Maximality and Totality of Stable Functions in the Category of Stable Bifinite Domains

YIXIANG CHEN
College of Mathematics, Physics and Informatics
Shanghai Normal University
100 Guilin Road, Shanghai 200234, P.R. China
ychen@shnu.edu.cn

GUO-QIANG ZHANG
Department of Electrical Engineering and Computer Science
Case Western Reserve University
10900 Euclid Avenue, Cleveland, OH 44106, U.S.A.
gqz@eecs.cwru.edu

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Abstract—This paper studies maximality and totality of stable functions in the category of stable bifinite domains. We present three main results as follows,

1. every maximum-preserving function is a maximal element in the stable function spaces;
2. a maximal stable function \( f : D \rightarrow E \) is maximum-preserving if \( D \) is maximum-separable and \( E \) is completely separable; and
3. a stable bifinite domain \( D \) is maximum-separable if and only if for any locally distributive stable bifinite domain \( E \), each maximal stable function \( f : D \rightarrow E \) is maximum-preserving.

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1. INTRODUCTION

Within semantic frameworks for programming languages, interesting connections between the operational notion of totality and the order-theoretic notion of maximality have been studied, such as the notion of totality using games [1], totality spaces [2], topological notions of totality [3], and the relationship between maximality and termination [4]. Much of this work is inspired by Girard’s basic notion of totality in the category of coherent spaces [2,5], a subcategory of \( \mathcal{D} \)-domains.

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The most relevant previous development related to the current paper is [6], in which Zhang provided a quite thorough connection between totality defined as maximal points and maximality defined by the Berry order in the category of dI-domains. The subtle differences between the two notions can be appreciated by observing the fact that a maximal element in stable function spaces (i.e., a maximal stable function under the Berry order) need not map maximals (i.e., total elements in the domain) to maximals (i.e., total elements in the codomain). The main results of [6] are, within dI-domains,

1. totality implies maximality but not vice versa;
2. maximum-separability for the domain is a necessary and sufficient condition for the two notions to coincide.

This paper generalizes the results of Zhang [6] to the category of stable bifinite domains. This more general category is interesting because it occupies a central place in the stable semantics landscape: it has been conjectured to be the largest cartesian closed category within the ambient category of ω-algebraic meet-cpos with stable functions as morphisms [7-10]. More recent results in [11] reinforce this belief. The lack of distributivity and the lack of consistent completeness are important technical challenges not present in dI-domains. New proof strategies and refined notions of maximum-separability are needed in this paper in order to overcome these challenges.

The rest of the paper is organized as follows. Background and notations for stable bifinite domains are briefly reviewed in Section 2. In each of the subsequent sections, the three main results indicated in the abstract are presented. We show in Section 3 that every maximum-preserving function is a maximal element in stable function spaces. We prove in Section 4 that a maximal stable function \( f : D \rightarrow E \) is maximum-preserving if \( D \) is maximum-separable and \( E \) is completely separable. Finally, we show in Section 5 that a stable bifinite domain \( D \) is maximum-separable if and only if for any locally distributive stable bifinite domain \( E \), each maximal stable function \( f : D \rightarrow E \) is maximum-preserving.

### 2. STABLE BIFINITE DOMAINS

The basic property of a conditional multiplicative (cm) function is that it preserves the meet of any pair of compatible elements. Berry distinguishes cm functions from stable functions, those for which least local input information can be found for an achievable target output. For the purpose of this paper, the notions of cm and stable functions are interchangeable, with conditional multiplicativity providing clean equational proofs, and minimal input providing intuition for stability. Sometimes we use the word cm simply to avoid a repetition of the word "stable".

Bounded meets should exist for stability to make sense (item (a) below). Meet should also interact smoothly with the join of any directed set (item (b) below). The Berry order then arises naturally from the minimal requirement that the evaluation map (for cartesian closure) is stable [7].

**Definition 2.1.** Let \( D \) be a dcpo (with bottom). It is called a meet-cpo if

1. for any \( x, y \in D \), \( x \sqcap y \) exists when \( \{x, y\} \) is bounded above (or compatible); and
2. if \( R \subseteq D \) is a directed set and \( x \) is compatible with the join of \( R \), then

\[
x \sqcap (\sqcup R) = \sqcup \{x \sqcap r \mid r \in R\}.
\]

**Definition 2.2.** Let \( D, E \) be meet-cpos. A Scott continuous function \( f : D \rightarrow E \) is called stable if it preserves meets of compatible pairs, i.e.,

\[
\forall x, y \in D, \quad x \uparrow y \Rightarrow f (x \sqcap y) = f (x) \cap f (y).
\]
The stable function space \([D \to E]\) consists of all stable functions from \(D\) to \(E\) under the Berry order: \(f\) is stably less than \(g\), written \(f \sqsubseteq s g\), if
\[
\forall x, y \in D, \quad x \sqsubseteq y \Rightarrow f(x) = f(y) \sqcap g(x).
\]

**NOTATION.** We deal exclusively with stable function spaces (i.e., cm functions under the Berry order) in this paper and use \([D \to E]\) as the default notation for it.

The following result can be found in [8].

**THEOREM 2.1.** (See [8].) The category of meet-cpos with stable functions is a Cartesian closed category (ccc).

We mention below some basic properties of stable functions, the proofs of which can be found in [7,12].

**LEMMA 2.1.** Let \(D, E\) be meet-cpos and \(f, g\) be compatible stable functions in \([D \to E]\). We have
1. if \(f(x) = g(x)\), then \(f(y) = g(y)\) for any \(y \sqsubseteq x\); and
2. if \(a \uparrow b\) then \(f(a) \sqcap g(b) = f(b) \sqcap g(a)\).

This lemma reveals a striking difference between the Berry order and the standard extensional order: if compatible stable functions have the same value at some point, they must be identical on the principal ideal determined by that point. The contrapositive of this observation is also useful.

If \(f(x) = g(x)\) but \(f(y) \not\sqsubseteq g(y)\) for some \(y \sqsubseteq x\), then \(f\) and \(g\) are incompatible with respect to the Berry order.

Meet-cpos need not be \(\omega\)-algebraic. Amadio [8] and Droste [9,10] showed that beyond Scott domains, there is the category of stable bifinite domains which also forms a ccc. However, readers should be aware of the small notational variations of the terminology stable bifinite domains in the literature. For example, [13, Section 12.4] refers to stable bifinite domains without requiring a countable basis while [9] takes this as a prerequisite. In this paper, we follow [9]: by stable bifinite domains, we mean \(\omega\)-algebraic meet-cpos for which the identity functions can be expressed as joins (under the Berry order) of directed sets of stable projections with finite images. However, all our results hold without the countable basis requirement.

**THEOREM 2.2.** (See [8,9].) \(\text{SB}\) is Cartesian closed, where \(\text{SB}\) denotes the category of stable bifinite domains with stable functions as morphisms.

The rest of the paper studies maximality and totality in the background category \(\text{SB}\). Thus, it is important to note that we do not assume distributivity or bounded completeness. We do, however, assume algebraicity and Axiom I (defined below). In fact, deriving similar results as those given in [5,14] in lack of distributivity is the key contribution of this paper.

**DEFINITION 2.3.** An algebraic cpo \(D\) satisfies Axiom I if for each compact element \(q\), the principal ideal \(\downarrow q = \{x \mid x \sqsubseteq q\}\) is finite.

**THEOREM 2.3.** (See [8].) Every stable bifinite domain satisfies Axiom I.

### 3. MAXIMUM-PRESERVING FUNCTIONS

Let \(D\) be a meet-cpo. An element \(m \in D\) is called maximal if there is no element in \(D\) that is bigger than it. We write \(M(D)\) for the set of maximal elements of \(D\). Clearly, distinct maximal elements are incompatible, and each element in \(D\) is dominated by some element in \(M(D)\). The following lemma can be readily checked according to the axiom of choice.
LEMMA 3.1. Let $D$ be a meet-cpo. For any $x \in D$, there is an $m \in M(D)$ such that $x \subseteq m$.

DEFINITION. Let $D$ and $E$ be meet-cpos. A function $f : D \to E$ is called maximum-preserving if $f(M(D)) \subseteq M(E)$, i.e., it maps maximal elements to maximal elements.

THEOREM 3.1. Let $D, E$ be meet-cpos. A stable function $f$ is maximal in $[D \to E]$ if it is a maximum-preserving function.

PROOF. Suppose $f : D \to E$ is maximum-preserving and $f \sqsubseteq g$. Then, $f(m) = g(m)$ for each $m \in M(D)$, since $f(m)$ is a maximal element in $D$. By Lemma 2.1 (or the remark after it), for any $x \sqsubseteq m$ with $m \in M(D)$, $f(x) = g(x)$. By Lemma 3.1, every element in $D$ is below some maximal element. Therefore, $f(x) = g(x)$ holds for each $x \in D$.

COROLLARY 3.1. Every identity is maximal since it is maximum-preserving.

The converse of Theorem 3.1 is not true. We have the following example. Let $N$ be the natural numbers \{0, 1, 2, \ldots, n, \ldots\}, and let $\mathbb{Z}$ the set of \{-20, -1, -2, \ldots, 2, \ldots\}.

Set $D_0 = N \cup \mathbb{Z} \cup \{\bot\}$. Define the order relationship on $D$ as in the following picture.

Then, $D_0$ is a meet-cpo. Let $2$ be the two-point domain \{0, 1\}, with $0 \sqsubseteq 1$. Now, we define the function $f : D_0 \to 2$ by letting $f(N) = \{1\}$, $f(D_0 - N) = \{0\}$. Then, $f$ is a maximal stable function. However, it maps the maximal element $\bot$ to the least element 0 of 2.

4. THE MAXIMUM SEPARABILITY CONDITION

The question becomes: for which meet-cpos $D$ and $E$ does the converse of Theorem 3.1 hold? Zhang [6] introduced the concept of maximum-separable domains for this purpose in the category of $dI$-domains. Later, Chen [14] introduced the notion of strongly maximum-separable (referred to in this paper as “completely separable”) domains in the category of $L$-domains. Without any changes, these definitions can be lifted to meet-cpos and stable bifinite domains.

Let $D$ be a domain, and $m \in M(D)$. Set

$$[m] := \{a \in K(D) \mid (\uparrow a) \cap M(D) = \{m\}\},$$

where $K(D)$ is the set of compact elements of $D$. If $[m] \neq \emptyset$, then $[m]$ is up-closed and directed restricted to compact elements, as well as $m = \bigcup [m]$. Clearly, $[m]$ is directed. To see $m = \bigcup [m]$, let $\downarrow^k x$ stand for the set of all compact elements below $x$, i.e.,

$$\downarrow^k x := \{q \mid q \text{ is a compact element with } q \subseteq x\}.$$

Since $[m] \neq \emptyset$, pick $a \in [m]$. For each $b \in \downarrow^k m$, there is a $c \in \downarrow^k m$ with $a \subseteq c$ and $b \subseteq c$. However, $c \in [m]$, for $a \in [m]$. So, $b \subseteq \bigcup [m]$. This proves $m = \bigcup [m]$. Now, consider the previous example $D_0$. Clearly, $M(D_0) = N \cup \{\bot\}$ and, for each $n \in N$, $[n] = \{n\}$. But, $[\bot] = \emptyset$, since each element dominated by $\bot$ is also dominated by other maximal elements.
**Definition 4.1.** Let $D$ be a meet-cpo.

1. $D$ is maximum-separable if for all maximal element $m$, $[m] \neq \emptyset$. (See [6].)
2. $D$ is completely separable (i.e., satisfying the complete separability condition) if for all maximal element $m$, $[m] \neq \emptyset$ and for any $d \in D$, $d \sqsubseteq m$ implies that there exists an $a \in [m]$ such that $d \sqsubseteq a$. (See [14].)

The example given above is not maximum-separable because the required property for $\perp_\omega$ fails. Obviously, complete separability is stronger than maximal separability, but the reverse is not true in general. The following is a counterexample.

\[
\begin{array}{cccccc}
0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow \cdots \rightarrow n & \rightarrow \cdots \rightarrow \omega \\
\uparrow & \uparrow & \uparrow & \cdots & \uparrow & \cdots & \uparrow \\
\perp_0 & \rightarrow & \perp_1 & \rightarrow & \perp_2 & \rightarrow \cdots & \rightarrow \perp_n & \rightarrow \cdots \rightarrow \perp_\omega
\end{array}
\]

In this example, $\perp_\omega$ violates the second condition in the definition of complete separability.

**Definition 4.2.** Let $D$ be a meet-cpo, $m \in M(D)$. An element in $[m]$ is called a separation points for $m$.

**Notation.** These separation points can be regarded intuitively as "name-tags" or "finite witness" for maximal elements. By the way, if $a$ is a separation point for $m$, then for any element $d \in D$, either $a \not\sqsubseteq d$, or $d \sqsubseteq m$.

The following lemma, which shows that the separation point $a$ has a nice feature, is useful in the proof of the main result (i.e., Theorem 4.1 of this section).

**Lemma 4.1.** Suppose $D$ and $E$ are stable bifinite domains, $f : D \rightarrow E$ is a stable function and $m \in M(D)$. If there is a compact element $p$ which covers $f(m)$, then there is a separation point $a$ for $m$ such that $f(a) = f(m)$. Thus, for any $y$ with $a \not\sqsubseteq y$, $f(y) = f(m)$.

**Proof.** Consider the two sets of compact elements $\downarrow k p$ and $\downarrow k f(m)$. Axiom I implies that $\downarrow k p$ is a finite set. Hence, $\downarrow k f(m)$ is a finite set because $\downarrow k f(m) \subseteq \downarrow k p$.

Now, take an arbitrary $q \in \downarrow k f(m)$. There is an $a \in \downarrow k m$ such that $q \sqsubseteq f(a)$ by the continuity of $f$. Moreover, a least such $a$ exists since $D$ satisfies Axiom I and $f$ is a stable function. This least element is uniquely determined by $q$. It is denoted as $a_q$. Thus, we get an one-to-one function from the set $\downarrow k f(m)$ to $\downarrow k m$. The finiteness of the set $\downarrow k f(m)$ implies that the following set $A$ is finite,

\[
A = \{a_q \mid q \in \downarrow k f(m)\}.
\]

By the algebraicity and maximum-separability of $D$, there is a separation point $a$ for $m$ which dominates every point in the set $A$ defined above. Since $a$ is a separation point of $m$, we have that $f(a) \sqsubseteq f(m)$. Now, we take any compact $q$ with $q \sqsubseteq f(m)$. Then, $a_q \in A$. So, $a_q \sqsubseteq a$ by the choice of $a$. It follows that

\[
q \sqsubseteq f(a_q) \sqsubseteq f(a).
\]

So, $f(a) = f(m)$. 

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![Figure 1](image-url)
THEOREM 4.1. Let $D$ be a maximum-separable stable bifinite domain and $E$ a nontrivial completely separable stable bifinite domain. Then $f \in [D \to E]$ is maximal if and only it is maximum-preserving.

PROOF. By Theorem 3.1, it suffices to show that if $f$ is maximal then it is maximum-preserving.

Suppose there exists a maximal element $m$ of $D$ such that $f(m)$ is not a maximal element of $E$. Then there exists a maximal element $e$ of $E$ with $f(m) \sqsubseteq e$ and a compact element $p \in [e]$ satisfying $f(m) \sqsubseteq p \sqsubseteq e$ by the complete separability of $E$. By Lemma 4.1, there is a separation point $a$ for $m$ such that $f(a) = f(m)$. Now, by using the separation point $a$, we define a function $g$ from $D$ to $E$ by letting

$$g(x) = \begin{cases} p, & a \sqsubseteq x, \\ f(x), & a \not\sqsubseteq x. \end{cases}$$

We prove that $g$ is a stable function from $D$ to $E$.

- Monotonicity. Suppose that $x \sqsubseteq y$. We consider three cases.

  Case 1: $a \sqsubseteq x$. In this case, we have that $g(x) = p \sqsubseteq p = g(y)$.

  Case 2: $a \not\sqsubseteq x$, but $a \not\sqsubseteq y$. Then, $g(x) = f(x)$, and $g(y) = p$. Note that $a \sqsubseteq y$ implies $y \sqsubseteq m$ due to the separability of $a$ for $m$. So, $x \sqsubseteq m$ because $x \sqsubseteq y$. Thus, we have that $g(x) = f(x) \sqsubseteq f(m) \sqsubseteq p = g(y)$.

  Case 3: $a \not\sqsubseteq x$, and $a \not\sqsubseteq y$. Hence, $g(x) = f(x) \sqsubseteq f(y) = g(y)$.

- Scott continuity. Suppose that $A$ is a directed subset of $D$. We consider two cases.

  Case 1: $a \sqsubseteq \sqcup A$. Then, $g(\sqcup A) = p$, whilst $a \sqsubseteq \sqcup A$ implies that there is an $x_0 \in A$ such that $a \sqsubseteq x_0$. So, $g(x_0) = p$. The condition $a \sqsubseteq \sqcup A$ implies also that any member $x \in A$ is covered by $m$. So, for any element $x \in A$, we have that $f(x) \sqsubseteq f(m) \sqsubseteq p$. Thus,

$$\sqcup g(A) = \sqcup \{(g(x) \mid x \in A, a \sqsubseteq x) \cup \{g(x) \mid x \in A, a \not\sqsubseteq x\}\}$$

$$= \sqcup \{p \cup \{f(x) \mid x \in A, a \not\sqsubseteq x\}\}$$

$$= p.$$

So, $\sqcup g(A) = g(\sqcup A)$.

Case 2: $a \not\sqsubseteq \sqcup A$. Then, $a \not\sqsubseteq x$ for all $x \in A$. So, we get that

$$g(\sqcup A) = f(\sqcup A) = \sqcup f(A) = \sqcup g(A).$$

Therefore, $g$ preserves the directed supremum.

- Stability. Let $x$ and $y$ be two compatible elements in $D$. We only need to verify that

$$g(x \sqcap y) = g(x) \sqcap g(y).$$

It is done by considering the following four cases.

Case 1: $a \sqsubseteq x$ and $a \sqsubseteq y$. In this case, we have that $a \sqsubseteq x \sqcap y$ and

$$g(x \sqcap y) = p = g(x) \sqcap g(y).$$

Case 2: $a \sqsubseteq x$, but $a \not\sqsubseteq y$. Then, $a \not\sqsubseteq x \sqcap y$. So,

$$g(x \sqcap y) = f(x \sqcap y) = f(x) \sqcap f(y).$$

Lemma 4.1 tells us that $f(x) = f(m)$, while the compatibility of $x$ and $y$ and $a \sqsubseteq x$ implies that $y \sqsubseteq m$, for $a$ is a separation point of $m$. So, $f(y) \sqsubseteq f(m) \sqsubseteq p$, and, $f(x) \sqcap f(y) = f(y)$. Hence,

$$g(x \sqcap y) = f(y).$$
Meanwhile,

\[ g(x) \cap g(y) = p \cap f(y) = f(y). \]

So,

\[ g(x) \cap y = g(x) \cap g(y). \]

Case 3: \( a \not\subseteq x \), but \( a \subseteq y \). The proof is the same as that in Case 2.

Case 4: \( a \not\subseteq x \) and \( a \not\subseteq y \). Then, \( a \not\subseteq x \cap y \). So,

\[ g(x \cap y) = f(x \cap y) = f(x) \cap f(y) = g(x) \cap g(y). \]

Hence, \( g \) is a stable function from \( D \) to \( E \). By the construction of \( g \), we can get that \( f \subseteq g \). But \( f \neq g \), because of \( g(a) = p \), and \( f(a) = f(m) \subseteq p \).

It remains to be shown that \( f \subseteq g \), which will contradict the assumption that \( f \) is a maximal element because \( g \) would be strictly bigger than \( f \).

Let \( x \subseteq y \) in \( D \). We check that \( f(x) = f(y) \cap g(x) \) by considering these two cases: \( a \subseteq x \) and \( a \not\subseteq x \). When \( a \subseteq x \), it follows that \( f(x) = f(y) = f(m) \subseteq p = g(x) \) by Lemma 4.1. Hence, \( f(x) = f(y) \cap g(x) \). When \( a \not\subseteq x \), we have \( g(x) = f(x) \) by the definition of \( g \). So, \( f(x) = f(y) \cap g(x) \).

5. MAXIMUM-SEPARABILITY OF DOMAINS

The aim of this section is to give characterizations of the maximum-separability in stable bifinite domains. A similar case was investigated by Zhang in [6]. Local distributivity is introduced in the sequel.

Amadio [8] showed that every stable bifinite domain is an \( L \)-domain. It is well known that \( L \)-domains are algebraic domains in which every principal ideal \( \downarrow x \) is a complete lattice. Therefore, arbitrary suprema and infima exist in \( \downarrow x \). Local infima are entirely exact ones (if exist). However, local suprema may be different from entire ones in general. We use the notation \( \cup^x \) for the local supremum in the principal ideal \( \downarrow x \). Amadio considered the local distributivity of stable bifinite domains.

**DEFINITION 5.1. LOCAL DISTRIBUTIVITY.** Given a stable bifinite domain \( D \), if for each principal ideal \( \downarrow x \), and \( a, c, d \in \downarrow x \),

\[ a \cap (c \cup^x d) = (a \cup^x c) \cap (a \cup^x d), \]

then \( D \) is said to have the local distributivity.

**THEOREM 5.1.** Fix a stable bifinite domain \( D \). \( D \) is maximum-separable if and only if for any locally distributive stable bifinite domain \( E \),

\[ f \in M([D \rightarrow E]) \Rightarrow \forall m \in M(D), f(m) \in M(E). \]

**PROOF.** (If) Suppose \( D \) is not maximum-separable. Then, there exists a maximal element \( m \) such that \( |(\uparrow a) \cap M(D)| > 1 \) for all compact element \( a \in \downarrow m \). Now, consider the set \( \mathcal{P} \) whose members are families

\[ F = \{ \uparrow a_i \mid i \in I \text{ and } \forall i \in I (a_i \in K(D) \text{ and } m \nsubseteq \uparrow a_i) \text{ and } \forall i \neq j (\uparrow a_i) \cap (\uparrow a_j) = \emptyset \}. \]

\( \mathcal{P} \) contains a nonempty family since otherwise \( m \) would be the largest element of \( D \), making it maximum-separable. The poset \( \mathcal{P} \), under inclusion, is also inductive, for the union of a directed family in \( \mathcal{P} \) is again a member in \( \mathcal{P} \).

Using the axiom of choice in form of Zorn’s lemma, we obtain a maximal element \( F_0 = \{ \uparrow b_j \mid j \in J \} \) in \( \mathcal{P} \), i.e., a maximal element in the poset \( \mathcal{P} \) (under inclusion). Note that \( \cup F_0 \) is a Scott
open set in $D$, since all $a_i$ are compact elements in $D$. Let $f$ be the Scott continuous function from $D$ to $2$ (by letting $E = 2$, where $2$ is the two element cpo) such that $f^{-1}(\{1\}) = \cup F_0$, i.e., $f(x) = 1$ if $x \in \cup F_0$ and $f(x) = 0$ otherwise. $f$ is stable because $\cup F_0$ is a stable neighborhood, where a stable neighborhood is a Scott open set whose minimal elements are pairwise incompatible (see [15], Theorem 2.1). We have $f(m) = 0$ because $m \not\in \cup F_0$. That is, $f$ maps the maximal element $m$ in $D$ to a nonmaximal element $0$ in $2$.

It remains to verify that $f$ is a maximal element in $[D \rightarrow 2]$.

Let $g$ be a maximal element, above $f$, in the stable function space $[D \rightarrow 2]$. Again by (the proof of) Theorem 2.1 in [15], we have $f^{-1}(\{1\}) \subseteq g^{-1}(\{1\})$, and each minimal element of $f^{-1}(\{1\})$ is a minimal element of $g^{-1}(\{1\})$, where $g^{-1}(\{1\})$ is again a stable neighborhood. These imply that any minimal element $a \in g^{-1}(\{1\})$ not in $f^{-1}(\{1\})$ gives rise to a principal ideal $\uparrow a$ which is pairwise disjoint from each member of $F_0$. If $f \neq g$ then there is a minimal element $a_0$ in $g^{-1}(\{1\})$ not in $f^{-1}(\{1\})$. So, $f(a_0) = 0$. Since $F_0$ is a maximal collection of pairwise nonoverlapping principal ideals not containing $m$, we must have $m \not\in \uparrow a_0$. By the assumption of $m$, we get an $m' \in M(D)$ such that $a_0 \subseteq m' \neq m$. Thus, there exists a $b_0 \subseteq m'$ such that $a_0 \subseteq b_0$, but $b_0 \not\subseteq m$. That is, $m \not\subseteq b_0$, and meanwhile, $\uparrow b_0$ is disjoint from each member of $F_0$. Indeed, if $\uparrow b_0 \cap \uparrow b_j \neq \emptyset$ for some $j \in J$, then $b_0$ and $b_j$ are compatible, and so are $a_0$ and $b_j$. Since $f(a_0) = 0$, we have $f(a_0) = f(a_0) \cap g(b_j)$. By Lemma 2.1, we have $f(a_0) \cap g(b_j) = f(b_j) \cap g(a_0) = 1$, contradicting with the choice of $a_0$. Consequently, the family $F_0 \cup \{\uparrow b_0\}$ belongs to $\mathcal{P}$, which is strictly greater than $F_0$, contradicting the maximality of the $F_0$. Thus, $f = g$ and $f$ is a maximal stable function from $D$ to $2$.

(Only if) Suppose $D$ is maximum-separable, $E$ is a locally distributive stable bifinite domain, and $f \in M([D \rightarrow E])$. We show that for any maximal element $m$ in $D$, $f(m)$ is maximal in $E$.

Now, let $e$ be a maximal element in $E$ such that $f(m) \subseteq e$, and let $p$ be a compact element below $e$. Since $E$ satisfies Axiom I, $p \cap f(m)$ is a compact element below $p$ and hence below $f(m)$. So, there is a compact element $a \subseteq m$ such that $p \cap f(m) \subseteq f(a)$. We can assume that such an $a$ is already a separation point for $m$. Construct a function $g$ such that

$$g(x) = \begin{cases} p \cup^e f(x), & \text{if } a \subseteq x, \\ f(x), & \text{otherwise.} \end{cases}$$

Note that since $a$ is a separation point for $m$, for all $a \subseteq x$, we have $f(x) \subseteq f(m)$. So, $g$ is well defined. Since $a$ is a compact element, $g$ remains continuous.

To check the stability of $g$, let $\{x, y\}$ be compatible elements in $D$. There are three cases.

1. If $a \subseteq x \cap y$, then $a \subseteq x$, $a \subseteq y$, $f(x) \subseteq e$, $f(y) \subseteq e$, and

$$g(x \cap y) = p \cup^e (f(x \cap y))$$

$$= (p \cup^e f(x)) \cap (p \cup^e f(y)) \text{ by the local distributivity of } E$$

$$= g(x) \cap g(y).$$

2. If $a \not\subseteq x$ and $a \not\subseteq y$, then $a \not\subseteq x \cap y$, and

$$g(x \cap y) = f(x \cap y) = f(x) \cap f(y) = g(x) \cap g(y).$$

3. Let $a \subseteq x$ and $a \not\subseteq y$. The condition $a \subseteq x$ implies that $m$ is an upper bound of $x, y$. So, we get that $f(a) \subseteq f(x) \subseteq f(m) \subseteq e$ and $f(y) \subseteq f(m) \subseteq e$. Thus,

$$g(x) \cap g(y) = (p \cup^e f(x)) \cap f(y)$$

$$= (p \cap f(y)) \cup^e (f(x) \cap f(y)) \text{ by the local distributivity of } E.$$
Maximality and Totality of Stable Functions

Since $p \cap f(m) \subseteq f(a)$, we have

\[ f(x) \cap f(y) \supseteq f(a) \cap f(y) \]
\[ \supseteq p \cap f(m) \cap f(y) \]
\[ = p \cap f(y). \]

Therefore,

\[ g(x) \cap g(y) = p \cap f(y). \]

Thus, we have

\[ g(x) \cap g(y) = p \cap f(y) \cap f(y) \]
\[ \subseteq p \cap f(m) \cap f(y) \]
\[ \subseteq f(x) \cap f(y) \]
\[ = f(x \cap y) \]
\[ = g(x \cap y). \]

This proves $g(x) \cap g(y) = g(x \cap y)$.  

Now, we show that $f \subseteq g$. Let $x, y \in D$, and $x \subseteq y$.

- If $a \nsubseteq x$, then $f(x) = f(y) \cap f(x) = f(y) \cap g(x)$.
- If $a \subseteq x$, then we have

\[ f(y) \cap g(x) = f(y) \cap (p \cup^e f(x)) \]
\[ = (f(y) \cap p) \cup^e (f(y) \cap f(x)) \text{ by the local distributivity of } E \]
\[ = (f(y) \cap p) \cup^e f(x) \]
\[ = f(x). \]

For the last step, note that the conditions $a \nsubseteq x$ and $x \subseteq y$ imply $y \subseteq m$. Hence,

\[ f(y) \cap p \subseteq f(m) \cap p \subseteq f(a) \subseteq f(x). \]

Since $f$ is a maximal stable function, we must have $f = g$. Thus, $f(m) = g(m) = p \cup^e f(m)$.

Therefore, $p \subseteq f(m)$. As a result, $f(m) = e$, a maximal element of $E$.  

REFERENCES


