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Computing center conditions for vector fields with constant angular speed

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Abstract

We investigate the planar analytic systems which have a center-focus equilibrium at the origin and whose angular speed is constant. The conditions for the origin to be a center (in fact, an isochronous center) are obtained. Concretely, we find conditions for the existence of a $\mathscr{C}^w$-commutator of the field. We cite several subfamilies of centers and obtain the centers of the cuartic polynomial systems and of the families $(-y + x(H_1 + H_m),\ x + y(H_1 + H_m))^t$ and $(-y + x(H_2 + H_{2n}),\ x + y(H_2 + H_{2n}))^t$, with H_i homogeneous polynomial in x, y of degree i. In these cases, the maximum number of limit cycles which can bifurcate from a fine focus is determined.

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1. Introduction

In this work, we study the problem of center in the planar analytic systems which have a center-focus equilibrium at the origin and whose angular speed is constant. In these systems, the origin is the only finite equilibrium and if it is a center, it will be automatically isochronous. These systems, up to a linear change, take the following expression:

$$\dot{x} = -y + xH(x, y),$$
$$\dot{y} = x + yH(x, y), \tag{1}$$

where H is an analytic function which vanishes at the origin.

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The interest in studying this family is due, on the one hand, to the importance of these systems in the general problem of the isochronicity, since any analytic system with linear part $(-y,x)^t$ has an isochronous center if and only if it is possible to transform it by means of specific analytic change $(x \to x + P(y^2), y \to y + Q(x,y))$ into a system (1) (see Rudenok [14]).

On the other hand, these systems in polar coordinates take the form

$$\dot{r} = \sum_{k \geqslant 1} H_k(\cos\theta, \sin\theta) r^{k+1},$$

$$\dot{\theta} = 1,$$

where H_k are the homogeneous parts of degree k of H. These systems can be written as a single equation in form Abel generalized

$$\partial_\theta r = \sum_{k \geqslant 1} H_k(\cos\theta, \sin\theta) r^{k+1}. \tag{2}$$

The study of this equation gives us information about the systems, and vice versa, since the constant solution $r=0$ of (2) corresponds to the critical point $x=y=0$ of (1), and the periodic solutions of (2) correspond to closed orbits of (1), (see [4,5,8–10]).

Let us do now a more detailed review of the works related to the family (1).

The first place where a subfamily of centers of (1) is characterized, is Conti [7]. Particularly, it is characterized the case when H is a homogeneous polynomial of arbitrary degree. It is proved that if the degree of H is odd then the origin is a center, but if it is even must satisfy one condition, which is equivalent to vanish the first coefficient of the normal form in the radial component. In Algaba et al. [1] we find the center condition in terms of the coefficients of the system.

In Mardesic et al. [11] we have linearizating changes for reversible systems of type (1) with $H = H_1 + H_2$. In fact, as it is proved later in Collins [6], all the centers of this subfamily are reversible. Up to rotation, its expression is $(\dot{x}, \dot{y})^t = (-y + x^2\sigma(y),\ x + xy\sigma(y))^t$, with $\sigma(y) = a + by$. Later, in [12], Mazzi and Sabatini study the system (1) when it commutes with a radial field, and they find integral first and a linearization for (1). In [3] are characterized the systems (1) which have polynomial commutator, appearing not reversible centers with H non-homogeneous.

This paper is divided into four section. In Section 2, we present the main result of this work, derive a few conditions that characterize the centers with angular constant speed. We show the equivalence between these conditions and the vanishing of the coefficients of the radial component of the normal form. In Section 3, we develop an recursive algorithm that allows us to obtain conditions on the coefficients of the system which they must hold in order that the origin is a center. In Section 4, we cite several subfamilies that have a center. Finally, we derivate the centers of the cuartic polynomial systems, and of the families $(-y + x(H_1 + H_m), x + y(H_1 + H_m))^t$ and $(-y + x(H_2 + H_{2n}), x + y(H_2 + H_{2n}))^t$, being H_i homogeneous polynomial in x, y of degree i, and determine the maximum number of limit cycles which can bifurcate from a fine focus.

2. Some properties of centers

We consider the following couple of differential systems defined in open set $\mathscr{U}$ of the plane

$$(\dot{x},\dot{y})^{\mathrm{t}}=X(x,y),\quad (\dot{x},\dot{y})^{\mathrm{t}}=U(x,y)$$

with X and U analytic functions in $\mathscr{U}$.

Definition 1. It is said that X and U commute if Lie's bracket of both fields is null, that is, $[X,U]=DX.U-DU.X=0$. Moreover, if X and U are transverse in a neighborhood of the origin, to U it names a commutator of X in the above mentioned neighborhood.

If we denote by $\Phi_X(t,(x,y))$, $\Phi_U(t,(x,y))$ with $\Phi_X(0,(x,y))=(x,y)$ and $\Phi_U(0,(x,y))=(x,y)$, the flows of the previous systems, respectively, it is known that X and U commute if and only if the local flows Φ_X and Φ_U verify

$$\Phi_X(t,\Phi_U(s,(x,y)))=\Phi_U(s,\Phi_X(t,(x,y))),$$

for every t and s such that $\Phi_X(t,\Phi_U(s,(x,y)))$ and $\Phi_U(s,\Phi_X(t,(x,y)))$ exist, see Olver [13]. It is known that the problem of the isochronicity of a vector field is equivalent to the existence of an analytic commutator of such vector field, more precisely, for any analytic system with linear part $(-y,x)^{\mathrm{t}}$, the existence of an analytic commutator with linear part $(x,y)^{\mathrm{t}}$ is a necessary and sufficient condition so that the origin be an isochronous center (see Algaba et al. [1] and Sabatini [15]).

We are interested in the centers of the analytic systems with constant angular speed,

$$(\dot{x},\dot{y})^{\mathrm{t}}=X(x,y)=(-y+xH(x,y),x+yH(x,y))^{\mathrm{t}}, \tag{3}$$

where H is an analytic function in a neighborhood of the origin and $H(0,0)=0$. In Algaba et al. [3] it is proved that if H is a polynomial, the polynomial commutators, in case of existing, are of the type

$$U(x,y)=(xK(x,y),yK(x,y))^{\mathrm{t}}, \tag{4}$$

where K is a polynomial of the same degree as H, and in the analytic case we can choose K an analytic function with $K(0,0)=1$. To make this section of the paper self-contained, we include the following proposition.

Proposition 2. *If the analytic system* (3) *is center*, *then there exists an analytic commutator* (4) *around the origin.*

Proof. We assume that X is a center. Let Ψ be, the local flow which defines the solutions of the differential equation $\dot{x}=\alpha(x)$, with α analytic, $\alpha(0)=0$ and $\alpha(x)<0$ in $(0,\varepsilon)$. Fixed $(x,y)\in\mathscr{U}$, let s_0 be the minimum value $s\geqslant 0$ such that $\Phi_X(-s,(x,y))$ is on the x-axis. We already define the analytic flow Φ_U as $\Phi_U(t,(x,y))=\Phi_X(s_0,(\Psi(t,\Phi_X(-s_0,(x,y)),0))$. Since X is uniformly isochronous, Φ_U does not depend of s_0. Moreover, it commutes with Φ_X and all the straight lines which through the origin are invariants to the flow. That is, the associated field U is analytic and of the form (4). □

Therefore, the problem of looking for those H for that (3) has a center in the origin is equivalent to the problem of looking for those H which exists U of the form (4) with $[X,U]=0$.

If we use complex coordinates, $z = x + \mathrm{i}y$, (3) turns out $\dot{z} = \mathrm{i}z + zH(z,\bar{z})$ with $H(z,\bar{z}) = \bar{H}(z,\bar{z})$, and the commutator (4) will be $zK(z,\bar{z})$ with $K(z,\bar{z}) = \bar{K}(z,\bar{z})$. If for all m we denote by H_m and K_m the homogeneous parts of degree m of H and K, respectively, the equation $[X,U]=0$ becomes

$$\left[\mathrm{i}z, z\sum_{m\geqslant 0} K_m\right] + \left[z\sum_{l\geqslant 1} H_l, z\sum_{j\geqslant 0} K_j\right] = 0.$$

The term of degree $m+1$ of the above expression gets

$$[\mathrm{i}z, zK_m] = \sum_{\substack{j+l=m\\ j\geqslant 0, l\geqslant 1}} [zK_j, zH_l]. \tag{5}$$

By definition, the Lie's bracket in complex coordinates is $[P,Q] = P_zQ + P_{\bar{z}}\bar{Q} - Q_zP - Q_{\bar{z}}\bar{P}$, being P_z, Q_z and $P_{\bar{z}}, Q_{\bar{z}}$ the first partial derivatives of P and Q respect to z and $\bar{z}$ respectively, and as $H = \bar{H}$ and $K = \bar{K}$, we obtain

$$[\mathrm{i}z, zK] = \mathrm{i}z(\bar{z}K_{\bar{z}} - zK_z),$$

$$[zK, zH] = z(H(zK_z + \bar{z}K_{\bar{z}}) - K(zH_z + \bar{z}H_{\bar{z}})).$$

From Euler's Theorem for homogeneous functions, we have that

$$[zK_j, zH_l] = z(jH_lK_j - lK_jH_l) = (j-l)zH_lK_j.$$

Let $m \geqslant 1$, we consider the linear vector space $\mathscr{H}_m$, consisting of homogeneous polynomials, H_m, of degree m in the variables z and $\bar{z}$. And we denote by $\tilde{\mathscr{H}}_m$, the linear vector subspace of $\mathscr{H}_m$ on $\mathbb{R}$ defined by

$$\tilde{\mathscr{H}}_m = \{P \in \mathscr{H}_m / P(z,\bar{z}) = \bar{P}(z,\bar{z})\} = \left\{\sum_{k=0}^{[m/2]} (a_k z^k \bar{z}^{m-k} + \bar{a}_k z^{m-k}\bar{z}^k),\ a_k \in \mathbb{C}\right\}.$$

A basis of $\tilde{\mathscr{H}}_m$ is

$$\tilde{\mathscr{B}}_m = \left\{u_k = z^k\bar{z}^{m-k} + z^{m-k}\bar{z}^k,\ 0 \leqslant k \leqslant \left[\frac{m}{2}\right]\right\}$$
$$\cup \left\{v_k = \mathrm{i}(z^k\bar{z}^{m-k} - z^{m-k}\bar{z}^k),\ 0 \leqslant k \leqslant \left[\frac{m-1}{2}\right]\right\}.$$

Lemma 3. *The map*

$$L_m\left(\sum_{k=0}^{[m/2]} (a_k z^k\bar{z}^{m-k} + \bar{a}_k z^{m-k}\bar{z}^k)\right) = \sum_{k=0}^{[m/2]} \mathrm{i}(2k-m)\big(a_k z^k\bar{z}^{m-k} - \bar{a}_k z^{m-k}\bar{z}^k\big),$$

is a linear map of $\tilde{\mathscr{H}}_m$ *into itself.*

Moreover, if m *is odd*, $L_m(\tilde{\mathscr{H}}_m) = \tilde{\mathscr{H}}_m$ *and* $\mathrm{Ker}(L_m) = \{\mathbf{0}\}$. *And if* m *is even*, $\dim(L_m(\tilde{\mathscr{H}}_m)) = m$ *and* $\mathrm{Ker}(L_m) = \langle z^{m/2}\bar{z}^{m/2}\rangle$.

Proof. The first part is easy to check and the second one it follows from the action of the map on each element of the basis $\tilde{\mathscr{B}}_m$,

$$L_m(u_k) = (m-2k)v_k, \quad 0 \leqslant k \leqslant \left[\frac{m}{2}\right],$$

$$L_m(v_k) = (2k-m)u_k, \quad 0 \leqslant k \leqslant \left[\frac{m-1}{2}\right]. \qquad \square$$

The Eqs. (5) can be rewritten as

$$L_m(K_m) = \sum_{j=0}^{m-1}(2j-m)K_jH_{m-j}, \quad m \geqslant 1. \tag{6}$$

And from Lemma 3, we have the following result which characterizes the systems (3) with center.

Theorem 4. *The analytic system* (3) *is a center if and only if there exists a* $\mathscr{C}^\infty$*-function defined in a neighborhood of the origin*, $K = \sum_{m\geqslant 0} K_m$, with $K_0 = 1$, *such that, for any* $m \geqslant 1$,

$$\mathrm{Proy}_{\mathrm{Cor}\,L_{2m}}\left(\sum_{j=0}^{2m-1} 2(j-m)K_jH_{2m-j}\right) = \{\mathbf{0}\},$$

being $L_{2m}(\tilde{\mathscr{H}}_{2m}) \oplus \mathrm{Cor}\,L_{2m} = \tilde{\mathscr{H}}_{2m}$, *and* $L_m(K_m) = \sum_{j=0}^{m-1}(2j-m)K_jH_{m-j}$.

For each m, we denote by Cond_m the above projection on $\mathrm{Cor}\,L_{2m}$. The expression of Cond_m is a polynomial in the coefficients of $H_1,\ldots,H_{2m}$ and $\mathrm{Cond}_m = 0$ is the condition that must satisfy so the homogeneous part of degree $2m$ of the commutator exists, that is, $\mathrm{Cond}_m = 0$ is the compatibility's condition of the system of linear equations whose unknown quantities are the coefficients of K_{2m}, which turn out of Eq. (6). The following result proves that the conditions of compatibility for the existence of the terms of degree even of a commutator up to a certain degree, is equivalent to the vanishing of the coefficients of radial component of the normal form of the system up to this degree.

Theorem 5. *Let fields* (3) *and* (4) *be such that* $[X,U] = \mathscr{O}(|z,\bar{z}|^{2m+1})$ (*i.e.*, $\mathrm{Cond}_1 = \cdots = \mathrm{Cond}_{m-1} = 0$). *Then*, $\mathrm{Cond}_m \neq 0$ *if and only if the normal form of the system* (3) *up to order* $2m$ *is* $\dot{z} = \mathrm{i}z + \sum_{j\geqslant m} a_j z(z\bar{z})^j$, *with* $a_m \neq 0$.

Proof. It is known if $[X,U]=0$, then $[\Phi_*X, \Phi_*U]=0$, where Φ is a diffeomorphism and Φ_*X, Φ_*U are the transformed vector fields of X, U, respectively. Besides, if $J^N[X,U] = 0$, then $J^N[\Phi_*X, \Phi_*U] = 0$, where J^NX represents the Taylor's N-jet of the vector field X.

(1) Let us $\tilde{X}(z,\bar{z}) = \Phi_*X(z,\bar{z}) = \mathrm{i}z + a_m z^{m+1}\bar{z}^m + \mathscr{O}(|z,\bar{z}|^{2m+3})$ a normal form of vector field X. From the structure of the homological operator L_m, $m \geqslant 2$, the change of variables can be chosen of the form $\Phi(z) = z(1 + \Psi(z,\bar{z}))$, i.e., it is radial change.

Since $[\tilde{X}, z] = \mathscr{O}(|z,\bar{z}|^{2m+1})$, then $[\Phi_*^{-1}X, \Phi_*^{-1}z] = \mathscr{O}(|z,\bar{z}|^{2m+1})$ and $U(z,\bar{z}) = \Phi_*^{-1}z$ is a radial type, i.e., $U(z,\bar{z}) = zK(z,\bar{z})$ with $K = \bar{K}$.

(2) Let us $\tilde{X}(z,\bar{z}) = \Phi_* X(z,\bar{z}) = \mathrm{i}z + \sum_{j=1}^{m} a_j z^{j+1}\bar{z}^j + \mathcal{O}(|z,\bar{z}|^{2m+3})$ a normal form of the vector field X. By hypothesis, we have $[X,U] = \mathcal{O}(|z,\bar{z}|^{2m+1})$, that is, $[\Phi_* X, \Phi_* U] = \mathcal{O}(|z,\bar{z}|^{2m+1})$ with $\tilde{U} = \Phi_* U = zK(z,\bar{z})$, $K = \bar{K}$ and $K(0,0) = 1$. Therefore, $a_1, a_2, \ldots, a_{m-1} = 0$, since $J^1[\tilde{X},\tilde{U}] = 0$, $J^2[\tilde{X},\tilde{U}] = 0, \ldots, J^{2m}[\tilde{X},\tilde{U}] = 0$. □

From Lemma 3, we know that the terms of even degree of the commutator are not uniquely determined, that is, if K_{2m}^* is a particular solution of $L_{2m}(K_{2m}) = P_{2m}$, then $K_{2m}^* + \gamma z^m \bar{z}^m$ also it is, being γ any real parameter.

With the following result, we can consider the above-mentioned null coefficients.

Lemma 6. *Let the fields* (3) *and* (4) *be, such that* $[X,U] = \mathcal{O}(|z,\bar{z}|^{2m+1})$, *where* $K_{2j} = K_{2j}^* + \gamma_j z^j \bar{z}^j$, *and let* Cond_m^* *be the value which comes from to substitute* $\gamma_1 = \gamma_2 = \cdots = \gamma_{m-1} = 0$ *in* Cond_m. *Then*, $\mathrm{Cond}_m = \mathrm{Cond}_m^*$.

Proof. By hypothesis, $\mathrm{Cond}_1 = \mathrm{Cond}_2 = \cdots = \mathrm{Cond}_{m-1} = 0$, and from Theorem 5, we know that the coefficients of radial component of the normal form of the system (3), $a_1, \ldots, a_{m-1}$, are nulls. Cond_m comes given by

$$\mathrm{Cond}_m = \mathrm{Cond}_m^* + \gamma_1 f_1 + \gamma_2 f_2 + \cdots + \gamma_{m-1} f_{m-1}$$

with Cond_m^* which does not depend of $\gamma_1, \ldots, \gamma_{m-1}$, and where $f_1, \ldots, f_{m-1}$ are polynomials in the coefficients of H.

We now see that these polynomials are nulls.

Suppose $a_m \neq 0$, if some f_i was not null, taking $\gamma_i = -(1/f_i)\mathrm{Cond}_m^*$, $\gamma_j = 0$, $\forall j \neq i$, we arrive at $\mathrm{Cond}_m = 0$ which contradicts to Theorem 5. Hence, $f_i = 0$, $i = 1, \ldots, m-1$, so that $\mathrm{Cond}_m = \mathrm{Cond}_m^*$.

Conversely, if $a_m = 0$, and if there existed an f_i not null, we can take γ_i such that Cond_m was nonzero, therefore, we again arrive at a contradiction. □

3. Recursive algorithm

We already present a recursive algorithm that allows to compute the conditions of compatibility that arise for the existence of the commutator, in function of the coefficients of the system.

To apply the algorithm, before we must fix the corrange of L_{2m}. For simplicity in the operations, we have chosen $\mathrm{Cor}(L_{2m}) = \mathrm{Ker}(L_{2m})$.

We assume that know the first ones $2m-2$ components homogeneous of the commutator K, which, obviously, satisfy the $m-1$ first conditions that are mentioned in Theorem 4.

The algorithm consists of two steps:

Step 1: Computation of odd component, K_{2m-1}.

From Lemma 3, this component is determined for the term that appears in the right-hand side of (6), i.e., for the homogeneous polynomials of H and K of low degree to $2m-2$. In fact, we have

Lemma 7. *Let* $P_{2m-1}(z,\bar{z})=\sum_{j=0}^{m-1} a_j z^j \bar{z}^{2m-j-1}+\bar{a}_j z^{2m-j-1}\bar{z}^j$ *be. The equation* $L_{2m-1}(K_{2m-1})=P_{2m-1}$ *has unique solution and it is given by*

$$K_{2m-1}=-\sum_{j=0}^{m-1}\frac{a_j}{2m-2j-1}\,\mathrm{i}z^j\bar{z}^{2m-j-1}-\frac{\bar{a}_j}{2m-2j-1}\,\mathrm{i}z^{2m-j-1}\bar{z}^j.$$

Step 2: Compatibility's condition and computation of the even term, K_{2m}.

Let $P_{2m}(z,\bar{z})$ be, a homogeneous polynomial of degree $2m$. It will exists a homogeneous polynomial $K_{2m}(z,\bar{z})$ verifying $L_{2m}(K_{2m})=P_{2m}$ if and only if P_{2m} is in $L_{2m}(\tilde{\mathscr{H}}_{2m})$. That is,

$$\mathrm{Proy}_{\mathrm{Cor}\,L_{2m}}P_{2m}=\mathrm{Proy}_{\mathrm{Ker}\,L_{2m}}P_{2m}=\{\mathbf{0}\}.$$

We now give a explicit form of the compatibility's condition and the value of K_{2m} which it is obtained by straightforward computation.

Lemma 8. *Let* $P_{2m}(z,\bar{z})=\sum_{j=0}^{m} a_j z^j\bar{z}^{2m-j}+\bar{a}_j z^{2m-j}\bar{z}^j$ *be.*

Then, $\mathrm{Proy}_{\mathrm{Cor}\,L_{2m}}P_{2m}=0$ *if and only if* $a_m=0$, *i.e.*, $\mathrm{Cond}_m=a_m$.

Lemma 9. *Let* $P_{2m}(z,\bar{z})=P_{2m}(z,\bar{z})=\sum_{j=0}^{m-1} a_j z^j\bar{z}^{2m-j}+\bar{a}_j z^{2m-j}\bar{z}^j$ *be. The equation* $L_{2m}(K_{2m})=P_{2m}$ *has solution and comes given by*

$$K_{2m}=-\sum_{j=0}^{m-1}\frac{a_j}{2m-2j}\,\mathrm{i}z^j\bar{z}^{2m-j}-\frac{\bar{a}_j}{2m-2j}\,\mathrm{i}z^{2m-j}\bar{z}^j+\gamma_m(z\bar{z})^m.$$

For the real case, we can write the Lemmas 7, 8 and 9 of the following way.

Lemma 10. *Let* $P_{2m+1}(x,y)=\sum_{j=0}^{m} a_{m,j}x^{2m-2j+1}y^{2j}+b_{m,j}x^{2m-2j}y^{2j+1}$ *be. The unique solution of the equation* $L^{(\mathrm{R})}_{2m+1}(K_{2m+1})=P_{2m+1}$ (*where* $L^{(\mathrm{R})}_{2m+1}$ *denotes the map* L_{2m+1} *in the real case*) *is* $K_{2m+1}(x,y)=\sum_{j=0}^{2m+1}\alpha_j x^{2m-j+1}y^j$, *with* α_j *verifying*

$$\begin{aligned}
\alpha_{2m+1}&=b_{m,m}, & \alpha_0&=-a_{m,0},\\
\alpha_{2j+1}&=\frac{b_{m,j}+(2j+2)\alpha_{2j+3}}{2m-2j+1}, & \alpha_{2j}&=\frac{-a_{m,j}+(2m-2j+2)\alpha_{2j-2}}{2j+1},\\
j&=m-1,\ldots,0. & j&=1,\ldots,m.
\end{aligned}$$

Lemma 11. *Let* $P_{2m}(x,y)=\sum_{j=0}^{m} a_{m,j}x^{2m-2j}y^{2j}+b_{m,j}x^{2m-2j-1}y^{2j+1}$ *be. The equation* $L^{(\mathrm{R})}_{2m}(K_{2m})=P_{2m}$ (*where* $L^{(\mathrm{R})}_{2m}$ *denotes the map* L_{2m} *in the real case*) *has solution if and only if*

$$a_{m,m}+a_{m,0}+\sum_{j=1}^{m-1}\frac{(2j-1)!!(2m-2j-1)!!}{(2m-1)!!}a_{m,m-j}=0.$$

In this case, it comes given by $K_{2m}(x,y)=\sum_{j=0}^{2m}\beta_j x^{2m-j}y^j$, *with* β_j *verifying*

$$\beta_1=-a_{m,0},\qquad \beta_{2m}=\gamma_m,$$

$$\beta_{2j+1}=-\frac{a_{m,j}-(2m-2j+1)\beta_{2j-1}}{2j+1},\qquad \beta_{2m-2j}=\frac{b_{m,m-j}+(2m-2j+2)\beta_{2m-2j+2}}{2j},$$

$$j=1,\ldots,m-1.\qquad j=1,\ldots,m.$$

4. Applications

The described procedure allows an easy proof of the following well-known result.

Theorem 12. *The reversible analytic vector fields with constant angular speed are centers.*

Proof. A vector field is called reversible if it has a symmetric phase portrait respect to a straight line which passes through the origin, inverting the time. Up to a linear change, we can assume that above-mentioned straight line is the y-axis, in this case, the reversible systems (3) respect to this straight line are those which verify $H(x,-y)=-H(x,y)$. In complex coordinates, this condition over H is equivalent to say that the coefficients of the monomials of H are imaginary.

Applying the recursive algorithm (Lemmas 7 and 9), we see that the coefficients of the monomials of K are real. Therefore, H_{2m} and K_jH_{2m-j} with $j=1,\ldots,2m-1$ have only imaginary coefficients, and since this expression is real, we deduce that the coefficient of $z^m\bar{z}^m$ is zero, that is, for each $m>1$, the projection over $\operatorname{Ker} L_{2m}$ is null, therefore, Theorem 4 holds. □

We next present the following family of centers.

Theorem 13. *The systems* $\dot{z}=\mathrm{i}z+z\sum_{r\geqslant 1}H_{rm}$ *with* $H_{rm}=(a_r/m)L_m(\beta_m)\beta_m^{r-1}$, *being* $\beta_m(z,\bar{z})\in\tilde{\mathscr{H}}_m$ *and* a_r *any real number, have a center at the origin.*

Proof. The proof consists of proving that the system commutes with (4), being $K=1+\sum_{r\geqslant 1}K_{rm}$ with $K_{rm}=-a_r\beta_m^r$, That is, we show that Eqs. (6) hold.

We see that first component K_m verifies $L_m(K_m)=\langle 1,H_m\rangle$. So,

$$L_m(K_m)=L_m(-a_1\beta_m)=-a_1L_m(\beta_m)=-mH_m.$$

We see that for $r>1$, (6) also holds.

On the one hand, using the fact that $L(M^h)=hM^{h-1}L(M)$, with h natural and M homogeneous polynomial in z, $\bar{z}$, the left-hand side of (6) becomes,

$$L_{rm}(K_{rm})=L_{rm}(-a_r\beta_m^r)=-ra_r\beta_m^{r-1}L_m(\beta_m).$$

On the other hand, given $j\geqslant 0$ and $l\geqslant 1$, with $j+l=r$, we have

$$\begin{aligned}(l-j)mK_{jm}H_{lm}&=-(l-j)ma_ja_l\beta_m^j\frac{1}{m}L_m(\beta_m)\beta_m^{l-1}\\&=(j-l)a_ja_l\beta_m^{r-1}L_m(\beta_m),\end{aligned}$$

therefore, the right-hand side of (6) turns out

$$-\frac{rm}{m}a_r\beta_m^{r-1}L_m(\beta_m)+\beta_m^{r-1}L_m(\beta_m)\sum_{h=1}^{r-1}(r-2h)a_ha_{r-h}=-ra_r\beta_m^{r-1}L_m(\beta_m). \qquad \square$$

Remarks. The considered systems in Theorem 13, in Cartesian coordinates, can be written as

$$\dot{x}=-y+x\sum_{r\geqslant 1}H_{rm}(x,y),$$

$$\dot{y}=x+y\sum_{r\geqslant 1}H_{rm}(x,y), \tag{7}$$

where $H_{rm}=(a_r/m)(y\partial_x\beta_m-x\partial_y\beta_m)\beta_m^{r-1}$.

If the system is polynomial, then the commutator is polynomial, and both have the same degree. Moreover, in [3] is proved that they are the only ones with constant angular speed which have polynomial commutator. There are reversible fields which cannot be expressed in form (7), for instance,

$$\dot{x}=-y+ax^2+bx^4,$$

$$\dot{y}=x+axy+bx^3y.$$

Besides, there are nonreversible systems of family (7), even not homogeneous, for instance,

$$\dot{x}=-y+xH_3+xH_3K_3,$$

$$\dot{y}=x+yH_3+yH_3K_3,$$

with H_3=$2x^3-6x^2y+2xy^2+2y^3$ and $K_3=2x^3+6x^2y-6xy^2+6y^3$. Its commutator is

$$U=x+xK_3+xK_3^2,$$

$$V=y+yK_3+yK_3^2.$$

We next see that the fields of this family contain all the polynomial systems (3) with centers that we know, except the reversible systems.

(1) The homogeneous systems (3), $\dot{z}=\mathrm{i}z+zH_m$, studied by Conti [7].
If m is odd, $L_m(\tilde{\mathscr{H}}_m)=\tilde{\mathscr{H}}_m$, therefore it is isochronous, since always exists $\beta_m\in\tilde{\mathscr{H}}_m$ such that $L_m(\beta_m)=H_m$. Thus, it is a particular case of the mentioned family. If m is even, the first condition different from zero is $\mathrm{Cond}_m=\mathrm{Proy}_{\mathrm{Ker}\,L_m}(-mH_m)$, i.e., we must impose that $H_m\in L_m(\tilde{\mathscr{H}}_m)$, and we again see that exists β_m such that $L_m(\beta_m)=H_m$, thus, it is a subfamily of the above theorem.

(2) When $m=1$ and $\beta_1=y$, we obtain the fields of the form $(-y+x^2\sigma(y),\ x+xy\sigma(y))^{\mathrm{t}}$ with $\sigma(y)$ polynomial in y, that were studied in [12].

(3) In [12] is proved that if H and K are conjugate harmonic and $H^2(x,y)+K^2(x,y)$ is a function of x^2+y^2, then systems (3) and (4) commute.

We know that they are of this family, since are polynomial systems (3) with polynomial commutator, being $H = H_m = (a/m)(y\partial_x\beta_m - x\partial_y\beta_m)$ and $K = -a\beta_m$, with β_m homogeneous polynomial of degree m, satisfying $(\partial_x\beta_m)^2 + (\partial_y\beta_m)^2 = P_m(x^2 + y^2)$, $H_x = -K_y$, $H_y = K_x$, being P_m any homogeneous polynomial of degree m.

4.1. Centers of the family $H = H_1 + H_m$

Next let characterize systems (3) with $H = H_1 + H_m$, being H_1 and H_m not null, which have commutator.

Since $H_1 \neq 0$, making a rotation and a rescaling on the state variables into the field, we can assume, without loss of generality, $H_1 = \mathrm{i}z - \mathrm{i}\bar{z}$.

The first ones terms of a commutator (4) of (3) with $K = 1 + \sum_{j\geqslant 1} K_j$ come given by:

As $L_1(K_1) = -H_1$, then $K_1 = z + \bar{z}$.
As $L_r(K_r) = 0$, $2 \leqslant r \leqslant m-1$, from Lemma 6, we can suppose that $K_r = 0$.
K_m satisfies the equation $L_m(K_m) = -mH_m$.
K_{m+1} verifies $L_{m+1}(K_{m+1}) = (1-m)(K_1H_m - K_mH_1)$.

Each term K_r, with $m+2 \leqslant r \leqslant 2m$, must satisfy the equation

$$L_r(K_r) = (r-2)\big(\mathrm{i}z - \mathrm{i}\bar{z}\big)K_{r-1}.$$

Evidently, these equations relate the coefficients of the consecutive terms of the commutator. Next we mention some properties.

Lemma 14. *Let $K_r = \sum_{j=0}^{r} q_j^r z^j \bar{z}^{r-j}$ be, with $r \geqslant 1$, the homogeneous part of K of degree r of a commutator of the form* (4) *of the system* (3), *with $H = H_1 + H_m$, $H_1H_m \neq 0$. Then,*

(1) *For l such that $m+1 \leqslant 2l \leqslant 2m-2$,*

$$q_0^{2l+1} = \frac{2l-1}{2l+1}\, q_0^{2l},$$

$$q_j^{2l+1} = \frac{2l-1}{2l-2j+1}\,(q_{j-1}^{2l} - q_j^{2l}), \quad 1 \leqslant j \leqslant l. \tag{8}$$

(2) *For l such that $m+2 \leqslant 2l \leqslant 2m$, $\mathrm{Cond}_l = -4(l-1)\,\mathrm{Im}\, q_{l-1}^{2l-1}$.*
If $\mathrm{Cond}_l = 0$,

$$q_0^{2l} = \frac{2l-2}{2l}\, q_0^{2l-1},$$

$$q_j^{2l} = \frac{2l-2}{2l-2j}\,(q_{j-1}^{2l-1} - q_j^{2l-1}), \quad 1 \leqslant j \leqslant l-1. \tag{9}$$

From Lemma 6, we can take $q_l^{2l} = 0$.

(3) *For l such that $m+2 \leqslant 2l \leqslant 2m$,*

$$
\begin{aligned}
q_0^{2l+1} &= \frac{(2l-1)(2l-2)}{(2l+1)(2l)}\, q_0^{2l-1},\\
q_1^{2l+1} &= -\frac{1}{l}\, q_0^{2l-1} + q_1^{2l-1},\\
q_j^{2l+1} &= \frac{(2l-1)(2l-2)}{2l-2j+1}\left[\frac{1}{2l-2j+2}\, q_{j-2}^{2l-1} - \left(\frac{1}{2l-2j+2} + \frac{1}{2l-2j}\right) q_{j-1}^{2l-1}\right.\\
&\quad \left. + \frac{1}{2l-2j}\, q_j^{2l-1}\right], \quad 2 \leqslant j \leqslant l-1,\\
q_l^{2l+1} &= (2l-1)(l-1)(q_{l-2}^{2l-1} - q_{l-1}^{2l-1}).
\end{aligned} \tag{10}
$$

Lemma 15. *Let $K_r = \sum_{j=0}^{r} q_j^r z^j \bar{z}^{r-j}$ be, with $r \geqslant 1$, the homogeneous part of K of degree r of a commutator of the form* (4) *of the system* (3), *with $H = H_1 + H_m$, $H_1 H_m \neq 0$ and $m = 2n$ (or $m = 2n-1$).*

If $\mathrm{Cond}_{n+1} = \cdots = \mathrm{Cond}_{n+j} = 0$, *with $1 \leqslant j \leqslant n-1$ (or $1 \leqslant j \leqslant n-2$), then*

$$\mathrm{Cond}_{n+j+1} = -4(n+j)\binom{2n+2j-1}{2j} \operatorname{Im} q_{n-j}^{2n+1}.$$

Proof. Using (8)–(10) it is easy to verify that:

$\mathrm{Cond}_{n+1} = 0$ arrives at $\operatorname{Im} q_n^{2n+1} = 0$,
$\mathrm{Cond}_{n+2} = 0$ implies that $\operatorname{Im} q_{n+1}^{2n+3} = \operatorname{Im} q_n^{2n+2} = \operatorname{Im} q_{n-1}^{2n+1} = 0$,
$\mathrm{Cond}_{n+3} = 0$ implies that $\operatorname{Im} q_{n+2}^{2n+5} = \operatorname{Im} q_{n+1}^{2n+4} = \operatorname{Im} q_n^{2n+3} = \operatorname{Im} q_{n-1}^{2n+2} = \operatorname{Im} q_{n-2}^{2n+1} = 0$,

and, finally,
$\mathrm{Cond}_{n+j} = 0$ arrives at $\operatorname{Im} q_{n+j-1}^{2(n+j-1)+1} = \operatorname{Im} q_{n+j}^{2(n+j-1)} = \cdots = \operatorname{Im} q_{n-j+1}^{2n+1} = 0$.

By (9), we deduce that $\mathrm{Cond}_{n+j+1} = -4(n+j)\operatorname{Im} q_{n+j}^{2n+2j+1}$. Using (10), we have

$$
\begin{aligned}
\operatorname{Im} q_{n+j}^{2n+2j+1} &= \frac{(2n+2j-1)(2n+2j-2)}{2} \operatorname{Im} q_{n+j-2}^{2n+2j-1}\\
&= \frac{(2n+2j-1)(2n+2j-2)(2n+2j-3)(2n+2j-4)}{2.3.4} \operatorname{Im} q_{n+j-4}^{2n+2j-3}\\
&\;\;\vdots\\
&= \binom{2n+2j-1}{2j} \operatorname{Im} q_{n-j}^{2n+1}. \qquad \square
\end{aligned}
$$

Theorem 16. *Let the field* (3) *be with $H = H_1 + H_m$ and $H_1 H_m \neq 0$, where $m = 2n$ (or $m = 2n-1$).*

Then,

(1) (3) *is center if and only if it is reversible*, *that is*, *up to rotation and a scaling*, *the system* (3) *comes given by* $\dot{z} = -\mathrm{i}z + z(\mathrm{i}z - \mathrm{i}\bar{z}) + \mathrm{i}z\,\sum_{j=0}^{n-1} b_j z^j \bar{z}^{m-j} - b_j z^{m-j}\bar{z}^j$, *with* b_j *real coefficients.*

(2) *The maximum order of a fine focus of the system* $\dot{z} = -\mathrm{i}z + z(\mathrm{i}z - \mathrm{i}\bar{z}) + z\,\sum_{j=0}^{[m/2]} (a_j + \mathrm{i}b_j) z^j \bar{z}^{m-j} + (a_j - \mathrm{i}b_j) z^{m-j}\bar{z}^j$, with a_j, b_j *any real numbers is* $[m/2]+1$. *Besides*, *it is of order* $[m/2]+1$ *if and only if* $a_{j-1}a_j < 0$, *with* $|a_j| \ll |a_{j-1}|$, $j = 1, \ldots, [m/2]$.

Proof. Given $P_m \in \tilde{\mathscr{H}}_m$, from now on, we will denote by P_m^{L} and P_m^{R} the homogeneous polynomials such that $P_m = P_m^{\mathrm{L}} + P_m^{\mathrm{R}}$, with $P_m^{\mathrm{L}} = \sum_{k=0}^{m} a_k z^k \bar{z}^{m-k}$.

We have seen that applying a rotation and a rescaling, we can assume $H_1 = \mathrm{i}z - \mathrm{i}\bar{z}$, $K_1 = z + \bar{z}$ and $K_j = 0$, $2 \leqslant j \leqslant m-1$.

We distinguish two cases, according to the parities of m:

(1) Let $m = 2n$. The term K_{2n} must be particular solution of $L_{2n}(K_{2n}) = -2nH_{2n}$. Thus, it must verify that $\mathrm{Proy}_{\mathrm{Ker}\,L_{2n}} H_{2n} = 0$. From Lemma 8 it follows that H_{2n} does not have the monomial $z^n\bar{z}^n$.

Let H_{2n} be with $H_{2n}^{\mathrm{L}}(z,\bar{z}) = \sum_{j=0}^{n-1} h_j z^j \bar{z}^{2n-j}$, applying Lemma 9 turn out $K_{2n}^{\mathrm{L}} = \sum_{j=0}^{n-1} \frac{2n}{2n-2j} \mathrm{i} h_j z^j \bar{z}^{2n-j}$.

The term K_{2n+1} is solution of the equation

$$L_{2n+1}(K_{2n+1}) = (1-2n)(K_1H_{2n} - K_{2n}H_1)$$

and from Lemma 7 we deduce that

$$K_{2n+1}^{\mathrm{L}} = -\sum_{j=1}^{n-1} \frac{2n-1}{2n-2j+1}\,\mathrm{i}\left(\frac{2n-j+1}{n-j+1}\,h_{j-1} + \frac{j}{n-j}\,h_j\right) z^j \bar{z}^{2n-j+1}$$
$$-(2n-1)(n+1)\mathrm{i}h_{n-1} z^n \bar{z}^{n+1}.$$

From Lemma 15, we obtain that the vanishing of $\mathrm{Cond}_{n+1}, \mathrm{Cond}_{n+2}, \ldots, \mathrm{Cond}_{2n}$ leads to that the coefficients of K_{2n+1} must be real, that is, the coefficients of H_{2n} must be imaginaries, thus, the system is reversible.

(2) We now suppose $m = 2n-1$ and let H_{2n-1} be with $H_{2n-1}^{\mathrm{L}}(z,\bar{z}) = \sum_{j=0}^{n-1} h_j z^j \bar{z}^{2n-j}$, from Lemma 7, $K_{2n-1}^{\mathrm{L}} = \sum_{j=0}^{n-1} \frac{2n-1}{2n-2j-1} \mathrm{i} h_j z^j \bar{z}^{2n-j-1}$.

We now consider the equation

$$L_{2n}(K_{2n}) = -2(n-1)(K_1H_{2n-1} - K_{2n-1}H_1).$$

It has solution if $\mathrm{Proy}_{\mathrm{Ker}\,L_{2n}}(K_1H_{2n-1} - K_{2n-1}H_1) = 0$, thus, from Lemma 8, it is clear that $h_{n-1} + \bar{h}_{n-1} = 0$, so that h_{n-1} is an imaginary number. By Lemma 9, we have $K_{2n} = K_{2n}^{\mathrm{L}} + K_{2n}^{\mathrm{R}} + \gamma_n z^n \bar{z}^n$, with

$$K_{2n}^{\mathrm{L}} = \sum_{j=1}^{n-1} \frac{2n-2}{2n-2j}\left(\frac{4n-j-1}{2n-j}\,h_{j-1} - \frac{j}{2n-j-1}\,h_j\right) \mathrm{i} z^j \bar{z}^{2n-j},$$

moreover, we can take $\gamma_n = 0$.

Notice that if $\operatorname{Im} q_n^{2n} = 0$, then, by (8) $\operatorname{Im} q_l^{2n+1} = (2n-1)\operatorname{Im} q_{n-1}^{2n}$. Moreover, from the expression of K_{2n} also is deduced that if $\operatorname{Im} q_n^{2n} = \operatorname{Im} q_{n-1}^{2n} = \cdots = \operatorname{Im} q_j^{2n} = 0$, with $1 \leqslant j \leqslant n-1$, then $\operatorname{Re} h_{n-1} = \operatorname{Re} h_{n-2} = \cdots = \operatorname{Re} h_{j-1} = 0$, therefore, the coefficients must be imaginary, and in consequence, the system must be reversible.

We see the second part.

If $m = 2n$, of the expression of K_{2n+1} and by (8), we have

$$q_{n-j}^{2n+1} = -\frac{(2n-1)(n+j+1)}{(2j+1)(j+1)}\, \mathrm{i} h_{n-j-1}, \quad 1 \leqslant j \leqslant n-1,$$

hence,

$$\mathrm{Cond}_{n+j+1} = \frac{2(2n-1)(n+j+1)}{j+1} \binom{2n+2j}{2j+1} \operatorname{Re} h_{n-j-1}, \quad 1 \leqslant j \leqslant n-1.$$

And if $m = 2n-1$, from the expression of K_{2n}, we have

$$q_{n-j-1}^{2n} = \frac{(2n-2)(3n+j)}{(2j+2)(n+j+1)}\, \mathrm{i} h_{n-j-2}, \quad 1 \leqslant j \leqslant n-2.$$

thus,

$$\mathrm{Cond}_{n+j+1} = -\frac{4(2n-1)(3n+j)(n+j)(n-1)}{(j+1)(2n-2j+1)(n+j+1)} \binom{2n+2j-1}{2j} \operatorname{Re} h_{n-j-2},$$

with $1 \leqslant j \leqslant n-2$. Therefore, if $a_0, \ldots, a_{[m/2]}$ satisfy $a_{j-1}a_j < 0$ with $|a_j| \ll |a_{j-1}|$ for $j = 1, \ldots, [m/2]$, the numbers Cond_{n+j} with $j = 0, \ldots, [m/2]$ are a Sturm's sequence, thus we have a system with $[m/2]+1$ limit cycles. □

4.2. Centers of the family $H = H_2 + H_{2n}$

We now characterize the systems (3), with $H = H_2 + H_{2n}$ and $H_2 H_{2n} \neq 0$, which have commutator.

Applying a rotation and a scaling into the field, we can assume that $H_2 = -\mathrm{i}z^2 + \mathrm{i}\bar{z}^2$.

The first terms of a commutator (4) of (3) with $K = 1 + \sum_{j \geqslant 1} K_j$ are:

As $L_1(K_1) = 0$, we have $K_1 = 0$. In general, the terms of odd degree of a commutator are null, since are solution of $L_{2r+1}(K_{2r+1}) = 0$ and applying Lemma 7 we deduce that $K_{2r+1} = 0$.

$K_2 = -z^2 - \bar{z}^2$ comes from $L_2(K_2) = -2H_2$.

By $L_{2r}(K_{2r}) = 0$, $2 \leqslant r \leqslant n-1$ (from Lemma 6, we can suppose null parameters) we can assume $K_{2r} = 0$, $2 \leqslant r \leqslant n-1$.

K_{2n} is solution of $L_{2n}(K_{2n}) = -2nH_{2n}$.

K_{2n+2} must be solution of $L_{2n+2}(K_{2n+2}) = 2(1-n)(K_2 H_{2n} - K_{2n} H_2)$.

Each term K_{2r+2}, with $n+1 \leqslant r \leqslant 2n-1$, satisfies

$$L_{2r+2}(K_{2r+2}) = -2(r-1)(\mathrm{i}z^2 - \mathrm{i}\bar{z}^2)K_{2r}.$$

Evidently, these equations relate the coefficients of the consecutive terms of the commutator. We arrive at the following result, which proof we omitted for being analogous to that of Lemma 15.

Lemma 17. *Let* $K_{2n+2}=\sum_{j=0}^{2n+2} q_j^{2n+2} z^j \bar{z}^{2n+2-j}$ *be, the homogeneous part of* K *of order* $2n+2$ *of a commutator of the form* (4) *of the system* (3), *with* $H=H_2+H_{2n}$, $H_2H_{2n}\neq 0$.

Then, if $\mathrm{Cond}_{n+1}=\cdots=\mathrm{Cond}_{n+j}=0$, *with* $1\leqslant j\leqslant n-1$, *it follows that*

$$\mathrm{Cond}_{n+j+1}=4(-1)^j(n+j-1)\binom{n+j-2}{j-1}\operatorname{Im} q_{n-j+1}^{2n+2}.$$

Using similar arguments that in the proof of Theorem 16, we obtain the following result.

Theorem 18. *Let field* (3) *be, with* $H=H_2+H_{2n}$ with $H_2H_{2n}\neq 0$. *Then,*

(1) (3) *is center if and only if it is reversible. That is, up to a rotation and scaling, the system* (3) *comes given by* $\dot{z}=-\mathrm{i}z+\mathrm{i}z(-z^2+\bar{z}^2)+\mathrm{i}z\sum_{j=0}^{n-1} b_j z^j\bar{z}^{2n-j}-b_j z^{2n-j}\bar{z}^j$, *with* b_j *real coefficients.*
(2) *The maximum order of a fine focus of the system* $\dot{z}=-\mathrm{i}z+\mathrm{i}z(-z^2+\bar{z}^2)+z\sum_{j=0}^{n}(a_j+\mathrm{i}b_j)z^j\bar{z}^{2n-j}+(a_j-\mathrm{i}b_j)z^{2n-j}\bar{z}^j$, *with* a_j, b_j *real constants is* $n+1$. *Moreover, it is of order* $n+1$ *if and only if* $a_{j-1}a_j>0$, *with* $|a_{j-1}|\ll|a_j|$, $j=1,\ldots,n$.

Remark. The Lemmas 15 and 17 give necessary conditions that guarantee us the existence of the commutator of the families (3) with $H=H_1+H_m$ and $H=H_2+H_{2n}$, respectively. In both cases, it arrives at the vanishing, term to term, of the real coefficients of the expression of H_m and H_{2n}, respectively, and it implies that the centers are the reversible systems. In the case of the family (3) with $H=H_2+H_{2n+1}$, with $H_2=-\mathrm{i}z^2+\mathrm{i}\bar{z}^2$, the first ones conditions nonnull are $\mathrm{Cond}_{2n+2},\mathrm{Cond}_{2n+3},\ldots,\mathrm{Cond}_{4n+2}$, which correspond to the conditions of compatibility of the equations $L(K_{4n+4})=-2K_{2n+3}H_{2n+1}$, $L(K_{4n+6})=-4K_{2n+5}H_{2n+1},\ldots,L(K_{8n+4})=-(4n+2)K_{6n+3}H_{2n+1}$, being $K_{2n+1},\ldots,K_{6n+3}$, the solutions of the functional equations

$$L(K_{2n+1})=-(2n+1)H_{2n+1},$$

$$L(K_{2n+3})=(2n-1)(K_2H_{2n+1}-K_{2n+1}H_2),$$

$$L(K_{2n+2l+1})=(3-2(n+l))K_{2(n+l)-1}H_2,\qquad l=2,\ldots,2n+1,$$

respectively. The main difference with the cases $H=H_1+H_m$ and $H=H_2+H_{2n}$ is that these conditions are not a linear expression of the coefficients $h_j=A_j+\mathrm{i}B_j$ of H_{2n+1}. In this case, they are bilinears expressions of the form

$$\sum_{j=1}^{n}\left(\sum_{l=1}^{n}\lambda_{r,l}(j)A_l\right)B_j=0,\qquad r=2n+2,\ldots,4n+2$$

with $\lambda_{r,l}(j)$ real. We have analyzed, the subfamilies $H=H_2+H_{2n+1}$ for $n=1,2,3$ and we have obtained that the only centers are the reversible ones.

4.3. Centers of the quartic systems

Next we derive the quartic systems (3), that is, $H(z,\bar{z})=H_1(z,\bar{z})+H_2(z,\bar{z})+H_3(z,\bar{z})$, with $H_3(z,\bar{z})\neq 0$, with at least one nonlinearity besides H_3.

Theorem 19. *Let* (3) *be with* $H = H_1 + H_2 + H_3$, $H_3 \neq 0$ *and* $H_1^2 + H_2^2 \neq 0$. *Then*, (3) *is center if and only if it is reversible.*

Proof. The proof is assisted by an algebraic computer, continuing the recursive algorithm described in the section above. The nonhomogeneous quartic systems (3) take the form

$$\dot{z} = \mathrm{i}z + z[Az + \bar{A}\bar{z} + Bz^2 + 2(b_1 + b_3)z\bar{z} + \bar{B}\bar{z}^2 + Cz^3 + Dz^2\bar{z} + \bar{D}z\bar{z}^2 + \bar{C}\bar{z}^3],$$

being $A = \frac{1}{2}(a_1 - \mathrm{i}a_2)$, $B = \frac{1}{4}(b_1 + b_3 - \mathrm{i}b_2)$, $C = \frac{1}{8}(d_1 - \mathrm{i}d_2)$ and $D = \frac{1}{8}(d_3 - \mathrm{i}d_4)$, where $a_1, a_2, b_1, b_2, b_3, d_1, d_2, d_3, d_4$ are real constants with $a_1a_2b_1b_2b_3 \neq 0$, and $d_1d_2d_3d_4 \neq 0$. Next we compute necessary conditions in order to the existence of a commutator of the system, i.e., we will vanish the compatibility conditions Cond_i. The first one is $\mathrm{Cond}_1 = b_1 - b_3 = 0$, that is $b_1 = b_3$.

If $H_1 \equiv 0$ and H_2 is different from zero, it is easily seen that, making a rotation, we can take, without loss of generality, $H_2 = -b_2\mathrm{i}/4(z^2 - \bar{z}^2)$, with $b_2 \neq 0$.

In this case, the first three conditions of compatibility are nulls. If the coefficient $d_4 = 0$, the fourth one turns out $\mathrm{Cond}_4 = d_2d_3b_2 = 0$. Thus, it must be vanished d_3 or d_2. In both cases, by means of a rotation, we can transform it into a field (3) reversible and from Theorem 12 we know that it is center.

If $d_4 \neq 0$, in having imposed $\mathrm{Cond}_4 = 0$, it turns out $d_1 = -(d_3(3d_4 - d_2))/(d_4)$, and the fifth condition becomes $\mathrm{Cond}_5 = d_3b_2^2(-2d_2 + 3d_4) = 0$. If $d_3 = 0$, we obtain again a reversible system, and if $d_3 \neq 0$, we must impose that $d_2 = \frac{3}{2}d_4$. In this case, the sixth one condition is not null, therefore the system does not has a center.

Finally, if H_1 is not null, by means of a rotation we can do $a_2 = 0$ and with a homotecia can take $a_1 = 1$. The second condition of compatibility is $\mathrm{Cond}_2 = d_4 - b_3 = 0$, that is $d_4 = b_3$. Substituting, the third condition becomes $\mathrm{Cond}_3 = 2b_3 + 2d_2 + 2b_3d_1 + b_2b_3 + b_2d_2 = 0$.

Let us suppose that $b_3 = 0$, in this case, $\mathrm{Cond}_3 = d_2(b_2 + 2) = 0$, hence, $d_2 = 0$ or $b_2 = -2$. In the first one, $d_2 = 0$, we obtain that H has its real coefficients, thus, it is center.

In second case, $b_2 = -2$, we have $\mathrm{Cond}_4 = d_2(-35 + 4d_3) = 0$, that is, $d_3 = \frac{35}{4}$, and now we have $\mathrm{Cond}_5 = d_2 = 0$, i.e., it corresponds to the case $d_4 = 0$.

We suppose now $b_3 \neq 0$. Let us notice that this condition implies that the system is not reversible.

Then, we have $d_1 = (2d_3b_3 + b_3b_2 - d_2b_2 + 2b_3 - 2d_2)/2b_3$. The condition Cond_4 becomes $\mathrm{Cond}_4 = \mathrm{Curve}_1(d_3, b_2, b_3)b_3 + \mathrm{Curve}_2(d_3, b_2, b_3)d_2 = 0$, being

$$\mathrm{Curve}_1(d_3, b_2, b_3) = 90 - 2d_3^2 + 52d_3 + 24b_2 + 26d_3b_2 + 7b_2^2 - 26b_3^2,$$

$$\mathrm{Curve}_2(d_3, b_2, b_3) = -90 - 24b_2 + 8d_3 - 28b_3^2 - 7b_2^2.$$

We distinguish the following situations separately:

(I) If $\mathrm{Curve}_2(d_3, b_2, b_3) = 0$, in order to vanish Cond_4, must verify that $\mathrm{Curve}_1(d_3, b_2, b_3) = 0$. The resultant of the curves $\mathrm{Curve}_1 = 0$ and $\mathrm{Curve}_2 = 0$ with respect to b_3 is $4\Omega_1^2(b_2, d_3)$, with

$$\Omega_1 = 189b_2^2 + 648b_2 + 364d_3b_2 + 624d_3 + 2430 - 28d_3^2. \tag{11}$$

By means of the affine change of coefficients

$$b_2 = \frac{1}{196}(7\xi + \eta - 336), \quad d_3 = \frac{1}{392}(189\xi - \eta),$$

turns out $\Omega_1(\xi, \eta) = 7\xi\eta + 13,122 = 0$. Since $\eta \neq 0$, we have that $\xi = -13122/7\eta$. The vanishing of $\mathrm{Cond}_5, \mathrm{Cond}_6$ and Cond_7 come given by the vanishing of $l_5 = l_5^*/b_3\eta^3, l_6 = l_6^*/b_3^2\eta^4$ and

$l_7 = l_7^*/b_3^2\eta^5$, being l_5^*, l_6^* and l_7^* polynomial expressions that come given in function of η, d_2 and b_3. The resultant of l_5^* and Curve$_2$, l_6^* and Curve$_2$ and l_7^* and Curve$_2$ with respect to b_3 are, respectively, $\eta^{12}r_5(\eta, d_2)$, $\eta^{22}r_6(\eta, d_2)$, $\eta^{24}r_7(\eta, d_2)$. In turn, the resultant of r_5 and r_6, and r_5 and r_7 with respect to η are, respectively, polynomial expressions of the form $d_2^{56}r_{56}(d_2)$ and $d_2^{56}r_{57}(d_2)$. Last on, the resultant of this with respect to d_2 is a number different from zero, therefore, we conclude that it cannot be a center.

(II) If Curve$_2 \neq 0$, in having imposed Cond$_4 = 0$, turns out

$$d_2 = -\frac{b_3(b_3^2 + 90 + 22d_3 + 7b_2^2 + 24b_2 - d_3^2 + 13d_3b_2)}{(90 - 8d_3 + 28b_3^2 + 7b_2^2 + 24b_2)}.$$

Similar to that we did in the previous paragraphs we obtain that in order that Cond$_5$, Cond$_6$, Cond$_7$ and Cond$_8$ are nulls, there must be vanished the polynomials l_5^*, l_6^*, l_7^* and l_8^*, which appear in the numerator of l_5, l_6, l_7 and l_8, respectively. The resultant of l_5^* and l_6^*, l_5^* and l_7^* and l_5^* and l_8^* with respect to b_3 are, respectively:

$$R_1(l_5^*, l_6^*, b_3) = d_3^4\Omega_1^{12}(b_2, d_3)\Omega_2^2(b_2, d_3)\lambda_1^2(b_2, d_3),$$

$$R_2(l_5^*, l_7^*, b_3) = d_3^4\Omega_1^{12}(b_2, d_3)\Omega_2^2(b_2, d_3)\lambda_2^2(b_2, d_3),$$

$$R_3(l_5^*, l_8^*, b_3) = d_3^4\Omega_1^{16}(b_2, d_3)\Omega_2^4(b_2, d_3)\lambda_3^2(b_2, d_3)$$

being $\Omega_2 = 2b_2 + 4d_3 - 31$ and Ω_1 the above expression gives in (11).

Therefore in order to vanish the resultants we have four options:

(a) If $d_3 = 0$, the coefficients l_5^*, l_6^* and l_7^* come given by

$$l_5^* = b_3^3F_1(b_2, b_3), \quad l_6^* = b_3^3G_1(b_2, b_3), \quad l_7^* = b_3^3H_1(b_2, b_3),$$

and the resultants are

$$R(F_1, G_1, b_2) = b_3^{10}(16b_3^2 + 1225)\Xi_1(b_3),$$

$$R(F_1, H_1, b_2) = b_3^{10}(16b_3^2 + 1225)\Gamma_1(b_3).$$

Since $b_3 \neq 0$, the first and second factors of the above resultants cannot be vanished. Moreover, $\Xi_1(b_3)$ and $\Gamma_1(b_3)$ are not vanished simultaneously, so its resultant with respect to b_3 is a constant differently from zero.

(b) If $\Omega_1 = 0$, making the same change of coefficients that in the paragraph I and similarity coming, we deduce that l_5, l_6 and l_7 cannot be vanished simultaneous.

(c) If $\Omega_2 = 0$, we have $d_3 = \frac{1}{4}(31 - 2b_2)$ and, in this case, we obtain $l_5^* = b_3F_2(b_2, b_3)$, $l_6^* = b_3G_2(b_2, b_3)$ and $l_7^* = b_3H_2(b_2, b_3)$, and the resultants are

$$R(F_2, G_2, b_2) = b_3^4(1,486,873b_3^2 + 442,225)^2\Xi_2(b_3),$$

$$R(F_2, H_2, b_2) = b_3^4(1,486,873b_3^2 + 442,225)^2\Gamma_2(b_3),$$

And since the resultant of $\Xi_2(b_3)$ and $\Gamma_2(b_3)$ with respect to b_3 is a not null constant, we come to that cannot be vanished simultaneously.

(d) Finally, the resultant of $\lambda_1(b_2, d_3)$ and $\lambda_2(b_2, d_3)$ and it of $\lambda_1(b_2, d_3)$ and $\lambda_3(b_2, d_3)$ with respect to b_2 are polynomials in d_3, and the resultant of both polynomials is a constant differently from zero. Therefore, λ_1, λ_2 and λ_3 do not have common real roots. □

In Algaba et al. [2] is studied the case $H = H_1 + H_2 + H_3 + H_4$ with $H_1^2 + H_2^2 + H_3^2 \neq 0$ and $H_4 \neq 0$, resulting that all this centers are reversibles.

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