JOURNAL OF ALGEBRA **183,** 929–948 (1996)
ARTICLE NO. 0245

Relative Projectivity and Ideals in Cohomology Rings

Jon F. Carlson*

Department of Mathematics, University of Georgia, Athens, Georgia 30602

and

Department of Mathematics, Morehouse College, Atlanta, Georgia 30314

Communicated by Walter Feit

Received August 29, 1995

In this paper we explore one aspect of the relationship between group cohomology and representation theory. For a finite group G and a field k in characteristic $p > 0$, ideals in the cohomology ring $H^*(G, k)$ can sometimes be characterized by exact sequences of kG-modules in much the same way that elements of Ext_{kG} are classically represented by exact sequences. Furthermore the maps on the stable category which represent the exact sequences have functorial properties which mimic the structure of the ideas in the ring. The natural setting for this study is the relative homological algebra for the module category, relative to the subcategory generated by a single module using the tensor product operation.

1. INTRODUCTION

The connection between cohomology and module theory goes back half a century now. Its basis was established in fundamental work of Yoneda. Briefly, the story is that elements of the cohomology, $\text{Ext}^n_R(M, N)$, are equivalence classes of exact sequences of R-modules, beginning with N and ending with M and having length n. For group algebras, there are numerous variations which can be made on the correspondence. Suppose that G is a finite group, k is a field of characteristic $p > 0$, and that M and N are finitely generated kG-modules. Then $\text{Ext}^*_{kG}(M, N)$ is a finitely

* Partially supported by a grant from NSF. E-mail: jfc@math.uga.edu.
† E-mail: peng@math.uga.edu.

0021-8693/96 $18.00

generated module over the group cohomology $\mathrm{Ext}^*_{kG}(k, k) = H^*(G, k)$. The action is made possible by the Hopf algebra structure in that the tensor product functor, $\otimes_k$, gives a sort of multiplicative operation on the module category. Thus one variation might be to assign to any element of $\mathrm{Ext}^*_{kG}(M, N)$ the ideal in the cohomology ring $H^*(G, k)$ which annihilates it. This is the essence of the theory of support varieties for modules.

Here we wish to consider some aspects of the inverse problem. Suppose that we are given an ideal $\mathscr{I}$ generated by a set S of homogeneous elements of the cohomology ring $H^*(G, k)$. We look at the full subcategory $\mathscr{M}_{\mathscr{I}}$ of all modules whose cohomology is annihilated by the ideal. One of the developments of the work in [8] is that the ideal itself can be represented by a single element of $\mathrm{Ext}^1_{kG}(U, k)$ for a suitably chosen $U = U(S)$. That is, a module M is in the subcategory if and only if it annihilates the given element of $\mathrm{Ext}^1_{kG}(U, k)$, or equivalently, the tensor product of M with a short exact sequence representing the given element splits. For an ideal generated by a single element ζ the result is obvious since we can take U to be the translate $\Omega^{n-1}(k)$ and then the cohomology element is just $\zeta \in \mathrm{Ext}^1_{kG}(\Omega^{n-1}(k), k) \cong \mathrm{Ext}^n_{kG}(k, k)$. But for ideals which are not principal, the module U is constructed as the homology of a truncated Koszul complex of modules. In [8] it is also shown that the infinitely generated idempotent modules discovered by Rickard [12] are colimits of the modules $U(S)$.

A key observation for this paper is that if $p > 2$ then the subcategory $\mathscr{M}_{\mathscr{I}}$ is functorially finite. That is, it permits the definition of a relative cohomology theory. In this context the sequences representing the ideal $\mathscr{I}$ are steps in the relatively projective resolution of the trivial module k. This has several interesting consequences. One is that the module $U(S)$ has an essential direct summand $U_{\mathscr{I}}$ which is independent of the choice of generators S and depends only on the ideal $\mathscr{I}$. Then the dual $U^*_{\mathscr{I}}$ is the cokernel of the relatively injective hull of the trivial module, and from this we conclude that it is indecomposable. The complete minimal relatively projective and injective resolutions involve taking tensor products of these essential pieces.

The relative projectivity that we define here was originally developed by Okuyama in an unpublished manuscript. It depends on several unique properties of group algebras, and specifically is based on the choice of a single module V which generates the projectivity by tensor products. A module M is relatively V-projective if it is a direct summand of a tensor product of V with some other module. In the case that $V \cong k\uparrow_H^G$ then the relative V-projectivity is the usual H-projectivity. In Section 3 we sketch the details of the relative projectivity and describe the basic facts of the relative cohomology. One of Okuyama's important observations was that the dual module V^* is always relatively V-projective. Hence V-projectivity and V-injectivity coincide.

In Section 4 we consider an ideal $\mathscr{I} = (\zeta_1, \ldots, \zeta_t)$ in the cohomology ring. A cohomology element $\zeta \in H^*(G, k)$ can be represented by a homomorphism $\hat{\zeta}\colon \Omega^n(k) \to k$ whose kernel is denoted by L_ζ. In the event that the cohomology ring $\operatorname{Ext}^{n_i}_{kG}(L_{\zeta_i}, L_{\zeta_i})$ is annihilated by ζ_i for every i then the tensor product $V = \otimes L_{\zeta_i}$ has a special property. Namely, the collection of V-projective modules is the same as the set of all kG-modules whose cohomology is annihilated by $\mathscr{I}$. The usual facts about indecomposability of the relative syzygies and the uniqueness of minimal projective resolutions are important tools in the analysis. Among other things the independence of the construction from the generators of the ideal is derived in Section 5. In Section 6 they aid in the development of the tensor product relations.

Unfortunately, to make things work right we need to assume that the ideals in $H^*(G, k)$ which we consider are generated by elements which satisfy the stated annihilator condition (ζ annihilates the cohomology of L_ζ). We call these *productive* elements and the ideal they generate is a *productive* ideal. In the case that $p > 2$ there is no ambiguity, since it is known that every element of even degree has this property. Elements of odd degree are nilpotent and consequently many of the ideals that one might be interested in are productive. For $p = 2$ the situation is not so clear and some more work needs to be done on this case. Evidence suggests that there may still be an abundance of productive elements, but there is no proof. The one case which has been fully analyzed is that of a fours group $G \cong (\mathbb{Z}/2)^2$. Some homogeneous elements which are not productive are the elements of degree 1 that are not $\mathbb{F}_2$-rational. Strangely, these elements do not annihilate the cohomology of any nonprojective module.

The result of all of this is a function which assigns to any productive ideal $\mathscr{I}$ of the cohomology ring a pair of the form $(U_\mathscr{I}, \theta_\mathscr{I})$ with $\theta_\mathscr{I}\colon U_\mathscr{I} \to k$. If $\mathscr{I} \subseteq \mathscr{J}$ then there is a morphism of the pairs $\theta_{\mathscr{I}\mathscr{J}}\colon (U_\mathscr{I}, \theta_\mathscr{I}) \to (U_\mathscr{J}, \theta_\mathscr{J})$. A module is $\mathscr{I}$-projective if and only if it annihilates by tensor product the class modulo projective homomorphisms of $\theta_\mathscr{I}$. In some sense, much of this could have been anticipated from the work in [8]. One of the surprises has been that the natural setting for the results is the relative projectivity. There may be many other secrets here.

2. NOTATIONS AND PRELIMINARIES

Throughout this paper, we assume that G is a finite group, and k is an algebraic closed field of characteristic $p > 0$. All kG-modules are assumed to be finitely generated, unless otherwise specified.

For any module M, let (P, ε) be a projective cover of M. That is, P is a projective module and $\varepsilon\colon P \to M$ is a surjective homomorphism whose kernel has no projective summands. So we have a short exact sequence,

$$0 \longrightarrow \Omega(M) \longrightarrow P \xrightarrow{\varepsilon} M \longrightarrow 0,$$

where $\Omega(M)$ has no projective submodules. Similarly, $\Omega^{-1}(M)$ can be defined in the following short exact sequence,

$$0 \longrightarrow M \longrightarrow Q \longrightarrow \Omega^{-1}(M) \longrightarrow 0,$$

where Q is the minimal injective hull of M. In general, let

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

be the minimal projective resolution of M; we write $\Omega^n(M)$ for the kernel of ∂_n and, dually, $\Omega^{-n}(M)$ for the nth cokernel in a minimal injective resolution. It holds that

$$\Omega^n(M) \otimes \Omega^m(N) = \Omega^{n+m}(M \otimes N) \oplus (\text{proj})$$

for any integers m and n. Here by $\oplus(\text{proj})$ we mean the direct sum with some projective module, and we will use this notation in the rest of the paper.

Let mod-kG be the category of finitely generated left kG-modules. For any modules M and N, let $\mathrm{PHom}_{kG}(M, N) \subseteq \mathrm{Hom}_{kG}(M, N)$ denote the subspace of all kG-homomorphisms which factor through a projective module. The stable category stmod-kG has the same objects as mod-kG, but the morphisms are defined by $\overline{\mathrm{Hom}}_{kG}(M, N) = \mathrm{Hom}_{kG}(M, N)/\mathrm{PHom}_{kG}(M, N)$. If $\alpha\colon M \to N$ is a kG-homomorphism, then $\bar{\alpha}$ will denote the corresponding morphism in stmod-kG. Although the stable category is not abelian, it is a triangulated category and the translation functor is given by Ω^{-1} (see [13]).

Our interest in the triangulated structure is more philosophical than substantive. Every short exact sequence of modules $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ corresponds to a triangle in stmod-kG of the form

$$X \xrightarrow{\bar{\alpha}} Y \xrightarrow{\bar{\beta}} Z \xrightarrow{\bar{\gamma}} \Omega^{-1}(X).$$

Moreover, every triangle in stmod-kG is isomorphic to one that arises from a short exact sequence in this way.

In addition, we can shift the triangle to get other associated exact sequences of modules

$$0 \longrightarrow Y \xrightarrow{\tilde{\beta}} Z \oplus (\text{proj}) \xrightarrow{\gamma} \Omega^{-1}(X) \longrightarrow 0,$$

$$0 \longrightarrow \Omega(Z) \xrightarrow{\tilde{\gamma}} X \oplus (\text{proj}) \xrightarrow{\tilde{\alpha}} Y \longrightarrow 0,$$

and $\alpha \equiv \tilde{\alpha}$, $\beta \equiv \tilde{\beta}$, $\gamma \equiv \tilde{\gamma}$ modulo homomorphisms which factor through projectives. Similar results hold for further translation in both directions.

Also, from the triangulation axioms, we get that any morphism of short exact sequences induces a morphism of triangles which then applies to give a morphism of the shifted short exact sequences. That is, if we are given the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \longrightarrow & 0, \end{array}$$

then we have the corresponding commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\tilde{\beta}} & Z \oplus (\text{proj}) & \xrightarrow{\gamma} & \Omega^{-1}(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \bar{h}} & & \downarrow{\scriptstyle \Omega^{-1}(f)} & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\tilde{\beta}'} & Z' \oplus (\text{proj}) & \xrightarrow{\gamma'} & \Omega^{-1}(X') & \longrightarrow & 0. \end{array}$$

The diagram can be shifted to the other direction as well. Readers might want to see [9] for further details of the triangulated categories.

3. PROJECTIVITY RELATIVE TO A MODULE

In this section we generate some general information concerning the homological algebra relative to a subcategory $\mathscr{P}(V)$ generated by a single module, V. Our initial introduction to this subject was through an unpublished manuscript of Okuyama [11]. The idea of the relative homological algebra is the same as that generated by a "projective class of epimorphisms" as expounded in Hilton and Stammbach's book [10, Chap. 9]. It also coincides with the relative homological algebra associated to a functorially finite subcategory [2]. Indeed the subcategory $\mathscr{P}(V)$ is functorially finite. Such a relative homological algebra can also be generated by a suitable pair of adjoint functors as in Section 9.4 of [10]. In the present situation the adjoint functors are $-\otimes_k V$ and $-\otimes_k V^*$. The relation which makes everything work is the observation of Okuyama in Theorem 3.2. For this paper we need to develop some of the relations among the homological algebras defined relative to different modules. Some of the material of the section is known, but is included for the sake of completeness.

DEFINITION 3.1 [11]. *Let V be a fixed module. A module m is said to be relatively V-projective* (*or injective*) *if M is a direct summand of $V \otimes A$ for* *some model A. An exact sequence* $E\colon 0 \to A \to B \to C \to 0$ *is said to be V-split if* $E \otimes V\colon 0 \to A \otimes V \to B \otimes V \to C \otimes V \to 0$ *is split. Let* $\mathscr{P}(V)$ *be the full subcategory of all relatively V-projective kG-modules.*

THEOREM 3.2 [11]. *V and its dual* V^* *generate the same relative projectivity, i.e.,* $\mathscr{P}(V) = \mathscr{P}(V^*)$.

Proof. Note that the map $1_V \otimes \alpha_V\colon V \otimes V^* \otimes V \to V$ (Proposition 4.8 of [1]) has a right inverse $\beta\colon V \to V \otimes V^* \otimes V$ given by $\beta(v) = \sum v_i \otimes v_i^* \otimes v$, where $\alpha_V\colon V^* \otimes V \to k$, $\lambda \otimes v \mapsto \lambda(v)$ is the evaluation map and $\{v_i\}, \{v_i^*\}$ are dual bases. So, V is a direct summand of $V \otimes V^* \otimes V$. Dually, V^* is a direct summand of $V^* \otimes V \otimes V^*$. ∎

Usually we will write V-projective to mean relatively V-projective. For group algebras, relative projectivity has been studied extensively in the situation of projectivity relative to a subgroup H of G. However, that sort of relativity is a special case of the one defined above. That is, if $V = k\uparrow_H^G$, where H is a subgroup of G, then from Frobenius reciprocity, we know that for any module M,

$$M \otimes k\uparrow_H^G \cong M\uparrow_H^G .$$

So the V-projectivity is equivalent to the projectivity relative to the subgroup H. Similarly, if $\mathscr{H}$ is a set of subgroups of G and $V = \bigoplus_{H \in \mathscr{H}} k\uparrow_H^G$, then V-projectivity is equivalent to the projectivity relative to $\mathscr{H}$.

The proofs of following relations are easy exercises which we leave to the reader.

LEMMA 3.3. $\mathscr{P}(V)$ *is closed under direct sums, summands, and tensor products. In addition, for modules U and V, we have*

(i) *if* $U \in \mathscr{P}(V)$, *then* $\mathscr{P}(U) \subseteq \mathscr{P}(V)$;

(ii) $\mathscr{P}(V) = \mathscr{P}(V^*) = \mathscr{P}(\Omega(V)) = \mathscr{P}(\Omega^{-1}(V))$;

(iii) $\mathscr{P}(U \otimes V) = \mathscr{P}(U) \otimes \mathscr{P}(V)$;

(iv) $\mathscr{P}(U \oplus V) = \mathscr{P}(U) \oplus \mathscr{P}(V)$.

LEMMA 3.4. (i) *If P is V-projective and if* $\theta\colon B \to C$ *is a right V-split surjection, then for any* $\alpha\colon P \to C$, *there is a* $\beta\colon P \to B$ *such that* $\theta\beta = \alpha$.

(ii) *Suppose that* $E\colon 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is exact. Let* $A \xrightarrow{\bar\alpha} B \xrightarrow{\bar\beta} C \xrightarrow{\bar\gamma} \Omega^{-1}(A)$ *be the triangle corresponding to E. Then E is V-split* *if and only if* $\bar{\gamma} \otimes \bar{1}_V$ *is zero in the stable category where* $1_V\colon V \to V$ *is the identity.*

(iii) *If* E *is* V*-split, then so also are* E^* *and* $\Omega^n(E)$ *for any* n.

Proof. (i) We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{kG}(P, B) & \xrightarrow{\theta_*} & \mathrm{Hom}_{kG}(P, C) \\ \| \wr & & \| \wr \\ \mathrm{Hom}_{kG}(k, P^* \otimes B) & \xrightarrow{(1_{P^*} \otimes \theta)_*} & \mathrm{Hom}_{kG}(k, P^* \otimes C), \end{array}$$

where the vertical maps are isomorphisms. But because $1_{P^*} \otimes \theta$ has a right inverse, θ_* is surjective.

(ii) This comes from the fact that a short exact sequence splits if and only if the third map in the corresponding triangle is zero in the stable category. (iii) is trivial. ∎

For any module M, (P, ε) is said to be a V-projective cover of M if P is V-projective and $\varepsilon\colon P \to M$ is a right V-split surjection and has no V-projective summands in its kernel. Let $\Omega_V(M)$ denote the kernel of ε. Dually, $\Omega_V^{-1}(M)$ is the cokernel of δ where (Q, δ) is the V-injective hull. In general, a long exact sequence

$$P_*\colon \cdots \to P_n \to \cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

is said to be a V-projective resolution of M if each P_i is V-projective and for each i, the short exact sequence

$$0 \to P_{n+1}/\mathrm{Ker}(\partial_{n+1}) \to P_n \to \mathrm{Im}(\partial_n) \to 0$$

is V-split A minimal V-projective resolution of M is the V-projective resolution in which each P_n is the minimal V-projective cover of $\mathrm{Ker}(\partial_n)$. We let $\Omega_V^n(M)$ denote the kernel of ∂_n in the minimal resolution. By the dual argument, we can define the V-injective resolution and the minimal V-injective resolution. We write $\Omega_V^{-n}(M)$ for the cokernel of ∂_{-n} in the minimal injective resolution.

THEOREM 3.5. *Every module has a minimal* V*-projective resolution, and it is unique up to isomorphisms.*

Proof. Again, the evaluation map $\alpha_V\colon V^* \otimes V \to k$, $\lambda \otimes v \mapsto \lambda(v)$ is V-split. Then for any module M, $\alpha_V \otimes 1_M\colon V^* \otimes V \otimes M \to M$ is a right V-split surjection, with $V^* \otimes V \otimes M$ being V-projective. So clearly M has a V-projective resolution. The minimality and uniqueness can be derived from the comparison theorem for projective resolution which is a consequence of Lemma 3.4(i). The minimal resolution is the one which at each stage has the smallest possible dimension.

An explicit construction for the V-projective cover of k [11, Proposition 3.2] can be given as follows in the case when V is indecomposable. We have an isomorphism of k-spaces

$$\varphi\colon \mathrm{End}_{kG}(V) \cong \mathrm{Hom}_{kG}(V^* \otimes V, k), \qquad \varphi(\sigma)(\lambda \otimes v) = \lambda(\sigma(v)).$$

Let $V^* \otimes V = \oplus X_i$ be a direct sum decomposition into indecomposable modules, and let $e_i \in \mathrm{End}_{kG}(V^* \otimes V)$ be the idempotent corresponding to X_i in the decomposition. For each i, there exists $\sigma_i \in \mathrm{End}_{kG}(V)$ such that $\alpha_V \cdot e_i = \varphi(\sigma_i)$, where α_V is the evaluation map. Then in the above decomposition, since $\sum \sigma_i = 1$ there exists j such that σ_j is an automorphism of V, and consequently,

$$0 \to \mathrm{Ker}(\alpha_V) \cap X_j \longrightarrow X_j \xrightarrow{\alpha_V|_{X_j}} k \longrightarrow 0$$

gives a V-projective cover of k, where an inverse map of $(\alpha_V|_{X_j}) \otimes 1_V$ is the composition of $\gamma\colon V \to V \otimes V^* \otimes V$, $v \mapsto \sum_i v_i \otimes v_i^* \otimes \sigma^{-1}(v)$ with the projection to X_j. ∎

Proposition 3.6. *Suppose that M and N are modules and m and n are integers. Then we have the following.*

(i) $\Omega_V^n(M \oplus N) = \Omega_V^n(M) \oplus \Omega_V^n(N)$.

In particular, if M is indecomposable, then both $\Omega_V^n(M)$ and $\Omega_V^{-n}(M)$ are indecomposable modules.

(ii) $\Omega_V^n(M) \otimes N = \Omega_V^n(M \otimes N) \oplus (\text{rel. proj})$,

or more generally,

$$\Omega_V^n(M) \otimes \Omega_V^m(N) = \Omega_V^{n+m}(M \otimes N) \oplus (\text{rel. proj}).$$

Proof. The first part is obvious. For the second part, note that if $(\mathbf{P}_*, \varepsilon)$ is a V-projective resolution of M, then for any module N, $(\mathbf{P}_* \otimes N, \varepsilon \otimes 1_N)$ gives a V-projective resolution of $M \otimes N$. Hence

$$\Omega_V^n(M) \otimes N \cong \Omega_V^n(M \otimes N) \oplus (\text{rel. proj}).$$

LEMMA 3.7. *Suppose we have an exact sequence* $0 \to A \to B \to C \to 0$, *and a map* $\phi\colon A \to A'$. *Then we can complete the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow & & \| & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0, \end{array}$$

where the left-hand square is a push-out diagram. Moreover, if the top row is V*-split, then the bottom row is also* V*-split.*

Proof. The first part is a well-known fact; only the splitting needs to be argued. Tensoring V with the above diagram, we obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \otimes V & \longrightarrow & B \otimes V & \longrightarrow & C \otimes V & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi \otimes 1_V} & & \downarrow & & \| & & \\ 0 & \longrightarrow & A' \otimes V & \longrightarrow & B' \otimes V & \longrightarrow & C \otimes V & \longrightarrow & 0, \end{array}$$

with both rows exact. By [10, Lemma III.1.3], the left-hand square is also a push-out diagram. Note that the push-out of a split map splits, and we have the desired result. ∎

The usual comparison theorem for projective resolutions also holds for the relative projectivity, and this enables us to define relative cohomology

$$\mathrm{Ext}^n_{G,V}(M, N) = H^n\big(\mathrm{Hom}_{kG}(\mathbf{P}, N), \partial^*\big),$$

where $\mathbf{P}$ is a V-projective resolution of M. The relative group cohomology of G can be defined as

$$H^n(G, V, M) = \mathrm{Ext}^n_{G,V}(k, M).$$

Moreover, from [10, Proposition IV 5.5], any element in $\mathrm{Ext}^n_{G,V}(M, N)$ can be represented by a homomorphism in $\mathrm{Hom}_{kG}(\Omega^n_V(M), N)$. The Yoneda composition is well defined on the V-split extensions by the standard homological algebra. Also, with Lemma 3.7, we have a bijection between the V-split n-extensions and $\mathrm{Ext}^n_{G,V}(M, N)$. The cup product can be defined in the usual way. As usual, Yoneda composition and the cup product agree on $\mathrm{Ext}^*_{G,V}(k, k)$. The degree shifting technique in the relative cohomology holds. Namely, for $n \neq 0$, we have the following

$$\mathrm{Ext}^n_{G,V}(M, N) \cong \mathrm{Ext}^n_{G,V}\big(\Omega_V(M), \Omega_V(N)\big).$$

The isomorphism is compatible with the Yoneda composition. So, in particular, we have a graded ring isomorphism

$$\mathrm{Ext}^*_{G,V}(M, M) \cong \mathrm{Ext}^*_{G,V}(\Omega_V(M), \Omega_V(M))$$

modulo homomorphisms in $\mathrm{Ext}^0_{G,V}(M, M) \cong \mathrm{Hom}_{kG}(M, M)$ which factor through relatively projective modules.

One remark here is that even when we choose the minimal V-projective resolution of k and apply the functor $\mathrm{Hom}(-, k)$, the corresponding differentials are not necessarily zero maps. So it is not clear at this point that the rate of growth of the relatively projective resolution is the same as that of the relative cohomology.

4. RELATIVE PROJECTIVITY BY ELEMENTS IN $H^*(G, k)$

In this section, we consider the projectivity relative to some modules determined by elements in the cohomology ring $H^*(G, k)$. For any homogeneous element $\zeta \in H^n(G, k)$, $\zeta \neq 0$, we know that ζ can be represented by a unique map $\Omega^n(k) \xrightarrow{\zeta} k$. Define L_ζ as the kernel of ζ, so that we have an exact sequence

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^n(k) \xrightarrow{\zeta} k \longrightarrow 0.$$

DEFINITION 4.1. A homogeneous element $0 \neq \zeta \in H^*(G, k)$ is said to be productive if ζ annihilates the cohomology $\mathrm{Ext}^*_{kG}(L_\zeta, L_\zeta)$ of the module L_ζ.

It is known that if p is odd then all elements of even degree in $H^*(G, k)$ are productive (see [5]). For $p = 2$, the existence of productive elements is not so well established. For $G \cong (\mathbb{Z}/2)^2$, squares and higher powers of homogeneous elements are productive. Also products of productive elements are productive. However, all but a few of the elements of degree 1 are not. We should note that for any module M, $\mathrm{Ext}^*_{kG}(M, L_\zeta)$ and $\mathrm{Ext}^*_{kG}(L_\zeta, M)$ are modules over the ring $\mathrm{Ext}^*_{kG}(L_\zeta, L_\zeta)$ (the first is a left module and the second is a right module). If ζ annihilates $\mathrm{Ext}^*_{kG}(L_\zeta, L_\zeta)$, then it annihilates the identity element of the ring and by associativity it annihilates $\mathrm{Ext}^*_{kG}(L_\zeta, M)$ and $\mathrm{Ext}^*_{kG}(M, L_\zeta)$.

LEMMA 4.2. *Suppose $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Omega^{-1}(A)$ is a triangle in the stable category. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals in $H^*(G, k)$. If $\mathcal{I}$ annihilates the cohomology of A and $\mathcal{J}$ annihilates the cohomology of B, then $\mathcal{I}\mathcal{J}$ annihilates the cohomology of C. In particular, the product of productive elements is productive.*

Proof. Let M be any module. From the triangle we have a corresponding long exact sequence on cohomology

$$\cdots \longrightarrow \widehat{\operatorname{Ext}}^r(A, M) \xrightarrow{\gamma^*} \widehat{\operatorname{Ext}}^{r+1}(C, M) \xrightarrow{\beta^*} \widehat{\operatorname{Ext}}^{r+1}(B, M) \longrightarrow \cdots .$$

By hypothesis, $\mathcal{J} \cdot \widehat{\operatorname{Ext}}^{r+1}(C, M)$ must be in the image of γ^*. But then we have $\mathcal{I}\mathcal{J} \cdot \widehat{\operatorname{Ext}}^{r+1}(C, M) = \{0\}$.

To prove the last statement, notice that for $\zeta, \gamma \in H^*(G, k)$, there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^m(L_\zeta) & \longrightarrow & L_{\gamma\zeta} \oplus (\text{proj}) & \longrightarrow & L_\gamma & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^m(L_\zeta) & \longrightarrow & \Omega^{m+n}(k) \oplus (\text{proj}) & \xrightarrow{\Omega^m(\zeta)} & \Omega^m(k) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \gamma\zeta} & & \downarrow{\scriptstyle \gamma} & & \\
 & & & & k & = & k & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The middle row is the Ω^m-shift ($m = \deg(\gamma)$) of the exact sequence defining L_ζ. From the triangle of the top row we see that $\gamma\zeta$ annihilates the cohomology of $L_{\gamma\zeta}$ if both γ and ζ are productive. ∎

LEMMA 4.3. *Suppose that* $\zeta \in \operatorname{Ext}^n_{kG}(k, k)$, $\zeta \neq 0$. *Then the triangle*

$$L_\zeta \xrightarrow{\sigma} \Omega^n(k) \xrightarrow{\zeta} k \xrightarrow{\tau} \Omega^{-1}(L_\zeta)$$

is associated to the exact sequence

$$E_\zeta : 0 \longrightarrow k \xrightarrow{\tau} \Omega^{-1}(L_\zeta) \longrightarrow \Omega^{n-1}(k) \to 0$$

which represents the cohomology class ζ *in* $\operatorname{Ext}^1_{kG}(\Omega^{n-1}(k), k) \cong \operatorname{Ext}^n_{kG}(k, k)$. *In particular, a module M has the property that* ζ *annihilates the cohomology ring* $\operatorname{Ext}^*_{kG}(M, M)$ *of M if and only if* $E_\zeta \otimes M$ *splits.*

Proof. Because $\zeta \neq 0$ there is no projective summand in the middle term of E_ζ (see [4, Lemma 5.9.4]). Moreover, $E_\zeta \otimes M$ represents $\zeta \otimes 1_M$ in $\operatorname{Ext}^n_{kG}(M, M) = \operatorname{Ext}^1_{kG}(\Omega^{n-1}(M), M)$. This element is zero if and only if the sequence splits. ∎

PROPOSITION 4.4. *Suppose that $\zeta \in H^*(G,k)$ is productive. Then in the sequence E_ζ, $(\Omega^{-1}(L_\zeta), \tau)$ is the L_ζ-injective hull of k. Also, the Ω^{1-n}-translation of the sequence E_ζ has the form*

$$\Omega^{1-n}(E_\zeta): 0 \to \Omega^{1-n}(k) \to \Omega^{-n}(L_\zeta) \xrightarrow{\Omega^{-n}(\sigma)} k \to 0$$

and $(\Omega^{-n}(L_\zeta), \Omega^{-n}(\sigma))$ is the L_ζ-projective cover of k.

Proof. Note that $\Omega^{-1}(L_\zeta)$ is L_ζ-projective and $\Omega^{n-1}(k)$ is indecomposable; it is easy to see that $(\Omega^{-1}(L_\zeta), \tau)$ is the L_ζ-injective hull of k.

Take the dual of the extension (E_ζ). We have that $((\Omega^{-1}(L_\zeta))^*, \tau^*)$ is an L_ζ-projective cover of k. On the other hand, if we use the Ω-operator to translate the extension (E_ζ) back by $n-1$, we obtain the exact sequence

$$\Omega^{1-n}(E_\zeta): 0 \to \Omega^{1-n}(k) \to \Omega^{-n}(L_\zeta) \oplus (\text{proj}) \xrightarrow{\Omega^{1-n}(\sigma)} k \to 0.$$

From the isomorphisms

$$\mathrm{Ext}^n(k,k) \cong \mathrm{Ext}^1_{kG}(\Omega^{n-1}(k), k) \cong \mathrm{Ext}^1_{kG}(k, \Omega^{1-n}(k))$$

we know that $\Omega^{1-n}(E_\zeta)$ still represents ζ. Hence the extension $\Omega^{1-n}(E_\zeta)$ is also L_ζ-split. Again, the indecomposability of $\Omega^{1-n}(k)$ implies that $(\Omega^{-n}(L_\zeta) \oplus (\text{proj}), \Omega^{1-n}(\sigma))$ is an L_ζ-projective cover of k. By the uniqueness of the relatively projective cover and the Krull–Schmit theorem, we have that

$$(\Omega^{-1}(L_\zeta))^* \cong \Omega^{-n}(L_\zeta)$$

and the possibility projective summand in $\Omega^{1-n}(E_\zeta)$ vanishes. Hence $(\Omega^{-n}(L_\zeta), \Omega^{-n}(\sigma))$ is the L_ζ-projective cover of k. ∎

One note here is that although both extensions E_ζ^* and $\Omega^{1-n}(E_\zeta)$ give the relatively projective cover of k, they might represent different elements in $H^*(G,k)$. In fact, by Poincaré duality, E_ζ^* represents $(-1)^{n(n+1)/2}\zeta$, where n is the degree of ζ (see [4]).

By applying the above proposition, we can get an L_ζ-projective resolution of k. Namely, if E_ζ is translated by the Ω-operator $n-1$ times, then we get

$$0 \to \Omega^{2-2n}(k) \to \Omega^{1-2n}(L_\zeta) \oplus (\text{proj}) \to \Omega^{1-n}(k) \to 0.$$

Generally, for any integer t, translating E_ζ $(t-1)(n-1)$ times produces an exact sequence

$$0 \longrightarrow \Omega^{t-tn}(k) \longrightarrow \Omega^{(t-1)-tn}(L_\zeta) \oplus (\text{proj}) \longrightarrow \Omega^{(t-1)(1-n)}(k) \longrightarrow 0.$$

Note that $\Omega^{t-tn}(k)$ is indecomposable. By splicing these sequences together we get the relatively projective resolution.

THEOREM 4.5. *The minimal L_ζ-projective resolution of k has the form*

$$\cdots \to P_t \xrightarrow{\partial_t} P_{t-1} \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} k \to 0,$$

where $P_t = \Omega^{(t-1)-tn}(L_\zeta) \oplus (\text{proj})$ *and* $\Omega^t_{L_\zeta}(k) = \Omega^{t-tn}(k)$, *and the rate of growth of the L_ζ-projective resolution is equal to that of* $\Omega^t(k)$, *which is the p-rank of G.*

The product structure on the cohomology is very special in this case. As we have discussed, any elements in $\operatorname{Ext}^t_{G,V}(k,k)$ can be represented by homomorphisms in

$$\operatorname{Hom}_{kG}\big(\Omega^t_V(k), k\big) = \operatorname{Hom}_{kG}\big(\Omega^{t-tn}(k), k\big) \cong \operatorname{Ext}^{t-tn}_{kG}(k,k).$$

In [3], it has been shown that all the products of elements in negative degrees are zero if the depth of the cohomology ring of the group is two or more. So, with this hypothesis, the relative cohomology ring is a zero multiplication ring.

In what follows we generalize some of the above argument.

LEMMA 4.6. *Assume that $\zeta_1, \ldots, \zeta_r \in H^*(G,k)$ are productive elements. Then*

$$\mathscr{P}\big(L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_r}\big) = \mathscr{P}\big(L_{\zeta_1}\big) \cap \cdots \cap \mathscr{P}\big(L_{\zeta_r}\big),$$

and an injection or surjection f is $\otimes L_{\zeta_i}$-split if and only if $f \otimes 1_M$ splits for any module M whose cohomology is annihilated by all ζ_i's.

Proof. Clearly it suffices to show this for a pair of elements ζ, η. Since $L_\zeta \otimes L_\eta$ has cohomology annihilated by both ζ and η, it is clear that $\mathscr{P}(L_\zeta \otimes L_\eta) \subseteq \mathscr{P}(L_\zeta) \cap \mathscr{P}(L_\eta)$. On the other hand, if $M \in \mathscr{P}(L_\zeta) \cap \mathscr{P}(L_\eta)$, then both ζ and η annihilate $\operatorname{Ext}^*_{kG}(M,M)$. Therefore

$$L_\zeta \otimes \Omega^{-1}(M) = M \oplus \Omega^{n-1}(M) \oplus (\text{proj}),$$

$$L_\eta \otimes M = \Omega(M) \oplus \Omega^n(M) \oplus (\text{proj}),$$

and then

$$L_\eta \otimes L_\zeta \otimes \Omega^{-1}(M) = L_\eta \otimes \big(M \oplus \Omega^{n-1}(M) \oplus (\text{proj})\big)$$
$$= L_\eta \otimes M \oplus L_\eta \otimes \Omega^{n-1}(M) \oplus (\text{proj})$$
$$= \Omega(M) \oplus \Omega^n(M) \oplus L_\eta \otimes \Omega^{n-1}(M) \oplus (\text{proj}).$$

So, $\Omega(M) \in \mathscr{P}(L_\zeta \otimes L_\eta)$ and hence $M \in \mathscr{P}(L_\zeta \otimes L_\eta)$. By applying Lemma 3.4 the fact about splitness follows. ∎

Let $V = \otimes L_{\zeta_i}$, and note that for each ζ_i we have an L_{ζ_i}-projective resolution

$$\cdots \to P_t^{(i)} \longrightarrow P_{t-1}^{(i)} \longrightarrow \cdots \longrightarrow P_2^{(i)} \longrightarrow P_1^{(i)} \longrightarrow P_0^{(i)} \xrightarrow{\varepsilon_i} k \to 0,$$

where $P_t^{(i)} = \Omega^{(t-1)-tn_i}(L_\zeta) \oplus (\text{proj})$ and the argumentation map is $\varepsilon_i = \Omega^{-n_i}(\sigma_i)$. Here n_i is the degree of ζ_i and σ_i is the embedding in the sequence

$$0 \longrightarrow L_{\zeta_i} \xrightarrow{\sigma_i} \Omega^{n_i}(k) \xrightarrow{\zeta_i} k \longrightarrow 0.$$

Take the tensor product of the resolutions of k relative to each L_{ζ_i}. We get a V-projective resolution of k

$$\cdots \to P_t \longrightarrow P_{t-1} \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\varepsilon} k \to 0,$$

where $P_0 = \otimes \Omega^{-n_i}(L_{\zeta_i})$ and the argumentation map $\varepsilon = \otimes \varepsilon_i$. Using the previous lemma we have the following.

PROPOSITION 4.7. *Let $\zeta_1, \dots, \zeta_t \in \mathrm{H}^*(G,k)$ be productive elements and $V = \otimes L_{\zeta_i}$. Then*

(i) *the V-projective resolution of k has polynomial growth*, *and*

(ii) *a module M is V-projective if and only if the cohomology of M is annihilated by all of the ζ_i's.*

Let $\mathscr{I}$ be an ideal of $\mathrm{H}^*(G,k)$, which is generated by productive elements $\zeta_1, \dots, \zeta_t$. Let $V = \otimes L_{\zeta_i}$. From the last proposition we can see that a module is V-projective if and only if its cohomology is annihilated by all elements in the ideal $\mathscr{I}$. Therefore, for any ideal $\mathscr{I}$, if we define a module to be *relatively $\mathscr{I}$-projective* (or abbreviate as *$\mathscr{I}$-projective*) if its cohomology is annihilated by all elements in $\mathscr{I}$, and likewise define a homomorphism f to be *$\mathscr{I}$-split* if $f \otimes 1_M$ splits for any module M whose cohomology is annihilated by the idea $\mathscr{I}$, then from what we have discussed, we know that the $\mathscr{I}$-projectivity is equivalent to the V-projectivity, and $\mathscr{I}$-split is equivalent to V-split. We write $\mathscr{P}(\mathscr{I})$ for all $\mathscr{I}$-projective modules. Therefore, we have the following results.

PROPOSITION 4.8. (i) *The $\mathscr{I}$-projectivity does not depend on the choice of the productive generators of $\mathscr{I}$.*

(ii) *If $\mathscr{I}$ has a set of productive generators, then the $\mathscr{I}$-projective resolution of k has polynomial growth.*

5. TRIANGLES DEFINED BY IDEALS IN THE COHOMOLOGY RING

We shall say that a homogeneous ideal $\mathscr{I} \subseteq \mathrm{H}^*(G, k)$ is *productive* if it has a set of productive generators. In this section we will demonstrate the relationship of the $\mathscr{I}$-projective resolution of modules to certain modules first encountered in the study of homomorphism in quotient categories in [8].

Let $\mathscr{I}$ be a productive ideal with productive generators $\zeta_1, \dots, \zeta_t$. We start from the exact sequence

$$0 \longrightarrow L_{\zeta_i} \longrightarrow \Omega^{n_i}(k) \xrightarrow{\zeta_i} k \longrightarrow 0,$$

and take the triangle shifting to get a sequence

$$0 \longrightarrow \Omega^{n_i}(k) \xrightarrow{\tilde{\zeta}_i} k \oplus (\mathrm{proj}) \xrightarrow{\hat{\tau}_i} \Omega^{-1}(L_{\zeta_i}) \longrightarrow 0, \tag{5.1}$$

where $\hat{\tau}_i$ is injective when restricted to k. So for each i, we have a complex

$$\mathscr{C}(\zeta_i): 0 \longrightarrow \Omega^{n_i}(k) \xrightarrow{\tilde{\zeta}_i} k \oplus (\mathrm{proj}) \longrightarrow 0.$$

The tensor product of these complexes is the complex $\mathscr{C}(\zeta_1, \dots, \zeta_t) = \mathscr{C}(\zeta_1) \otimes \cdots \otimes \mathscr{C}(\zeta_t)$ given by

$$\mathscr{C}(\zeta_1, \dots, \zeta_t): 0 \to C_t \longrightarrow C_{t-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \to 0,$$

where $C_0 \cong k \oplus (\mathrm{proj})$ and $C_1 \cong \oplus \Omega^{n_i}(k) \oplus (\mathrm{proj})$.

By Künneth's formula, the homology of $\mathscr{C}$ is $\bigotimes_{i=1}^{t} \Omega^{-1}(L_{\zeta_i})$ in degree zero and is zero in all other degrees. Hence we have an exact sequence

$$0 \to C_t \longrightarrow C_{t-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\otimes \hat{\tau}_i} \otimes \Omega^{-1}(L_{\zeta_i}) \to 0.$$

The cohomology class of ∂_1 on $\Omega^{n_i}(k)$ is ζ_i. In the above sequence, the module U is defined in [8] as

$$U = U(\zeta_1, \dots, \zeta_t) = \partial_1(C_1)$$

and there is an exact sequence

$$(5.2) \qquad 0 \longrightarrow U \xrightarrow{\theta} k \oplus (\text{proj}) \xrightarrow{\otimes \hat{\tau}_i} \otimes\, \Omega^{-1}(L_{\zeta_i}) \to 0,$$

where θ is the embedding.

Now shift the sequence (5.2), and note that the tensor product of splitting injectives splits, $\otimes\, \hat{\tau}_i|_k = \otimes(\hat{\tau}_i|_k) = \otimes\, \tau_i$ is a left $\mathscr{I}$-split injection. So we have an exact sequence in the form

$$E_{\mathscr{I}}\colon 0 \longrightarrow k \xrightarrow{\otimes \tau_i = (\otimes \hat{\tau}_i)|_k} \otimes\, \Omega^{-1}(L_{\zeta_i}) \longrightarrow \Omega^{-1}(U) \oplus (\text{proj}) \longrightarrow 0$$

and $E_{\mathscr{I}}$ is $\mathscr{I}$-split. Also note that the projective summand in the right-hand term is also a direct summand of the middle term and hence it can be removed from the sequence. Thus we have an injective resolution for k,

$$(5.3) \qquad 0 \longrightarrow k \xrightarrow{\tau} P(\mathscr{I}) \longrightarrow \Omega^{-1}(U) \longrightarrow 0,$$

where $P(\mathscr{I})$ is $\mathscr{I}$-projective and τ is left $\mathscr{I}$-split.

Since U may have some projective summand, we let $\tilde{U}$ be the nonprojective part of U. Namely, $\tilde{U}$ is the direct summand of U which has no projective submodule and has the largest dimension with this property. From (5.2) we have a map

$$\tilde{U} \xrightarrow{\theta'} k,$$

where θ' is the projection of the original θ to k, restricted to $\tilde{U}$. The following lemma enables us to define a new module from this map. It can be proved by an argument similar to the proof of the Krull–Schmidt Theorem in [7].

LEMMA 5.4. *Suppose that A is a finite-dimensional k-algebra, M, S are modules over A, and $\varphi\colon M \to S$ is a homomorphism. Then, up to isomorphism, M has a unique direct sum decomposition $M = M' \oplus M''$ such that*

(i) $\varphi(M'') = 0$;

(ii) *M' has no direct summand on which φ vanishes.*

DEFINITION 5.5. *For ideal $\mathscr{I} \subseteq \mathrm{H}^*(G, k)$ be generated by productive elements $\zeta_1, \ldots, \zeta_t$, let the homomorphism $\theta'\colon \tilde{U} \to k$ be as defined before. We define the module $U_{\mathscr{I}}$ as the unique submodule satisfying the conditions,*

(i) *$\tilde{U} = U_{\mathscr{I}} \oplus U'$ with $\theta'(U') = 0$;*

(ii) *$U_{\mathscr{I}}$ has no direct summand on which θ' vanishes.*

For every such ideal $\mathscr{I}$, we define $\theta_{\mathscr{I}} = \theta'|_{U_{\mathscr{I}}}$. Note that from Lemma 5.4 the pair $(U_{\mathscr{I}}, \theta_{\mathscr{I}})$ is well defined up to isomorphism.

THEOREM 5.6. *Suppose that $\mathcal{I} \subseteq \mathrm{H}^*(G,k)$ is a productive ideal. Then*

$$U_{\mathcal{I}} \cong \Omega\left(\Omega_{\mathcal{I}}^{-1}(k)\right),$$

where $\Omega_{\mathcal{I}}^{-1}(k)$ is the relative syzygy with respect to the ideal $\mathcal{I}$.

Proof. First from the $\mathcal{I}$-split sequence (5.3)

$$0 \longrightarrow k \xrightarrow{\tau} P(\mathcal{I}) \longrightarrow \Omega^{-1}(U) \longrightarrow 0$$

we have that

$$\Omega^{-1}(U) \cong \Omega_{\mathcal{I}}^{-1}(k) \oplus (\text{rel. proj}).$$

Note that the translation functor Ω^{-1} preserves the relative projectivity. So it suffices to show that any $\mathcal{I}$-projective summand N of U can be made to satisfy $\theta'(N) = 0$. But this is a consequence of the next lemma. ▮

LEMMA 5.7. *Suppose that we have an exact sequence $0 \to X \xrightarrow{\alpha} Y \oplus P \xrightarrow{\beta} Z \to 0$ where P is a projective module and X has no projective summands. Let N be a direct summand of Z. If the composition of β with the projection to N splits, then, in the shifted sequence*

$$0 \longrightarrow \Omega(Z) \xrightarrow{\delta} X \oplus (\text{proj}) \longrightarrow Y \longrightarrow 0,$$

we can assume that $\delta'(\Omega(N)) = 0$, where δ' is the composition of δ with the projection to X.

Proof. Let $\Omega(Z) \to X \to Y \xrightarrow{\bar{\beta}} Z$ be the triangle of β. Then by hypothesis we have a homomorphism of triangles

$$\begin{array}{ccccccc} \Omega(N) & \longrightarrow & 0 & \longrightarrow & N & \longrightarrow & N \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega(Z) & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

which can be lifted to a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(N) & \longrightarrow & (\text{proj}) & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \\ 0 & \longrightarrow & \Omega(Z) & \longrightarrow & X \oplus (\text{proj}) & \longrightarrow & Y & \longrightarrow & 0. \end{array}$$

However, ϕ is injective and hence split. So $\phi(\text{proj}) \cap X = \{0\}$. ▮

The following is an immediate corollary to Theorem 5.6 and Proposition 3.6 and 4.8.

COROLLARY 5.8. (i) *$U_{\mathscr{J}}$ depends only on the ideal $\mathscr{I}$, not on the choice of productive generators*;

(ii) *$U_{\mathscr{J}}$ is indecomposable*;

(iii) *the other summand U' in $\tilde{U}$ has cohomology annihilated by the ideal $\mathscr{I}$.*

The next result is very similar to Theorem 3.8 of [8]. The proof given there is based on the construction of certain chain maps of modular Koszul complexes. The notions of relativity used here make the presentation much more direct and simple.

PROPOSITION 5.9. *For productive ideals $\mathscr{I}, \mathscr{J} \in \mathrm{H}^*(G, k)$ with $\mathscr{I} \subseteq \mathscr{J}$, there is a homomorphism $\phi_{\mathscr{I}\mathscr{J}}\colon U_{\mathscr{J}} \to U_{\mathscr{J}}$ such that the following diagram commutes*

$$\begin{array}{ccc} U_{\mathscr{J}} & \xrightarrow{\theta_{\mathscr{J}}} & k \\ \Big\downarrow{\scriptstyle \phi_{\mathscr{I}\mathscr{J}}} & & \Big\| \\ U_{\mathscr{J}} & \xrightarrow{\theta_{\mathscr{I}}} & k. \end{array}$$

The third object in the triangle of $\phi_{\mathscr{I}\mathscr{J}}$ is $\mathscr{I}\mathscr{J}$-projective. $\phi_{\mathscr{I}\mathscr{J}}$ is unique up to a homomorphism which can be factored through an $\mathscr{I}$-projective module. Moreover, if $\mathscr{I}_1 \subseteq \mathscr{I}_2 \subseteq \mathscr{I}_3$, then

$$\phi_{\mathscr{I}_2\mathscr{I}_3} \circ \phi_{\mathscr{I}_1\mathscr{I}_2} = \phi_{\mathscr{I}_1\mathscr{I}_3}; \tag{5.10}$$

the equation holds up to a homomorphism which factors through a relatively $\mathscr{I}_1$-projective module.

Proof. Let $P_{\mathscr{J}}$ and $P_{\mathscr{J}}$ be the minimal relatively $\mathscr{I}$- and $\mathscr{J}$-projective covers of k, respectively. Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{\mathscr{I}}(k) & \longrightarrow & P_{\mathscr{J}} & \longrightarrow & k & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \varphi_{\mathscr{J}\mathscr{I}}} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\| & & \\ 0 & \longrightarrow & \Omega_{\mathscr{J}}(k) & \longrightarrow & P_{\mathscr{J}} & \longrightarrow & k & \longrightarrow & 0. \end{array} \tag{5.11}$$

Note that since $\mathscr{I} \subseteq \mathscr{J}$, the relative $\mathscr{J}$-projectivity implies the relative $\mathscr{I}$-projectivity. So the bottom row is $\mathscr{I}$-split and there exists a map $\alpha\colon P_{\mathscr{J}} \to P_{\mathscr{J}}$ which in turn induces a homomorphism $\varphi_{\mathscr{J}\mathscr{I}}\colon \Omega_{\mathscr{J}}(k) \to \Omega_{\mathscr{J}}(k)$ such that the diagram commutes.

Translating in the triangles, we have that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_{\mathscr{J}} & \longrightarrow & k \oplus (\text{proj}) & \longrightarrow & \Omega^{-1}(\Omega_{\mathscr{J}}(k)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \Omega^{-1}(\varphi_{\mathscr{I}\mathscr{J}})} & & \\ 0 & \longrightarrow & P_{\mathscr{J}} & \longrightarrow & k \oplus (\text{proj}) & \longrightarrow & \Omega^{-1}(\Omega_{\mathscr{J}}(k)) & \longrightarrow & 0, \end{array}$$

where the restriction of β to the k component is the identity map. In the dual diagram, let $\phi_{\mathscr{I}\mathscr{J}} = (\Omega^{-1}(\varphi_{\mathscr{I}\mathscr{J}}))^*$, and then we have the commutative diagram

$$\begin{array}{ccc} U_{\mathscr{J}} & \xrightarrow{\theta_{\mathscr{J}}} & k \\ \downarrow{\scriptstyle \phi_{\mathscr{I}\mathscr{J}}} & & \| \\ U_{\mathscr{J}} & \xrightarrow{\theta_{\mathscr{J}}} & k. \end{array}$$

From Lemma 4.2, we know that the third object of the triangle of α is $\mathscr{I}\mathscr{J}$-projective. Using the Snake lemma in diagram (5.11), the third object in the triangle of $\varphi_{\mathscr{I}\mathscr{J}}$ is $\mathscr{I}\mathscr{J}$-projective. Therefore that of $\phi_{\mathscr{I}\mathscr{J}}$ is also $\mathscr{I}\mathscr{J}$-projective.

The uniqueness comes from translating back the diagram. Namely, if we have another homomorphism $\phi'_{\mathscr{I}\mathscr{J}}$ with the analogous commutative diagram, then we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{\mathscr{J}}(k) & \longrightarrow & P_{\mathscr{J}} & \longrightarrow & k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi'_{\mathscr{I}\mathscr{J}}} & & \downarrow{\scriptstyle \alpha'} & & \| & & \\ 0 & \longrightarrow & \Omega_{\mathscr{J}}(k) & \longrightarrow & P_{\mathscr{J}} & \xrightarrow{\varepsilon_{\mathscr{J}}} & k & \longrightarrow & 0. \end{array}$$

So $\operatorname{Im}(\alpha - \alpha') \subseteq \operatorname{Ker}(\varepsilon_{\mathscr{J}}) = \Omega_{\mathscr{J}}(k)$ which implies that $\varphi_{\mathscr{I}\mathscr{J}} - \varphi'_{\mathscr{I}\mathscr{J}}$ can be factored through the $\mathscr{J}$-projective module $P_{\mathscr{J}}$, and so can $\phi_{\mathscr{I}\mathscr{J}} - \phi'_{\mathscr{I}\mathscr{J}}$ since both the dual and the Ω functors preserve the relative projectivity and the splitness.

The proof of the transitivity is similar. ∎

REFERENCES

1. M. Auslander and J. Carlson, Almost-split sequence and group rings, *J. Algebra* **103** (1986), 122–140.
2. M. Auslander and S. Smalø, Preprojective modules over Artin algebras, *J. Algebra* **66** (1980), 61–122.
3. D. J. Benson and J. F. Carlson, Products in negative homology, *J. Pure Appl. Algebra* **82** (1992), 107–129.

4. D. Benson, ‘‘Representations and Cohomology,’’ Cambridge Univ. Press, Cambridge, UK, 1991.
5. J. Carlson, Products and projective resolutions, *Proc. Sympos. Pure Math.* **47** (1987), 339–408.
6. J. Carlson, P. Donovan, and W. Wheeler, Complexity and quotient categories for group algebras, *J. Pure Appl. Algebra* **93** (1994), 147–167.
7. C. Curtis and I. Reiner, ‘‘Representation Theory of Finite Groups and Associative Algebras,’’ Wiley, New York, 1981.
8. J. Carlson and W. Wheeler, Homomorphisms in higher complexity quotient categories, preprint.
9. D. Happel, ‘‘Triangulated Categories in the Representation Theory of Finite Dimensional Algebras,’’ Cambridge Univ. Press, Cambridge, UK, 1988.
10. P. J. Hilton and U. Stammbach, ‘‘A Course in Homological Algebra,’’ Springer-Verlag, Berlin/New York, 1971.
11. T. Okuyama, A generalization of projective covers of modules over finite group algebras, unpublished manuscript.
12. J. Rickard, Idempotent modules in the stable category, *J. London Math. Soc.*, to appear.
13. C. A. Weibel, ‘‘An introduction to homological algebra,’’ Cambridge Univ. Press, Cambridge, UK, 1994.