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A family of indecomposable positive linear maps based on entangled quantum states

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Abstract

We introduce a new family of indecomposable positive linear maps based on entangled quantum states. Central to our construction is the notion of an unextendible product basis. The construction lets us create indecomposable positive linear maps in matrix algebras of arbitrary high dimension.

Keywords: Quantum information theory; Quantum entanglement; Positive linear maps

1. Introduction

One of the central problems in the emergent field of quantum information theory [1] is the classification and characterization of the entanglement (to be defined in Section 2) of quantum states. Entangled quantum states have been shown to be valuable resources in (quantum) communication and computation protocols. In this context it has been shown [2] that there exists a strong connection between the classification of the entanglement of quantum states and the structure of positive linear maps. Very little is known about the structure of positive linear maps even on low-dimensional matrix algebras, in particular the structure of indecomposable positive linear maps. We denote the $n \times n$ matrix algebra as $M_n(\mathbb{C})$. The first example of an indecomposable positive linear map in $M_3(\mathbb{C})$ was found by Choi [3]. There have been only

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several other examples of indecomposable positive linear maps (see [4] for some recent literature); they seem to be hard to find and no general construction method is available. In this paper we make use of the connection with quantum states to develop a method to create indecomposable positive linear maps on matrix algebras $M_n(\mathbb{C})$ for any $n > 2$. Central in this construction is the notion of an unextendible product basis, of which there exist examples in arbitrary high dimensions [5,6]. Unextendible product bases have turned out to be mathematically rich objects which can be understood with the use of graph theoretic and linear algebraic tools [7].

In Section 2 we present the general construction. In Section 3 we present two examples and discuss various open problems.

2. Unextendible product bases and indecomposable maps

An n-dimensional complex Hilbert space is denoted as $\mathscr{H}_n$. The set of linear operators on a Hilbert space $\mathscr{H}_n$ will be denoted as $B(\mathscr{H}_n)$. The subset of Hermitian positive semidefinite operators is denoted as $B(\mathscr{H}_n)^+$. We will use the conventional bra and ket notation in quantum mechanics, i.e. a vector ψ in $\mathscr{H}_n$ is written as a ket,

$$|\psi\rangle \in \mathscr{H}_n \tag{1}$$

and the Hermitian conjugate of ψ, ψ^*, is denoted as a bra $\langle\psi|$. The complex inner-product between vectors $|\psi\rangle$ and $|\phi\rangle$ in $\mathscr{H}_n$ is denoted as

$$\langle\psi|\phi\rangle \equiv \psi^*\phi. \tag{2}$$

The vectors $|\psi\rangle \in \mathscr{H}$ are usually normalized, $\langle\psi|\psi\rangle = 1$. Elements of $B(\mathscr{H}_n)^+$ can be denoted as

$$\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|, \tag{3}$$

where $|\psi_i\rangle$ are the normalized eigenvectors of ρ and $\lambda_i \geqslant 0$ are the eigenvalues. When ρ has trace equal to 1, the matrix ρ is said to be a density matrix. The physical state of a quantum mechanical system is given by its density matrix. If a density matrix ρ has rank 1, ρ is called a pure state and can be written as

$$\rho = |\psi\rangle\langle\psi|. \tag{4}$$

Let $\mathscr{S} : B(\mathscr{H}_n) \to B(\mathscr{H}_m)$ be a linear map. The map $\mathscr{S}$ is positive when $\mathscr{S} : B(\mathscr{H}_n)^+ \to B(\mathscr{H}_m)^+$. Let id_k be the identity map on $B(\mathscr{H}_k)$. We define the map $\mathrm{id}_k \otimes \mathscr{S} : B(\mathscr{H}_k \otimes \mathscr{H}_n) \to B(\mathscr{H}_k \otimes \mathscr{H}_m)$ for $k = 1, 2, \ldots$ by

$$(\mathrm{id}_k \otimes \mathscr{S})\left(\sum_i \sigma_i \otimes \tau_i\right) = \sum_i \sigma_i \otimes \mathscr{S}(\tau_i), \tag{5}$$

where $\sigma_i \in B(\mathscr{H}_k)$ and $\tau_i \in B(\mathscr{H}_n)$. The map $\mathscr{S}$ is k-positive when $\mathrm{id}_k \otimes \mathscr{S}$ is positive. The map $\mathscr{S}$ is completely positive when $\mathscr{S}$ is k-positive for all $k = 1, 2, \ldots$ Following Lindblad [8], the set of physical operations on a density matrix $\rho \in B(\mathscr{H}_n)^+$

is given by the set of completely positive trace-preserving maps $\mathscr{S}:B(\mathscr{H}_n)\to B(\mathscr{H}_m)$. Similarly as k-positive, one can define a k-copositive map. Let $T:B(\mathscr{H}_n)\to B(\mathscr{H}_n)$ be defined as matrix transposition in a chosen basis for $\mathscr{H}_n$, i.e.

$$(T(A))_{ij} = A_{ji} \tag{6}$$

on a matrix $A \in B(\mathscr{H}_n)$. The map $\mathscr{S}$ is k-copositive when $\mathrm{id}_k \otimes [\mathscr{S} \circ T]$ is positive. A positive linear map $\mathscr{S} : B(\mathscr{H}_n) \to B(\mathscr{H}_m)$ is decomposable if it can be written as

$$\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2 \circ T, \tag{7}$$

where

$$\mathscr{S}_1 : B(\mathscr{H}_n) \to B(\mathscr{H}_m) \quad \text{and} \quad \mathscr{S}_2 : B(\mathscr{H}_n) \to B(\mathscr{H}_m)$$

are completely positive maps and T is matrix transposition relative to some basis. It has been shown by Woronowicz [9] that all positive linear maps $\mathscr{S} : B(\mathscr{H}_2) \to B(\mathscr{H}_2)$ and $\mathscr{S} : B(\mathscr{H}_2) \to B(\mathscr{H}_3)$ are decomposable.

Definition 1. Let ρ be a density matrix on a finite-dimensional Hilbert space $\mathscr{H}_A \otimes \mathscr{H}_B$. A state $|\psi\rangle$ of the form $|\psi^A\rangle \otimes |\psi^B\rangle$ is a (pure) product state in $\mathscr{H}_A \otimes \mathscr{H}_B$. The density matrix ρ is entangled iff ρ cannot be written as a nonnegative combination of pure product states, i.e. there does not exist an ensemble $\{p_i \geqslant 0, |\psi_i^A \otimes \psi_i^B\rangle\}$ such that

$$\rho = \sum_i p_i \, |\psi_i^A\rangle\langle\psi_i^A| \otimes |\psi_i^B\rangle\langle\psi_i^B|. \tag{8}$$

When ρ is not entangled, then the density matrix ρ is called separable.

The problem of deciding whether a bipartite density matrix ρ on $\mathscr{H}_A \otimes \mathscr{H}_B$ is entangled can be quite hard. The following theorem by M., P. and R. Horodecki [2] formulates a necessary and sufficient condition for a density matrix ρ to be entangled:

Theorem 1 (Horodecki). *A density matrix ρ on $\mathscr{H}_A \otimes \mathscr{H}_B$ is entangled iff there exists a positive linear map $\mathscr{S} : \mathscr{H}_B \to \mathscr{H}_A$ such that*

$$(\mathrm{id}_A \otimes \mathscr{S})\,(\rho) \tag{9}$$

is not positive semidefinite. Here id_A *denotes the identity map on $B(\mathscr{H}_A)$.*

Remark. An equivalent statement as Theorem 1 holds for positive linear maps $\mathscr{S} : \mathscr{H}_A \to \mathscr{H}_B$ and the positivity of $\mathscr{S} \otimes \mathrm{id}_B$.

The consequences of Theorem 1 and Woronowicz' result are that a bipartite density matrix ρ on $\mathscr{H}_2 \otimes \mathscr{H}_2$ and $\mathscr{H}_2 \otimes \mathscr{H}_3$ is entangled iff $(\mathrm{id}_A \otimes [\mathscr{S}_1 + \mathscr{S}_2 \circ T])\,(\rho)$ is not positive semidefinite for some $\mathscr{S}_1$ and $\mathscr{S}_2$. Since $\mathscr{S}_1$ and $\mathscr{S}_2$ are completely positive maps this is equivalent to saying that $(\mathrm{id}_A \otimes T)(\rho)$ is not positive semidefinite.

The more complicated structure of the positive linear maps in higher-dimensional matrix algebras, namely the existence of indecomposable positive maps, is reflected in the existence of entangled density matrices ρ on $\mathscr{H}_A \otimes \mathscr{H}_B$ for which $(\mathrm{id}_A \otimes T)(\rho)$ *is* positive semidefinite.

The first example of such a density matrix on $\mathscr{H}_2 \otimes \mathscr{H}_4$ and $\mathscr{H}_3 \otimes \mathscr{H}_3$ was found by P. Horodecki [10]. In [5] a method was discovered to construct entangled density matrices ρ with positive semidefinite $(\mathrm{id}_A \otimes T)(\rho)$ in various dimensions $\dim \mathscr{H}_A > 2$ and $\dim \mathscr{H}_B > 2$. The construction was based on the notion of an unextendible product basis. Let us give the definition.

Definition 2. Let $\mathscr{H}$ be a finite-dimensional Hilbert space of the form $\mathscr{H}_A \otimes \mathscr{H}_B$. A partial product basis is a set S of mutually orthonormal pure product states spanning a proper subspace $\mathscr{H}_S$ of $\mathscr{H}$. An unextendible product basis is a partial product basis whose complementary subspace $\mathscr{H}_S^{\perp}$ contains no product state.

Remark. This definition can be extended to product bases in $\mathscr{H} = \bigotimes_{i=1}^{m} \mathscr{H}_i$ with arbitrary m. Note that we restrict ourselves to orthonormal sets S.

With this notion we can construct the following density matrix:

Theorem 2 [5]. *Let S be a bipartite unextendible product basis $\{|\alpha_i\rangle \otimes |\beta_i\rangle\}_{i=1}^{|S|}$ in $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$. We define a density matrix ρ as*

$$\rho = \frac{1}{\dim \mathscr{H} - |S|} \left(\mathrm{id}_{AB} - \sum_i |\alpha_i\rangle\langle\alpha_i| \otimes |\beta_i\rangle\langle\beta_i| \right), \tag{10}$$

where id_{AB} is the identity operator on $\mathscr{H}$. The density matrix ρ is entangled. Furthermore, the state $(\mathrm{id}_A \otimes [\mathscr{S}_1 + T \circ \mathscr{S}_2])(\rho) \geqslant 0$ for all completely positive maps $\mathscr{S}_1$ and $\mathscr{S}_2$.

Proof. The density matrix ρ is proportional to the projector on the complementary subspace $\mathscr{H}_S^{\perp}$. Since S is unextendible, the subspace $\mathscr{H}_S^{\perp}$ contains no product states. Therefore the density matrix is entangled. It is not hard to see that $(\mathrm{id}_A \otimes T)(\rho)$ is positive semidefinite. It has been proved in [11] that when $(\mathrm{id}_A \otimes T)(\rho)$ is positive semidefinite then $(\mathrm{id}_A \otimes [T \circ \mathscr{S}_2])(\rho) \geqslant 0$, where $\mathscr{S}_2$ can be any completely positive map. Therefore, $(\mathrm{id}_A \otimes [\mathscr{S}_1 + T \circ \mathscr{S}_2])(\rho) \geqslant 0$ for all completely positive maps $\mathscr{S}_1$ and $\mathscr{S}_2$. $\square$

We are now ready to present our results relating these density matrices obtained from the construction in Theorem 2 to indecomposable positive linear maps. We will need the definition of a maximally entangled pure state:

Definition 3. Let $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$. Let $|\psi\rangle$ be a normalized state in $\mathscr{H}$ and

$$\rho_{A,\psi} = \mathrm{Tr}_B\, |\psi\rangle\langle\psi|, \tag{11}$$

where Tr_B indicates that the trace is taken with respect to Hilbert space $\mathscr{H}_B$ only. The state $|\psi\rangle \in \mathscr{H}$ is maximally entangled when

$$S(\rho_{A,\psi}) = -\mathrm{Tr}\, \rho_{A,\psi} \log_2 \rho_{A,\psi} = \log_2 \min(\dim \mathscr{H}_A, \dim \mathscr{H}_B). \tag{12}$$

The function $S(\rho_{A,\psi})$ is the von Neumann entropy of the density matrix $\rho_{A,\psi}$.

Remark. For pure states $|\psi\rangle$ the von Neumann entropy of $\rho_{A,\psi}$ is always less than or equal to $d \equiv \log_2 \min(\dim \mathscr{H}_A, \dim \mathscr{H}_B)$. For maximally entangled states we will have $\rho_{A,\psi} = \mathrm{diag}(1/d, \ldots, 1/d, 0, \ldots, 0)$ so that the maximum von Neumann entropy, Eq. (12), is achieved. When $\dim \mathscr{H}_A = \dim \mathscr{H}_B$ one can always make an orthonormal basis for $\mathscr{H}$ with maximally entangled states [12].

The following lemma bounds the innerproduct between a maximally entangled state and any product state.

Lemma 1. *Let $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$. Let $|\Psi\rangle \in \mathscr{H}$ be a maximally entangled state. Let $d = \min(\dim \mathscr{H}_A, \dim \mathscr{H}_B)$. For all (normalized) product states $|\phi_A\rangle \otimes |\phi_B\rangle$,*

$$|\langle\Psi|\phi_A\rangle \otimes |\phi_B\rangle|^2 \leqslant \frac{1}{d}. \tag{13}$$

Proof. We write the maximally entangled state $|\Psi\rangle$ in the Schmidt polar form [13] as

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |a_i\rangle \otimes |b_i\rangle, \tag{14}$$

where $\langle a_i|a_j\rangle = \delta_{ij}$ and $\langle b_i|b_j\rangle = \delta_{ij}$. Thus we can write

$$|\langle\Psi|\phi_A\rangle \otimes |\phi_B\rangle|^2 = \frac{1}{d} \left| \sum_{i=1}^{d} \langle\phi_A|a_i\rangle\langle\phi_B|b_i\rangle \right|^2 \leqslant \frac{1}{d}, \tag{15}$$

using the Schwarz inequality and

$$\sum_{i=1}^{d} |\langle\phi_A|a_i\rangle|^2 \leqslant 1 \quad \text{and} \quad \sum_{i=1}^{d} |\langle\phi_B|b_i\rangle|^2 \leqslant 1. \qquad \square$$

We will also need the following lemma:

Lemma 2. *Let S be an unextendible product basis $\{|\alpha_i\rangle \otimes |\beta_i\rangle\}_{i=1}^{|S|}$ in $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$. Let*

$$f(|\phi_A\rangle, |\phi_B\rangle) = \sum_{i=1}^{|S|} |\langle\phi_A|\alpha_i\rangle|^2 |\langle\phi_B|\beta_i\rangle|^2. \tag{16}$$

The minimum of f over all pure states $|\phi_A\rangle \in \mathscr{H}_A$ and $|\phi_B\rangle \in \mathscr{H}_B$ exists and is strictly larger than 0.

Proof. The set of all pure product states $|\phi_A\rangle \otimes |\phi_B\rangle$ on $\mathscr{H}$ is a compact set. The function f is a continuous function on this set. Therefore, if there exists a set of states $|\phi_A\rangle \otimes |\phi_B\rangle$ for which f is arbitrarily small then there would also exist a pair $|\phi'_A\rangle \otimes |\phi'_B\rangle$ for which $f = 0$. This contradicts the fact that S is an unextendible product basis. $\square$

The following two theorems contain the main result of this paper.

Theorem 3. *Let S be an unextendible product basis $\{|\alpha_i\rangle \otimes |\beta_i\rangle\}_{i=1}^{|S|}$ in $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$. Let ρ be the density matrix*

$$\rho = \frac{1}{\dim \mathscr{H} - |S|} \left(\mathrm{id}_{AB} - \sum_{i=1}^{|S|} |\alpha_i\rangle\langle\alpha_i| \otimes |\beta_i\rangle\langle\beta_i| \right). \tag{17}$$

Let $d = \min(\dim \mathscr{H}_A, \dim \mathscr{H}_B)$. Let H be a Hermitian operator given by

$$H = \sum_{i=1}^{|S|} |\alpha_i\rangle\langle\alpha_i| \otimes |\beta_i\rangle\langle\beta_i| - \mathrm{d}\epsilon |\Psi\rangle\langle\Psi|, \tag{18}$$

where $|\Psi\rangle$ is a maximally entangled state such that

$$\langle\Psi|\, \rho\, |\Psi\rangle > 0 \tag{19}$$

and

$$\epsilon = \min_{|\phi_A\rangle\otimes|\phi_B\rangle} \sum_{i=1}^{|S|} |\langle\phi_A|\alpha_i\rangle|^2 |\langle\phi_B|\beta_i\rangle|^2, \tag{20}$$

where the minimum is taken over all pure states $|\phi_A\rangle \in \mathscr{H}_A$ and $|\phi_B\rangle \in \mathscr{H}_B$. For any unextendible product basis S it is possible to find a maximally entangled state $|\Psi\rangle$ such that Eq. (19) *holds. The operator H has the following properties*:

$$\mathrm{Tr}\, H\, \rho < 0, \tag{21}$$

and for all product states $|\phi_A\rangle \otimes |\phi_B\rangle \in \mathscr{H}$,

$$\mathrm{Tr}\, H\, |\phi_A\rangle\langle\phi_A| \otimes |\phi_B\rangle\langle\phi_B| \geqslant 0. \tag{22}$$

Proof. Eq. (22) follows from the definition of ϵ, Eq. (20), and Lemma 1. Consider Eq. (21). As the density matrix ρ is proportional to the projector on $\mathscr{H}_S^{\perp}$, one has

$$\mathrm{Tr}\, H\, \rho = -\mathrm{d}\epsilon\, \langle\Psi|\, \rho\, |\Psi\rangle, \tag{23}$$

which is strictly smaller than zero by Lemma 2 and the choice of the maximally entangled state, Eq. (19). When $\dim \mathscr{H}_A = \dim \mathscr{H}_B$, there exist bases of maximally entangled states and thus there will be a basis vector $|\Psi\rangle$ for which $\langle\Psi|\, \rho\, |\Psi\rangle$

is nonzero. In case, say, $\dim \mathscr{H}_A > \dim \mathscr{H}_B$, the maximally entangled states form bases of subspaces $\mathscr{H}' = \mathscr{H}'_A \otimes \mathscr{H}_B$ with $\mathscr{H}'_A \subset \mathscr{H}_A$ and $\dim \mathscr{H}'_A = \dim \mathscr{H}_B$. This completes the proof. □

Theorem 4. *Let S be an unextendible product basis* $\{|\alpha_i\rangle \otimes |\beta_i\rangle\}_{i=1}^{|S|}$ *in* $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$. *Let H be defined as in Theorem* 3, *Eq.* (18). *Choose an orthonormal basis* $\{|i\rangle\}_{i=1}^{\dim \mathscr{H}_A}$ *for* $\mathscr{H}_A$. *Let* $\mathscr{S} : B(\mathscr{H}_A) \to B(\mathscr{H}_B)$ *be a linear map defined by*

$$\mathscr{S}(|i\rangle\langle j|) = \langle i|\, H\, |j\rangle. \tag{24}$$

Then $\mathscr{S}$ *is positive*, *but not completely positive. The map* $\mathscr{S}$ *is indecomposable.*

Proof. The relation between $\mathscr{S}$ and H, Eq. (24), follows from the isomorphism between Hermitian operators on $\mathscr{H}_A \otimes \mathscr{H}_B$ with the property of Eq. (22) and linear positive maps, see [2,14]. In particular, iff a Hermitian H operator on $\mathscr{H}_A \otimes \mathscr{H}_B$ has the property of Eq. (22), then the linear map $\mathscr{S} : B(\mathscr{H}_A) \to B(\mathscr{H}_B)$ defined by

$$H = (\mathrm{id}_A \otimes \mathscr{S})(|\Psi^+\rangle\langle\Psi^+|), \tag{25}$$

where $|\Psi^+\rangle$ is equal to the (unnormalized) maximally entangled state $\sum_{i=1}^{\dim \mathscr{H}_A} |i\rangle \otimes |i\rangle$, is positive for any choice of the orthonormal basis $\{|i\rangle\}_{i=1}^{\dim \mathscr{H}_A}$.

We will show how the density matrix ρ derived from the unextendible product basis, Eq. (17), shows that $\mathscr{S}$ is not completely positive. At the same time we prove that the assumption that $\mathscr{S}$ is decomposable leads to a contradiction. Note that Eq. (24) is equivalent to Eq. (25).

Let $\mathscr{S}^* : B(\mathscr{H}_B) \to B(\mathscr{H}_A)$ be the Hermitian conjugate of $\mathscr{S}$. We use the definition of the map $\mathscr{S}^*$

$$\mathrm{Tr}\, \mathscr{S}^*(A^*)\, B = \mathrm{Tr} A^*\, \mathscr{S}(B) \tag{26}$$

and Eq. (25) to derive that Eq. (21) can be rewritten as

$$\mathrm{Tr}\, H\, \rho = \langle\Psi^+| \left(\mathrm{id}_A \otimes \mathscr{S}^*\right)(\rho)|\Psi^+\rangle < 0, \tag{27}$$

Thus $\mathscr{S}^*$ cannot be completely positive and therefore $\mathscr{S}$ itself is not completely positive. If $\mathscr{S}$ were decomposable, then $\mathscr{S}^*$ would be of the form $\mathscr{S}_1 + T \circ \mathscr{S}_2$, where $\mathscr{S}_1$ and $\mathscr{S}_2$ are completely positive maps. The density matrix ρ is positive semidefinite under any linear map of the form $\mathscr{S}_1 + T \circ \mathscr{S}_2$ by Theorem 2. This is in contradiction with Eq. (27) and therefore $\mathscr{S}$ cannot be decomposable. □

We will now show how one can determine a lower bound on the value of ϵ, Eq. (20). Note that when we determine a lower bound $\epsilon \geqslant \epsilon_{\max}$, then all operators H, as in Eq. (18) of the form

$$H = \sum_{i=1}^{|S|} |\alpha_i\rangle\langle\alpha_i| \otimes |\beta_i\rangle\langle\beta_i| - \mathrm{d}\mu|\Psi\rangle\langle\Psi|, \tag{28}$$

where $\mu \in (0, \epsilon_{\max}]$, correspond to positive maps.

Let $\{|\alpha_i\rangle \otimes |\beta_i\rangle\}_{i=1}^{|S|}$ be an unextendible product basis in $\mathscr{H}_A \otimes \mathscr{H}_B$ with $d_A = \dim \mathscr{H}_A$ and $d_B = \dim \mathscr{H}_B$. Let $S_A = \{|\alpha_i\rangle\}_{i=1}^{|S|}$ and $S_B = \{|\beta_i\rangle\}_{i=1}^{|S|}$. We pick a vector $|\phi_A\rangle$ and order the innerproducts $|\langle\alpha_i|\phi_A\rangle|^2$ in an increasing sequence; let us call them $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{|S|}$. Then we select vectors $|\alpha_i\rangle$ corresponding to the smallest innerproducts in this sequence up to the point where the set of selected vectors $|\alpha_i\rangle$ spans the full d_A-dimensional Hilbert space $\mathscr{H}_A$. Let us call this set $S_A^P \in S_A$. If we would take away anyone state from S_A^P, the remaining vectors would no longer span $\mathscr{H}_A$. As the vectors in the set S_A^P span $\mathscr{H}_A$, it must be that $x_{|S_A^P|} > 0$. Let us label this corresponding vector as $|\alpha_{i_{\max}}\rangle$, i.e. $x_{|S_A^P|} = |\langle\alpha_{i_{\max}}|\phi_A\rangle|^2$. We now construct a subset of S_B in the following way; we define $S_B^P = \{|\beta_i\rangle \mid |\alpha_i\rangle \notin S_A^P\} \cup \{|\beta_{i_{\max}}\rangle\}$. We note that the vectors in the set S_B^P span the Hilbert space $\mathscr{H}_B$; if not, then there would exist a vector $|\phi_B\rangle$ which is orthogonal to all vectors in S_B^P *and* a vector $|\phi_A\rangle$ which is orthogonal to all vectors in $S_A^P \setminus \{|\alpha_{i_{\max}}\rangle\}$, which would in turn imply that $\epsilon = 0$, in other words, the set S would be extendible. Let us pick a vector $|\phi_B\rangle$ and denote the innerproducts $|\langle\beta_i|\phi_B\rangle|^2$ with $|\beta_i\rangle \in S_B^P$ as $y_1 \leqslant y_2 \leqslant \cdots \leqslant y_{|S_B^P|}$. As the vectors in S_B^P span $\mathscr{H}_B$, we know that $y_{|S_B^P|} > 0$ for any state $|\phi_B\rangle$. This implies that for a particular choice of $|\phi_A\rangle$ and $|\phi_B\rangle$ we can bound

$$\sum_i |\langle\alpha_i|\phi_A\rangle|^2 |\langle\beta_i|\phi_B\rangle|^2 \geqslant x_{|S_A^P|}\, y_{|S_B^P|}, \tag{29}$$

the product of the two largest innerproducts of the vectors $|\phi_A\rangle$ and $|\phi_B\rangle$ with the vectors from S_A^P and S_B^P, respectively. Therefore ϵ itself, Eq. (20), can be bounded as

$$\epsilon \geqslant \min_{|\phi_A\rangle \to S_A^P, |\phi_B\rangle \to S_B^P} x_{|S_A^P|} y_{|S_B^P|}, \tag{30}$$

where $x_{|S_A^P|}$ denotes the largest innerproduct between $|\phi_A\rangle$ and a state in the set S_A^P and similarly for $y_{|S_B^P|}$. We minimize over $|\phi_A\rangle \to S_A^P$ and $|\phi_B\rangle \to S_B^P$, where the arrow denotes that a state $|\phi_A\rangle$ gives rise to a set S_A^P as in the construction given above. A set S_A^P (and similarly S_B^P) might not be uniquely defined given the vector $|\phi_A\rangle$, for example when several innerproducts of the state $|\phi_A\rangle$ with states $|\alpha_i\rangle$ are identical. Since the lowerbound given in Eq. (29) works for all sets S_A^P and S_B^P which are constructed with the method given above, we could do an extra maximization over S_A^P and S_B^P, given the states $|\phi_A\rangle$ and $|\phi_B\rangle$, but for the sake of clarity this maximization is omitted in Eq. (30).

We have the following proposition that can be used to bound $x_{|\mathrm{S}_A^P|}$ and $y_{|S_B^P|}$ given the sets S_A^P and S_B^P:

Proposition 1. *Let $\{|\psi_i\rangle\}_{i=1}^n$ be a set of n vectors in $\mathscr{H}$ such that the set $\{|\psi_i\rangle\}_{i=1}^n$ spans the Hilbert space $\mathscr{H}$. Then for any vector $|\phi\rangle$ we have*

$$n \max_i |\langle\phi|\psi_i\rangle|^2 \geqslant \sum_i |\langle\phi|\psi_i\rangle|^2 \geqslant \lambda_{\min}, \tag{31}$$

where $\lambda_{\min}$ *is the smallest eigenvalue of the Hermitian matrix* $P = \sum_i |\psi_i\rangle\langle\psi_i|$.

Using Proposition 1, we get the following:

$$\epsilon \geqslant \min_{S_B^P, S_B^P} \frac{\lambda_{\min, S_A^P}}{|S_A^P|} \frac{\lambda_{\min, S_B^P}}{|S_B^P|} \equiv \epsilon_{\max}. \tag{32}$$

In order to carry out this calculation, we first find all minimal 'full rank' subsets S_A^P of S_A. Then for each of these sets S_A^P we compute the smallest eigenvalue of $\sum_{i \in S_A^P} |\alpha_i\rangle\langle\alpha_i|$. Also for each set S_A^P, we construct complementary sets S_B^P which contain all the vectors $|\beta_i\rangle$ such that $|\alpha_i\rangle \notin S_A^P$ and a single state $|\beta_i\rangle$ such $|\alpha_i\rangle \in S_A^P$. For each set S_A^P there will be $|S_A^P|$ of such sets S_B^P. Then for each S_B^P we compute the smallest eigenvalue of $\sum_{i \in S_B^P} |\beta_i\rangle\langle\beta_i|$. Then we can take the minimum over all these values to obtain a bound on ϵ. Note that this is now a minimization over a discrete number of values. If the set S has few symmetries and is defined in a high-dimensional space, the procedure will be cumbersome. In small dimensions or for unextendible product bases which do have many symmetries, the calculation will be much simpler. In Section 3 we present two examples of positive maps based on the construction in Theorem 4 and for one of them we will explicitly compute a lower bound on ϵ.

3. Examples and discussion

As we have shown the structure of unextendible product bases carries over to indecomposable positive linear maps. In this section we will list some of the results that have been obtained about unextendible product bases. We will take two examples of unextendible product bases and demonstrate the construction of Theorems 3 and 4.

1. In [5] it was shown that there exist no unextendible product bases in $\mathcal{H}_2 \otimes \mathcal{H}_n$ for any $n \geqslant 2$.
2. In [6] it was shown how to parametrize *all* possible unextendible product bases in $\mathcal{H}_3 \otimes \mathcal{H}_3$ as a six-parameter family.
3. In [6] a family of unextendible product bases, based on quadratic residues, in $\mathcal{H}_n \otimes \mathcal{H}_n$, where n is any odd number and $2n-1$ is a prime of the form $4m+1$ has been found.
4. In [6] a family of unextendible product bases $\mathcal{H}_n \otimes \mathcal{H}_m$ ($m > 2, n > 2$) for arbitary $m \neq n$ as well as even $n = m$ has been conjectured. The conjecture was proved in $\mathcal{H}_3 \otimes \mathcal{H}_n$ and $\mathcal{H}_4 \otimes \mathcal{H}_4$ (The full conjecture (arbitrary n and m) has recently been proved by Terhal and DiVincenzo and will be presented in a forthcoming paper.)

5. In [6] it was shown that when S_1 and S_2 are unextendible product bases on $\mathscr{H}_A^1 \otimes \mathscr{H}_B^1$ and $\mathscr{H}_A^2 \otimes \mathscr{H}_B^2$, respectively, then the tensor product of the two sets, $S_1 \otimes S_2$, is again an unextendible product bases on $(\mathscr{H}_A^1 \otimes \mathscr{H}_A^2) \otimes (\mathscr{H}_B^1 \otimes \mathscr{H}_B^2)$.

Example 1. One of the first examples of an unextendible product basis in $\mathscr{H}_3 \otimes \mathscr{H}_3$ was the following set of states [5]. Consider five vectors in real three-dimensional space forming the apex of a regular pentagonal pyramid, the height h of the pyramid being chosen such that nonadjacent apex vectors are orthogonal. The vectors are

$$|v_i\rangle = N\left(\cos\frac{2\pi\mathrm{i}}{5}, \sin\frac{2\pi\mathrm{i}}{5}, h\right), \quad i = 0, \ldots, 4 \tag{33}$$

with

$$h = \frac{1}{2}\sqrt{1+\sqrt{5}} \quad \text{and} \quad N = 2/\sqrt{5+\sqrt{5}}.$$

Then the following five states in $\mathscr{H}_3 \otimes \mathscr{H}_3$ form an unextendible product basis:

$$|p_i\rangle = |v_i\rangle \otimes |v_{2i \bmod 5}\rangle, \quad i = 0, \ldots, 4. \tag{34}$$

Let ρ be the entangled state derived from this unextendible product basis as in Eq. (10) Theorem 2. We choose a maximally entangled state $|\Psi\rangle$, here named $|\Psi^+\rangle$,

$$|\Psi^+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle). \tag{35}$$

One can easily compute that

$$\langle\Psi^+|\,\rho\,|\Psi^+\rangle = \frac{1}{4}\left(1 - \frac{7+\sqrt{5}}{3(3+\sqrt{5})}\right) > 0. \tag{36}$$

Let us now compute a lower bound on ϵ, as in Eq. (32). Due to the high symmetry of this set of states, we will only need to compute the minimum eigenvalue of the Hermitian matrix

$$P_1 = |v_0\rangle\langle v_0| + |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|$$

and

$$P_2 = |v_0\rangle\langle v_0| + |v_1\rangle\langle v_1| + |v_3\rangle\langle v_3|,$$

all other subsets of three vectors, either on Bob's or Alice's side, correspond to matrices with the same eigenvalues as P_1 or P_2. Easy computation shows that P_1 has the smallest eigenvalue which is equal to

$$\lambda_{\min} = \frac{2+\sqrt{2}-\sqrt{10}}{2}. \tag{37}$$

Then as the states on Bob's side are identical, we get

$$\epsilon \geqslant \frac{\lambda_{\min}^2}{9} = \frac{4+\sqrt{2}-\sqrt{5}-\sqrt{10}}{9}. \tag{38}$$

The map $\mathscr{S}$ as defined in Eq. (24) Theorem 4, follows directly:

$$\mathscr{S}(|i\rangle\langle j|) = \sum_{k=0}^{4} \langle i|v_k\rangle\langle v_k|j\rangle |v_{2k \bmod 5}\rangle\langle v_{2k \bmod 5}| - \mu|i\rangle\langle j|, \tag{39}$$

where

$$\mu \in \left(0, \frac{4+\sqrt{2}-\sqrt{5}-\sqrt{10}}{9}\right]. \tag{40}$$

A positive linear map $\mathscr{S} : B(\mathscr{H}_n) \to B(\mathscr{H}_m)$ is unital if $\mathscr{S}(\mathrm{id}_n) = \mathrm{id}_m$. We will demonstrate that $\mathscr{S}$ is not unital. One can write

$$\mathscr{S}(\mathrm{id}_A) = \mathrm{Tr}_A\, H = \sum_{k=0}^{4} |v_{2k \bmod 5}\rangle\langle v_{2k \bmod 5}| - 3\mu\ \mathrm{Tr}_A|\Psi^+\rangle\langle\Psi^+|, \tag{41}$$

which in turn is equal to

$$\mathscr{S}(\mathrm{id}_A) = \mathrm{diag}\left(\frac{10}{5+\sqrt{5}}, \frac{10}{5+\sqrt{5}}, \sqrt{5}\right) - \mu\, \mathrm{id}_B. \tag{42}$$

The next example is based on a more general unextendible product basis that was presented in [6].

Example 2. The states of S in $\mathscr{H}_3 \otimes \mathscr{H}_n$ are:

$$|F_k^0\rangle = \frac{1}{\sqrt{n-2}}|0\rangle \otimes \left(|1\rangle + \sum_{l=3}^{n-1} \omega^{k(l-2)}|l\rangle\right), \quad 1 \leqslant k \leqslant n-3, \tag{43}$$

$$|F_k^1\rangle = \frac{1}{\sqrt{n-2}}|1\rangle \otimes \left(|2\rangle + \sum_{l=3}^{n-1} \omega^{k(l-2)}|l\rangle\right), \quad 1 \leqslant k \leqslant n-3, \tag{44}$$

$$|F_k^2\rangle = \frac{1}{\sqrt{n-2}}|2\rangle \otimes \left(|0\rangle + \sum_{l=3}^{n-1} \omega^{k(l-2)}|l\rangle\right), \quad 1 \leqslant k \leqslant n-3, \tag{45}$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |0\rangle, \tag{46}$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \otimes |1\rangle, \tag{47}$$

$$|\psi_5\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |0\rangle) \otimes |2\rangle, \tag{48}$$

$$|\psi_6\rangle = \frac{1}{\sqrt{3n}} \sum_{i=0}^{2} \sum_{j=0}^{n-1} |i\rangle \otimes |j\rangle \tag{49}$$

and we have $\omega = \exp(2\pi \mathrm{i}/(n-2))$. Here the states $\{|k\rangle\}_{k=0}^{n-1}$ form an orthonormal basis. In total there are $3n-5$ states in the basis. We choose a maximally entangled state, again we take $|\Psi^+\rangle$, Eq. (35). One can show that

$$\langle\Psi^+|\,\rho\,|\Psi^+\rangle = \frac{1}{5}\left(\frac{1}{2} - \frac{1}{3n}\right) > 0. \tag{50}$$

The map $\mathscr{S} : B(\mathscr{H}_3) \to B(\mathscr{H}_n)$ is given as

$$\mathscr{S}(|i\rangle\langle j|) = \sum_{k=1}^{n-3}\sum_{p=0}^{2}\langle i|F_k^p\rangle\langle F_k^p|j\rangle + \sum_{p=3}^{6}\langle i|\psi_p\rangle\langle\psi_p|j\rangle - \epsilon\,|i\rangle\langle j|. \tag{51}$$

The following questions concerning the positive maps that were introduced in this paper are left open.

1. Is $\mathscr{S}$ always nonunital? We conjecture it is. As we showed, see Eq. (41), the answer to this question depends on whether

$$\sum_{i=1}^{|S|} |\beta_i\rangle\langle\beta_i| \propto \mathrm{id}_B, \tag{52}$$

where the set of states $\{|\beta_i\rangle\}_{i=1}^{|S|}$ is one side of the unextendible product basis. The states $|\beta_i\rangle$ will span $\mathscr{H}_B$ but they will not be all orthogonal, nor all nonorthogonal.
2. It was shown in Theorem 4 that the new indecomposable positive linear maps $\mathscr{S} : B(\mathscr{H}_m) \to B(\mathscr{H}_n)$ are not m-positive, as they are not completely positive. Are these maps $\mathscr{S}$ k-positive with $1 < k < m$? The answer to this question will rely on a better understanding of the structure of unextendible product bases.
3. In [5] a single example was given of a entangled density matrix on $\mathscr{H}_3 \otimes \mathscr{H}_4$, which stayed positive semidefinite under the action of $\mathrm{id}_3 \otimes T$. The density matrix was based, not on an unextendible product basis, but on a 'strongly uncompletable' product basis S. It could be shown that the Hilbert space $\mathscr{H}_S^\perp$ had a product state *deficit*, i.e. the number of product states in $\mathscr{H}_S^\perp$ was less than $\dim \mathscr{H}_S^\perp$. It is a open question on how to generalize this example and whether these kinds of density matrices will give rise to more general family of indecomposable positive linear maps, see [15,16] for progress in this direction.

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