On Regular Semigroups*

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1. Introduction and Summary

A very general question in the theory of semigroups is: From information about the idempotents of a semigroup what information can be deduced about the semigroup? This question takes its importance from the fact that in many semigroups the idempotents are easily recognized. A powerful tool for attacking this question for inverse semigroups has been constructed by Munn in [13]; for each semilattice $E$ there is constructed a semigroup $T_E$ which is a "maximal" fundamental inverse semigroup with semilattice isomorphic to $E$ (a semigroup is called fundamental if its only congruence contained in $\mathcal{H}$ is the trivial one). The advantage held by $T_E$ is that not only is its semilattice (isomorphic to) $E$, but for each inverse semigroup with semilattice $E$, $T_E$ contains a closely related subsemigroup. Some of the consequences of this construction are given in papers [13, 15–18], and in Sections 6 and 8 below.

The unifying result of this paper is the construction in Section 5 of the corresponding semigroups for regular semigroups. More explicitly, for each fundamental regular semigroup, $B$, say, generated by its set of idempotents, $E$, say, we construct a "maximal" fundamental regular semigroup, denoted by $T_{\langle E \rangle}$, whose subsemigroup generated by its idempotents is isomorphic to $B = \langle E \rangle$ (note that the subsemigroup generated by the idempotents of a regular semigroup is always regular, from Result 1, due to Fitz-Gerald). When $B$ is a band, $T_{\langle E \rangle}$ reduces to the semigroup $W(B)$ of [5]. In Sections 4, 6–9 we give consequences of the construction of $T_{\langle E \rangle}$, or of related concepts, in five different areas in the theory of regular semigroups.

After Section 2 which gives necessary preliminaries, we study in Section 3 the subsemigroup of a regular semigroup generated by the set of all idem-

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potents, one result being that a regular semigroup is generated by its idempotents if and only if each principal factor is generated by its own idempotents.

In Section 4 we give a determination of the maximum congruence contained in $\mathcal{H}$ on a regular semigroup. Determinations for inverse and orthodox semigroups have been given by Howie [8] (see [1, Section 7.6]) and Meakin [12], respectively.

In Section 6 we extend to fundamental regular semigroups the results of Munn [13], and Howie and Schein [10], concerning bi-simple inverse semigroups, semilattices of groups and uniform and antiuniform semilattices. We prove also analogous results for completely semisimple semigroups and regular semigroups on which $\mathcal{H}$ is a congruence.

In Section 7 we continue work begun by Reilly and Scheiblich [20], by determining the maximum regular subsemigroup of a semigroup, containing a given set of idempotents as its set of all idempotents.

Section 8 concerns completely semisimple semigroups in which each $\mathcal{D}$-class contains at most $m$ idempotents, for some integer $m$. It is shown that for each element $a$ in such a semigroup, $a^{m}$ is an element of a subgroup. Hence each homomorphic image of such a semigroup is also completely semisimple (this can also be shown by proving that each homomorphic image contains at most $m$ idempotents per $\mathcal{D}$-class [7]). In general, a homomorphic image of a completely semisimple inverse semigroup with finitely many idempotents in each $\mathcal{D}$-class is not necessarily itself completely semisimple [7].

In Section 9 we show that a fundamental regular semigroup generated by its idempotents is uniquely determined, to within isomorphism, by the partial groupoid of its idempotents. Section 10 continues a study of the semigroup $T_{(E)}$.

2. Preliminaries

For any semigroup $S$ and for any element $a$ of $S$ and any subset $A$ of $S$, we put

$V(a) = \{x \in S : axa = a \text{ and } xax = x\}$,

$V(A) = \{x \in S : \text{ for some } a \in A, axa = a \text{ and } xax = x\}$.

We use, whenever possible, and often without comment, the notations of Clifford and Preston [1]. When there might otherwise be an ambiguity, we shall denote Green’s relations on the semigroup $S$ by $\mathcal{H}(S)$, $\mathcal{L}(S)$, etc.

Result 1 (from Fitz-Gerald [3]). Let $S$ be any regular semigroup and $E$ its set of idempotents. Then for $n = 1, 2, 3, \ldots$, $V(E^n) = E^{n+1}$. Hence $\langle E \rangle$, the subsemigroup of $S$ generated by $E$, is regular.
RESULT 2. Let $U$ be any semigroup and let $T$ be any regular subsemigroup of $U$. Then

(i) $eL_f$ in $U$ if and only if $ef = e$, $fe = f$, for any idempotents $e, f$ in $U$,

(ii) $aL_b$ in $T$ if (and only if) $aL_b$ in $U$, for any elements $a, b$ in $T$,

(iii) any element $a$ of $T$ is an element of a subgroup of $T$ if (and only if) it is an element of a subgroup of $U$.

Proof. Part (i) is a corollary of [1, Lemma 2.14]. Part (ii), which is well-known, is proved in [5, Result 9]. Part (iii), probably also well-known, is proved as follows: $a$ is an element of a subgroup of $T$ if and only if $aL_a = e$ in $T$, which holds if and only if $aL_a = e$ in $U$ (from part (ii) and its dual result), and this is true if and only if $a$ is an element of a subgroup of $U$.

The following well-known result [14, Lemma 1] is a routine generalization of a result due to Preston [19, Lemma 2] and of Theorem 2.20 [1].

RESULT 3. Let $S$ be any semigroup and let $e, f$ be any $D$-related idempotents of $S$. Take any element $a$ in $S$ and any inverse $a'$ of $a$, such that $aa' = e$ and $a'a = f$ ($a$ and $a'$ exist, from Section 2.3 of [1]). Define mappings $\theta_{a', a} : eSe \to fSf$ and $\theta_{a, a'} : fSf \to eSe$ by, for each $x \in eSe$, $x\theta_{a', a} = a'xa$, and for each $y \in fSf$, $y\theta_{a, a'} = aya'$. Then $\theta_{a', a}$ is a $D$-class preserving isomorphism from $eSe$ onto $fSf$ and $\theta_{a, a'} = \theta_{a', a}^{-1}$.

Proof. For each $x \in S$ it is clear that $a'xa \in fSf$, and for any $x, y \in eSe$, since $ey = y$, we have

$$(xy)\theta_{a', a} = a'xya = a'xa'a'ya = (x\theta_{a', a})(y\theta_{a', a}),$$

and so $\theta_{a', a}$ is a homomorphism of $eSe$ into $fSf$. Also, for any $x \in eSe$,

$$x\theta_{a', a}\theta_{a, a'} = (a'xa)\theta_{a, a'} = aa'xa' = x.$$

Similarly $\theta_{a, a'}\theta_{a', a}$ is the identity map of $fSf$, and so $\theta_{a', a}$ is an isomorphism onto $fSf$ and $\theta_{a, a'} = \theta_{a', a}^{-1}$.

Now since, for any $x \in eSe$, $xa' = x$ and $aa'xa = xa$, we have $xaxaLa'xa$, whence $\theta_{a', a}$ is $D$-class preserving.

RESULT 4 [6, Remark 2]. Let $a$ and $b$ be any regular elements in any semigroup $S$ such that $L_a \geq L_b$. Then for each idempotent $e \in L_a$ there exists an idempotent $f \in L_b$ such that $e \geq f$.

Proof. Take any idempotent $e \in L_a$ and any inverse $b'$ of $b$. Using $be = b$ we may routinely show that $eb'$ is also an inverse of $b$, whence $eb'b \in L_b$ and moreover $eb'b$ is an idempotent and $eb'b \leq e$. 

RESULT 5 (from [1, Exercise 3, Section 8.4]). A regular semigroup $S$ is $[0,1]$ simple if and only if for any nonzero idempotents $e, f$ there exists an idempotent $g \leq f$ such that $f \not\in e$.

Since for any idempotents $e, f$ in any semigroup $S, e \in SfS$ if and only if $J_e \leq J_f$, we may restate the first part of [1, Exercise 3, Section 8.4] as follows: Let $S$ be a semigroup and $a, b$ any elements of $S$ such that $Ja \leq Jb$. Then for each idempotent $e \in J_a$ and regular $\beta$-class $D$ of $S$ contained in $J_b$ (if such exist) there is an idempotent $f \in D$ such that $f \leq e$ (cf. [6, Theorem 1]).

RESULT 6. A regular semigroup $S$ is completely semisimple if (and clearly only if) no pair of distinct comparable idempotents are $\beta$-related.

Proof. This is probably well-known. Let us prove the following stronger statement, which is probably also well-known. If a $\beta$-class, $D$, say, of a semigroup $S$, contains an idempotent, $e$, say, which is minimal among the idempotents of $D$, then $D = D_e = J_e$ and $J(e) \cap I(e)$ is completely $0$-simple or completely simple. If $e$ is minimal among the idempotents of $J_a$, then $J(e) \cap I(e)$ contains a primitive idempotent, namely $e$, and is also $0$-simple or simple, giving the required conclusion. Take then any idempotent $f \in J_e$ such that $f \leq e$. From above there exists an idempotent $g \leq f \leq e$ such that $g \not\in e$, whence $g = f = e$ and so $e$ is minimal among the idempotents of $J_e$, and the result follows.

RESULT 7 (a corollary to [1, Theorem 2.4]). The regular elements of a semigroup $S$ form a subsemigroup if (and clearly only if) the product of any two idempotents of $S$ is a regular element.

RESULT 8. Let $S$ be a completely semisimple semigroup and $a$ any element of $S$. If, for some positive integers $m, n, a^m \not\in a^{m+n}$, then $a^m \not\in a^{m+n+1} \cdots$, and $H_{am}$ is a subgroup. Consequently, if every strictly descending chain of $\mathcal{J}$-classes of $S$ has length at most $m$, then for each element $a$ in $S$, $a^m$ is an element of a subgroup of $S$.

Proof. This result is probably well-known but we give a proof nevertheless. Now $L_{am} \geq L_{am+n}$ and $R_{am} \geq R_{am+n}$ and $S$ satisfies both $M_L^*$ and $M_R^*$ [1, Section 6.6]. Hence $a^m \not\in a^{m+n}$ implies that $a^m \not\in a^{m+n+1}$; also

\[ H_{am} \geq H_{am+1} \geq \cdots \geq H_{am+n} = H_{am}. \]

Since $L$ is a right congruence, $a^m \not\in a^{m+n}$ implies $a^{m+1} \not\in a^{m+n+1}$ and similarly $a^{m+1} \not\in a^{m+n+1}$ whence $a^m \not\in a^{m+1} \not\in a^{m+n+1}$. We easily see now that $a^m \not\in a^{m+1} \not\in a^{m+2} \cdots$ and that $H_{am}$ is a group. To prove the remaining statement we note that $Ja \geq J_{a^2} \geq \cdots \geq J_{a^n} \geq J_{a^{n+1}}$ and so two of these $\mathcal{J}$-classes are equal.
Let $\mathcal{I}_m$ and $\mathcal{G}_m$ denote the symmetric inverse semigroup and the symmetric group, respectively, on the set $\{1, 2, \ldots, m\}$. Being finite and inverse, $\mathcal{I}_m$ and the ideals of $\mathcal{I}_m$ are of course completely semisimple. Also $\mathcal{G}_m \subseteq \mathcal{I}_m$ and $\mathcal{I}_m \setminus \mathcal{G}_m$ is an ideal of $\mathcal{I}_m$ containing $m$ $\mathcal{J}$-classes (of $\mathcal{I}_m \setminus \mathcal{G}_m$). Thus for any element $a$ in $\mathcal{I}_m$, $a^m$ is an element of a subgroup of $\mathcal{I}_m$.

**RESULT 9** [7, Theorem 15]. Let $S$ be a regular semigroup and let $\rho$ be any congruence contained in $\mathcal{L}$ on $S$. For each element $a$ in $S$, $a$ is an element of a subgroup of $S$ if (and only if) $a\rho$ is an element of a subgroup of $S/\rho$.

**RESULT 10** [7, Theorem 13]. Let $S$ be any semigroup and let $\rho$ be any congruence contained in $\mathcal{L}$ on $S$. For any elements $a, b \in S$, $a \equiv_\rho b$ in $S$ if (and only if) $a\rho b\rho$ in $S/\rho$.

In the notation of [20] the conclusion of Result 10 becomes

$$\mathcal{L}(S/\rho) = \mathcal{L}(S)/\rho.$$ 

**RESULT 11** (from Tully [21, Proposition 1.2]). Let $S$ be any semigroup and for each element $a \in S$ define $\sigma_a \in \mathcal{F}_{S \setminus \mathcal{L}}$ by $L_x \sigma_a = L_x a$ for each $x \in S^1$. Then the mapping $\sigma$ which maps each element $a \in S$ to $\sigma_a$ is a homomorphism, and the congruence $\sigma \circ \sigma^{-1}$ is the maximum congruence contained in $\mathcal{L}(S)$.

### 3. Idempotents in a Regular Semigroup

We use the method of Fitz-Gerald [3] to prove the following lemma.

**LEMMA 1.** Let $S$ be any regular semigroup and let $A_1, A_2, \ldots, A_n$ be any elements of $S$. Put $A_1 A_2 \cdots A_n = a$. Then there exist elements $a_1, a_2, \ldots, a_n$ in $S$ such that

1. $a = a_1 a_2 \cdots a_n$ and $a \equiv_\sigma a_i$, $i = 1, 2, \ldots, n$;
2. $H(a_i) \leq H(a_i)$ for $i = 1, 2, \ldots, n$;
3. if, for some $i$, $A_i$ is an idempotent, then $a_i$ is also an idempotent, whence $a_i \leq A_i$ from (ii).

**Proof.** The only element of which we need the regularity is the element $a$. Let $x$ be any inverse of $a$ and define, for $i = 1, 2, \ldots, n$,

$$a_i = A_1 A_{i+1} \cdots A_n A_i A_1 \cdots A_i.$$

Then $a_1 = axA_1$; assume for some $k < n$ that

$$a_1 a_2 \cdots a_k = (ax)^k A_1 A_2 \cdots A_k.$$


Then

$$a_1 a_2 \cdots a_k a_{k+1} = [(ax)^k A_1 A_2 \cdots A_k] A_{k+1} \cdots A_n x A_1 A_2 \cdots A_{k+1}$$

$$= (ax)^{k+1} A_1 A_2 \cdots A_{k+1}.$$ 

By induction $a_1 a_2 \cdots a_n = (ax)^n a = a$. Now for each $i$,

$$(a_i A_{i+1} A_{i+2} \cdots A_n) x A_1 A_2 \cdots A_i = A_i A_{i+1} \cdots A_n x a A_1 A_2 \cdots A_i = a_i$$

and so $a_i a A_i A_{i+1} \cdots A_n = A_i \cdots A_n x a$. Also

$$A_1 A_2 \cdots A_{i-1}(A_i \cdots A_n x a) = ax a = a,$$

and so $A_i \cdots A_n x a = a_i$. This proves part (i); part (ii) is obvious.

Suppose for some $i$ that $A_i$ is an idempotent. Then

$$a_i^2 = (A_i A_{i+1} \cdots A_n x A_1 A_2 \cdots A_i) (A_i A_{i+1} \cdots A_n x A_1 A_2 \cdots A_i) = a_i,$$

giving part (iii).

The following theorem is a corollary to Lemma 1.

**Theorem 2.** Let $S$ be a regular semigroup with set of idempotents $E$ say. For each $x \in S$ let $E(J(x))$ be the set of idempotents of $J(x)$ and let $\langle E \rangle$ and $\langle E(J(x)) \rangle$ be the subsemigroups of $S$ generated by $E$ and $E(J(x))$, respectively. Let $\langle E(J(x)/I(x)) \rangle$ be the subsemigroup of the principal factor $J(x)/I(x)$ generated by its set of idempotents $E(J(x)) \cup \{I(x)\}$. Then

(i) $\langle E \rangle = \bigcup_{x \in S} (\langle E(J(x)/I(x)) \rangle \setminus \{I(x)\}) = \bigcup_{x \in S} (\langle E(J(x)) \rangle \setminus I(x))$.

(ii) $S$ is generated by its idempotents if and only if each principal factor is generated by its own idempotents,

(iii) the idempotents of $S$ form a subsemigroup if and only if the idempotents of each principal factor form a subsemigroup of that principal factor.

Part (iii) has already appeared in [4].

Remark 1. From Howie [9], we deduce that if $X$ is a finite set, then any element $t$ of $\mathcal{F}_X$ not a permutation, is a product of idempotents of $\mathcal{F}_X$ each having the same rank as $t$. From Erdos [2] we deduce that any $n \times n$ matrix of rank $r < n$ over any field, is a product of idempotent matrices each of rank $r$.

The author is grateful to C. Eberhart for bringing the semigroups satisfying condition (ii) of the next theorem to his attention.

**Theorem 3** (This was found independently by D. G. Fitz-Gerald). Let $S$ be a regular semigroup with set of idempotents $E$ say. The following conditions are equivalent:
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(i) \langle E \rangle is a union of groups;

(ii) For any \( e, f, g \in E \) such that \( e \mathcal{L} f \mathcal{R} g \), there exists \( h \in E \) such that \( e \mathcal{L} h \mathcal{R} g \).

Proof. Suppose condition (i) holds and take any \( e, f, g \in E \) such that \( e \mathcal{L} f \mathcal{R} g \) in \( S \). Then \( eg \in R_e \cap L_g \) [1, Section 2.3] and from (i), \( eg \) is an element of a subgroup of \( \langle E \rangle \), which is of course also a subgroup of \( S \). Hence \( R_e \cap L_g \) contains an idempotent, giving condition (ii).

Conversely, suppose condition (ii) holds. Now the elements of \( E \) are contained in the subgroups of \( S \), so suppose for some \( k \) that the elements of \( E^k \) are contained in the subgroups of \( S \). Take any element \( a \in E^{k+1} \). Since \( V(E^k) = E^{k+1} \) (Result 1) \( a \) has an inverse \( a' \), say, in \( E^k \). By assumption, \( H_{a'} \) contains an idempotent, \( f \), say. Then \( aa'L_{a'a'a} \) whence \( aa'L_{a'a'a} \) and so by condition (ii) \( R_{aa'} \cap L_{a'a} = H_{a'} \) contains an idempotent and is a group. Thus the elements of \( E^{k+1} \), and by induction those of \( \langle E \rangle \) also, are contained in the subgroups of \( S \). By Results 1 and 2 \( \langle E \rangle \) is a union of groups.

Remark 2. Condition (ii) is of course equivalent to the following condition.

(iii) If \( \{R_i : i \in I\} \) and \( \{L_\lambda : \lambda \in \Lambda\} \) are the sets of \( \mathcal{R} \)-classes and of \( \mathcal{L} \)-classes, respectively, of \( S \), for some index sets \( I, \Lambda \), then for some index set \( J \) there exist partitions \( \mathcal{P}(I) = \{I_j : j \in J\} \) and \( \mathcal{P}(\Lambda) = \{\Lambda_j : j \in J\} \) of \( I \) and \( \Lambda \), respectively, such that any \( \mathcal{R} \)-class \( R_i \cap L_\lambda \), where \( i \in I_j, \lambda \in \Lambda_j \) say, contains an idempotent if and only if \( j = k \).

Compare Theorem 3 above with [4] Theorem 6, Lemma 3, and Remark 2. Under the stronger condition that there are no three distinct idempotents \( e, f, g \) in \( S \) such that \( e \mathcal{L} f \mathcal{R} g \), the idempotents of \( S \) form a subsemigroup; this is easily shown using condition (2) of [20, Lemma 1.3].

4. THE MAXIMUM CONGRUENCE IN \( \mathcal{H} \)

Let \( S \) be any regular semigroup and, for each element \( a \in S \), define transformations \( \rho_a \) and \( \lambda_a \) of \( S/\mathcal{L} \) and \( S/\mathcal{R} \), respectively, by \( L_s \rho_a = L_{sa} \), \( R_x \lambda_a = R_{ax} \), for each \( x \in S \). We note that \( \rho_a, \lambda_a \) are single-valued because \( \mathcal{L} \) and \( \mathcal{R} \) are right and left congruences, respectively; and that clearly the mapping \( \rho \) which maps each element \( a \) in \( S \) to \( \rho_a \), is a homomorphism of \( S \) into \( \mathcal{T}_{S/\mathcal{L}} \), and the mapping \( \lambda \) which maps each \( a \) in \( S \) to \( \lambda_a \), is an antihomomorphism of \( S \) into \( \mathcal{T}_{S/\mathcal{R}} \).

**Lemma 4.** The congruence \( \rho \circ \rho^{-1} \) is the maximum congruence on \( S \) contained in \( \mathcal{L} \).
**Proof.** This lemma follows very easily from Result 11. From Result 11 we have that the maximum congruence contained in $\mathcal{L}$, $\mathcal{L}^C$ say, is given by

$$\mathcal{L}^C = \{(a, b) \in S \times S : L_{ua} = L_{ub} \text{ for each } u \in S^1\}.$$ 

However, from the regularity of $S$, we see, as follows, that $uaL^cub$ for each $u \in S^1$ if and only if $xaL^xb$ for each $x \in S$: for if $xaL^xb$ for each $x \in S$, then by choosing $x = aa'$ for any $a' \in V(a)$, we have $L_a = L_{aa'a} = L_{aa'b} \leq L_b$, and similarly $L_b \leq L_a$, whence $a \mathcal{L} b$ and the "if" statement follows; the "only if" statement is obvious. Thus

$$\mathcal{L}^C = \{(a, b) \in S \times S : L_{xa} = L_{xb} \text{ for each } x \in S\} = \rho \circ \rho^{-1},$$

giving the lemma.

Clearly, it follows that $(\rho \circ \rho^{-1}) \cap (\lambda \circ \lambda^{-1})$ is the maximum congruence contained in $\mathcal{H}$.

Remark 3. The lemma may also of course be deduced from the material of [1, Section 10.1] which gives a determination of the maximum congruence contained in any equivalence relation on a semigroup.

Remark 4. Now by Section 2.3 [1], for each $x \in S$ the set of idempotents in $L_x$ (i.e., $L_x \cap E$, where $E$ is the set of idempotents of $S$) is $V(x) x$, and of course $\{(V(x) x, L_x) : x \in S\}$ is a natural one-to-one correspondence between the sets $\{V(x) x : x \in S\}$ and $\{L_x : x \in S\} = S/\mathcal{L}$. Suppose now that we make the mapping $\rho_a \in \mathcal{F}_{S/\mathcal{L}}$, for each $a \in S$, an operator on the set $\{V(x) x : x \in S\}$ by defining, (for all $x \in S$) $(L_x \cap E) \rho_a = (L_{xa} \cap E)$, or equivalently, $V(x) x^{\rho_a} = V(xa) xa$. Then we see that $\rho_a$ is essentially the same as the $\rho_a$ of [5].

**Theorem 5.** The maximum congruence contained in $\mathcal{H}$ on any regular semigroup $S$, $\mu = \mu(S)$ say, is given by

$$\mu = \{(a, b) \in \mathcal{H} : \text{for some [each pair of] $\mathcal{H}$-related inverses } a' \text{ of } a \text{ and } b' \text{ of } b,$$

$$a'e = b'eb \text{ for each idempotent } e \leq aa'\};$$

and equivalently by

$$\mu = \{(a, b) \in S \times S : \text{for some inverses } a' \text{ of } a \text{ and } b' \text{ of } b, aa' = bb', a'a = b'b,$$

$$\text{and } a'e = b'eb \text{ for each idempotent } e \leq aa'\}.$$ 

**Proof.** (i) Take any pair $(a, b) \in \mu$, the maximum congruence contained in $\mathcal{H}$. Since $a \mathcal{H} b$ there exist (and so let us take any) inverses $a'$ of $a$ and $b'$ of $b$ such that $a' \mathcal{H} b'$; then $aa' = bb'$, $a'a = b'b$ [1, Section 2.3]. Since $a'\mu$ and $b'\mu$ are $\mathcal{H}$-related in $S/\mu$, and both are inverses of $a\mu b = b\mu b$ in $S/\mu$, we
have $a'\mu^n = b'\mu^n$ [1, Section 2.3]; i.e., $a'\mu b'$. Take any idempotent $e \in S$ such that $e \leq aa'$. Since $\mu \subseteq \mathcal{H}$ we have $ea \mathcal{H} eb$ and $a' \mathcal{H} b' e$; but routinely one gets $a'e \in V(\mu a)$, $b'e \in V(\mu b)$, whence $(a'e)(\mu a)$ and $(b'e)(\mu b)$ are $\mathcal{H}$-related idempotents, and so $a'ea = (a'e)(\mu a) = (b'e)(\mu b) = b'eb$. Therefore

$$\mu \subseteq \{(a, b) \in \mathcal{H}: \text{for each pair of } \mathcal{H}\text{-related inverses } a' \text{ of } a \text{ and } b' \text{ of } b,$$

$$a'ea = b'eb \text{ for each idempotent } e \leq aa'\}$$

(ii) Take now any pair $(a, b) \in \mathcal{H}$ such that for some $\mathcal{H}$-related inverses $a'$ of $a$, $b'$ of $b$, $a'ea = b'eb$ for each idempotent $e \leq aa'$. Take any element $x \in S$. Then $L_xaa' \leq L_xa'a$; so by Result 4 there exists an idempotent $e \in L_xaa'$ such that $e \leq aa'$. Then $a'e \in V(\mu a)$, $b'e \in V(\mu b)$ (easily checked) and so

$$L_xp_a = L_xa'a = L_{e'a} = L_{e'a}$$

$$= L_{a'ea} = L_{b'eb} = L_xp_b,$$

similarly. Therefore $\rho_a = \rho_b$.

From Result 3, and from $a'a = b'b$, $aa' = bb'$ and $a'ea = b'eb$ for each idempotent $e \leq aa'$, we deduce that $afa' = bfb'$ for each idempotent $f \leq a'a$. We now deduce dually that $\lambda_a = \lambda_b$, and so $(a, b) \in (\rho \circ \rho^{-1}) \cap (\lambda \circ \lambda^{-1}) = \mu$. This proves the first formula and its alternative reading. The second formula is easily deduced, by use of Section 2.3 [1].

**Corollary 6.** Let $T$ be a regular subsemigroup of a regular semigroup $S$ such that for any idempotents $e, f$ in $S$ with $f \leq e$, $e \in T$ implies $f \in T$. For example, let $T$ be an ideal, a subsemigroup of the form $eSe$ for some idempotent $e \in S$, or any regular subsemigroup containing all the idempotents of $S$. Then $\mu(T) = \mu(S) \cap (T \times T)$. In particular, if $S$ is fundamental, then so is $T$.

**Remark 5.** As stated in [17, Theorem 2.1] and [7, Corollary 14] if $\mu$ is the maximum congruence contained in $\mathcal{H}$ on any semigroup $S$, then $S/\mu$ is fundamental.

**5. The Construction of $T_{\langle E \rangle}$**

Let $S$ be any regular semigroup, with set of idempotents, $E$, say, and let $\langle E \rangle = B$, say, be the subsemigroup of $S$ generated by $E$. From Result 1 $B$ is regular. In this section the common notations ($\mathcal{L}, \mathcal{R}, L_a$, etc.) associated with Green's relations will be with respect to the semigroup $B$; we shall use $\mathcal{L}(S)$ etc. in connection with $S$. For each $a \in S$, we define $\rho_a$ and $\lambda_a$ slightly differently from the way in Section 4, as follows: for each $x \in B$,

$$L_xp_a = B \cap L(S)_{xa}, \quad R_x\lambda_a = B \cap R(S)_{ax}.$$
From Result 2, we have that $\rho_a \in \mathcal{T}_{\mathcal{B}/\mathcal{L}}$, $\lambda_a \in \mathcal{T}_{\mathcal{B}/\mathcal{R}}$, and because of the natural one-to-one correspondence between the $\mathcal{L}$ and $\mathcal{R}$-classes of $\mathcal{B}$ and those of $\mathcal{S}$, $\rho_a$ and $\lambda_a$ are essentially the same as the $\rho_a$ and $\lambda_a$ of Section 4. In particular, the mapping $(\rho, \lambda)$, which maps each $a \in \mathcal{S}$ to $(\rho_a, \lambda_a) \in \mathcal{T}_{\mathcal{B}/\mathcal{L}} \times \mathcal{T}_{\mathcal{B}/\mathcal{R}}$ (where $\mathcal{T}_{\mathcal{B}/\mathcal{R}}^*$ is the semigroup dual to $\mathcal{T}_{\mathcal{B}/\mathcal{R}}$) is a homomorphism.

For each idempotent $e \in \mathcal{E}$, let $<e>$ denote the subsemigroup of $e\mathcal{B}e$ generated by the idempotents of the semigroup $e\mathcal{B}e$, which is easily shown to be regular, whence $<e>$ is also regular (Result 1). Thus

$$<e> = \langle E(e\mathcal{B}e) \rangle = \langle \{ f \in \mathcal{E} : f \leq e \} \rangle.$$  

Define

$$\mathcal{U} = \{(e, f) \in \mathcal{E} \times \mathcal{E} : <e> \subseteq <f>\}$$

and for each $(e, f) \in \mathcal{U}$ let $T_{e,f}$ be the set of all isomorphisms from $<e>$ onto $<f>$, and for each $\alpha \in T_{e,f}$ define the further mappings $\tilde{\alpha}$ and $\check{\alpha}$ by (for each idempotent $g \leq e$)

$$L_g \tilde{\alpha} = L_{g\alpha}, \quad R_g \check{\alpha} = R_{g\alpha}.$$  

From Result 2 and the fact that $\alpha$ is an isomorphism, we see that $\tilde{\alpha} \in \mathcal{J}_{\mathcal{B}/\mathcal{L}}$ and $\check{\alpha} \in \mathcal{J}_{\mathcal{B}/\mathcal{R}}$. For each $\alpha \in T_{e,f}$, consider $\rho_\alpha \tilde{\alpha}$ and $\lambda_\alpha \check{\alpha}^{-1}$, products being taken in $\mathcal{P}\mathcal{T}_{\mathcal{B}/\mathcal{L}}$ and $\mathcal{P}\mathcal{T}_{\mathcal{B}/\mathcal{R}}$, respectively, and put

$$T_{<E>} = \bigcup_{(e,f) \in \mathcal{U}} \{(\rho_\alpha \tilde{\alpha}, \lambda_\alpha \check{\alpha}^{-1}) : \alpha \in T_{e,f}\}.$$  

For convenience we shall denote $(\rho_\alpha \tilde{\alpha}, \lambda_\alpha \check{\alpha}^{-1})$ simply by $\phi(\alpha)$. That

$$T_{<E>} \subseteq \mathcal{T}_{\mathcal{B}/\mathcal{L}} \times \mathcal{T}_{\mathcal{B}/\mathcal{R}}^*$$

is easily shown.

Theorem 7.  

(i) The set $T_{<E>}$ is a subsemigroup of $\mathcal{T}_{\mathcal{B}/\mathcal{L}} \times \mathcal{T}_{\mathcal{B}/\mathcal{R}}^*$.  

(ii) Moreover, $T_{<E>}$ is a regular semigroup whose subsemigroup generated by its idempotents is isomorphic to $\langle E \rangle/\mu(\langle E \rangle)$.  

(iii) Let $(\rho, \lambda)$ be the mapping which maps each element $a$ in the regular semigroup $\mathcal{S}$ to $(\rho_a, \lambda_a)$ as defined above. Then $(\rho, \lambda)$ is a homomorphism of $\mathcal{S}$ into $T_{<E>}$ which maps $E$ onto the set of idempotents of $T_{<E>}$.  

(iv) For any $e, f \in \mathcal{E}$, if $<e>$ is isomorphic to $<f>$, then $(\rho_e, \lambda_e)$ and $(\rho_f, \lambda_f)$ are $\mathcal{D}$-related in $T_{<E>}$.

Proof. After the lemma we proceed straight to the proof.
**Lemma 8.** If $\alpha \in T_{e,f}$ and $\beta \in T_{f,g}$, for some $e, f, g \in E$, then $\vec{\alpha \beta} = \vec{\alpha \beta}$ and $\overline{\alpha \beta} = \overline{\alpha \beta}$.

**Proof.** Now $\alpha \beta \in T_{e, g}$, so take any idempotent $h \in \langle e \rangle$. Then since $h \alpha \in \langle f \rangle$, we have

$$L_h \vec{\alpha \beta} = L_h \vec{\alpha \beta} = L_h \vec{\alpha \beta} = L_h \vec{\alpha \beta},$$

whence $\vec{\alpha \beta} = \vec{\alpha \beta}$, since the domains of $\vec{\alpha \beta}$ and $\overline{\alpha \beta}$ are both equal to $\{L_x \in B | \mathcal{L} : L_x \leq L_e\}$. Similarly $\overline{\alpha \beta} = \overline{\alpha \beta}$.

(i) Take any pairs $(e, f), (g, h)$ in $\mathcal{U}$, and any elements $\alpha \in T_{e, f}, \beta \in T_{g, h}$. We wish to show $(\alpha \beta) \in \mathcal{U}$.

Take any inverse $(fg)'$ of $fg$ in $\langle E \rangle$, and put $i = g(fg)'f$, an idempotent which is also an inverse of $fg$ [11, Lemma 1.1]. Now define the mappings of Result 3 for the semigroup $B = \langle E \rangle$. Then $\theta_{i,fg} | \langle fgi \rangle = \gamma$ say, maps $\langle fgi \rangle$ isomorphically onto $\langle ifg \rangle$,

$$\theta_{i,fg} | \langle fgi \rangle = (\theta_{i,fg} | \langle fgi \rangle)^{-1} = \gamma^{-1},$$

and also, for any idempotent $l \leq fgi$,

$$L_{i,fg} = L_{i,fg} = L_{i,fg} \quad \text{(since } i \in V(fg), l \leq fgi)$$

Hence

$$\rho_{fg} \{L_1 : l \in E, l \leq fgi\} = \gamma,$$  \hspace{1cm} (1)

and similarly

$$\lambda_{fg} \{R_1 : l \in E, l \leq ifg\} = \gamma^{-1}.$$

Put $(fg)' \alpha^{-1} = j$ and $(ifg) \beta = k$. (Note that $j^2 = j \leq e$ and $k^2 = k \leq h$.)

Now $(\alpha \beta) \gamma = \delta$ say, is an isomorphism from $\langle j \rangle$ onto $\langle k \rangle$; we shall show that $\phi(\alpha) \phi(\beta) = \phi(\delta)$. We see that $\delta^{-1} = (\beta^{-1} | \langle k \rangle) \gamma^{-1} \alpha^{-1}$.

Take now any element $x \in \langle E \rangle$. Then $L_{xe} = L_1$ for some idempotent $l \leq e$ (Result 4), whence (from $\rho_r \rho_j = \rho_{rj} = \rho_r$)

$$L_{xe} \overline{\alpha \beta} = L_{xe} \overline{\alpha \beta} = L_{xe} \overline{\alpha \beta} = L_{xe} \overline{\alpha \beta} = L_{xe} \overline{\alpha \beta},$$

and hence

$$\rho_{\overline{\alpha \beta}} = \rho_{\overline{\alpha \beta}},$$  \hspace{1cm} (2)

and similarly

$$\lambda_{\overline{\alpha \beta}} = \lambda_{\overline{\alpha \beta}} \lambda_{ifg}.$$
Now we have
\[
\rho_e \tilde{\rho}_i \tilde{\beta} = (\rho_e \tilde{\rho}_i) \rho_j \tilde{\beta} = \rho_e \tilde{\rho}_i \rho_j \tilde{\beta}
\]
\[
= (\rho_e \tilde{\rho}_i \rho_j) \tilde{\beta} = \rho_i \tilde{\rho}_j \tilde{\beta} \quad \text{(from (2))}
\]
\[
= \rho_i \tilde{\gamma} \tilde{\beta} \quad \text{(from (1))}
\]
\[
= \rho_i \tilde{\delta},
\]
since

\[
\text{range}(\rho_f \tilde{\alpha}) = \{L_1 : l \in E, l \leq fg\}
\]
and

\[
\text{range } \rho_f = \{L_1 : l \in E, l \leq j\}.
\]

Similarly

\[
\lambda_i \tilde{\beta}^{-1} \lambda_j \tilde{\alpha}^{-1} = \lambda_i \tilde{\delta}^{-1},
\]
and so \(\phi(\alpha) \phi(\beta) = \phi(\delta)\). This completes the proof of the closure of \(T_{\langle E \rangle}\).

(ii) To show that \(T_{\langle E \rangle}\) is regular, let us take any element, \(\phi(\alpha)\), say, in \(T_{\langle E \rangle}\), and consider \(\phi(\alpha^{-1})\). If \(\alpha \in T_{\langle E \rangle}\), say, then using \(\tilde{\rho}_f = \tilde{\alpha}\) and \(\alpha^{-1} \lambda_e = \alpha^{-1}\), we may routinely show that \(\phi(\alpha^{-1})\) is an inverse of \(\phi(\alpha)\) in \(T_{\langle E \rangle}\). We proceed now to find the idempotents of \(T_{\langle E \rangle}\).

Take any element of \(T_{\langle E \rangle}\) which is idempotent, \(\phi(\alpha)\), say, where \(\alpha \in T_{\langle E \rangle}\) for some \((e, f) \in \mathbb{E}\). Then \(\rho_e \tilde{\alpha}\) is an idempotent, and since \(L_f\) is in its range, we have \(L_f \rho_e \tilde{\alpha} = L_f\). But also

\[
L_e \tilde{\alpha} = L_e = L_f \tilde{\alpha} 
\]
(from above),

and since \(\tilde{\alpha}\) is one-to-one, we have \(L_e = L_{fe}\). Using the idempotence of \(\lambda_f \alpha^{-1}\) we similarly obtain \(R_f = R_{fe}\). From [1, Section 2.3] there exists an inverse \((fe)^*\), say, of \(fe\), in \(R_e \cap L_f\), and then \(e(fe)^* = (fe)^* = (fe)^* f\), and so \((fe)^* = i\), say, is an idempotent. (In fact, from \(L_e = L_{fe}\) and \(R_f = R_{fe}\) we may show that \((fe)^* = e(fe)^* f\) for any inverse \((fe)^*\) of \(fe\).) Now \(i \in R_e\) implies \(ie = e\), and so, for each \(x \in \langle E \rangle\),

\[
L_{x \rho e \tilde{\alpha}} = L_{x \tilde{\alpha} \tilde{\alpha}} = L_{x \tilde{\alpha} \tilde{\alpha} \tilde{\alpha}} = L_{x \tilde{\alpha}}
\]
since \(L_{x \tilde{\alpha}} \leq L_i = L_f\) implies that \(L_{x \tilde{\alpha}}\) is in the range of \(\rho_e \tilde{\alpha}\). Therefore, \(\rho_e \tilde{\alpha} = \rho_i\) and similarly \(\lambda_\alpha^{-1} = \lambda_i\), giving \(\phi(\alpha) = (\rho_i, \lambda_i)\). Thus the set of idempotents of \(T_{\langle E \rangle}\) is \(\{(\rho_i, \lambda_i) : i \in E\}\), the set of images of the idempotents of \(S\) under the homomorphism \((\rho, \lambda)\) of \(S\) into \(\mathcal{F}_{1/\mathbb{Q}} \times \mathcal{F}_{1/\mathbb{R}}\). Thus \((\rho, \lambda)\) maps \(\langle E \rangle\) onto the subsemigroup of \(T_{\langle E \rangle}\) generated by its idempotents, which is therefore isomorphic to \(\langle E \rangle / \mu(\langle E \rangle)\), by Corollary 6.
(iii) To prove part (iii) of the theorem it suffices to prove that, for each \(a \in S\), \((\rho_a, \lambda_a) \in T_{\langle E \rangle}\) (since \((\rho, \lambda)\) is a homomorphism of \(S\) into \(\mathcal{F}_{B(\mathbb{F})} \times \mathcal{F}_{B(\mathbb{F})}^*\)).

Take any \(a \in S\), any \(a' \in \mathcal{V}(a)\) and any \(x \in \langle E \rangle\). Then \(L_{xa'a} \leq L_{aa'}\) and so \(L_{xa'a} = L_x\) for some idempotent \(e \leq aa'\). Also \(\theta_{a',a} | <aa'> = x\) say, is an isomorphism from \(<aa'>\) onto \(<a'a>\), i.e., \(x \in T_{aa',aa'}\). Now

\[
L_{x\rho_a} - L_{x\rho_{aa'}p_a} - L_e\rho_a - B \cap L(S)_{ea} = B \cap L(S)_{a'ea} = L_{a'ea} = L_{ea} = L_{xa'a}^{-1} = L_{x\rho_{aa'}\alpha},
\]

Therefore, \(\rho_a = \rho_{aa'}\alpha^{-1}\) and similarly \(\lambda_a = \lambda_{a'aa'}\alpha^{-1}\), giving

\[
(\rho_a, \lambda_a) = \varphi(a) \in T_{\langle E \rangle}.
\]

(iv) Take any \((e,f) \in \U\) and any \(a \in T_{e,f}\). It is quite routine to show that in \(T_{\langle E \rangle}\) we have \(\varphi(a) \varphi(a^{-1}) = (\rho_e, \lambda_e)\) and \(\varphi(a^{-1}) \varphi(a) = (\rho_f, \lambda_f)\). But since \(\varphi(a^{-1})\) is an inverse of \(\varphi(a)\) [from (ii)] this gives that \((\rho_e, \lambda_e)\) and \((\rho_f, \lambda_f)\) are \(\mathcal{D}\)-related in \(T_{\langle E \rangle}\).

6. SOME CONSEQUENCES

Let \(B = \langle E \rangle\) be any fundamental regular semigroup generated by its set of idempotents \(E\), and let us define \(\U\) and \(T_{\langle E \rangle}\) as in Section 5. In this case the subsemigroup generated by the idempotents of \(T_{\langle E \rangle}\) is isomorphic to \(\langle E \rangle\), and by renaming elements one could construct a semigroup isomorphic to \(T_{\langle E \rangle}\) in which \(\langle E \rangle\) is the actual subsemigroup generated by the idempotents. We will not do this.

For the fixed semigroup \(B = \langle E \rangle\) we consider the class of semigroups, \(\mathcal{C}(B)\) say, determined by the following: A semigroup \(S\) is a member of \(\mathcal{C}(B)\) if and only if \(S\) is a regular semigroup whose subsemigroup generated by its idempotents is isomorphic to \(B\).

We consider and give answers to the following questions: What conditions on \(B\) are equivalent to the following conditions:

(a) some member of \(\mathcal{C}(B)\) is \([0-]\) bisimple,
(b) some member of \(\mathcal{C}(B)\) is \([0-]\) simple,
(c) every member of \(\mathcal{C}(B)\) is completely semisimple [a union of groups],
(d) Green's relation \(\mathcal{H}\) is a congruence on every member of \(\mathcal{C}(B)\)?
Theorem 9. Some member of \( \mathcal{C}(B) \) is \([0-] \) bisimple (i.e., the semigroup \( B \) is the subsemigroup generated by the idempotents of a \([0-] \) bisimple semigroup) if and only if \( \mathcal{U} = E \times E [[(0, 0)] \cup (E \{\{0\}) \times (E \{\{0\})] \), in which case we call \( \langle E \rangle \) \([0-] \) uniform.

The corresponding statement for semilattices and inverse semigroups is due to Munn [13] (see also [14]), and that for bands and orthodox semigroups (regular semigroups whose idempotents form a band) is due to the author [5].

Proof. From Result 3, if \( \langle E \rangle \) is the subsemigroup generated by the idempotents of a bisimple semigroup, \( S \), say, then \( \langle E \rangle \) is uniform, since for any \( e, f \in E, eSe \cong fSf \) implies \( \sim e > > \), the subsemigroup generated by the idempotents of \( eSe \), is isomorphic to \( \langle f > > \rangle \), the corresponding subsemigroup of \( fSf \). Conversely, if \( \langle E \rangle \) is uniform, then \( T = \langle E \rangle \) is bisimple, since all its idempotents will be \( D \)-related from Theorem 7 part (iv), and each \( D \)-class of a regular semigroup contains an idempotent. The proof of the bracketed statement is entirely similar; we note that if \( \langle E \rangle \) has a zero \( 0 \), then \( T \) has a zero, namely, \((\rho_0, \lambda_0)\).

Theorem 10. The semigroup \( B = \langle E \rangle \) is the subsemigroup generated by the idempotents of a \([0-] \) simple regular semigroup if and only if, for any \([\text{non-zero}] \) \( e, f \in E \), there exists an idempotent \( g < f \) such that \( (e, g) \in \mathcal{U} \), in which case we call \( \langle E \rangle \) \([0-] \) subuniform.

The corresponding statement for semilattices and inverse semigroups is due to Munn [16].

Proof. From Results 3 and 5, we have the “only if” statement. Conversely, if \( \langle E \rangle \) is subuniform then \( T = \langle E \rangle \) is simple, since for any idempotents \((\rho, \lambda)_i \), \((\rho_f, \lambda_f) \) in \( T = \langle E \rangle \), there exists \( g \in \langle E \rangle \) such that \( g < f \) and \( (e, g) \in \mathcal{U} \), whence \( (\rho, \lambda) \leq (\rho_f, \lambda_f) \) and \( (\rho, \lambda) \) is \( D \)-related to \( (\rho, \lambda) \). The proof of the bracketed statement is entirely similar.

Theorem 11. Every regular semigroup, whose subsemigroup generated by its idempotents is (isomorphic to) \( B \) is completely semisimple, (i.e., every member of \( \mathcal{C}(B) \) is completely semisimple), if and only if \( \mathcal{U} \cap \leq = \iota \), where \( \leq \) is the relation \( \{ (f, e) \in E \times E : fe = ef \} \) and \( \iota \) is the identity relation on \( E \).

Proof. If \( \mathcal{U} \cap \leq = \iota \), then in any regular semigroup with \( \langle E \rangle \) as its subsemigroup generated by its idempotents, no two distinct comparable idempotents can be \( D \)-related, so from Result 6, the semigroup is completely semisimple.

Conversely, if \( T = \langle E \rangle \) is completely semisimple, then for any \((f, e) \in \mathcal{U} \), since \((\rho_f, \lambda_f) \) and \((\rho, \lambda) \) are \( D \)-related in \( T = \langle E \rangle \), we have \( f \leq e \), giving \( \mathcal{U} \cap \leq = \iota \).
Remark 1. Let $S$ be any regular semigroup whose subsemigroup generated by its idempotents is $B = \langle E \rangle$. If each $\mathcal{H}$-class of $E$ contains a finite number of elements, then $S$ is completely semisimple. If moreover the number of elements in each $\mathcal{H}$-class is bounded by $m$, some positive integer, then for each $a \in S$, $a^m$ is an element of a subgroup (see Theorem 15).

In the next theorem, by $\mathcal{D}(B)$ we mean of course Green's relation $\mathcal{D}$ on $B$.

Theorem 12. Every regular semigroup, whose subsemigroup generated by its idempotents is $B = \langle E \rangle$, is a union of groups if and only if both $B$ is a union of groups and $\mathcal{U} = \mathcal{D}(B) \cap (E \times E)$.

The corresponding statement for semilattices and inverse semigroups is due to Howie and Schein [10], and that for bands and orthodox semigroups is due to the author [5].

Proof. Suppose $B$ is a union of groups and $\mathcal{U} = \mathcal{D}(B) \cap (E \times E)$. Let $S$ be any regular semigroup whose subsemigroup generated by its idempotents is $\langle E \rangle$ and take any $a \in S$ and any inverse $a'$ of $a$. Then $(aa', a'a) \in \mathcal{U} \subseteq \mathcal{D}(B)$ and so $R(B)_{aa'} \cap L(B)_{a'a} \neq \emptyset$; and $B$ being a union of groups this $\mathcal{H}$-class of $B$ contains an idempotent, $e$, say. But then $e\mathcal{L}(S)a' e\mathcal{R}(S)a$ and $e\mathcal{H}(S)aa'e\mathcal{R}(S)a$, whence $e\mathcal{H}(S)a$ and $a$ is an element of a subgroup of $S$. Thus $S$ is a union of groups.

To prove the converse, we really only need $T_{\langle E \rangle}$ to be a union of groups. First of all, since $B$ is isomorphic to a regular subsemigroup of $T_{\langle E \rangle}$, we have $B$ is a union of groups (Result 2). Secondly, let us take any $(e,f) \in \mathcal{U}$ and $\alpha \in T_{\langle E \rangle}$. Then $(\rho_\alpha, \lambda_\alpha) \mathcal{R}(\alpha) \mathcal{L}(\rho_f, \lambda_f)$ in $T_{\langle E \rangle}$ (Section 5, proof of Theorem 7 part (iv)) and $\phi(\alpha)$ is $\mathcal{H}$-related to an idempotent in $T_{\langle E \rangle}$, say, $(\rho_\alpha, \lambda_g)$, where $g \in E$. Then $e\mathcal{H}g \mathcal{L}f$ in $B$ whence $(e,f) \in \mathcal{D}(B)$ and $\mathcal{U} \subseteq \mathcal{D}(B)$. But we always have $\mathcal{D}(B) \cap (E \times E) \subseteq \mathcal{U}$. Therefore $\mathcal{U} = \mathcal{D}(B) \cap (E \times E)$ and the theorem is proved.

Theorem 13. Green's relation $\mathcal{H}$ is a congruence on every regular semigroup, whose subsemigroup generated by its idempotents is $B = \langle E \rangle$, if and only if for each $e \in E$, $\langle e \rangle$ has only one automorphism, the trivial one, in which case we say that $\langle E \rangle$ is taut.

The corresponding statement for semilattices and inverse semigroups was proved by Munn in the proof of Theorem 3.2 [13]. We delay a proof of Theorem 13 until Section 10, where we study $T_{\langle E \rangle}$ further (we could prove the "if" statement now from Result 3 and Theorem 5; our proof in Section 10 is more conceptual). If for each $e \in E$, $\{ f \in E : f \ll e \}$ is inversely well-ordered (see Munn [13, p. 157]) then $E$ is taut; of course each $\langle e \rangle$ is then necessarily a semilattice but $\langle E \rangle$ need not even be a band (consider a suitable completely 0-simple semigroup).
Let us consider the statements of the theorems of this section without the restriction that $B$ is fundamental i.e., with $B$ as any regular semigroup generated by its idempotents. Then the proofs given here are sufficient to prove the following statements without the mentioned restriction: the "only if" statements of Theorems 9 and 10 and the "if" statements of Theorems 11–13. The author does not know if the remaining statements without the restriction are also true.

7. MAXIMAL REGULAR SUBSEMGROUFS

THEOREM 14. Let $E$ be any set of idempotents of a semigroup $S$.

(i) There is a regular subsemigroup of $S$ with $E$ as its set of idempotents if and only if $(E)$, the subsemigroup generated by $E$, is such a semigroup, i.e., a regular semigroup with $E$ as its set of idempotents.

(ii) If $(E)$ is a regular subsemigroup with $E$ as its set of idempotents then

$$EC = \{a \in S : \text{for some } a' \in V(a), aa', a'a, a'ea, afa' \in E$$

for all $e, f \in E$ such that $e \leq aa', f \leq a'a$}

is the maximum regular subsemigroup of $S$ with $E$ as its set of idempotents.

The corresponding statement of part (ii) for $E$ a subsemigroup of $S$ is equivalent to the statement due to Reilly and Scheiblich [20, Theorem 1.5].

Proof. Part (i) is a corollary of Result 1, so we now consider part (ii). It is easily seen that $EC$ contains any regular subsemigroup of $S$ with $E$ as its set of idempotents. We now let

$$X = \{a \in S : \text{for some } a' \in V(a), aa', a'a \in E \text{ and } L_Ea \subseteq L_E, aR_E \subseteq R_E\},$$

where

$$L_E = \bigcup \{L_e \in S/\mathcal{L} : e \in E\} \quad \text{and} \quad R_E = \bigcup \{R_e \in S/\mathcal{R} : e \in E\};$$

we shall show that $X$ is a subsemigroup of $S$ with $E$ as its set of idempotents. Take any $a, b \in X$. Clearly $L_Eab \subseteq L_Eb \subseteq L_E$ and $abR_E \subseteq R_E$. Now there are $a' \in V(a), b' \in V(b)$ such that $a'a, aa', b'b, bb' \in E$. Since $L_Eb \subseteq L_E$ and $\mathcal{L}$ is a right congruence on $S$, we have $ab \in L_0b = L_{a'b}b \subseteq L_e$ for some $e \in E$ and similarly $ab \in aR_{b'} \subseteq R_e$ for some $f \in E$. Hence $ab \in L_e \cap R_e$ and so there is an inverse $(ab)'$ say, of $ab$, in $R_e \cap L_e$, , and then $ab(ab)' = f \in E$ and $(ab)'ab = e \in E$. Thus $ab \in X$ and $X$ is a subsemigroup. Take now any idempotent $a \in X$. Then there is $a' \in V(a)$ such that $a'a, aa' \in E \subset \langle E \rangle$ and
then \(a' = a'aa' = a'aaa' \in \langle E \rangle\). Now \(aa'L'aRa'a \in \langle E \rangle\) and so there is an inverse \(x\) of \(a'\) in \(\langle E \rangle\) such that \(aa'L'aRa'a \in \langle E \rangle\) and hence also in \(S\). But \(aa'L'aRa'a \in S\) and \(a \in V(a')\) and so \(a = x \in \langle E \rangle\) [1, Section 2.3], whence \(a \in E\) and the set of idempotents of \(X\) is \(E\) (since clearly \(E \subseteq X\)).

Since \(E\) generates a regular subsemigroup of \(X\), by Result 7 we have that the set of regular elements of \(X\), \(Y\) say, forms a subsemigroup (with set of idempotents \(E\)); from above, \(Y \subseteq E^c\). Clearly

\[Y = \{a \in S : \text{for some} \ a' \in V(a), \ aa', \ a'a \in E, \text{and} \ L_{Ea} \subseteq L_E, \ L_{Ea'} \subseteq L_E, \text{and} \ aR_E \subseteq R_E, \ a'R_E \subseteq R_E\}.\]

We show eventually that \(Y = E^c\).

Take then any \(a \in E^c\) and any \(a' \in V(a)\) satisfying the conditions for the membership of \(a\) in \(E^c\). Take any \(e \in E\). Now \(eaa' \in \langle E \rangle\) and so \(eaa'Laf\) in \(\langle E \rangle\) for some \(f \in E\) such that \(f \leq aa'\) (Result 4). Considering the \(L\)-classes of \(S\) we now have

\[L_{ea}a \subseteq L_{ea} = L_{eaa'} = L_{fa} = L_{a'fa}\]

since \(L\) is a right congruence and \(a(a'fa) = fa\). But \(a'fa \in E\) and so \(L_{ea} \subseteq L_E\) and \(L_{Ea} \subseteq L_E\). Similarly, \(aR_E \subseteq R_E\), \(L_{Ea'} \subseteq L_E\) and \(a'R_E \subseteq R_E\), giving \(a \in Y\). Therefore, \(E^c \subseteq Y\) and, then, \(E^c = Y\).

Thus \(E^c\) is the maximum regular subsemigroup of \(S\) with \(E\) as its set of idempotents.

8. Completely Semisimple Semigroups

Theorem 15. Let \(S\) be a regular semigroup such that each \(D\)-class of \(S\) contains at most \(m\) \(L\)-classes of \(S\), for some integer \(m\) \((S\) is thus completely semisimple). Then for each element \(a \in S\), \(a^m\) is an element of a subgroup of \(S\).

Proof. Consider the homomorphism \(\rho\) of \(S\) into \(\mathcal{T}_{S/L}\) defined in Section 4. Take any element \(a \in S\). Note that \(\rho_{am} = \rho_a^m\). For each \(D\)-class \(D\) of \(S\) denote \(\{L_x \in S/L : x \in D\}\), the set of \(L\)-classes of \(S\) contained in \(D\), by \(D/L\).

Take any \(D\)-class \(D\) of \(S\) such that \((D/L) \cap \text{range} \ \rho_a^m \neq \emptyset\) and take any \(L\)-class, \(L_y\) say, in \((D/L) \cap \text{range} \ \rho_a^m\). Then there exists \(x \in S\) such that

\[L_y = L_x \rho_a^m = L_x a^m = L_x a^m (a^m)^{-1} = L_x a^m (a^m)^{-1} \rho_a^m\]

for any inverse \((a^m)'\) of \(a^m\) in \(S\). Put \(x a^m (a^m)' = b\). Then

\[bRba \ Rba^2 \cdots Rba^m L_y.\]
But $D$ contains at most $m \mathcal{L}$-classes of $S$ and so two of $L_{ba}, L_{ba}, ..., L_{ba^n}$ are equal, i.e., for some $i < j, i, j \in \{0, 1, ..., m\}$ we have $L_{ba^i} = L_{ba^j}$ where $ba^a$ means just $b$. By repeated multiplication by $a$ on the right of the subscripts in $L_{ba^i} = L_{ba^j}$, we obtain (since $\mathcal{L}$ is a right congruence) that

$$L_{ba^n} \in \{L_{ba}, L_{ba}, ..., L_{ba^n}\} \quad \text{for } n = 1, 2, 3, ... \quad (3)$$

and that also, there are arbitrarily large $n$ for which $L_{ba^n} = L_{ba^n}$. Choose such an $n > 2m$; then $n = 2m + p$ for some positive integer $p$. We now have

$$L_{ba} = L_{ba^n} = L_{ba^n} = (L_{ba^n} \rho_a^m) \rho_a^m$$

and $L_{ba^n} \in (D/\mathcal{L}) \cap \text{range } \rho_a^m$ from (3). Now $L_{ba}^n$ was an arbitrary element of $(D/\mathcal{L}) \cap \text{range } \rho_a^m$ so $(D/\mathcal{L}) \cap \text{range } \rho_a^m \subseteq (D/\mathcal{L}) \cap \text{range } \rho_a^m \rho_a^m$. But $(D/\mathcal{L}) \cap \text{range } \rho_a^m$ is finite so $\rho_a^m$ maps it one-to-one onto itself. It follows that for every $D \in S[\mathcal{J}] \rho_a^m \in (D/\mathcal{L}) \cap \text{range } \rho_a^m$ one-to-one onto itself (which may be empty). From this it follows that $\rho_a^m$ maps its entire range one-to-one onto itself, and thus is an element of a subgroup of $\mathcal{T}_{S/\mathcal{L}}$ [1, Theorem 2.10]. By Result 2 part (iii), $\rho_a^m$ is an element of a subgroup of $\rho(S) = \{\rho_s : s \in S\}$ and by Result 9, since $\rho \circ \rho^{-1} \subseteq \mathcal{L}$, $a^m$ is an element of a subgroup of $S$, giving the theorem.

Result 8 and Theorem 15 make a nice pair; the former involves a concept known as the height of $S$ and the latter involves a concept sometimes called the width of $S$. From the two results (not numbered) adjacent to Result 6 it follows that a regular semigroup $S$ is completely semisimple and each strictly descending chain of $\mathcal{J}$-classes has length at most $m$, if and only if each strictly descending chain of idempotents has length at most $m$ (due to W. D. Munn, unpublished). This enables a restatement of Result 8.

The proof of Theorem 15 was designed so that minor variations in it give one a proof of the following. Let $m$ be an integer, let $a$ be an element of a regular semigroup $S$, and let $\rho_a$ be defined as in Section 4. If

(i) the range of $\rho_a$ contains less than $m \mathcal{L}$-classes of $S$ from each $\mathcal{D}$-class of $S$

Then $a^m$ is an element of a subgroup of $S$. Now let $a'$ be any inverse of $a$ and put $a'a = e$. Then the range of $\rho_a$ is $\{L_x \in S[\mathcal{L}] : x \in eSe\}$ (Result 4) and each of Green's relations on $eSe$ is simply the restriction to $eSe$ of the corresponding Green's relation on $S$ (this is easily shown for any semigroup $S$). Hence condition (i) is equivalent to the following condition:

(ii) each $\mathcal{D}$-class of $eSe$ contains at most $m - 1 \mathcal{L}$-classes of $eSe$.

This further enables us to replace the hypothesis of Theorem 15 with the hypothesis that for each idempotent $e$ in a regular semigroup $S$, condition (ii) holds.
There is a proof of Theorem 15 that uses Result 8. We present this proof for the case when S is an inverse semigroup, because of its simplicity. In this case each \( \mathcal{R} \)-class of \( S \) contains at most \( m \) idempotents. For each \( a \in S \) define \( \alpha_a : aa^{-1}E \to a^{-1}aE \) by \( \alpha_a e = a^{-1}ea \) for each \( e \in aa^{-1}E \), where \( E \) is the semilattice of \( S \), as in [13]. Then \( \alpha_a \) is \( \mathcal{R} \)-class preserving and so \( \alpha_a = \bigcup_{x \in a} \alpha_a \cap (D_x \times D_x) \) whence \( \alpha_a^m = \bigcup_{x \in a} (D_x \times D_x)^m \). But for each \( D \in S \mathcal{D} \), \( \alpha_a \cap (D \times D) \in \mathcal{I}_E(D) \) and \( |E(D)| \leq m \), where \( E(D) \) is the set of idempotents of \( D \). Thus \( [\alpha_a \cap (D \times D)]^m \) is an element of a subgroup of \( \mathcal{I}_E \) (Section 2) and hence its domain equals its range. Hence the domain of \( \alpha_a^m \) equals the range of \( \alpha_a^m \) and so \( \alpha_a^m \) is an element of a subgroup of \( S \).

The methods of this section also yield the following result. Let \( S \) be a fundamental regular semigroup in which each \( \mathcal{D} \)-class contains at most \( m \) idempotents. Then there is an integer \( n \) such that \( a^n \) is an idempotent for each \( a \) in \( S \).

9. PARTIAL GROUPOIDS OF IDEMPOTENTS

Let \( S \) be a fundamental regular semigroup generated by its set of idempotents, \( E \), say. Then \( S = \langle E \rangle \). We show in this section that \( S \) is uniquely determined (to within isomorphism) by its partial groupoid \([1, p. 1]\) of idempotents \( E \); by the partial groupoid \( E \) we mean the set \( E \) together with the partial binary operation \( (\cdot) \) say, that \( S \) induces on \( E \), namely, that defined by: for all \( e, f \in E \), \( e \cdot f = ef \) if \( ef \in E \), \( e \cdot f \) is undefined if \( ef \in S \setminus E \).

We show this uniqueness by reconstructing \( S \) from \( E(\cdot) \).

Define the following equivalences (see below) on \( E(\cdot) \):

\[
\mathcal{L}' = \{(e, f) \in E \times E : e \cdot f = e, f \cdot e = f\},
\]

\[
\mathcal{R}' = \{(e, f) \in E \times E : e \cdot f = f, f \cdot e = e\}.
\]

For each \( e \in E \), let \( L_e[R_e] \) denote the \( \mathcal{L}' \)[\( \mathcal{R}' \)]-class of \( E \) containing \( e \). Let us partially order the sets \( E/\mathcal{L}' \) and \( E/\mathcal{R}' \) by defining, for any \( e, f \in E \), \( L_e \leq L_f \) if and only if \( e \cdot f = e \) and \( R_e \leq R_f \) if and only if \( f \cdot e = e \). That these give well-defined partial orderings, and in fact that \( \mathcal{L}' \) and \( \mathcal{R}' \) are equivalences, we may easily deduce from facts concerning Green's relations \( \mathcal{D} \) and \( \mathcal{R} \) on \( S \).
For each \( e \in E \) we define a transformation \( \rho_e \) in \( \mathcal{T}_{E_1 \mathcal{L}_\rho} \) as follows: For each \( x \in E \), let \( L_x \rho_e \) be the unique maximum \( \mathcal{L}' \)-class of \( E \) in the set

\[
X = \{ L'_{(h \cdot x), e} \in E | \mathcal{L}' : h \in E, h \cdot x \in E, (h \cdot x) \cdot e \in E \};
\]

we use \( S \) again to show the existence of such an \( \mathcal{L}' \)-class, as follows. Take any \( x \in E \) and let \((xe)\)' be any inverse of \( xe \) in \( S \). Put \( g = e(xe)' x \), an idempotent which is also an inverse of \( xe \) [11, Lemma 1.1]. Then \( gx = g \in E \) and \( gxe \in E \) whence \( X \) is nonempty. Take any \( h \in E \) such that \((h \cdot x) \cdot e \in E \). Then

\[
[(h \cdot x) \cdot e] \cdot [(g \cdot x) \cdot e] = (hxe) \cdot (gxe) = hxe,
\]

since \((hxe)(gxe) = hxe \in E \). Thus \( X \) contains a maximum member, \( L'_{(g \cdot x), e} \), which equals \( E \cap L_x = E \cap L_{xe} = V(xe) xe \), where we use the usual notations associated with Green's relations on \( S \). Hence, for any \( x, y \in E, L_x' = L_y' \) implies \( L_x = L_y \) and \( E \cap L_x = E \cap L_y \), i.e., \( L_y \rho_e = L_y \rho_e \). Thus \( \rho_e \) is a transformation.

Similarly we define \( \lambda_e \in \mathcal{T}_{E_1 \mathcal{R}_\rho} \) by, for each \( x \in E \), \( R_x \lambda_e \) is the unique maximum \( \mathcal{R}' \)-class in \( \{ R'_{x \cdot (h \cdot x) : h \in E, x \cdot h \in E, e \cdot (x \cdot h) \in E \} \}. \) Let us finally define \( T \) to be the subsemigroup of \( \mathcal{T}_{E_1 \mathcal{L}_\rho} \times \mathcal{T}_{E_1 \mathcal{R}_\rho} \), generated by \( \{ (\rho_e, \lambda_e) : e \in E \} \), where \( \mathcal{T}_{E_1 \mathcal{L}_\rho} \) is the semigroup dual to \( \mathcal{T}_{E_1 \mathcal{R}_\rho} \).

**Theorem 16.** The semigroup \( T \), defined in terms of \( E(\cdot) \), is isomorphic to \( S \).

**Proof.** For each element \( a \in S \), define \( (\rho'_a, \lambda'_a) \in \mathcal{T}_{E_1 \mathcal{L}_\rho} \times \mathcal{T}_{E_1 \mathcal{R}_\rho} \) by, for each \( x \in S \), \( V(x) x p_a' = V(xa) x a, x V(x) \lambda_a' = a x V(ax) \). By Remark 4 \( (\rho', \lambda') \), the mapping which maps each \( a \) in \( S \) to \( (\rho'_a, \lambda'_a) \) is an isomorphism of \( S \) into \( \mathcal{T}_{E_1 \mathcal{L}_\rho} \times \mathcal{T}_{E_1 \mathcal{R}_\rho} \). From above we see that, for each \( e \in E, \rho_e = \rho'_e \), and similarly \( \lambda_e = \lambda'_e \). Now \( \{ (\rho_e, \lambda_e) : e \in E \} \) generates the isomorphic image of \( S \) under \( (\rho', \lambda') \), but \( \{ (\rho'_e, \lambda'_e) : e \in E \} \) generates \( T \), and so \( T \) is isomorphic to \( S \).

**Corollary 17.** Let \( S_1 = \langle E_1 \rangle \) and \( S_2 = \langle E_2 \rangle \) be fundamental regular semigroups generated by their sets of idempotents \( E_1 \) and \( E_2 \), respectively.

(i) If the partial groupoid \( E_1 \) is isomorphic to the partial groupoid \( E_2 \), then \( S_1 \) is isomorphic to \( S_2 \).

(ii) Further, any isomorphism of \( E_1 \) onto \( E_2 \) is extendible to an isomorphism of \( S_1 \) onto \( S_2 \).

(iii) The group of automorphisms of \( S_1 \) is isomorphic to the group of automorphisms of \( E_2 \).
Remark 8. If we let \( S \) be any regular semigroup, with set of idempotents \( E \) say, and construct \( T \) as above, then \( T \) is isomorphic to \( \langle E \rangle / \mu(\langle E \rangle) \), where \( \mu(\langle E \rangle) \) is the maximum congruence contained in \( \mathcal{H} \) on \( \langle E \rangle \), the subsemigroup of \( S \) generated by \( E \).

Remark 9. From Lemma 10.64 [1], for any set \( X \), \( \mathcal{T}_X \) is fundamental. When \( X \) is a finite set, any proper ideal of \( \mathcal{T}_X \) is fundamental, regular, and generated by its idempotents (from Remark 1 and Corollary 6).

10. Further Results on \( T_{\langle E \rangle} \)

Let \( B = \langle E \rangle \) be any regular semigroup generated by its set of idempotents \( E \) and consider \( \mathcal{U} \) and \( T_{\langle E \rangle} \) as defined in Section 5.

**Theorem 18.** (i) Let \( (e, f) \in \mathcal{U} \), \( \alpha \in T_{e,f} \), and \( a \in \langle e \rangle \). Then 
\[
\phi(a^{-1}) (\mu_a, \lambda_a) \phi(a) = (\mu_{\alpha a}, \lambda_{\alpha a}).
\]

(ii) Let \( (e, f) \in \mathcal{U} \) and \( \alpha, \beta \in T_{e,f} \). Then \( \phi(\alpha) = \phi(\beta) \) if and only if \( \alpha = \beta \).

(iii) Let \( (e, f), (g, h) \in \mathcal{U} \), \( \alpha \in T_{e,f} \), and \( \beta \in T_{g,h} \). Then \( \phi(\alpha) = \phi(\beta) \) if and only if \( e \mathcal{H} g, f \mathcal{H} h \) in \( B \), and \( \alpha = \theta_{e,g} \theta_{h,f} \), where \( \theta_{e,g} \) and \( \theta_{h,f} \) are defined for \( B \) as in Result 3.

(iv) Take any pair of \( \mathcal{D} \)-related idempotents of \( T_{\langle E \rangle} \), say, \( (\rho_e, \lambda_e) \), \( (\rho_f, \lambda_f) \), where \( e, f \in E \); and let us denote by \( H \) the \( \mathcal{H} \)-class of \( T_{\langle E \rangle} \) which is the intersection of its \( \mathcal{R} \)-class containing \( (\rho_e, \lambda_e) \) and its \( \mathcal{L} \)-class containing \( (\rho_f, \lambda_f) \). Then \( (e, f) \in \mathcal{U} \) and \( \phi(T_{e,f}) = H \).

(v) For any \( e \in E \), \( \phi \) maps \( T_{e,\alpha} \), the group of automorphisms of \( \langle e \rangle \), isomorphically onto \( \phi(T_{e,\alpha}) \), the maximal subgroup of \( T_{\langle E \rangle} \) containing \( (\rho_e, \lambda_e) \).

(vi) Take any element \( \phi(\beta) \), say, in \( T_{\langle E \rangle} \) and any inverse \( \phi(\beta)' \) of \( \phi(\beta) \) in \( T_{\langle E \rangle} \). Let \( e, f \) be the elements of \( E \) such that \( \phi(\beta) \phi(\beta)' = (\rho_e, \lambda_e) \) and \( \phi(\beta)' \phi(\beta) = (\rho_f, \lambda_f) \). Then for some \( \alpha \in T_{e,f} \), \( \phi(\beta) = \phi(\alpha) \) and \( \phi(\beta)' = \phi(\alpha^{-1}) \).

(vii) \( T_{\langle E \rangle} \) is fundamental.

**Proof.** Let us assume the notations of Section 5 concerning Green's relations.

(i) Take any \( x \) in \( B \). Then \( L_{xf} \leq L_f \) and so \( L_{xf} = L_g \) for some idempotent \( g \leq f \) (Result 4). Then \( g \alpha^{-1} \leq e \) whence \( (g \alpha^{-1}) a \in \langle e \rangle \). We now obtain
\[
L_{xf} \rho_{\alpha^{-1}} \rho_a \rho_e \tilde{\alpha} = L_{xf} \rho_{\alpha^{-1}} \rho_a \tilde{\alpha} = L_{g(a^{-1})a} \tilde{\alpha} = L_{g(a)} = L_{g \alpha} \rho_{ae} = L_{xf} \rho_{ae} = L_{xf} \rho_{ae}.
\]
since \( ax \in \langle f \rangle \). Therefore \( p_\alpha^{-1} p_\alpha p_\alpha x = p_\alpha x \) and similarly \( H_\alpha^{-1} H_\alpha y \in \alpha \), giving \( \phi(x^{-1})(p_\alpha, \lambda_\alpha) \phi(\alpha) = (p_\alpha, \lambda_\alpha) \) as required.

(ii) The "if" statement is trivial. Suppose \( \phi(\alpha) = \phi(\beta) \). Then \( \phi(\alpha) \phi(x^{-1}) = (p_\alpha, \lambda_\alpha) = \phi(\beta) \phi(x^{-1}), \phi(x^{-1}) \phi(\alpha) = (p_{\beta^*}, \lambda_{\beta^*}) = \phi(\beta^{-1}) \phi(\beta) \) and so \( \phi(x^{-1}) H_\phi H(\beta^{-1}) \) in \( T(\langle x \rangle) \). But \( \phi(\alpha^{-1}), \phi(\beta^{-1}) \) are inverses of \( \phi(\alpha) = \phi(\beta) \) and so \( \phi(x^{-1}) = \phi(\beta^{-1}) \). Now by (i), for any idempotent \( g \leq e \) in \( \langle E \rangle \), we have

\[
(\rho_{\alpha x}, \lambda_{\alpha x}) = \phi(x^{-1})(\rho_\alpha, \lambda_\alpha) \phi(\alpha) = (\rho_\beta, \lambda_\beta).
\]

Now \( (\rho, \lambda) \) maps \( E \) one-to-one onto the set of idempotents of \( T(\langle E \rangle) \), since \( (\rho, \lambda) \circ (\rho, \lambda)^{-1} \subseteq H(S) \), and therefore \( g \alpha = g \beta \) for all idempotents \( g \leq e \). But \( \langle e \rangle \) is generated by \( \{g \in E : g \leq e\} \) and so \( \alpha x = \alpha \beta \) for all \( a \in \langle e \rangle \). Hence \( x = \beta \).

(iii) Suppose that \( e \theta g, f \theta h \) and \( \alpha = \theta_{e, \theta} \theta_{h, f} \). Take any element \( x \in B \). Then \( L_{x e} = L_{x f} \) for some idempotent \( l \leq e \), and then

\[
lx = \theta_{e, \theta} \theta_{h, f} [h[(elg) \beta] f = [(elg) \beta] h f = [(elg) \beta] h = (lg) \beta.
\]

Therefore

\[
L_{x e} p_\alpha = L_{x f} = L_{(x f) \beta} = L_{x e} p_\gamma = L_{x e} p_\alpha,
\]

giving \( p_\alpha \beta = p_\alpha \beta \). Similarly \( \lambda_\alpha^{-1} = \lambda_\alpha^{-1} \) and so \( \phi(\alpha) = \phi(\beta) \).

Suppose conversely that \( \phi(\alpha) = \phi(\beta) \). Then \( (\rho_\alpha, \lambda_\alpha) = \phi(\alpha) \phi(\alpha^{-1}) \) is \( \mathcal{R} \)-related in \( T(\langle x \rangle) \) to \( (\rho_\beta, \lambda_\beta) = \phi(\beta) \phi(\beta^{-1}) \) and since \( (\rho, \lambda) \circ (\rho, \lambda)^{-1} \subseteq H(S) \) we have by the result dual to Result 10 that \( e \mathcal{H}(S) g \) and then \( e \mathcal{H}(S) \) (Result 2).

Similarly \( f \mathcal{L} h \).

Take any \( x \in B \). Then \( L_{x e} = L_{x f} \) for some idempotent \( l \leq e \), and then \( \theta_{e, \theta} \theta_{h, f} = (lg) \beta \) as above. We need to note that \( \theta_{e, \theta} \theta_{h, f} = \gamma \) say, is an element of \( T_{e, f} \); this is because \( \theta_{e, \theta} \theta_{h, f} = (\theta_{e, \theta} \mid \langle e \rangle \rangle) \theta_{h, f} \) \( \langle e \rangle \). Now

\[
L_{x e} p_\gamma = L_{x f} = L_{(x f) \beta} = L_{x e} p_\beta \Rightarrow L_{x e} p_\alpha = L_{x e} p_\beta = L_{x e} p_\alpha.
\]

Therefore \( p_\alpha \gamma = p_\alpha \beta \) and similarly \( \lambda_\alpha^{-1} = \lambda_\alpha^{-1} \), and so \( \phi(\alpha) = \phi(\gamma) \), and then from (ii) we have \( \alpha = \gamma = \theta_{e, \theta} \theta_{h, f} \).

(iv) Take any element \( \phi(\beta) \in H \), where \( \beta \in T_{e, h} \), \( (g, h) \in \mathcal{H} \). Then \( \phi(\beta) \phi(\beta^{-1}) = (\rho_\beta, \lambda_\beta) \) whence \( (\rho_\beta, \lambda_\beta) \mathcal{H}(\rho_\beta, \lambda_\beta) \) in \( T(\langle x \rangle) \). Therefore \( e \mathcal{H} g \) as in the proof of (ii). Similarly \( f \mathcal{L} h \). For each element \( a \) in \( B \) and inverse \( a' \) of \( a \) in \( B \) define \( \theta_{a, a} \) as in Result 3, for the semigroup \( B \). Then \( \theta_{a, a}(\theta_{e, \theta} \theta_{h, f}) \theta_{f, h} = \beta \) and \( \theta_{e, \theta} \theta_{h, f} \) maps \( \langle e \rangle \) isomorphically onto \( \langle f \rangle \).
whence \((e, f) \in \mathcal{U}\) and \(\phi(\beta) = \phi(\theta, e, g, \theta, h, r) \in \phi(T_{e,f})\) from (iii). This gives 
\(H \subseteq \phi(T_{e,f})\), and by considering, for each \(\alpha \in T_{e,f}\), the products \(\phi(\alpha) \phi(\alpha^{-1})\) and \(\phi(\alpha^{-1}) \phi(\alpha)\) we see that \(\phi(T_{e,f}) \subseteq H\).

(v) We shall in fact prove, for any \((e, f), (f, g) \in \mathcal{U}\), for any \(\alpha \in E_{e,f}\) and \(\beta \in T_{f,g}\), that \(\phi(\alpha) \phi(\beta) = \phi(\alpha \beta)\). Using \(\alpha \beta = \alpha\) we have, from Lemma 8,
\[
\rho_{\alpha} \tilde{\alpha} = \rho_{\beta} \tilde{\beta} = \rho_{\alpha \beta}
\]
and similarly \(\tilde{\alpha} \tilde{\beta}^{-1} \rho_{\alpha} = \rho_{\alpha^{-1}} \rho_{\beta}^{-1}\) whence \(\phi(\alpha) \phi(\beta) = \phi(\alpha \beta)\). From this, and from (ii) and (iv), we deduce (v).

(vi) From (iv) we see that \(\phi(\beta) \in \phi(T_{e,f})\), \(\phi(\beta') \in \phi(T_{e,f})\), whence \(\phi(\beta) = \phi(\alpha)\) for some \(\alpha \in T_{e,f}\). Then \(\phi(\alpha^{-1}) \in \phi(T_{e,f})\) and \(\phi(\alpha^{-1})\) is also an inverse of \(\phi(\beta) = \phi(\alpha)\), whence \(\phi(\beta') = \phi(\alpha^{-1})\).

(vii) Let \(\mu\) denote the maximum congruence contained in \(\mathcal{H}\) on \(T_{e}\) and let \(t_1, t_2\) be any \(\mu\)-related elements of \(T_{e}\). From (iv) there exist \((e, f) \in \mathcal{U}, \alpha, \beta \in T_{e,f}\) such that \(t_1 = \phi(\alpha), t_2 = \phi(\beta)\). Moreover, \(\phi(\alpha^{-1})\) and \(\phi(\beta^{-1})\) are \(\mathcal{H}\)-related inverses of \(\phi(\alpha)\) and \(\phi(\beta)\), respectively. Take any idempotent \(g \in <e>\). Then \((\rho_g, \lambda_g) \leq (\rho_e, \lambda_e)\) in \(T_{e}\) so from Theorem 5 and part (i) above
\[
(\rho_{\alpha \beta} \lambda_{\alpha} \lambda_{\beta}) = \phi(\alpha^{-1}) (\rho_g \lambda_g) \phi(\beta) = \phi(\beta^{-1}) (\rho_g \lambda_g) \phi(\alpha) = (\rho_{\alpha \beta} \lambda_{\alpha} \lambda_{\beta}).
\]
Now \((\rho, \lambda) \circ (\rho, \lambda) \leq H(S)\) and \(g\alpha, g\beta\) are idempotents, so we have that \(g\alpha = g\beta\) for each idempotent \(g \in <e>\). But \(<e>\) is generated by its idempotents so \(\alpha \beta = x\beta\) for each \(x \in <e>\), whence \(\alpha = \beta, \phi(\alpha) = \phi(\beta)\) and \(\mu\) is the trivial relation. Thus \(T_{e}\) is fundamental.

We give now an outline of a longer, though conceptually simpler, proof that \(T_{e}\) is fundamental. Take any element \(t \in T_{e}\); then \(t = \phi(\alpha)\) for some \((e, f) \in \mathcal{U}, \alpha \in T_{e,f}\). Define \((\rho_t, \lambda_t)\) as in Section 4. From Theorem 18(i) it can be shown that \((\rho_t, \lambda_t)\) and \(t = \phi(\alpha)\) are essentially the same pairs of transformations, and so \((\rho, \lambda)\) essentially maps \(T_{e}\) identically onto itself (note though that in some cases \(T_{e}\) is properly "contained" in \(T_{e}\)). In particular, \(T_{e}\) is fundamental.

Proof of Theorem 13. If \(\mathcal{H}\) is a congruence on \(T_{e}\), then \(\mathcal{H}\) is trivial on \(T_{e}\), since \(T_{e}\) is fundamental. From Theorem 18, part (v), \(T_{e,e}\) consists of only the trivial automorphism of \(<e>\), for each \(e \in E\), and so \(<e>\) is taut.

Conversely, suppose \(<e>\) is taut. Then for any \(e, f \in E\) there is at most one isomorphism of \(<e>\) onto \(<f>\), and so from Theorem 18 part (iv) \(\mathcal{H}\) on \(T_{e}\) is trivial. Let \(S\) be any regular semigroup with \(<e>\) as its subsemigroup generated by its idempotents and define \((\rho, \lambda)\) as in Section 5. Then \((\rho, \lambda)\) is a homomorphism of \(S\) into \(T_{e}\) (Theorem 7 part (iii)) and so \(\mathcal{H}\) on this image
of $S$ is also trivial (Result 2 and its dual result). This quickly gives that $\mathcal{H}(S) \subseteq (\rho, \lambda) \circ (\rho, \lambda)^{-1} = \mu(S)$ (Result 10 and its dual result). But $\mu(S) \subseteq \mathcal{H}(S)$ and so $\mathcal{H}(S) = \mu(S)$, a congruence.

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