JOURNAL OF ALGEBRA 24, 1–24 (1973)

On Regular Semigroups*

T. E. HALL

Department of Mathematics, Monash University, Clayton, Victoria, Australia

Communicated by G. B. Preston

Received December 12, 1970

1. INTRODUCTION AND SUMMARY

A very general question in the theory of semigroups is: From information about the idempotents of a semigroup what information can be deduced about the semigroup? This question takes its importance from the fact that in many semigroups the idempotents are easily recognized. A powerful tool for attacking this question for inverse semigroups has been constructed by Munn in [13]; for each semilattice E there is constructed a semigroup T_E which is a "maximal" fundamental inverse semigroup with semilattice isomorphic to E (a semigroup is called fundamental if its only congruence contained in $\mathscr{H}$ is the trivial one). The advantage held by T_E is that not only is its semilattice (isomorphic to) E, but for each inverse semigroup with semilattice E, T_E contains a closely related subsemigroup. Some of the consequences of this construction are given in papers [13, 15–18], and in Sections 6 and 8 below.

The unifying result of this paper is the construction in Section 5 of the corresponding semigroups for regular semigroups. More explicitly, for each fundamental regular semigroup, B, say, generated by its set of idempotents, E, say, we construct a "maximal" fundamental regular semigroup, denoted by $T_{\langle E \rangle}$, whose subsemigroup generated by its idempotents is isomorphic to $B = \langle E \rangle$ (note that the subsemigroup generated by the idempotents of a regular semigroup is always regular, from Result 1, due to Fitz-Gerald). When B is a band, $T_{\langle E \rangle}$ reduces to the semigroup $W(B)$ of [5]. In Sections 4, 6–9 we give consequences of the construction of $T_{\langle E \rangle}$, or of related concepts, in five different areas in the theory of regular semigroups.

After Section 2 which gives necessary preliminaries, we study in Section 3 the subsemigroup of a regular semigroup generated by the set of all idem-

* This research was supported by a Nuffield Travelling Fellowship, and was carried out at Stirling, Scotland.

potents, one result being that a regular semigroup is generated by its idempotents if and only if each principal factor is generated by its own idempotents.

In Section 4 we give a determination of the maximum congruence contained in $\mathscr{H}$ on a regular semigroup. Determinations for inverse and orthodox semigroups have been given by Howie [8] (see [1, Section 7.6]) and Meakin [12], respectively.

In Section 6 we extend to fundamental regular semigroups the results of Munn [13], and Howie and Schein [10], concerning bisimple inverse semigroups, semilattices of groups and uniform and antiuniform semilattices. We prove also analogous results for completely semisimple semigroups and regular semigroups on which $\mathscr{H}$ is a congruence.

In Section 7 we continue work begun by Reilly and Scheiblich [20], by determining the maximum regular subsemigroup of a semigroup, containing a given set of idempotents as its set of all idempotents.

Section 8 concerns completely semisimple semigroups in which each $\mathscr{D}$-class contains at most m idempotents, for some integer m. It is shown that for each element a in such a semigroup, a^m is an element of a subgroup. Hence each homomorphic image of such a semigroup is also completely semisimple (this can also be shown by proving that each homomorphic image contains at most m idempotents per $\mathscr{D}$-class [7]). In general, a homomorphic image of a completely semisimple inverse semigroup with finitely many idempotents in each $\mathscr{D}$-class is not necessarily itself completely semisimple [7].

In Section 9 we show that a fundamental regular semigroup generated by its idempotents is uniquely determined, to within isomorphism, by the partial groupoid of its idempotents. Section 10 continues a study of the semigroup $T_{\langle E \rangle}$.

2. Preliminaries

For any semigroup S and for any element a of S and any subset A of S, we put

$$V(a) = \{x \in S : axa = a \text{ and } xax = x\},$$

$$V(A) = \{x \in S : \text{for some } a \in A,\ axa = a \text{ and } xax = x\}.$$

We use, whenever possible, and often without comment, the notations of Clifford and Preston [1]. When there might otherwise be an ambiguity, we shall denote Green's relations on the semigroup S by $\mathscr{H}(S)$, $\mathscr{L}(S)$, etc.

Result 1 (from Fitz-Gerald [3]). *Let S be any regular semigroup and E its set of idempotents. Then for $n = 1, 2, 3, \ldots$, $V(E^n) = E^{n+1}$. Hence $\langle E \rangle$, the subsemigroup of S generated by E, is regular.*

RESULT 2. *Let U be any semigroup and let T be any regular subsemigroup of U. Then*

(i) *$e\mathscr{L}f$ in U if and only if $ef = e$, $fe = f$, for any idempotents e, f in U,*

(ii) *$a\mathscr{L}b$ in T if (and only if) $a\mathscr{L}b$ in U, for any elements a, b in T,*

(iii) *any element a of T is an element of a subgroup of T if (and only if) it is an element of a subgroup of U.*

Proof. Part (i) is a corollary of [1, Lemma 2.14]. Part (ii), which is well-known, is proved in [5, Result 9]. Part (iii), probably also well-known, is proved as follows: a is an element of a subgroup of T if and only if $a\mathscr{H}a^2$ in T, which holds if and only if $a\mathscr{H}a^2$ in U (from part (ii) and its dual result), and this is true if and only if a is an element of a subgroup of U.

The following well-known result [14, Lemma 1] is a routine generalization of a result due to Preston [19, Lemma 2] and of Theorem 2.20 [1].

RESULT 3. *Let S be any semigroup and let e, f be any $\mathscr{D}$-related idempotents of S. Take any element a in S and any inverse a' of a, such that $aa' = e$ and $a'a = f$ (a and a' exist, from Section 2.3 of [1]). Define mappings $\theta_{a',a} : eSe \to fSf$ and $\theta_{a,a'} : fSf \to eSe$ by, for each $x \in eSe$, $x\theta_{a',a} = a'xa$, and for each $y \in fSf$, $y\theta_{a,a'} = aya'$. Then $\theta_{a',a}$ is a $\mathscr{D}$-class preserving isomorphism from eSe onto fSf and $\theta_{a,a'} = \theta_{a',a}^{-1}$.*

Proof. For each $x \in S$ it is clear that $a'xa \in fSf$, and for any $x, y \in eSe$, since $ey = y$, we have

$$(xy)\,\theta_{a',a} = a'xya = a'xaa'ya = (x\theta_{a',a})\,(y\theta_{a',a}),$$

and so $\theta_{a',a}$ is a homomorphism of eSe into fSf. Also, for any $x \in eSe$,

$$x\theta_{a',a}\theta_{a,a'} = (a'xa)\,\theta_{a,a'} = aa'xaa' = x.$$

Similarly $\theta_{a,a'}\theta_{a',a}$ is the identity map of fSf, and so $\theta_{a',a}$ is an isomorphism onto fSf and $\theta_{a,a'} = \theta_{a',a}^{-1}$.

Now since, for any $x \in eSe$, $xaa' = x$ and $aa'xa = xa$, we have $x\mathscr{R}xa\mathscr{L}a'xa$, whence $\theta_{a',a}$ is $\mathscr{D}$-class preserving.

RESULT 4 [6, Remark 2]. *Let a and b be any regular elements in any semigroup S such that $L_a \geqslant L_b$. Then for each idempotent $e \in L_a$ there exists an idempotent $f \in L_b$ such that $e \geqslant f$.*

Proof. Take any idempotent $e \in L_a$ and any inverse b' of b. Using $be = b$ we may routinely show that eb' is also an inverse of b, whence $eb'b \in L_b$ and moreover $eb'b$ is an idempotent and $eb'b \leqslant e$.

RESULT 5 (from [1, Exercise 3, Section 8.4]). *A regular semigroup S is* [0-] *simple if and only if for any nonzero idempotents e, f there exists an idempotent $g \leqslant f$ such that $f\mathscr{D}e$.*

Since for any idempotents e, f in any semigroup S, $e \in SfS$ if and only if $J_e \leqslant J_f$, we may restate the first part of [1, Exercise 3, Section 8.4] as follows: *Let S be a semigroup and a, b any elements of S such that $J_b \leqslant J_a$. Then for each idempotent $e \in J_a$ and regular $\mathscr{D}$-class D of S contained in J_b (if such exist) there is an idempotent $f \in D$ such that $f \leqslant e$* (cf. [6, Theorem 1]).

RESULT 6. *A regular semigroup S is completely semisimple if (and clearly only if) no pair of distinct comparable idempotents are $\mathscr{D}$-related.*

Proof. This is probably well-known. Let us prove the following stronger statement, which is probably also well-known. *If a $\mathscr{D}$-class, D, say, of a semigroup S, contains an idempotent, e, say, which is minimal among the idempotents of D, then $D = D_e = J_e$ and $J(e)/I(e)$ is completely 0-simple or completely simple.* If e is minimal among the idempotents of J_e, then $J(e)/I(e)$ contains a primitive idempotent, namely e, and is also 0-simple or simple, giving the required conclusion. Take then any idempotent $f \in J_e$ such that $f \leqslant e$. From above there exists an idempotent $g \leqslant f \leqslant e$ such that $g\mathscr{D}e$, whence $g = f = e$ and so e is minimal among the idempotents of J_e, and the result follows.

RESULT 7 (a corollary to [1, Theorem 2.4]). *The regular elements of a semigroup S form a subsemigroup if (and clearly only if) the product of any two idempotents of S is a regular element.*

RESULT 8. *Let S be a completely semisimple semigroup and a any element of S. If, for some positive integers m, n, $a^m \mathscr{J} a^{m+n}$, then $a^m \mathscr{H} a^{m+1} \mathscr{H} a^{m+2} \cdots$, and H_{a^m} is a subgroup. Consequently, if every strictly descending chain of $\mathscr{J}$-classes of S has length at most m, then for each element a in S, a^m is an element of a subgroup of S.*

Proof. This result is probably well-known but we give a proof nevertheless. Now $L_{a^m} \geqslant L_{a^{m+n}}$ and $R_{a^m} \geqslant R_{a^{m+n}}$ and S satisfies both $M_L{}^*$ and $M_R{}^*$ [1, Section 6.6]. Hence $a^m \mathscr{J} a^{m+n}$ implies that $a^m \mathscr{H} a^{m+n}$; also

$$H_{a^m} \geqslant H_{a^{m+1}} \geqslant \cdots \geqslant H_{a^{m+n}} = H_{a^m}.$$

Since $\mathscr{L}$ is a right congruence, $a^m \mathscr{L} a^{m+n}$ implies $a^{m+1} \mathscr{L} a^{m+n+1}$ and similarly $a^{m+1} \mathscr{R} a^{m+n+1}$ whence $a^m \mathscr{H} a^{m+1} \mathscr{H} a^{m+n+1}$. We easily see now that $a^m \mathscr{H} a^{m+1} \mathscr{H} a^{m+2} \cdots$ and that H_{a^m} is a group. To prove the remaining statement we note that $J_a \geqslant J_{a^2} \geqslant \cdots \geqslant J_{a^m} \geqslant J_{a^{m+1}}$ and so two of these $\mathscr{J}$-classes are equal.

Let $\mathscr{I}_m$ and $\mathscr{G}_m$ denote the symmetric inverse semigroup and the symmetric group, respectively, on the set $\{1, 2,..., m\}$. Being finite and inverse, $\mathscr{I}_m$ and the ideals of $\mathscr{I}_m$ are of course completely semisimple. Also $\mathscr{G}_m \subseteq \mathscr{I}_m$ and $\mathscr{I}_m \backslash \mathscr{G}_m$ is an ideal of $\mathscr{I}_m$ containing m $\mathscr{J}$-classes (of $\mathscr{I}_m \backslash \mathscr{G}_m$). *Thus for any element a in $\mathscr{I}_m$, a^m is an element of a subgroup of $\mathscr{I}_m$.*

RESULT 9 [7, Theorem 15]. *Let S be a regular semigroup and let ρ be any congruence contained in $\mathscr{L}$ on S. For each element a in S, a is an element of a subgroup of S if (and only if) $a\rho$ is an element of a subgroup of S/ρ.*

RESULT 10 [7, Theorem 13]. *Let S be any semigroup and let ρ be any congruence contained in $\mathscr{L}$ on S. For any elements $a, b \in S$, $a\mathscr{L}b$ in S if (and only if) $a\rho\mathscr{L}b\rho$ in S/ρ.*

In the notation of [20] the conclusion of Result 10 becomes

$$\mathscr{L}(S/\rho) = \mathscr{L}(S)/\rho.$$

RESULT 11 (from Tully [21, Proposition 1.2]). *Let S be any semigroup and for each element $a \in S$ define $\sigma_a \in \mathscr{T}_{S^1/\mathscr{L}}$ by $L_x\sigma_a = L_{xa}$ for each $x \in S^1$. Then the mapping σ which maps each element $a \in S$ to σ_a is a homomorphism, and the congruence $\sigma \circ \sigma^{-1}$ is the maximum congruence contained in $\mathscr{L}(S)$.*

3. IDEMPOTENTS IN A REGULAR SEMIGROUP

We use the method of Fitz-Gerald [3] to prove the following lemma.

LEMMA 1. *Let S be any regular semigroup and let $A_1, A_2,..., A_n$ be any elements of S. Put $A_1A_2 \cdots A_n = a$. Then there exist elements $a_1, a_2,..., a_n$ in S such that*

(i) *$a = a_1a_2 \cdots a_n$ and $a\mathscr{D}a_i$, $i = 1, 2,..., n$;*

(ii) *$H_{a_i} \leqslant H_{A_i}$ for $i = 1, 2,..., n$;*

(iii) *if, for some i, A_i is an idempotent, then a_i is also an idempotent, whence $a_i \leqslant A_i$ from* (ii).

Proof. The only element of which we need the regularity is the element a. Let x be any inverse of a and define, for $i = 1, 2,..., n$,

$$a_i = A_iA_{i+1} \cdots A_nxA_1A_2 \cdots A_i .$$

Then $a_1 = axA_1$; assume for some $k < n$ that

$$a_1a_2 \cdots a_k = (ax)^k A_1A_2 \cdots A_k .$$

Then

$$a_1a_2 \cdots a_k a_{k+1} = [(ax)^k A_1A_2 \cdots A_k]\, A_{k+1} \cdots A_n x A_1 A_2 \cdots A_{k+1}$$
$$= (ax)^{k+1} A_1A_2 \cdots A_{k+1} .$$

By induction $a_1a_2 \cdots a_n = (ax)^n a = a$. Now for each i,

$$(a_i A_{i+1}A_{i+2} \cdots A_n)\, x A_1 A_2 \cdots A_i = A_i A_{i+1} \cdots A_n xax A_1A_2 \cdots A_i = a_i$$

and so $a_i \mathscr{R} a_i A_{i+1} \cdots A_n = A_i \cdots A_n xa$. Also

$$A_1A_2 \cdots A_{i-1}(A_i \cdots A_n xa) = axa = a,$$

and so $A_i \cdots A_n xa \mathscr{L} a$, giving $a_i \mathscr{D} a$. This proves part (i); part (ii) is obvious.

Suppose for some i that A_i is an idempotent. Then

$$a_i^2 = (A_iA_{i+1} \cdots A_n x A_1A_2 \cdots A_i)\,(A_iA_{i+1} \cdots A_n x A_1 A_2 \cdots A_i) = a_i ,$$

giving part (iii).

The following theorem is a corollary to Lemma 1.

THEOREM 2. *Let S be a regular semigroup with set of idempotents E say. For each $x \in S$ let $E(J_x)$ be the set of idempotents of J_x and let $\langle E \rangle$ and $\langle E(J_x) \rangle$ be the subsemigroups of S generated by E and $E(J_x)$, respectively. Let $\langle E(J(x)/I(x)) \rangle$ be the subsemigroup of the principal factor $J(x)/I(x)$ generated by its set of idempotents $E(J_x) \cup \{I(x)\}$. Then*

(i) $\langle E \rangle = \bigcup_{x \in S} (\langle E(J(x)/I(x)) \rangle \backslash \{I(x)\}) = \bigcup_{x \in S} (\langle E(J_x) \rangle \backslash I(x))$,

(ii) *S is generated by its idempotents if and only if each principal factor is generated by its own idempotents,*

(iii) *the idempotents of S form a subsemigroup if and only if the idempotents of each principal factor form a subsemigroup of that principal factor.*

Part (iii) has already appeared in [4].

Remark 1. *From Howie* [9], *we deduce that if X is a finite set, then any element t of $\mathscr{T}_X$ not a permutation, is a product of idempotents of $\mathscr{T}_X$ each having the same rank as t. From Erdos* [2] *we deduce that any $n \times n$ matrix of rank $r < n$ over any field, is a product of idempotent matrices each of rank r.*

The author is grateful to C. Eberhart for bringing the semigroups satisfying condition (ii) of the next theorem to his attention.

THEOREM 3 (This was found independently by D. G. Fitz-Gerald). *Let S be a regular semigroup with set of idempotents E say. The following conditions are equivalent:*

(i) $\langle E \rangle$ *is a union of groups*;

(ii) *For any* $e, f, g \in E$ *such that* $e\mathscr{L}f\mathscr{R}g$, *there exists* $h \in E$ *such that* $e\mathscr{R}h\mathscr{L}g$.

Proof. Suppose condition (i) holds and take any $e, f, g \in E$ such that $e\mathscr{L}f\mathscr{R}g$ in S. Then $eg \in R_e \cap L_g$ [1, Section 2.3] and from (i), eg is an element of a subgroup of $\langle E \rangle$, which is of course also a subgroup of S. Hence $R_e \cap L_g$ contains an idempotent, giving condition (ii).

Conversely, suppose condition (ii) holds. Now the elements of E are contained in the subgroups of S, so suppose for some k that the elements of E^k are contained in the subgroups of S. Take any element $a \in E^{k+1}$. Since $V(E^k) = E^{k+1}$ (Result 1) a has an inverse a', say, in E^k. By assumption, $H_{a'}$ contains an idempotent, f, say. Then $aa'\mathscr{L}a'\mathscr{R}a'a$ whence $aa'\mathscr{L}f\mathscr{R}a'a$ and so by condition (ii) $R_{aa'} \cap L_{a'a} = H_a$ contains an idempotent and is a group. Thus the elements of E^{k+1}, and by induction those of $\langle E \rangle$ also, are contained in the subgroups of S. By Results 1 and 2 $\langle E \rangle$ is a union of groups.

Remark 2. *Condition* (ii) *is of course equivalent to the following condition.*

(iii) *If* $\{R_i : i \in I\}$ *and* $\{L_\lambda : \lambda \in \Lambda\}$ *are the sets of* $\mathscr{R}$*-classes and of* $\mathscr{L}$*-classes, respectively, of* S, *for some index sets* I, Λ, *then for some index set* J *there exist partitions* $\mathscr{P}(I) = \{I_j : j \in J\}$ *and* $\mathscr{P}(\Lambda) = \{\Lambda_j : j \in J\}$ *of* I *and* Λ, *respectively, such that any* $\mathscr{H}$*-class* $R_i \cap L_\lambda$, *where* $i \in I_j$, $\lambda \in \Lambda_k$ *say, contains an idempotent if and only if* $j = k$.

Compare Theorem 3 *above with* [4] *Theorem* 6, *Lemma* 3, *and Remark* 2. *Under the stronger condition that there are no three distinct idempotents* e, f, g *in* S *such that* $e\mathscr{L}f\mathscr{R}g$, *the idempotents of* S *form a subsemigroup; this is easily shown using condition* (2) *of* [20, Lemma 1.3].

4. The Maximum Congruence in $\mathscr{H}$

Let S be any regular semigroup and, for each element $a \in S$, define transformations ρ_a and λ_a of $S/\mathscr{L}$ and $S/\mathscr{R}$, respectively, by $L_x\rho_a = L_{xa}$, $R_x\lambda_a = R_{ax}$, for each $x \in S$. We note that ρ_a, λ_a are single-valued because $\mathscr{L}$ and $\mathscr{R}$ are right and left congruences, respectively; and that clearly the mapping ρ which maps each element a in S to ρ_a, is a homomorphism of S into $\mathscr{T}_{S/\mathscr{L}}$, and the mapping λ which maps each a in S to λ_a, is an antihomomorphism of S into $\mathscr{T}_{S/\mathscr{R}}$.

LEMMA 4. *The congruence* $\rho \circ \rho^{-1}$ *is the maximum congruence on* S *contained in* $\mathscr{L}$.

Proof. This lemma follows very easily from Result 11. From Result 11 we have that the maximum congruence contained in $\mathscr{L}$, $\mathscr{L}C$ say, is given by

$$\mathscr{L}C = \{(a, b) \in S \times S\colon L_{ua} = L_{ub} \text{ for each } u \in S^1\}.$$

However, from the regularity of S, we see, as follows, that $ua\mathscr{L}ub$ for each $u \in S^1$ if and only if $xa\mathscr{L}xb$ for each $x \in S$: for if $xa\mathscr{L}xb$ for each $x \in S$, then by choosing $x = aa'$ for any $a' \in V(a)$, we have $L_a = L_{aa'a} = L_{aa'b} \leqslant L_b$, and similarly $L_b \leqslant L_a$, whence $a\mathscr{L}b$ and the "if" statement follows; the "only if" statement is obvious. Thus

$$\mathscr{L}C = \{(a, b) \in S \times S : L_{xa} = L_{xb} \text{ for each } x \in S\} = \rho \circ \rho^{-1},$$

giving the lemma.

Clearly, it follows that $(\rho \circ \rho^{-1}) \cap (\lambda \circ \lambda^{-1})$ is the maximum congruence contained in $\mathscr{H}$.

Remark 3. *The lemma may also of course be deduced from the material of* [1, Section 10.1] *which gives a determination of the maximum congruence contained in any equivalence relation on a semigroup.*

Remark 4. *Now by Section* 2.3 [1], *for each $x \in S$ the set of idempotents in L_x (i.e., $L_x \cap E$, where E is the set of idempotents of S) is $V(x)\,x$, and of course $\{(V(x)\,x, L_x) : x \in S\}$ is a natural one-to-one correspondence between the sets $\{V(x)\,x : x \in S\}$ and $\{L_x : x \in S\} = S/\mathscr{L}$. Suppose now that we make the mapping $\rho_a \in \mathscr{T}_{S/\mathscr{L}}$, for each $a \in S$, an operator on the set $\{V(x)\,x : x \in S\}$ by defining, (for all $x \in S$) $(L_x \cap E)\,\rho_a = (L_x\rho_a \cap E)$, or equivalently, $V(x)\,x\rho_a = V(xa)\,xa$. Then we see that ρ_a is essentially the same as the ρ_a of* [5].

THEOREM 5. *The maximum congruence contained in $\mathscr{H}$ on any regular semigroup S, $\mu = \mu(S)$ say, is given by*

$$\mu = \{(a, b) \in \mathscr{H} : \textit{for some [each pair of] } \mathscr{H}\textit{-related inverses } a' \textit{ of } a \textit{ and } b' \textit{ of } b,\ a'ea = b'eb \textit{ for each idempotent } e \leqslant aa'\};$$

and equivalently by

$$\mu = \{(a, b) \in S \times S\colon \textit{for some inverses } a' \textit{ of } a \textit{ and } b' \textit{ of } b,\ aa' = bb',\ a'a = b'b,\ \textit{and } a'ea = b'eb \textit{ for each idempotent } e \leqslant aa'\}.$$

Proof. (i) Take any pair $(a, b) \in \mu$, the maximum congruence contained in $\mathscr{H}$. Since $a\mathscr{H}b$ there exist (and so let us take any) inverses a' of a and b' of b such that $a'\mathscr{H}b'$; then $aa' = bb'$, $a'a = b'b$ [1, Section 2.3]. Since $a'\mu^\natural$ and $b'\mu^\natural$ are $\mathscr{H}$-related in S/μ, and both are inverses of $a\mu^\natural = b\mu^\natural$ in S/μ, we have $a'\mu^\natural = b'\mu^\natural$ [1, Section 2.3]; i.e., $a'\mu b'$. Take any idempotent $e \in S$ such that $e \leqslant aa'$. Since $\mu \subseteq \mathscr{H}$ we have $ea\mathscr{H}eb$ and $a'e\mathscr{H}b'e$; but routinely one gets $a'e \in V(ea)$, $b'e \in V(eb)$, whence $(a'e)(ea)$ and $(b'e)(eb)$ are $\mathscr{H}$-related idempotents, and so $a'ea = (a'e)(ea) = (b'e)(eb) = b'eb$. Therefore

$\mu \subseteq \{(a, b) \in \mathscr{H}$: for each pair of $\mathscr{H}$-related inverses a' of a and b' of b, $a'ea = b'eb$ for each idempotent $e \leqslant aa'\}$

(ii) Take now any pair $(a, b) \in \mathscr{H}$ such that for *some* $\mathscr{H}$-related inverses a' of a, b' of b, $a'ea = b'eb$ for each idempotent $e \leqslant aa'$. Take any element $x \in S$. Then $L_{xaa'} \leqslant L_{aa'}$; so by Result 4 there exists an idempotent $e \in L_{xaa'}$ such that $e \leqslant aa'$. Then $a'e \in V(ea)$, $b'e \in V(eb)$ (easily checked) and so

$$\begin{aligned} L_x\rho_a &= L_{xaa'a} = L_e\rho_a = L_{ea} \\ &= L_{a'ea} = L_{b'eb} = L_x\rho_b , \end{aligned}$$

similarly. Therefore $\rho_a = \rho_b$.

From Result 3, and from $a'a = b'b$, $aa' = bb'$ and $a'ea = b'eb$ for each idempotent $e \leqslant aa'$, we deduce that $afa' = bfb'$ for each idempotent $f \leqslant a'a$. We now deduce dually that $\lambda_a = \lambda_b$, and so $(a, b) \in (\rho \circ \rho^{-1}) \cap (\lambda \circ \lambda^{-1}) = \mu$. This proves the first formula and its alternative reading. The second formula is easily deduced, by use of Section 2.3 [1].

COROLLARY 6. *Let T be a regular subsemigroup of a regular semigroup S such that for any idempotents e, f in S with $f \leqslant e$, $e \in T$ implies $f \in T$. For example, let T be an ideal, a subsemigroup of the form eSe for some idempotent $e \in S$, or any regular subsemigroup containing all the idempotents of S. Then $\mu(T) = \mu(S) \cap (T \times T)$. In particular, if S is fundamental, then so is T.*

Remark 5. *As stated in* [17, Theorem 2.1] *and* [7, Corollary 14] *if μ is the maximum congruence contained in $\mathscr{H}$ on any semigroup S, then S/μ is fundamental.*

5. The Construction of $T_{\langle E\rangle}$

Let S be any regular semigroup, with set of idempotents, E, say, and let $\langle E\rangle = B$, say, be the subsemigroup of S generated by E. From Result 1 B is regular. In this section the common notations ($\mathscr{L}, \mathscr{R}, L_g$, etc.) associated with Green's relations will be with respect to the semigroup B; we shall use $\mathscr{L}(S)$ etc. in connection with S. For each $a \in S$, we define ρ_a and λ_a slightly differently from the way in Section 4, as follows: for each $x \in B$,

$$L_x\rho_a = B \cap L(S)_{xa} , \qquad R_x\lambda_a = B \cap R(S)_{ax} .$$

From Result 2, we have that $\rho_a \in \mathscr{T}_{B/\mathscr{L}}$, $\lambda_a \in \mathscr{T}_{B/\mathscr{R}}$, and because of the natural one-to-one correspondence between the $\mathscr{L}$ and $\mathscr{R}$-classes of B and those of S, ρ_a and λ_a are essentially the same as the ρ_a and λ_a of Section 4. In particular, the mapping (ρ, λ), which maps each $a \in S$ to $(\rho_a, \lambda_a) \in \mathscr{T}_{B/\mathscr{L}} \times \mathscr{T}^*_{B/\mathscr{R}}$ (where $\mathscr{T}^*_{B/\mathscr{R}}$ is the semigroup dual to $\mathscr{T}_{B/\mathscr{R}}$) is a homomorphism.

For each idempotent $e \in E$, let $<e>$ denote the subsemigroup of eBe generated by the idempotents of the semigroup eBe, which is easily shown to be regular, whence $<e>$ is also regular (Result 1). Thus

$$<e> = \langle E(eBe) \rangle = \langle \{ f \in E : f \leqslant e \} \rangle.$$

Define

$$\mathscr{U} = \{ (e, f) \in E \times E : <e> \cong <f> \}$$

and for each $(e, f) \in \mathscr{U}$ let $T_{e,f}$ be the set of all isomorphisms from $<e>$ onto $<f>$, and for each $\alpha \in T_{e,f}$ define the further mappings $\bar{\alpha}$ and $\bar{\bar{\alpha}}$ by (for each idempotent $g \leqslant e$)

$$L_g \bar{\alpha} = L_{g\alpha}, \qquad R_g \bar{\bar{\alpha}} = R_{g\alpha}.$$

From Result 2 and the fact that α is an isomorphism, we see that $\bar{\alpha} \in \mathscr{I}_{B/\mathscr{L}}$ and $\bar{\bar{\alpha}} \in \mathscr{I}_{B/\mathscr{R}}$. For each $\alpha \in T_{e,f}$, consider $\rho_e \bar{\alpha}$ and $\lambda_f \overline{\overline{\alpha^{-1}}}$, products being taken in $\mathscr{P}\mathscr{T}_{B/\mathscr{L}}$ and $\mathscr{P}\mathscr{T}_{B/\mathscr{R}}$, respectively, and put

$$T_{\langle E \rangle} = \bigcup_{(e,f) \in \mathscr{U}} \{ (\rho_e \bar{\alpha}, \lambda_f \overline{\overline{\alpha^{-1}}}) : \alpha \in T_{e,f} \}.$$

For convenience we shall denote $(\rho_e \bar{\alpha}, \lambda_f \overline{\overline{\alpha^{-1}}})$ simply by $\phi(\alpha)$. That

$$T_{\langle E \rangle} \subseteq \mathscr{T}_{B/\mathscr{L}} \times \mathscr{T}^*_{B/\mathscr{R}}$$

is easily shown.

THEOREM 7. (i) *The set $T_{\langle E \rangle}$ is a subsemigroup of $\mathscr{T}_{B/\mathscr{L}} \times \mathscr{T}^*_{B/\mathscr{R}}$.*

(ii) *Moreover, $T_{\langle E \rangle}$ is a regular semigroup whose subsemigroup generated by its idempotents is isomorphic to $\langle E \rangle / \mu(\langle E \rangle)$.*

(iii) *Let (ρ, λ) be the mapping which maps each element a in the regular semigroup S to (ρ_a, λ_a) as defined above. Then (ρ, λ) is a homomorphism of S into $T_{\langle E \rangle}$ which maps E onto the set of idempotents of $T_{\langle E \rangle}$.*

(iv) *For any $e, f \in E$, if $<e>$ is isomorphic to $<f>$, then (ρ_e, λ_e) and (ρ_f, λ_f) are $\mathscr{D}$-related in $T_{\langle E \rangle}$.*

Proof. After the lemma we proceed straight to the proof.

LEMMA 8. *If* $\alpha \in T_{e,f}$ *and* $\beta \in T_{f,g}$, *for some* e, f, $g \in E$, *then* $\bar{\alpha}\bar{\beta} = \overline{\alpha\beta}$ *and* $\bar{\bar{\alpha}}\bar{\bar{\beta}} = \overline{\overline{\alpha\beta}}$.

Proof. Now $\alpha\beta \in T_{e,g}$, so take any idempotent $h \in \langle e\rangle$. Then since $h\alpha \in \langle f\rangle$, we have

$$L_h\bar{\alpha}\bar{\beta} = L_{h\alpha}\bar{\beta} = L_{h\alpha\beta} = L_h\overline{\alpha\beta},$$

whence $\bar{\alpha}\bar{\beta} = \overline{\alpha\beta}$, since the domains of $\bar{\alpha}\bar{\beta}$ and $\overline{\alpha\beta}$ are both equal to $\{L_x \in B/\mathscr{L} : L_x \leqslant L_e\}$. Similarly $\bar{\bar{\alpha}}\bar{\bar{\beta}} = \overline{\overline{\alpha\beta}}$.

(i) Take any pairs (e, f), (g, h) in $\mathscr{U}$, and any elements $\alpha \in T_{e,f}$, $\beta \in T_{g,h}$. We wish to show $\phi(\alpha)\phi(\beta) \in T_{\langle E\rangle}$.

Take any inverse $(fg)'$ of fg in $\langle E\rangle$, and put $i = g(fg)'f$, an idempotent which is also an inverse of fg [11, Lemma 1.1]. Now define the mappings of Result 3 for the semigroup $B = \langle E\rangle$. Then $\theta_{i,fg} \mid \langle fgi\rangle = \gamma$ say, maps $\langle fgi\rangle$ isomorphically onto $\langle ifg\rangle$,

$$\theta_{fg,i} \mid \langle ifg\rangle = (\theta_{i,fg} \mid \langle fgi\rangle)^{-1} = \gamma^{-1},$$

and also, for any idempotent $l \leqslant fgi$,

$$\begin{aligned} L_l\rho_{fg} &= L_{lfg} = L_{ilfg} \qquad (\text{since } i \in V(fg),\ l \leqslant fgi) \\ &= L_{l\gamma} = L_l\bar{\gamma}. \end{aligned}$$

Hence

$$\rho_{fg} \mid \{L_l : l \in E,\ l \leqslant fgi\} = \bar{\gamma}, \tag{1}$$

and similarly

$$\lambda_{fg} \mid \{R_l : l \in E,\ l \leqslant ifg\} = \overline{\overline{\gamma^{-1}}}.$$

Put $(fgi)\alpha^{-1} = j$ and $(ifg)\beta = k$. (Note that $j^2 = j \leqslant e$ and $k^2 = k \leqslant h$.)

Now $(\alpha \mid \langle j\rangle)\gamma\beta = \delta$ say, is an isomorphism from $\langle j\rangle$ onto $\langle k\rangle$; we shall show that $\phi(\alpha)\phi(\beta) = \phi(\delta)$. We see that $\delta^{-1} = (\beta^{-1} \mid \langle k\rangle)\gamma^{-1}\alpha^{-1}$.

Take now any element $x \in \langle E\rangle$. Then $L_{xe} = L_l$ for some idempotent $l \leqslant e$ (Result 4), whence (from $\rho_e\rho_j = \rho_{ej} = \rho_j$)

$$\begin{aligned} L_x\rho_j\bar{\alpha} &= L_x\rho_e\rho_j\bar{\alpha} = L_l\rho_j\bar{\alpha} = L_{(lj)\alpha} \\ &= L_{(l\alpha)(j\alpha)} = L_{xe}\bar{\alpha}\rho_{j\alpha} = L_x\rho_e\bar{\alpha}\rho_{fgi}, \end{aligned}$$

and hence

$$\rho_j\bar{\alpha} = \rho_e\bar{\alpha}\rho_{fgi}, \tag{2}$$

and similarly

$$\lambda_k\overline{\overline{\beta^{-1}}} = \lambda_h\overline{\overline{\beta^{-1}}}\lambda_{ifg}.$$

Now we have

$$\begin{aligned}\rho_e\bar{\alpha}\rho_g\bar{\beta} = (\rho_e\bar{\alpha}\rho_f)\,\rho_g\bar{\beta} &= \rho_e\bar{\alpha}\rho_{fgifg}\bar{\beta}\\ &= (\rho_e\bar{\alpha}\rho_{fgi})\,\rho_{fg}\bar{\beta}\\ &= \rho_j\bar{\alpha}\rho_{fg}\bar{\beta} \qquad \text{(from (2))}\\ &= \rho_j\bar{\alpha}\bar{\gamma}\bar{\beta} \qquad \text{(from (1))}\\ &= \rho_j\bar{\delta},\end{aligned}$$

since

$$\operatorname{range}(\rho_j\bar{\alpha}) = \{L_l : l \in E,\, l \leqslant fgi\}$$

and

$$\operatorname{range}\rho_j = \{L_l : l \in E,\, l \leqslant j\}.$$

Similarly

$$\lambda_h\overline{\overline{\beta^{-1}}}\lambda_f\overline{\overline{\alpha^{-1}}} = \lambda_k\overline{\overline{\delta^{-1}}},$$

and so $\phi(\alpha)\phi(\beta) = \phi(\delta)$. This completes the proof of the closure of $T_{\langle E\rangle}$.

(ii) To show that $T_{\langle E\rangle}$ is regular, let us take any element, $\phi(\alpha)$, say, in $T_{\langle E\rangle}$, and consider $\phi(\alpha^{-1})$. If $\alpha \in T_{e,f}$, say, then using $\bar{\alpha}\rho_f = \bar{\alpha}$ and $\overline{\overline{\alpha^{-1}}}\lambda_e = \overline{\overline{\alpha^{-1}}}$, we may routinely show that $\phi(\alpha^{-1})$ is an inverse of $\phi(\alpha)$ in $T_{\langle E\rangle}$. We proceed now to find the idempotents of $T_{\langle E\rangle}$.

Take any element of $T_{\langle E\rangle}$ which is idempotent, $\phi(\alpha)$, say, where $\alpha \in T_{e,f}$ for some $(e, f) \in \mathscr{U}$. Then $\rho_e\bar{\alpha}$ is an idempotent, and since L_f is in its range, we have $L_f\rho_e\bar{\alpha} = L_f$. But also

$$L_e\bar{\alpha} = L_{e\alpha} = L_f = L_{fe}\bar{\alpha} \qquad \text{(from above)},$$

and since $\bar{\alpha}$ is one-to-one, we have $L_e = L_{fe}$. Using the idempotence of $\lambda_f\overline{\overline{\alpha^{-1}}}$ we similarly obtain $R_f = R_{fe}$. From [1, Section 2.3] there exists an inverse $(fe)^*$, say, of fe, in $R_e \cap L_f$, and then $e(fe)^* = (fe)^* = (fe)^*f$, and so $(fe)^* = i$, say, is an idempotent. (In fact, from $L_e = L_{fe}$ and $R_f = R_{fe}$ we may show that $(fe)^* = e(fe)'f$ for any inverse $(fe)'$ of fe.) Now $i \in R_e$ implies $ie = e$, and so, for each $x \in \langle E\rangle$,

$$L_x\rho_e\bar{\alpha} = L_{xie}\bar{\alpha} = L_{xi}\rho_e\bar{\alpha} = L_{xi}$$

since $L_{xi} \leqslant L_i = L_f$ implies that L_{xi} is in the range of $\rho_e\bar{\alpha}$. Therefore, $\rho_e\bar{\alpha} = \rho_i$ and similarly $\lambda_f\overline{\overline{\alpha^{-1}}} = \lambda_i$, giving $\phi(\alpha) = (\rho_i, \lambda_i)$. Thus the set of idempotents of $T_{\langle E\rangle}$ is $\{(\rho_i, \lambda_i) : i \in E\}$, the set of images of the idempotents of S under the homomorphism (ρ, λ) of S into $\mathscr{T}_{B/\mathscr{L}} \times \mathscr{T}^*_{B/\mathscr{R}}$. Thus (ρ, λ) maps $\langle E\rangle$ onto the subsemigroup of $T_{\langle E\rangle}$ generated by its idempotents, which is therefore isomorphic to $\langle E\rangle/\mu(\langle E\rangle)$, by Corollary 6.

(iii) To prove part (iii) of the theorem it suffices to prove that, for each $a \in S$, $(\rho_a, \lambda_a) \in T_{\langle E\rangle}$ (since (ρ, λ) is a homomorphism of S into $\mathscr{T}_{B/\mathscr{L}} \times \mathscr{T}^*_{B/\mathscr{R}}$).

Take any $a \in S$, any $a' \in V(a)$ and any $x \in \langle E\rangle$. Then $L_{xaa'} \leqslant L_{aa'}$ and so $L_{xaa'} = L_e$ for some idempotent $e \leqslant aa'$. Also $\theta_{a',a} \mid \langle aa'\rangle = \alpha$ say, is an isomorphism from $\langle aa'\rangle$ onto $\langle a'a\rangle$, i.e., $\alpha \in T_{aa',a'a}$. Now

$$\begin{aligned} L_x\rho_a &= L_x\rho_{aa'}\rho_a = L_e\rho_a = B \cap L(S)_{ea} \\ &= B \cap L(S)_{a'ea} = L_{a'ea} = L_{ea} \\ &= L_{xaa'}\bar{\alpha} = L_x\rho_{aa'}\bar{\alpha}. \end{aligned}$$

Therefore, $\rho_a = \rho_{aa'}\bar{\alpha}$ and similarly $\lambda_a = \lambda_{a'a}\overline{\overline{\alpha^{-1}}}$, giving

$$(\rho_a, \lambda_a) = \phi(\alpha) \in T_{\langle E\rangle}.$$

(iv) Take any $(e, f) \in \mathscr{U}$ and any $\alpha \in T_{e,f}$. It is quite routine to show that in $T_{\langle E\rangle}$ we have $\phi(\alpha)\phi(\alpha^{-1}) = (\rho_e, \lambda_e)$ and $\phi(\alpha^{-1})\phi(\alpha) = (\rho_f, \lambda_f)$. But since $\phi(\alpha^{-1})$ is an inverse of $\phi(\alpha)$ [from (ii)] this gives that (ρ_e, λ_e) and (ρ_f, λ_f) are $\mathscr{D}$-related in $T_{\langle E\rangle}$.

6. Some Consequences

Let $B = \langle E\rangle$ be any fundamental regular semigroup generated by its set of idempotents E, and let us define $\mathscr{U}$ and $T_{\langle E\rangle}$ as in Section 5. In this case the subsemigroup generated by the idempotents of $T_{\langle E\rangle}$ is isomorphic to $\langle E\rangle$, and by renaming elements one could construct a semigroup isomorphic to $T_{\langle E\rangle}$ in which $\langle E\rangle$ is the actual subsemigroup generated by the idempotents. We will not do this.

For the fixed semigroup $B = \langle E\rangle$ we consider the class of semigroups, $\mathscr{C}(B)$ say, determined by the following: A semigroup S is a member of $\mathscr{C}(B)$ if and only if S is a regular semigroup whose subsemigroup generated by its idempotents is isomorphic to B.

We consider and give answers to the following questions: What conditions on B are equivalent to the following conditions:

(a) some member of $\mathscr{C}(B)$ is [0–] bisimple,

(b) some member of $\mathscr{C}(B)$ is [0–] simple,

(c) every member of $\mathscr{C}(B)$ is completely semisimple [a union of groups],

(d) Green's relation $\mathscr{H}$ is a congruence on every member of $\mathscr{C}(B)$?

THEOREM 9. *Some member of $\mathscr{C}(B)$ is* [0–] *bisimple* (*i.e., the semigroup B is the subsemigroup generated by the idempotents of a* [0–] *bisimple semigroup*) *if and only if* $\mathscr{U} = E \times E[\{(0,0)\} \cup (E\backslash\{0\}) \times (E\backslash\{0\})]$, *in which case we call* $\langle E \rangle$ [0–] *uniform.*

The corresponding statement for semilattices and inverse semigroups is due to Munn [13] (see also [14]), and that for bands and orthodox semigroups (regular semigroups whose idempotents form a band) is due to the author [5].

Proof. From Result 3, if $\langle E \rangle$ is the subsemigroup generated by the idempotents of a bisimple semigroup, S, say, then $\langle E \rangle$ is uniform, since for any $e, f \in E$, $eSe \cong fSf$ implies $\langle e \rangle$, the subsemigroup generated by the idempotents of eSe, is isomorphic to $\langle f \rangle$, the corresponding subsemigroup of fSf. Conversely, if $\langle E \rangle$ is uniform, then $T_{\langle E \rangle}$ is bisimple, since all its idempotents will be $\mathscr{D}$-related from Theorem 7 part (iv), and each $\mathscr{D}$-class of a regular semigroup contains an idempotent. The proof of the bracketed statement is entirely similar; we note that if $\langle E \rangle$ has a zero 0, then $T_{\langle E \rangle}$ has a zero, namely, (ρ_0, λ_0).

THEOREM 10. *The semigroup* $B = \langle E \rangle$ *is the subsemigroup generated by the idempotents of a* [0–] *simple regular semigroup if and only if, for any* [*non-zero*] $e, f \in E$, *there exists an idempotent* $g \leqslant f$ *such that* $(e, g) \in \mathscr{U}$, *in which case we call* $\langle E \rangle$ [0–] *subuniform.*

The corresponding statement for semilattices and inverse semigroups is due to Munn [16].

Proof. From Results 3 and 5, we have the "only if" statement. Conversely, if $\langle E \rangle$ is subuniform then $T_{\langle E \rangle}$ is simple, since for any idempotents (ρ_e, λ_e), (ρ_f, λ_f) in $T_{\langle E \rangle}$ there exists $g \in \langle E \rangle$ such that $g \leqslant f$ and $(e, g) \in \mathscr{U}$, whence $(\rho_g, \lambda_g) \leqslant (\rho_f, \lambda_f)$ and (ρ_g, λ_g) is $\mathscr{D}$-related to (ρ_e, λ_e). The proof of the bracketed statement is entirely similar.

THEOREM 11. *Every regular semigroup, whose subsemigroup generated by its idempotents is* (*isomorphic to*) *B is completely semisimple,* (*i.e., every member of* $\mathscr{C}(B)$ *is completely semisimple*), *if and only if* $(\mathscr{U} \cap \leqslant) = \iota$, *where* $\leqslant$ *is the relation* $\{(f, e) \in E \times E : fe = f = ef\}$ *and* ι *is the identity relation on E.*

Proof. If $\mathscr{U} \cap \leqslant = \iota$, then in any regular semigroup with $\langle E \rangle$ as its subsemigroup generated by its idempotents, no two distinct comparable idempotents can be $\mathscr{D}$-related, so from Result 6, the semigroup is completely semisimple.

Conversely, if $T_{\langle E \rangle}$ is completely semisimple, then for any $(f, e) \in \mathscr{U}$, since (ρ_f, λ_f) and (ρ_e, λ_e) are $\mathscr{D}$-related in $T_{\langle E \rangle}$ we have $f \nless e$, giving $\mathscr{U} \cap \leqslant = \iota$.

Remark 7. *Let S be any regular semigroup whose subsemigroup generated by its idempotents is $B = \langle E \rangle$. If each $\mathscr{U}$-class of E contains a finite number of elements, then S is completely semisimple. If moreover the number of elements in each $\mathscr{U}$-class is bounded by m, some positive integer, then for each $a \in S$, a^m is an element of a subgroup* (see Theorem 15).

In the next theorem, by $\mathscr{D}(B)$ we mean of course Green's relation $\mathscr{D}$ on B.

THEOREM 12. *Every regular semigroup, whose subsemigroup generated by its idempotents is $B = \langle E \rangle$, is a union of groups if and only if both B is a union of groups and $\mathscr{U} = \mathscr{D}(B) \cap (E \times E)$.*

The corresponding statement for semilattices and inverse semigroups is due to Howie and Schein [10], and that for bands and orthodox semigroups is due to the author [5].

Proof. Suppose B is a union of groups and $\mathscr{U} = \mathscr{D}(B) \cap (E \times E)$. Let S be any regular semigroup whose subsemigroup generated by its idempotents is $\langle E \rangle$ and take any $a \in S$ and any inverse a' of a. Then $(aa', a'a) \in \mathscr{U} \subseteq \mathscr{D}(B)$ and so $R(B)_{aa'} \cap L(B)_{a'a} \neq \square$; and B being a union of groups this $\mathscr{H}$-class of B contains an idempotent, e, say. But then $e\mathscr{L}(S)\, a'a\mathscr{L}(S)\, a$ and $e\mathscr{R}(S)\, aa'\mathscr{R}(S)\, a$, whence $e\mathscr{H}(S)\, a$ and a is an element of a subgroup of S. Thus S is a union of groups.

To prove the converse, we really only need $T_{\langle E \rangle}$ to be a union of groups. First of all, since B is isomorphic to a regular subsemigroup of $T_{\langle E \rangle}$, we have B is a union of groups (Result 2). Secondly, let us take any $(e, f) \in \mathscr{U}$ and $\alpha \in T_{e,f}$. Then $(\rho_e, \lambda_e)\, \mathscr{R}\phi(\alpha)\, \mathscr{L}(\rho_f, \lambda_f)$ in $T_{\langle E \rangle}$ (Section 5, proof of Theorem 7 part (iv)) and $\phi(\alpha)$ is $\mathscr{H}$-related to an idempotent in $T_{\langle E \rangle}$, say, (ρ_g, λ_g), where $g \in E$. Then $e\mathscr{R}g\mathscr{L}f$ in B whence $(e, f) \in \mathscr{D}(B)$ and $\mathscr{U} \subseteq \mathscr{D}(B)$. But we always have $\mathscr{D}(B) \cap (E \times E) \subseteq \mathscr{U}$. Therefore $\mathscr{U} = \mathscr{D}(B) \cap (E \times E)$ and the theorem is proved.

THEOREM 13. *Green's relation $\mathscr{H}$ is a congruence on every regular semigroup, whose subsemigroup generated by its idempotents is $B = \langle E \rangle$, if and only if for each $e \in E$, $\prec e \succ$ has only one automorphism, the trivial one, in which case we say that $\langle E \rangle$ is taut.*

The corresponding statement for semilattices and inverse semigroups was proved by Munn in the proof of Theorem 3.2 [13]. We delay a proof of Theorem 13 until Section 10, where we study $T_{\langle E \rangle}$ further (we could prove the "if" statement now from Result 3 and Theorem 5; our proof in Section 10 is more conceptual). *If for each $e \in E$, $\{f \in E : f \leqslant e\}$ is inversely well-ordered* (*see Munn* [13, p. 157]) *then E is taut;* of course each $\prec e \succ$ is then necessarily a semilattice but $\langle E \rangle$ need not even be a band (consider a suitable completely 0-simple semigroup).

Let us consider the statements of the theorems of this section without the restriction that B is fundamental i.e., with B as any regular semigroup generated by its idempotents. Then the proofs given here are sufficient to prove the following statements without the mentioned restriction: the "only if" statements of Theorems 9 and 10 and the "if" statements of Theorems 11–13. The author does not know if the remaining statements without the restriction are also true.

7. Maximal Regular Subsemigroups

THEOREM 14. *Let E be any set of idempotents of a semigroup S.*

(i) *There is a regular subsemigroup of S with E as its set of idempotents if and only if $\langle E \rangle$, the subsemigroup generated by E, is such a semigroup, i.e., a regular semigroup with E as its set of idempotents.*

(ii) *If $\langle E \rangle$ is a regular subsemigroup with E as its set of idempotents then*

$$E^C = \{a \in S : \textit{for some } a' \in V(a),\ aa',\ a'a,\ a'ea,\ afa' \in E \\ \textit{for all } e, f \in E \textit{ such that } e \leqslant aa', f \leqslant a'a\}$$

is the maximum regular subsemigroup of S with E as its set of idempotents.

The corresponding statement of part (ii) for E a subsemigroup of S is equivalent to the statement due to Reilly and Scheiblich [20, Theorem 1.5].

Proof. Part (i) is a corollary of Result 1, so we now consider part (ii). It is easily seen that E^C contains any regular subsemigroup of S with E as its set of idempotents. We now let

$$X = \{a \in S : \text{for some } a' \in V(a),\ aa',\ a'a \in E \text{ and } L_E a \subseteq L_E\ ,\ aR_E \subseteq R_E\},$$

where

$$L_E = \bigcup \{L_e \in S/\mathscr{L} : e \in E\} \qquad \text{and} \qquad R_E = \bigcup \{R_e \in S/\mathscr{R} : e \in E\};$$

we shall show that X is a subsemigroup of S with E as its set of idempotents. Take any a, $b \in X$. Clearly $L_E ab \subseteq L_E b \subseteq L_E$ and $abR_E \subseteq R_E$. Now there are $a' \in V(a)$, $b' \in V(b)$ such that $a'a$, aa', $b'b$, $bb' \in E$. Since $L_E b \subseteq L_E$ and $\mathscr{L}$ is a right congruence on S, we have $ab \in L_a b = L_{a'a} b \subseteq L_e$ for some $e \in E$ and similarly $ab \in aR_{bb'} \subseteq R_f$ for some $f \in E$. Hence $ab \in L_e \cap R_f$ and so there is an inverse $(ab)'$ say, of ab, in $R_e \cap L_f$, and then $ab(ab)' = f \in E$ and $(ab)'\,ab = e \in E$. Thus $ab \in X$ and X is a subsemigroup. Take now any idempotent $a \in X$. Then there is $a' \in V(a)$ such that $a'a$, $aa' \in E \subseteq \langle E \rangle$ and then $a' = a'aa' = a'aaa' \in \langle E \rangle$. Now $aa'\mathscr{L}a'\mathscr{R}a'a$ in $\langle E \rangle$ and so there is an inverse x of a' in $\langle E \rangle$ such that $aa'\mathscr{R}x\mathscr{L}a'a$ in $\langle E \rangle$ and hence also in S. But $aa'\mathscr{R}a\mathscr{L}a'a$ in S and $a \in V(a')$ and so $a = x \in \langle E \rangle$ [1, Section 2.3], whence $a \in E$ and the set of idempotents of X is E (since clearly $E \subseteq X$).

Since E generates a regular subsemigroup of X, by Result 7 we have that the set of regular elements of X, Y say, forms a subsemigroup (with set of idempotents E); from above, $Y \subseteq E^C$. Clearly

$$Y = \{a \in S : \text{ for some } a' \in V(a),\ aa',\ a'a \in E, \text{ and } L_E a \subseteq L_E,\ L_E a' \subseteq L_E, \text{ and } aR_E \subseteq R_E,\ a'R_E \subseteq R_E\}.$$

We show eventually that $Y = E^C$.

Take then any $a \in E^C$ and any $a' \in V(a)$ satisfying the conditions for the membership of a in E^C. Take any $e \in E$. Now $eaa' \in \langle E \rangle$ and so $eaa'\mathscr{L}f$ in $\langle E \rangle$ for some $f \in E$ such that $f \leqslant aa'$ (Result 4). Considering the $\mathscr{L}$-classes of S we now have

$$L_e a \subseteq L_{ea} = L_{eaa'a} = L_{fa} = L_{a'fa}$$

since $\mathscr{L}$ is a right congruence and $a(a'fa) = fa$. But $a'fa \in E$ and so $L_e a \subseteq L_E$ and $L_E a \subseteq L_E$. Similarly, $aR_E \subseteq R_E$, $L_E a' \subseteq L_E$ and $a'R_E \subseteq R_E$, giving $a \in Y$. Therefore, $E^C \subseteq Y$ and, then, $E^C = Y$.

Thus E^C is the maximum regular subsemigroup of S with E as its set of idempotents.

8. Completely Semisimple Semigroups

THEOREM 15. *Let S be a regular semigroup such that each $\mathscr{D}$-class of S contains at most m $\mathscr{L}$-classes of S, for some integer m (S is thus completely semisimple). Then for each element $a \in S$, a^m is an element of a subgroup of S.*

Proof. Consider the homomorphism ρ of S into $\mathscr{T}_{S/\mathscr{L}}$ defined in Section 4. Take any element $a \in S$. Note that $\rho_{a^m} = {\rho_a}^m$. For each $\mathscr{D}$-class D of S denote $\{L_x \in S/\mathscr{L} : x \in D\}$, the set of $\mathscr{L}$-classes of S contained in D, by $D/\mathscr{L}$.

Take any $\mathscr{D}$-class D of S such that $(D/\mathscr{L}) \cap \text{range } {\rho_a}^m \neq \square$ and take any $\mathscr{L}$-class, L_y say, in $(D/\mathscr{L}) \cap \text{range } {\rho_a}^m$. Then there exists $x \in S$ such that

$$L_y = L_x {\rho_a}^m = L_{xa^m} = L_{xa^m(a^m)'a^m} = L_{xa^m(a^m)'} {\rho_a}^m$$

for any inverse $(a^m)'$ of a^m in S. Put $xa^m(a^m)' = b$. Then

$$b\mathscr{R}ba\mathscr{R}ba^2 \cdots \mathscr{R}ba^m\mathscr{L}y.$$

But D contains at most m $\mathscr{L}$-classes of S and so two of L_b, L_{ba},..., L_{ba^m} are equal, i.e., for some $i < j$, $i, j \in \{0, 1,..., m\}$ we have $L_{ba^i} = L_{ba^j}$ where ba^0 means just b. By repeated multiplication by a on the right of the subscripts in $L_{ba^i} = L_{ba^j}$, we obtain (since $\mathscr{L}$ is a right congruence) that

$$L_{ba^n} \in \{L_b, L_{ba},..., L_{ba^m}\} \qquad \text{for } n = 1, 2, 3,... \tag{3}$$

and that also, there are arbitrarily large n for which $L_{ba^m} = L_{ba^n}$. Choose such an $n > 2m$; then $n = 2m + p$ for some positive integer p. We now have

$$L_y = L_{ba^m} = L_{ba^n} = (L_{ba^p} \rho_a{}^m) \, \rho_a{}^m$$

and $L_{ba^{p+m}} \in (D/\mathscr{L}) \cap$ range $\rho_a{}^m$ from (3). Now L_y was an arbitrary element of $(D/\mathscr{L}) \cap$ range $\rho_a{}^m$ so $(D/\mathscr{L}) \cap$ range $\rho_a{}^m \subseteq [(D/\mathscr{L}) \cap$ range $\rho_a{}^m] \, \rho_a{}^m$. But $(D/\mathscr{L}) \cap$ range $\rho_a{}^m$ is finite so $\rho_a{}^m$ maps it one-to-one onto itself. It follows that for every $D \in S/\mathscr{D}$ $\rho_a{}^m$ maps$(D/\mathscr{L}) \cap$ range $\rho_a{}^m$ one-to-one onto itself (which may be empty). From this it follows that ρ_{a^m} maps its entire range one-to-one onto itself, and thus is an element of a subgroup of $\mathscr{T}_{S/\mathscr{L}}$ [1, Theorem 2.10]. By Result 2 part (iii), ρ_{a^m} is an element of a subgroup of $\rho(S) = \{\rho_s : s \in S\}$ and by Result 9, since $\rho \circ \rho^{-1} \subseteq \mathscr{L}$, a^m is an element of a subgroup of S, giving the theorem.

Result 8 and Theorem 15 make a nice pair; the former involves a concept known as the height of S and the latter involves a concept sometimes called the width of S. From the two results (not numbered) adjacent to Result 6 it follows that *a regular semigroup S is completely semisimple and each strictly descending chain of $\mathscr{J}$-classes has length at most m, if and only if each strictly descending chain of idempotents has length at most m* (due to W. D. Munn, unpublished). This enables a restatement of Result 8.

The proof of Theorem 15 was designed so that minor variations in it give one a proof of the following. *Let m be an integer, let a be an element of a regular semigroup S, and let ρ_a be defined as in Section 4. If*

(i) *the range of ρ_a contains less than m $\mathscr{L}$-classes of S from each $\mathscr{D}$-class of S*

Then a^m is an element of a subgroup of S. Now let a' be any inverse of a and put $a'a = e$. Then the range of ρ_a is $\{L_x \in S/\mathscr{L} : x \in eSe\}$ (Result 4) and each of Green's relations on eSe is simply the restriction to eSe of the corresponding Green's relation on S (this is easily shown for any semigroup S). *Hence condition* (i) *is equivalent to the following condition:*

(ii) *each $\mathscr{D}$-class of eSe contains at most $m - 1$ $\mathscr{L}$-classes of eSe.*

This further enables us to replace the hypothesis of Theorem 15 with the hypothesis that for each idempotent e in a regular semigroup S, condition (ii) holds.

There is a proof of Theorem 15 that uses Result 8. We present this proof for the case when S is an inverse semigroup, because of its simplicity. In this case each $\mathscr{D}$-class of S contains at most m idempotents. For each $a \in S$ define $\alpha_a : aa^{-1}E \to a^{-1}aE$ by $e\alpha_a = a^{-1}ea$ for each $e \in aa^{-1}E$, where E is the semilattice of S, as in [13]. Then α_a is $\mathscr{D}$-class preserving and so $\alpha_a = \bigcup_{x\in S} \alpha_a \cap (D_x \times D_x)$ whence $\alpha_a{}^m = \bigcup_{x\in S} [\alpha_a \cap (D_x \times D_x)]^m$. But for each $D \in S/\mathscr{D}$, $\alpha_a \cap (D \times D) \in \mathscr{I}_{E(D)}$ and $|E(D)| \leqslant m$, where $E(D)$ is the set of idempotents of D. Thus $[\alpha_a \cap (D \times D)]^m$ is an element of a subgroup of $\mathscr{I}_{E(D)}$ (Section 2) and hence its domain equals its range. Hence the domain of $\alpha_a{}^m$ equals the range of $\alpha_a{}^m = \alpha_{a^m}$ and so α_{a^m} is an element of a subgroup of $\mathscr{I}_E$ (easily shown). Using $\alpha \circ \alpha^{-1} \subseteq \mathscr{H}$ one easily obtains that a^m is an element of a subgroup of S.

The methods of this section also yield the following result. *Let S be a fundamental regular semigroup in which each $\mathscr{D}$-class contains at most m idempotents. Then there is an integer n such that a^n is an idempotent for each a in S.* We may take n to be any integer such that t^n is an idempotent, for each t in $\mathscr{P}\mathscr{T}_m$, the semigroup of all partial transformations of the set $\{1, 2,\ldots, m\}$. In general, a fundamental regular semigroup in which each $\mathscr{D}$-class contains one $\mathscr{L}$-class (i.e., $\mathscr{D} = \mathscr{L}$) is not necessarily periodic.

9. Partial Groupoids of Idempotents

Let S be a fundamental regular semigroup generated by its set of idempotents, E, say. Then $S = \langle E \rangle$. We show in this section that S is uniquely determined (to within isomorphism) by its partial groupoid [1, p. 1] of idempotents E; by the partial groupoid E we mean the set E together with the partial binary operation $(\cdot)$ say, that S induces on E, namely, that defined by: for all $e, f \in E$, $e \cdot f =$ if $ef \in E$, $e \cdot f$ is undefined if $ef \in S\backslash E$. We show this uniqueness by reconstructing S from $E(\cdot)$.

Define the following equivalences (see below) on $E(\cdot)$:

$$\mathscr{L}' = \{(e, f) \in E \times E : e \cdot f = e, f \cdot e = f\},$$

$$\mathscr{R}' = \{(e, f) \in E \times E : e \cdot f = f, f \cdot e = e\}.$$

For each $e \in E$, let $L_e'[R_e']$ denote the $\mathscr{L}'$ — $[\mathscr{R}'$-]class of E containing e. Let us partially order the sets $E/\mathscr{L}'$ and $E/\mathscr{R}'$ by defining, for any $e, f \in E$, $L_e' \leqslant L_f'$ if and only if $e \cdot f = e$ and $R_e' \leqslant R_f'$ if and only if $f \cdot e = e$. That these give well-defined partial orderings, and in fact that $\mathscr{L}'$ and $\mathscr{R}'$ are equivalences, we may easily deduce from facts concerning Green's relations $\mathscr{L}$ and $\mathscr{R}$ on S.

For each $e \in E$ we define a transformation ρ_e in $\mathscr{T}_{E/\mathscr{L}'}$ as follows: For each $x \in E$, let $L_x'\rho_e$ be the unique maximum $\mathscr{L}'$-class of E in the set

$$X = \{L'_{(h\cdot x)\cdot e} \in E/\mathscr{L}' : h \in E,\, h \cdot x \in E,\, (h \cdot x) \cdot e \in E\};$$

we use S again to show the existence of such an $\mathscr{L}'$-class, as follows. Take any $x \in E$ and let $(xe)'$ be any inverse of xe in S. Put $g = e(xe)'\,x$, an idempotent which is also an inverse of xe [11, Lemma 1.1]. Then $gx = g \in E$ and $gxe \in E$ whence X is nonempty. Take any $h \in E$ such that $(h \cdot x) \cdot e \in E$. Then

$$[(h \cdot x) \cdot e] \cdot [(g \cdot x) \cdot e] = (hxe) \cdot (gxe) = hxe,$$

since $(hxe)\,(gxe) = hxe \in E$. Thus X contains a maximum member, $L'_{(g\cdot x)\cdot e}$, which equals $E \cap L_{gxe} = E \cap L_{xe} = V(xe)\,xe$, where we use the usual notations associated with Green's relations on S. Hence, for any $x, y \in E$, $L_x' = L_y'$ implies $L_x = L_y$ and $E \cap L_{xe} = E \cap L_{ye}$, i.e., $L_x'\rho_e = L_y'\rho_e$. Thus ρ_e is a transformation.

Similarly we define $\lambda_e \in \mathscr{T}_{E/\mathscr{R}'}$ by, for each $x \in E$, $R_x'\lambda_e$ is the unique maximum $\mathscr{R}'$-class in $\{R'_{e\cdot(x\cdot h)} : h \in E,\ x \cdot h \in E,\ e \cdot (x \cdot h) \in E\}$. Let us finally define T to be the subsemigroup of $\mathscr{T}_{E/\mathscr{L}'} \times \mathscr{T}^*_{E/\mathscr{R}'}$ generated by $\{(\rho_e, \lambda_e) : e \in E\}$, where $\mathscr{T}^*_{E/\mathscr{R}'}$ is the semigroup dual to $\mathscr{T}_{E/\mathscr{R}'}$.

THEOREM 16. *The semigroup T, defined in terms of $E(\cdot)$, is isomorphic to S.*

Proof. For each element $a \in S$, define $(\rho_a', \lambda_a') \in \mathscr{T}_{E/\mathscr{L}'} \times \mathscr{T}^*_{E/\mathscr{R}'}$ by, for each $x \in S$, $V(x)\,x\rho_a' = V(xa)\,xa$, $xV(x)\,\lambda_a' = axV(ax)$. By Remark 4 (ρ', λ'), the mapping which maps each a in S to (ρ_a', λ_a') is an isomorphism of S into $\mathscr{T}_{E/\mathscr{L}'} \times \mathscr{T}^*_{E/\mathscr{R}'}$. From above we see that, for each $e \in E$, $\rho_e = \rho_e'$, and similarly $\lambda_e = \lambda_e'$. Now $\{(\rho_e', \lambda_e') : e \in E\}$ generates the isomorphic image of S under (ρ', λ'), but $\{(\rho_e', \lambda_e') : e \in E\} = \{(\rho_e, \lambda_e) : e \in E\}$ generates T, and so T is isomorphic to S.

COROLLARY 17. *Let $S_1 = \langle E_1 \rangle$ and $S_2 = \langle E_2 \rangle$ be fundamental regular semigroups generated by their sets of idempotents E_1 and E_2, respectively.*

(i) *If the partial groupoid E_1 is isomorphic to the partial groupoid E_2, then S_1 is isomorphic to S_2.*

(ii) *Further, any isomorphism of E_1 onto E_2 is extendible to an isomorphism of S_1 onto S_2.*

(iii) *The group of automorphisms of S_1 is isomorphic to the group of automorphisms of E_1.*

Remark 8. *If we let S be any regular semigroup, with set of idempotents E say, and construct T as above, then T is isomorphic to $\langle E\rangle/\mu(\langle E\rangle)$, where $\mu(\langle E\rangle)$ is the maximum congruence contained in $\mathscr{H}$ on $\langle E\rangle$, the subsemigroup of S generated by E.*

Remark 9. *From Lemma* 10.64 [1], *for any set X, $\mathscr{T}_X$ is fundamental. When X is a finite set, any proper ideal of $\mathscr{T}_X$ is fundamental, regular, and generated by its idempotents* (from Remark 1 and Corollary 6).

10. Further Results on $T_{\langle E\rangle}$

Let $B = \langle E\rangle$ be any regular semigroup generated by its set of idempotents E and consider $\mathscr{U}$ and $T_{\langle E\rangle}$ as defined in Section 5.

THEOREM 18. (i) *Let $(e, f) \in \mathscr{U}$, $\alpha \in T_{e,f}$ and $a \in \langle e\rangle$. Then*

$$\phi(\alpha^{-1})(\rho_a, \lambda_a)\phi(\alpha) = (\rho_{a\alpha}, \lambda_{a\alpha}).$$

(ii) *Let $(e, f) \in \mathscr{U}$ and $\alpha, \beta \in T_{e,f}$. Then $\phi(\alpha) = \phi(\beta)$ if and only if $\alpha = \beta$.*

(iii) *Let $(e, f), (g, h) \in \mathscr{U}$, $\alpha \in T_{e,f}$ and $\beta \in T_{g,h}$. Then $\phi(\alpha) = \phi(\beta)$ if and only if $e\mathscr{R}g$, $f\mathscr{L}h$ in B, and $\alpha = \theta_{e,g}\beta\theta_{h,f}$, where $\theta_{e,g}$ and $\theta_{h,f}$ are defined for B as in Result* 3.

(iv) *Take any pair of $\mathscr{D}$-related idempotents of $T_{\langle E\rangle}$, say, (ρ_e, λ_e), (ρ_f, λ_f), where $e, f \in E$; and let us denote by H the $\mathscr{H}$-class of $T_{\langle E\rangle}$ which is the intersection of its $\mathscr{R}$-class containing (ρ_e, λ_e) and its $\mathscr{L}$-class containing (ρ_f, λ_f). Then $(e, f) \in \mathscr{U}$ and $\phi(T_{e,f}) = H$.*

(v) *For any $e \in E$, ϕ maps $T_{e,e}$, the group of automorphisms of $\langle e\rangle$, isomorphically onto $\phi(T_{e,e})$, the maximal subgroup of $T_{\langle E\rangle}$ containing (ρ_e, λ_e).*

(vi) *Take any element $\phi(\beta)$, say, in $T_{\langle E\rangle}$ and any inverse $\phi(\beta)'$ of $\phi(\beta)$ in $T_{\langle E\rangle}$. Let e, f be the elements of E such that $\phi(\beta)\phi(\beta)' = (\rho_e, \lambda_e)$ and $\phi(\beta)'\phi(\beta) = (\rho_f, \lambda_f)$. Then for some $\alpha \in T_{e,f}$ $\phi(\beta) = \phi(\alpha)$ and $\phi(\beta)' = \phi(\alpha^{-1})$.*

(vii) *$T_{\langle E\rangle}$ is fundamental.*

Proof. Let us assume the notations of Section 5 concerning Green's relations.

(i) Take any x in B. Then $L_{xf} \leqslant L_f$ and so $L_{xf} = L_g$ for some idempotent $g \leqslant f$ (Result 4). Then $g\alpha^{-1} \leqslant e$ whence $(g\alpha^{-1})a \in \langle e\rangle$. We now obtain

$$\begin{aligned} L_x\rho_f\overline{\alpha^{-1}}\rho_a\rho_e\bar{\alpha} &= L_{xf}\overline{\alpha^{-1}}\rho_a\bar{\alpha} = L_{(g\alpha^{-1})a}\bar{\alpha} \\ &= L_{g(a\alpha)} = L_{xf}\rho_{a\alpha} \\ &= L_x\rho_f\rho_{a\alpha} = L_x\rho_{a\alpha} \end{aligned}$$

since $a\alpha \in \langle f \rangle$. Therefore $\rho_f\overline{\alpha^{-1}}\rho_a\rho_e\bar{\alpha} = \rho_{a\alpha}$ and similarly $\lambda_f\overline{\overline{\alpha^{-1}}}\lambda_a\lambda_e\bar{\bar{\alpha}} = \lambda_{a\alpha}$, giving $\phi(\alpha^{-1})(\rho_a, \lambda_a)\phi(\alpha) = (\rho_{a\alpha}, \lambda_{a\alpha})$ as required.

(ii) The "if" statement is trivial. Suppose $\phi(\alpha) = \phi(\beta)$. Then $\phi(\alpha)\phi(\alpha^{-1}) = (\rho_e, \lambda_e) = \phi(\beta)\phi(\beta^{-1}), \phi(\alpha^{-1})\phi(\alpha) = (\rho_f, \lambda_f) = \phi(\beta^{-1})\phi(\beta)$ and so $\phi(\alpha^{-1})\mathscr{H}\phi(\beta^{-1})$ in $T_{\langle E\rangle}$. But $\phi(\alpha^{-1}), \phi(\beta^{-1})$ are inverses of $\phi(\alpha) = \phi(\beta)$ and so $\phi(\alpha^{-1}) = \phi(\beta^{-1})$. Now by (i), for any idempotent $g \leqslant e$ in $\langle E\rangle$, we have

$$(\rho_{g\alpha}, \lambda_{g\alpha}) = \phi(\alpha^{-1})(\rho_g, \lambda_g)\phi(\alpha) = (\rho_{g\beta}, \lambda_{g\beta}).$$

Now (ρ, λ) maps E one-to-one onto the set of idempotents of $T_{\langle E\rangle}$, since $(\rho, \lambda) \circ (\rho, \lambda)^{-1} \subseteq \mathscr{H}(S)$, and therefore $g\alpha = g\beta$ for all idempotents $g \leqslant e$. But $\langle e \rangle$ is generated by $\{g \in E : g \leqslant e\}$ and so $a\alpha = a\beta$ for all $a \in \langle e \rangle$. Hence $\alpha = \beta$.

(iii) Suppose that $e\mathscr{R}g$, $f\mathscr{L}h$ and $\alpha = \theta_{e,g}\beta\theta_{h,f}$. Take any element $x \in B$. Then $L_{xe} = L_l$ for some idempotent $l \leqslant e$, and then

$$l\alpha = l\theta_{e,g}\beta\theta_{h,f} = h[(elg)\beta]f = [(lg)\beta]hf = [(lg)\beta]h = (lg)\beta.$$

Therefore

$$L_x\rho_e\bar{\alpha} = L_{l\alpha} = L_{(lg)\beta} = L_l\rho_g\bar{\beta} = L_x\rho_e\rho_g\bar{\beta} = L_x\rho_g\bar{\beta},$$

giving $\rho_e\bar{\alpha} = \rho_g\bar{\beta}$. Similarly $\lambda_f\overline{\overline{\alpha^{-1}}} = \lambda_h\overline{\overline{\beta^{-1}}}$ and so $\phi(\alpha) = \phi(\beta)$.

Suppose conversely that $\phi(\alpha) = \phi(\beta)$. Then $(\rho_e, \lambda_e) = \phi(\alpha)\phi(\alpha^{-1})$ is $\mathscr{R}$-related in $T_{\langle E\rangle}$ to $(\rho_g, \lambda_g) = \phi(\beta)\phi(\beta^{-1})$ and since $(\rho, \lambda) \circ (\rho, \lambda)^{-1} \subseteq \mathscr{H}(S)$ we have by the result dual to Result 10 that $e\mathscr{R}(S)g$ and then $e\mathscr{R}g$ (Result 2). Similarly $f\mathscr{L}h$.

Take any $x \in B$. Then $L_{xe} = L_l$ for some idempotent $l \leqslant e$, and then $l\theta_{e,g}\beta\theta_{h,f} = (lg)\beta$ as above. We need to note that $\theta_{e,g}\beta\theta_{h,f} = \gamma$ say, is an element of $T_{e,f}$; this is because $\theta_{e,g}\beta\theta_{h,f} = (\theta_{e,g} \mid \langle e \rangle)\beta(\theta_{h,f} \mid \langle h \rangle)$. Now

$$\begin{aligned} L_x\rho_e\bar{\gamma} &= L_{l\gamma} = L_{(lg)\beta} = L_{xe}\rho_g\bar{\beta} \\ &= L_x\rho_e\rho_g\bar{\beta} = L_x\rho_g\bar{\beta} = L_x\rho_e\bar{\alpha}. \end{aligned}$$

Therefore $\rho_e\bar{\gamma} = \rho_e\bar{\alpha}$ and similarly $\lambda_f\overline{\overline{\gamma^{-1}}} = \lambda_f\overline{\overline{\alpha^{-1}}}$, and so $\phi(\alpha) = \phi(\gamma)$, and then from (ii) we have $\alpha = \gamma = \theta_{e,g}\beta\theta_{f,h}$.

(iv) Take any element $\phi(\beta) \in H$, where $\beta \in T_{g,h}$, $(g, h) \in \mathscr{U}$. Then $\phi(\beta)\phi(\beta^{-1}) = (\rho_g, \lambda_g)$ whence $(\rho_e, \lambda_e)\mathscr{R}(\rho_g, \lambda_g)$ in $T_{\langle E\rangle}$. Therefore $e\mathscr{R}g$ as in the proof of (ii). Similarly $f\mathscr{L}h$. For each element a in B and inverse a' of a in B define $\theta_{a',a}$, as in Result 3, for the semigroup B. Then $\theta_{g,e}(\theta_{e,g}\beta\theta_{h,f})\theta_{f,h} = \beta$ and $\theta_{e,g}\beta\theta_{h,f}$ maps $\langle e \rangle$ isomorphically onto $\langle f \rangle$, whence $(e, f) \in \mathscr{U}$ and $\phi(\beta) = \phi(\theta_{e,g}\beta\theta_{h,f}) \in \phi(T_{e,f})$ from (iii). This gives $H \subseteq \phi(T_{e,f})$, and by considering, for each $\alpha \in T_{e,f}$, the products $\phi(\alpha)\,\phi(\alpha^{-1})$ and $\phi(\alpha^{-1})\,\phi(\alpha)$ we see that $\phi(T_{e,f}) \subseteq H$.

(v) We shall in fact prove, for any $(e, f), (f, g) \in \mathscr{U}$, for any $\alpha \in T_{e,f}$ and $\beta \in T_{f,g}$, that $\phi(\alpha)\,\phi(\beta) = \phi(\alpha\beta)$. Using $\bar{\alpha}\rho_f = \bar{\alpha}$ we have, from Lemma 8,

$$\rho_e\bar{\alpha}\rho_f\bar{\beta} = \rho_e\bar{\alpha}\bar{\beta} = \rho_e\overline{\alpha\beta}$$

and similarly $\lambda_g\overline{\overline{\beta^{-1}}}\lambda_f\overline{\overline{\alpha^{-1}}} = \lambda_g\overline{\overline{\beta^{-1}\alpha^{-1}}}$ whence $\phi(\alpha)\,\phi(\beta) = \phi(\alpha\beta)$. From this, and from (ii) and (iv), we deduce (v).

(vi) From (iv) we see that $\phi(\beta) \in \phi(T_{e,f})$, $\phi(\beta)' \in \phi(T_{f,e})$, whence $\phi(\beta) = \phi(\alpha)$ for some $\alpha \in T_{e,f}$. Then $\phi(\alpha^{-1}) \in \phi(T_{f,e})$ and $\phi(\alpha^{-1})$ is also an inverse of $\phi(\beta) = \phi(\alpha)$, whence $\phi(\beta)' = \phi(\alpha^{-1})$.

(vii) Let μ denote the maximum congruence contained in $\mathscr{H}$ on $T_{\langle E\rangle}$ and let t_1, t_2 be any μ-related elements of $T_{\langle E\rangle}$. From (iv) there exist $(e, f) \in \mathscr{U}$, α, $\beta \in T_{e,f}$ such that $t_1 = \phi(\alpha)$, $t_2 = \phi(\beta)$. Moreover, $\phi(\alpha^{-1})$ and $\phi(\beta^{-1})$ are $\mathscr{H}$-related inverses of $\phi(\alpha)$ and $\phi(\beta)$, respectively. Take any idempotent $g \in \prec e \succ$. Then $(\rho_g, \lambda_g) \leqslant (\rho_e, \lambda_e)$ in $T_{\langle E\rangle}$ so from Theorem 5 and part (i) above

$$(\rho_{g\alpha}, \lambda_{g\alpha}) = \phi(\alpha^{-1})\,(\rho_g, \lambda_g)\,\phi(\alpha) = \phi(\beta^{-1})\,(\rho_g, \lambda_g)\,\phi(\beta) = (\rho_{g\beta}, \lambda_{g\beta}).$$

Now $(\rho, \lambda) \circ (\rho, \lambda)^{-1} \subseteq \mathscr{H}(S)$ and $g\alpha$, $g\beta$ are idempotents, so we have that $g\alpha = g\beta$ for each idempotent $g \in \prec e \succ$. But $\prec e \succ$ is generated by its idempotents so $x\alpha = x\beta$ for each $x \in \prec e \succ$, whence $\alpha = \beta$, $\phi(\alpha) = \phi(\beta)$ and μ is the trivial relation. Thus $T_{\langle E\rangle}$ is fundamental.

We give now an outline of a longer, though conceptually simpler, proof that $T_{\langle E\rangle}$ is fundamental. Take any element $t \in T_{\langle E\rangle}$; then $t = \phi(\alpha)$ for some $(e, f) \in \mathscr{U}$, $\alpha \in T_{e,f}$. Define (ρ_t, λ_t) as in Section 4. From Theorem 18(i) it can be shown that (ρ_t, λ_t) and $t = \phi(\alpha)$ are essentially the same pairs of transformations, and so (ρ, λ) essentially maps $T_{\langle E\rangle}$ identically onto itself (note though that in some cases $T_{\langle E\rangle}$ is properly "contained" in $T_{\langle E\rangle/\mu(\langle E\rangle)}$). In particular, $T_{\langle E\rangle}$ is fundamental.

Proof of Theorem 13. If $\mathscr{H}$ is a congruence on $T_{\langle E\rangle}$, then $\mathscr{H}$ is trivial on $T_{\langle E\rangle}$, since $T_{\langle E\rangle}$ is fundamental. From Theorem 18, part (v), $T_{e,e}$ consists of only the trivial automorphism of $\prec e \succ$, for each $e \in E$, and so $\langle E\rangle$ is taut.

Conversely, suppose $\langle E\rangle$ is taut. Then for any $e, f \in E$ there is at most one isomorphism of $\prec e \succ$ onto $\prec f \succ$, and so from Theorem 18 part (iv) $\mathscr{H}$ on $T_{\langle E\rangle}$ is trivial. Let S be any regular semigroup with $\langle E\rangle$ as its subsemigroup generated by its idempotents and define (ρ, λ) as in Section 5. Then (ρ, λ) is a homomorphism of S into $T_{\langle E\rangle}$ (Theorem 7 part (iii)) and so $\mathscr{H}$ on this image of S is also trivial (Result 2 and its dual result). This quickly gives that $\mathscr{H}(S) \subseteq (\rho, \lambda) \circ (\rho, \lambda)^{-1} = \mu(S)$ (Result 10 and its dual result). But $\mu(S) \subseteq \mathscr{H}(S)$ and so $\mathscr{H}(S) = \mu(S)$, a congruence.

References

1. A. H. Clifford and G. B. Preston, "The Algebraic Theory of Semigroups," Math. Surveys No. 7, Amer. Math. Soc., Providence, RI, Vol. I, 1961; Vol. II, 1967.
2. J. A. Erdos, On products of idempotent matrices, *Glasgow Math. J.* **8** (1967), 118–122.
3. D. G. Fitz-Gerald, On inverses of products of idempotents in regular semigroups, *J. Austral. Math. Soc.* **13** (1972), 335–337.
4. T. E. Hall, On regular semigroups whose idempotents form a subsemigroup, *Bull. Austral. Math. Soc.* **1** (1969), 195–208.
5. T. E. Hall, On orthodox semigroups and uniform and antiuniform bands, *J. Algebra* **16** (1970), 204–217.
6. T. E. Hall, On the natural ordering of $\mathscr{J}$-classes and of idempotents in a regular semigroup, *Glasgow Math. J.* **11** (1970), 167–168.
7. T. E. Hall, Congruences and Green's relations on regular semigroups, *Glasgow Math. J.*, to appear.
8. J. M. Howie, The maximum idempotent-separating congruence on an inverse semigroup, *Proc. Edinburgh Math. Soc.* **14** (1964), 71–79.
9. J. M. Howie, The subsemigroup generated by the idempotents of a full transformation semigroup, *J. London Math. Soc.* **41** (1966), 707–716.
10. J. M. Howie and B. M. Schein, Anti-uniform semilattices, *Bull. Austral. Math. Soc.* **1** (1969), 263–268.
11. J. M. Howie and G. Lallement, Certain fundamental congruences on a regular semigroup, *Proc. Glasgow Math. Assoc.* **7** (1966), 145–159.
12. J. C. Meakin, Congruences on orthodox semigroups, *J. Austral. Math. Soc.* **12** (1971), 323–341.
13. W. D. Munn, Uniform semilattices and bisimple inverse semigroups, *Quart. J. Math. Oxford Ser.* **17** (1966), 151–159.
14. W. D. Munn, The idempotent-separating congruences on a regular 0-bisimple semigroup, *Proc. Edinburgh Math. Soc.* **15** (1967), 233–240.
15. W. D. Munn, Some recent results on the structure of inverse semigroups, *in* "Semigroups: Proceedings of a Symposium," Academic Press, New York, 1969.
16. W. D. Munn, Fundamental inverse semigroups, *Quart. J. Math. Oxford Ser.* **21** (1970), 157–170.
17. W. D. Munn, On simple inverse semigroups, "Semigroup Forum" **1** (1970), 63–74.
18. W. D. Munn, 0-bisimple inverse semigroups, *J. Algebra* **15** (1970), 570–588.
19. G. B. Preston, A note on representations of inverse semigroups, *Proc. Amer. Math. Soc.* **8** (1957), 1144–1147.
20. N. R. Reilly and H. E. Scheiblich, Congruences on regular semigroups, *Pacific J. Math.* **23** (1967), 349–360.
21. E. J. Tully, Representation of a semigroup by transformations acting transitively on a set, *Amer. J. Math.* **83** (1961), 533–541.