The mapping ψ is *upper semicontinuous from the right* if whenever $\{u_n\}$ and $\{\psi u_n\}$ are both convergent (in norm), decreasing sequences then $\lim \psi u_n \leqslant \psi \lim u_n$.

Let u, v belong to a cone k. The strict inequality $u < v$ means that for any decreasing sequence $\{u_m\}$ which converges to θ, $u_m \leqslant v - u$ for m sufficiently large.

The norm in E is said to be *semimonotonic* if there is a numerical constant N such that $x \leqslant y$ implies $\| x \| \leqslant N \| y \|$.

Let x be a set and k be a cone. A function $\rho: X \times X \to k$ is said to be a *k-metric* on x if

$$\rho(x, y) = \rho(y, x), \qquad \rho(x, y) = \theta \quad \text{iff} \quad x = y,$$
$$\rho(x, y) \leqslant \rho(x, z) + \rho(z, y).$$

A sequence $\{x_n\}$ in a k-metric space is said to be a Cauchy sequence if

$$\limsup_{m \geqslant n \to \infty} \| \rho(x_n, x_m) \| = 0.$$

The sequence $\{x_n\}$ is said to be convergent if there is a $\bar{x} \in X$ such that

$$\lim_{n \to \infty} \| \rho(x_n, \bar{x})) \| = 0.$$

A k-metric space is complete if every Cauchy sequence is convergent.

3. A Bellman–Gronwall–Reid Type Inequality

We shall make the following hypotheses pertaining to an element $u_0 \neq \theta$ belonging to a cone k and to a mapping ψ of the segment $\theta \leqslant u \leqslant u_0$ into itself.

(H_1). $\psi^{n+1} u_0 \leqslant \psi^n u_0$, $n = 0, 1, \ldots$; if $\bar{u} = \lim_{n \to \infty} \psi^n u_0$ exists then $\psi \bar{u} \leqslant \bar{u}$.

(H_2). Either (i) ψ is upper semicontinuous from the right, k is regular or (ii) ψ is completely continuous, k is normal (or both).

LEMMA 3.1. *If*

$$\psi u_0 \leqslant u_0 \tag{1}$$

and ψ is monotone on $\theta \leqslant u \leqslant u_0$ then (H_1) *is satisfied.*

Proof. By induction $\psi^{n+1}u_0 \leqslant \psi^n u_0$, $n = 0, 1,\ldots$. If $\bar{u} = \lim_{n\to\infty} \psi^n u_0$ exists then, since $\{\psi^n u_0\}$ is decreasing, $\bar{u} \leqslant \psi^n u_0$ and by monotonicity $\psi\bar{u} \leqslant \psi^{n+1}u_0$, $n = 0, 1,\ldots$. Thus $\psi\bar{u} \leqslant \bar{u}$.

LEMMA 3.2. *Let* (H_2) *hold. If* $\{r_n\}$ *is a sequence in the conic segment* $\theta \leqslant u \leqslant u_0$ *such that* $\{\psi r_n\}$ *is decreasing, then* $\{\psi r_n\}$ *is convergent. In particular, if* (H_1) *holds,* $w = \lim \psi^n u_0$ *exists and* $\psi w = w$.

Proof. If (H_2) (i) holds then $\{\psi r_n\}$ is convergent by regularity. If (H_2) (ii) holds then we use the semimonotonic property of the norm in a normal cone [1, p. 24] to show that $\{\psi u_n\}$ is bounded. By complete continuity of ψ, $\{\psi r_n\}$ has a convergent subsequence, say $\{v_n\}$. Let $z = \lim v_n$. Suppose $\{\psi r_n\}$ is not convergent to z. Then there exists $\delta > 0$ and a subsequence $\{w_n\}$ of $\{\psi r_n\}$ such that $\| w_n - z \| \geqslant \delta$ and $w_n \leqslant v_n$. By semimonotonicity of the norm there is a numerical constant M such that

$$\| w_n - z \| \leqslant M \| v_n - z \| \to 0$$

as $n \to \infty$ and a contradiction is reached.

Let (H_1) hold. Then $w = \lim \psi^n u_0$ exists and $\psi w \leqslant w$. If (H_2) holds then also $w \leqslant \psi w$ by the semicontinuity condition.

THEOREM 3.1. *Let* ψ *be a monotone mapping of a cone segment* $\theta \leqslant u \leqslant u_0$ *into itself and let* (H_2) *be satisfied. Then the sequence of iterates* $\{\psi^n u_0\}$ *is decreasing and convergent to a fixed point* w *of* ψ*, i.e.,* $\psi u = w$. *Moreover,* $v \leqslant u_0$, $v \leqslant \psi v$ *implies* $v \leqslant w$. *In particular,* w *is the maximal fixed point of* ψ *in the segment.*

Proof. Lemmas 3.1, 3.2 apply to obtain the first part. Suppose $v \leqslant u_0$, $v \leqslant \psi v$. Then by the monotone property of ψ, $v \leqslant \psi^n v \leqslant \psi^n u_0$. Hence $v \leqslant w$.

COROLLARY 3.1. *Let the hypothesis of Theorem* 3.1 *be satisfied and let* p *be a mapping of a set* X *into the segment* $\theta \leqslant u \leqslant u_0$. *Suppose* T *is a mapping of* X *into itself such that* $pTx \leqslant \psi px$. *Then if* y *is a fixed point of* T, $py \leqslant w$ *where* w *is the maximal fixed point of* ψ *on the segment.*

Proof. Set $m = py$. Then $m = pTy \leqslant \psi py = \psi m$. By Theorem 3.1, $m \leqslant w$.

4. SUCCESSIVE ITERATES IN k-METRIC SPACES

THEOREM 4.1. *Let* ψ *be a mapping of a cone segment,* $\theta \leqslant u \leqslant u_0$ *into itself satisfying* (H_2). *Suppose further:*

(H_3) $\psi u_0 < u_0$;

(H_4) $\psi u = u$ *iff* $u = \theta$;

(H_5) ψ *is monotone.*

Let X be a complete k-metric space and let T be a mapping of X into itself such that

(H_6) $\rho(Tx, Ty) \leqslant \psi\rho(x, y)$, *when* $\rho(x, y) \leqslant u_0$.

Then if x_0 is any member of X for which $\rho(Tx_0, x_0) \leqslant u_0$, the sequence iterates $\{T^n x_0\}$ converges to a fixed point w, i.e., $Tw = w$. The fixed point w is unique in $B_w = \{y \mid \rho(y, w) \leqslant u_0\}$. If given any $x, y \in X$ there is a $u_0 \geqslant \rho(x, y)$ for which (H_2)–(H_6) is satisfied then the choice x_0 is completely arbitrary and the fixed point w is unique in X.

Proof. Let $x_n = T^n x_0$, $u_n = \psi^n u_0$. In view of Lemmas 3.1, 3.2, $\{u_n\}$ decreases and converges to a fixed point of ψ, which by (H_4), is θ using (H_6), the monotonicity of ψ, and $\rho(x_1, x_0) \leqslant u_0$ one shows further, by induction, that

$$\rho(x_{n+1}, x_n) \leqslant u_n \to \theta, \qquad n = 0, 1, \ldots \tag{1}$$

For $m \geqslant n$,

$$\begin{aligned}\rho(x_n, x_m) &\leqslant \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{m+1}) + \rho(x_{m+1}, x_m)\\ &\leqslant u_n + \rho(Tx^n, Tx_m) + u_m \leqslant \psi\rho(x_n, x_m) + 2u_n.\end{aligned}$$

As long as there is an integer $N \geqslant 1$ such that for $n \geqslant N$ the maps $\psi_n \equiv \psi + 2u_n$ satisfy the hypothesis of Theorem 3.1.

$$\rho(x_n, x_m) \leqslant r_n, \qquad m \geqslant n \geqslant N, \tag{2}$$

where r_n is the maximal fixed point of ψ_n. The existence of such an integer N is assured by (H_3) and $r_n = \lim_{k\to\infty} \psi_n{}^k u_0$. Since $\psi_j u \geqslant \psi_i u$ for $j \geqslant i \geqslant N$, $\theta \leqslant u \leqslant u_0$, $\{r_n\}$ is decreasing and hence $\{\psi r_n\}$ is also decreasing. From Lemma 3.2, $\{\psi r_n\}$ is convergent. Since

$$r_n = \psi r_n + 2u_n, \qquad n \geqslant N, \tag{3}$$

$\{r_n\}$ is also convergent to the same limit, say z. By semicontinuity, $\psi z \geqslant z$. In view of Theorem 3.1 and (H_4),

$$r_n \to \theta, \qquad \psi r_n \to \theta. \tag{4}$$

The sequence $\{x_n\}$ is, by (2), a Cauchy sequence. Let $y = \lim x_n$, i.e., $\rho(y, x_n) \to \theta$. From (2), $\rho(y, x_n) \leqslant r_n$, from which results $\psi\rho(y, x_n) \leqslant \psi r_n$. Thus

$$\rho(y, Ty) \leqslant \rho(y, x_n) + \rho(x_n, x_{n+1}) + \rho(Tx_n, Ty) \leqslant r_n + u_n + \psi r_n \to \theta.$$

Hence $\rho(y, Ty) = \theta$ and y is a fixed point of T.

Suppose $Ts = s$ and $\rho(s, y) \leqslant u_0$. Then $\rho(s, y) = \rho(Ts, Ty) \leqslant \psi\rho(s, y)$ and by Theorem 3.1 and (H_4), $\rho(s, y) = \theta$.

This completes the argument except for the obvious statements concerning global uniqueness and the choise of x_0.

Remarks 4.1. The above result is patterned after a theorem in [3, Vol. 1, p. 60] concerning the existence and uniqueness of a system of ordinary differential equations. (H_6) will be recognized by experts in differential equations as a generalization of Perron's uniqueness condition. The above result contains, as special cases, this theorem as well as a similar, new theorem which will be proved below.

If one specializes to the cone of nonnegative real numbers, R^+, then one obtains a generalization of Banach's Contraction Mapping Theorem. Observe that the linear map $\psi u = \alpha u$ of R^+ into itself where, $0 < \alpha < 1$, satisfies all the hypotheses (H_2)–(H_6). However, when one works in a metric space i.e., when k is completely ordered, then one can replace the monotonic condition by $\psi u < u$, (cf. [4, p. 459]).

The first part of the proof of Theorem 4.1 resembles the first part of the proof in [4]. First, note that the condition

$$\psi u < u, \qquad \theta \leqslant u \leqslant u_0, \qquad u \neq \theta, \tag{5}$$

is stronger than our condition (H_1). Hence condition (5), along with the semicontinuity condition, suffices for the use of Lemma 3.2. Let $c_n = \rho(x_n, x_{n+1})$. Then from (5) and (H_6) one obtains $c_{n+1} \leqslant \psi c_n \leqslant c_n$ which implies, by Lemma 3.2, that the sequences $\{c_n\}$, $\{\psi c_n\}$ converges to the same limit, which by semicontinuity and (5), is θ. As in the proof of Theorem 4.1, one may show further that:

$$\rho(x_m, x_n) \leqslant \psi\rho(x_m, x_n) + c_n + c_m, \qquad m \geqslant n, \quad c_n \to \theta. \tag{6}$$

Remarks 4.2. The inequality (6) may be used to obtain an estimate of rapidity of convergence in the case when (H_6) holds and

$$\psi u \leqslant \alpha u, \qquad \theta \leqslant u \leqslant u_0, \qquad u \neq \theta, \qquad 0 < \alpha < 1. \tag{7}$$

First, observe that the usual proof of the Contraction Mapping Principle, e.g., [5, p. 151], shows that $\rho(Tx, Ty) \leqslant \alpha\rho(x, y)$ implies that the iterates

$\{T_n x_0\}$ converge to a fixed point y, $x_0 \in X$. The fact that ρ is a k-metric does not cause difficulty. Since (7) is stronger than (5), we have, letting $m \to \infty$ in (6),

$$\rho(y, x_n) \leqslant \psi\rho(y, x_n) + c_n \leqslant \alpha\rho(y, x_n) + c_n \,.$$

This establishes the estimate

$$\rho(y, x_n) \leqslant (1 - \alpha)^{-1} \rho(x_n , x_{n+1}). \tag{8}$$

Furthermore, if ψ is monotone, then from (1),

$$\rho(y, x_n) \leqslant (1 - \alpha)^{-1} \psi^n u_0 \,. \tag{9}$$

That is, convergence of $T^n x_0 \to y$ occurs at least as rapid as convergence of $\psi^n u_0 \to \theta$.

5. Applications

Corollary 3.1 and Theorem 4.1 are useful tools in the theory of differential equations. We offer two representative results as examples of their applicability. Other results of similar type are found in [1–3] and [6].

Let $J = [t_0 , t_0 + a]$ be a compact ($a < \infty$) or an infinite interval ($a = \infty$). Define the partial ordering "$\leqslant$" between any two points $x = (x_1 ,..., x_n)$, $y = (y_1 ,..., y_n)$ in R^n by

$$x \leqslant y \quad \text{iff} \quad x_i \leqslant y_i \,, \qquad i = 1, 2,..., n. \tag{10}$$

Let the mapping g: $J \times R^n \to R^n$ be continuous and satisfy for any $x \leqslant y \in R^n$

$$x \leqslant y \Rightarrow g(t, x) \leqslant g(t, y), \qquad t \in J. \tag{11}$$

Theorem 5.1. *Consider a system of differential equations*

$$x' = f(t, x), \qquad x(t_0) = c, \tag{12}$$

Suppose that

$$| f(t, x)| \leqslant g(t, | x |), \qquad x \in R^n, \quad t \in J, \tag{13}$$

there exists $u \in C^n[J]$ *such that*

$$u(t) \geqslant | c | + \int_0^t g(t, u(s))\, ds, \qquad t \in J. \tag{14}$$

Then there is a maximal solution r of the integral equation

$$x(t) = |c| + \int_0^t g(s, x(s))\, ds, \tag{15}$$

in the segment $0 \leqslant x(t) \leqslant u(t)$, $t \in J$ and if $y(t)$ is any solution of (12) such that $|y(t)| \leqslant u(t)$, $t \in J$, then $|y(t)| \leqslant r(t)$.

Proof. We need only consider the case $a < \infty$ for $a = \infty$ is obtained by letting $a \to \infty$.

Define the partial ordering in $C^n[J]$ by $x \geqslant y$, $x, y \in C^n[J]$ iff $x(t) \geqslant y(t)$ for $t \in J$. Let

$$\psi u = |c| + \int_0^t g(s, u(s))\, ds, \qquad pu = |u| = (|u_1|, \ldots, |u_n|). \tag{16}$$

Clearly ψ is monotone, $\psi u \leqslant u$. Also the cone is regular and ψ is completely continuous from Arzela's Theorem [1].

If $y(t)$ is a solution of (12) then $y = Ty$ where the mapping T is given by

$$Tx(t) = c + \int_0^t f(s, xs))\, ds, \qquad t \in J. \tag{17}$$

The theorem is thus a special case of Corollary 3.1.

To obtain existence and uniqueness for the system (12), we assume a uniqueness condition of Perron type:

$$|f(t, x) - f(t, y)| \leqslant g(t, |x - y|), \qquad t \in J, \quad x, y \in R^n. \tag{18}$$

In applying Theorem 4.1 we take

$$\psi x(t) = \int_0^t g(s, x(s))\, ds \tag{19}$$

and assume, for some $u \in C^n[J]$,

$$u(t) > \int_0^t g(s, u(s))\, ds, \qquad t \in J. \tag{20}$$

The condition (H_4) may be expressed in terms of the uniqueness of the system of differential equations.

$$x' = g(t, x), \qquad x(t_0) = 0. \tag{21}$$

The following theorem is a special case of Theorem 4.1 and a generalization of Perron's Theorem [3, Vol. 1, p. 48].

THEOREM 5.2. *Let the system* (21) *have only the trivial solution where g is a nonnegative function from* $J \times R^n$ *into* R^n *which is monotonic in the sense of* (11). *Suppose a function* $u \in C^n[J]$ *can be found to satisfy* (20). *Let* (18) *hold. Then for any* $x_0 \in C^n[J]$ *such that* $| Tx_0 - x_0 | \leqslant u$, *where T is given by* (17), *the sequence of iterates* $\{T^n x_0\}$ *converge componentwise and pointwise to a solution*, y, *of* (12), *unique in*

$$B_y = \{x \in C^n[J] \mid , \mid x - y \mid \leqslant u\}.$$

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