Available online at www.sciencedirect.com
SCIENCE DIRECT®
An International Journal
computers & mathematics with applications
ELSEVIER Computers and Mathematics with Applications 47 (2004) 1517–1533
www.elsevier.com/locate/camwa

Identification of Source Terms in 2-D IHCP

Z. Yi and D. A. Murio
Department of Mathematical Sciences, University of Cincinnati
Cincinnati, OH 45221-0025, U.S.A.
diego@dmurio.csm.uc.edu
http://myrtle.csm.uc.edu

(Received December 2002; revised and accepted January 2004)

Abstract—We introduce a stable numerical space marching scheme based on discrete mollification—implemented as an automatic adaptive filter—for the approximate identification of temperature, temperature gradient, and source terms in the two-dimensional inverse heat conduction problem (IHCP).

The stability and error analysis of the algorithm, together with some numerical examples, are provided.

Keywords—Discrete mollification, Identification of source terms, Finite differences, Space marching schemes, Parabolic equations, IHCP.

1. INTRODUCTION

In this paper, we study the simultaneous identification of general source terms, temperature distribution and temperature gradient distribution in two-dimensional parabolic equations by mollification techniques provided that suitable noisy data is available only at the active boundary, generalizing the recent one-dimensional work of Yi and Murio [1], to two space dimensions.

Space marching schemes along with the method of discrete mollification—implemented as an automatic iterative filter—and generalized cross validation (GCV), has proven to be an effective way for solving these problems [2,3]. For an up to date detailed description of these techniques see [4] (Inverse Engineering Handbook, Chapter 4: Mollification and Space Marching).

In particular, the determination of source terms in the one-dimensional inverse heat conduction problem (IHCP) is a parameter identification type of problem that has been extensively explored. However, the available results are based on the assumptions that the source term depends only on one variable [5] or that it can be separated into spatial and temporal components [2,6,7]. An historical and technical review of inverse source problems can be found in the classical book of Isakov [8].

The manuscript is organized as follows. In Section 2, for completeness, we state basic properties and estimates corresponding to mollification in $\mathbf{R}^2$. In Section 3, the original ill-posed problem and the associated regularized (mollified) problem, respectively, are formulated and the two-dimensional numerical procedure is introduced. In Section 4, the stability and error analysis of the algorithm are investigate and numerical computations of interest are provided.

The authors would like to thank the C.W. Taft Foundation for partially supporting this research.

0898-1221/04/$ - see front matter Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
doi:10.1016/j.camwa.2004.06.004

2. MOLLIFICATION IN R^2

Complete proofs of the propositions in this section can be found in [9].

2.1. Abstract Setting

We introduce the δ-mollification for functions of two independent variables.

Let $x = (x_1, x_2)$, $p = (p_1, p_2)$, $\delta = (\delta_1, \delta_2)$, $p_i > 0$, $\delta_i > 0$, $x_i \in \mathbf{R}^1$ $(i = 1, 2)$, and introduce the following notation

$$\begin{aligned} I &= [0,1] \times [0,1], \\ |\delta|_\infty &= \max(\delta_1, \delta_2), \\ |\delta|_{-\infty} &= \min(\delta_1, \delta_2), \\ I_p &= [-p_1, p_1] \times [-p_2, p_2], \\ I_{p\delta} &= [-p_1\delta_1, p_1\delta_1] \times [-p_2\delta_2, p_2\delta_2], \\ I_\delta &= [p_1\delta_1, 1 - p_1\delta_1] \times [p_2\delta_2, 1 - p_2\delta_2]. \end{aligned}$$

We consider the following two-dimensional Gaussian kernel

$$\rho_{\delta,p}(x) = \begin{cases} A_p \delta_1^{-1} \delta_2^{-1} \exp\left(-\left(\dfrac{x_1^2}{\delta_1^2} + \dfrac{x_2^2}{\delta_2^2}\right)\right), & x \in I_{p\delta}, \\ 0, & \text{otherwise}, \end{cases}$$

where $A_p = (\int_{I_p} \exp(-\|x\|^2)\, dx)^{-1}$, $\|x\|^2 = x_1^2 + x_2^2$.

Note that $\rho_{\delta,p}$ is a nonnegative $C^\infty(I_{p\delta})$ function vanishing outside $I_{p\delta}$ and satisfying $\int_{I_{p\delta}} \rho_{\delta,p}(x)\, dx = 1$.

If $f(x)$ is integrable on I, we define its δ-mollification on I_δ by the convolution

$$J_\delta f(x) = \int_I \rho_\delta(x - s) f(s)\, ds,$$

where the p-dependency on the kernel has been dropped for simplicity.

Notice that $J_\delta f(x) = J_{\delta_1}(J_{\delta_2} f(x_1, x_2)) = J_{\delta_2}(J_{\delta_1} f(x_1, x_2))$, where $J_{\delta_i} f(x_1, x_2)$ $(i = 1, 2)$ denotes the δ-mollification of f with parameters δ_i, p_i with respect to the variable x_i.

The δ-mollification of an integrable function satisfies well-known consistency and stability estimates.

THEOREM 1. L^2 NORM CONVERGENCE IN $\mathbf{R}^2$. *If $f(x) \in L^2(I)$, then*

$$\lim_{\delta \to (0,0)} \|J_\delta f - f\|_{L^2(I_\delta)} = 0.$$

Moreover, if $\nabla f(x) \in L^2(I) \times L^2(I)$, then

$$\lim_{\delta \to (0,0)} \|\nabla(J_\delta f) - \nabla f\|_{L^2(I_\delta) \times L^2(I_\delta)} = 0.$$

THEOREM 2. CONSISTENCY, STABILITY, AND CONVERGENCE OF MOLLIFICATION IN $\mathbf{R}^2$.

(1) *If $f(x) \in C^1(I)$, then there exists a constant C, independent of δ, such that*

$$\|J_\delta f - f\|_{\infty, I_\delta} \le C|\delta|_\infty.$$

Moreover, if $\frac{\partial}{\partial x_1} f(x), \frac{\partial}{\partial x_2} f(x) \in C^1(I)$, then

$$\|\nabla(J_\delta f) - \nabla f\|_{\infty, I_\delta} \le C|\delta|_\infty,$$

where for $(f_1, f_2) \in C(I) \times C(I)$, the norm is defined by $\|(f_1, f_2)\|_{\infty, I} = \max(\|f_1\|_{\infty, I}, \|f_2\|_{\infty, I})$.

(2) *If* $f(x),\ f^{\epsilon}(x) \in C^0(I)$, *and* $\|f(x) - f^{\epsilon}(x)\|_{\infty,I} \le \epsilon$, *then*

$$\|J_\delta f - J_\delta f^{\epsilon}\|_{\infty,I_\delta} \le \epsilon \qquad \text{and} \qquad \|\nabla(J_\delta f) - \nabla(J_\delta f^{\epsilon})\|_{\infty,I_\delta} \le C\frac{\epsilon}{|\delta|_{-\infty}}.$$

(3) *If* $f(x) \in C^1(I)$ *and* $f^{\epsilon}(x) \in C^0(I)$ *with* $\|f(x) - f^{\epsilon}(x)\|_{\infty,I} \le \epsilon$, *then*

$$\|J_\delta f^{\epsilon} - f\|_{\infty,I_\delta} \le C|\delta|_\infty + \epsilon.$$

Moreover, if $\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x) \in C^1(I)$, *then*

$$\|\nabla(J_\delta f^{\epsilon}) - \nabla f\|_{\infty,I_\delta} \le C\left(|\delta|_\infty + \frac{\epsilon}{|\delta|_{-\infty}}\right).$$

We next consider the δ-mollification of a discrete function defined on the discrete set $K = \{(x_1^{(i)},\ x_2^{(j)}) : 1 \le i \le m,\ 1 \le j \le n\} \subset I$, with

$$0 \le x_1^{(1)} < x_1^{(2)} < \cdots < x_1^{(m)} \le 1, \qquad 0 \le x_2^{(1)} < x_2^{(2)} < \cdots < x_2^{(n)} \le 1,$$

and

$$\begin{aligned} s_1^{(0)} &= 0, \qquad s_1^{(m)} = 1, \qquad s_2^{(0)} = 0, \qquad s_2^{(n)} = 1,\\ s_1^{(i)} &= \frac{1}{2}\left(x_1^{(i)} + x_1^{(i+1)}\right) \qquad (i = 1, 2, \dots, m-1),\\ s_2^{(j)} &= \frac{1}{2}\left(x_2^{(j)} + x_2^{(j+1)}\right) \qquad (j = 1, 2, \dots, n-1),\\ \Delta x &= \max_{1 \le i \le m-1, 1 \le j \le n-1} \sqrt{\left|x_1^{(i+1)} - x_1^{(i)}\right|^2 + \left|x_2^{(j+1)} - x_2^{(j)}\right|^2}. \end{aligned}$$

Let $G = \{g_{ij} = g(x_1^i, x_2^j) : 1 \le i \le m,\ 1 \le j \le n\}$ be a discrete function defined on K. The discrete δ-mollification of G is then defined as follows.

For $x \in I_\delta$,

$$J_\delta G(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\int_{s_1^{(i-1)}}^{s_1^{(i)}} \int_{s_2^{(j-1)}}^{s_2^{(j)}} \rho_\delta(x - s)\, ds_1\, ds_2 \right) g_{ij}.$$

Notice that $\sum_{i=1}^m \sum_{j=1}^n (\int_{s_1^{(i-1)}}^{s_1^{(i)}} \int_{s_2^{(j-1)}}^{s_2^{(j)}} \rho_\delta(x-s)\, ds_1\, ds_2) = \int_{I_{p\delta}} \rho_\delta(-s)\, ds = 1$.

The consistency, stability, and convergence of the discrete δ-mollification are presented in the following theorem.

THEOREM 3. CONSISTENCY, STABILITY, AND CONVERGENCE OF DISCRETE MOLLIFICATION IN $\mathbf{R}^2$.

(1) *Let* $g(x) \in C^1(I)$ *and let* $G = \{g_{ij} : 1 \le i \le m,\ 1 \le j \le n\}$ *be the discrete version of* g *with* $g_{ij} = g(x_1^{(i)}, x_2^{(j)})$. *Then, there exists a constant* C *such that*

$$\|J_\delta G - g\|_{\infty,I_\delta} \le C(|\delta|_\infty + \Delta x).$$

Moreover, if $\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_1} \in C^1(I)$, *then there exists a constant* C *such that*

$$\|\nabla(J_\delta G) - \nabla g\|_{\infty,I_\delta} \le C\left(|\delta|_\infty + \frac{\Delta x}{|\delta|_{-\infty}}\right).$$

(2) *If the discrete functions* $G = \{g_{ij} : 1 \le i \le m,\ 1 \le j \le n\}$ *and* $G^\epsilon = \{g^\epsilon_{ij} : 1 \le i \le m,\ 1 \le j \le n\}$, *defined on* K, *satisfy* $\|G - G^\epsilon\|_\infty \le \epsilon$, *then*

$$\|J_\delta G^\epsilon - J_\delta G\|_{\infty, I_\delta} \le \epsilon \qquad \text{and} \qquad \|\nabla(J_\delta G^\epsilon) - \nabla(J_\delta G)\|_{\infty, I_\delta} \le C\frac{\epsilon}{|\delta|_{-\infty}}.$$

(3) *Let* $g(x) \in C^1(I)$, *let* $G = \{g_{ij} = g(x_1^{(i)}, x_2^{(j)}) : 1 \le i \le m,\ 1 \le j \le n\}$ *be the discrete version of* g *and let* $G^\epsilon = \{g^\epsilon_{ij} : 1 \le i \le m,\ 1 \le j \le n\}$ *be the perturbed discrete version of* g *with* $\|G - G^\epsilon\|_\infty \le \epsilon$. *Then, there exists a constant* C *such that*

$$\|J_\delta G^\epsilon - J_\delta g\|_{\infty, I_\delta} \le C(\epsilon + \Delta x) \qquad \text{and} \qquad \|J_\delta G^\epsilon - g\|_{\infty, I_\delta} \le C(\epsilon + |\delta|_\infty + \Delta x).$$

Moreover, if $\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x) \in C^1(I)$, *then*

$$\|\nabla(J_\delta G^\epsilon) - \nabla(J_\delta g)\|_{\infty, I_\delta} \le \frac{C}{|\delta|_{-\infty}}(\epsilon + \Delta x)$$

and

$$\|\nabla(J_\delta G^\epsilon) - \nabla g\|_{\infty, I_\delta} \le C\left(|\delta|_\infty + \frac{\epsilon}{|\delta|_{-\infty}} + \frac{\Delta x}{|\delta|_{-\infty}}\right).$$

2.2. Numerical Gradient Computation

This section discusses the main results on stable numerical computation of gradients by the mollification method.

Assume that

$$\begin{aligned} x_1^{(i)} - x_1^{(i-1)} &= x_1^{(i+1)} - x_1^{(i)} = \Delta x_1, \qquad i = 2, \dots, m-1, \\ x_2^{(j)} - x_2^{(j-1)} &= x_2^{(j+1)} - x_2^{(j)} = \Delta x_2, \qquad j = 2, \dots, n-1, \end{aligned}$$

and

$$\tilde{I}_\delta = [p_1\delta_1 + \Delta x_1, 1 - p_1\delta_1 - \Delta x_1] \times [p_2\delta_2 + \Delta x_2, 1 - p_2\delta_2 - \Delta x_2].$$

Given G^ϵ, a perturbed discrete version of g, in order to approximate ∇g, we compute the centered differences of $J_\delta G^\epsilon$. That is, we use $\mathbf{D}(J_\delta G^\epsilon)$ to approximate $\nabla(J_\delta G^\epsilon)$ in $\tilde{I}_\delta$. Here $\mathbf{D} = (\mathbf{D}_{x_1}, \mathbf{D}_{x_2})$, $\mathbf{D}_{x_i}$ $(i = 1, 2)$ denotes the centered difference operator with respect to the variable x_i.

In the proposition and theorems that follow, the generic constant C is independent of δ.

PROPOSITION 1. *Let* $\nabla g \in C^1(I) \times C^1(I)$, *let* $G = \{g_{i,j} = g(x_1^{(i)}, x_2^{(j)}) : 1 \le i \le m,\ 1 \le j \le n\}$ *be the discrete version of* g *and let* G *and* G^ϵ *satisfy* $\|G - G^\epsilon\|_{\infty, K} \le \epsilon$. *Then,*

$$\|\mathbf{D}(J_\delta G^\epsilon) - \nabla g\|_{\infty, \tilde{I}_\delta} \le C\left(|\delta|_\infty + \frac{\epsilon}{|\delta|_{-\infty}} + \frac{\Delta x}{|\delta|_{-\infty}}\right) + C_\delta(\Delta x)^2$$

and

$$\|\mathbf{D}(J_\delta G^\epsilon) - \nabla(J_\delta g)\|_{\infty, \tilde{I}_\delta} \le \frac{C}{|\delta|_{-\infty}}(\epsilon + \Delta x) + C_\delta(\Delta x)^2.$$

If G is a discrete function on K, we define $\mathbf{D}_0^\delta G \equiv \mathbf{D}(J_\delta G)|_K$. The next theorem establishes a uniform bound for $\mathbf{D}_0^\delta$.

THEOREM 4.

$$\|\mathbf{D}_0^\delta G\|_{\infty, K \cap \tilde{I}_\delta} \le \frac{C}{|\delta|_{-\infty}}\|G\|_{\infty, K}.$$

THEOREM 5. *If $g \in C^1(I)$ and G is a discrete version of g, then for $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$,*

$$\left|\mathbf{D}^2_{x_a}(J_\delta g)\left(x_1^{(i)}, x_2^{(j)}\right) - \mathbf{D}^2_{x_a}(J_\delta G)\left(x_1^{(i)}, x_2^{(j)}\right)\right| \le C\frac{\Delta x}{|\delta|^2_{-\infty}}, \qquad a = 1, 2,$$

where $\mathbf{D}^2_{x_a}(f)(x_1, x_2)$ denotes the centered difference approximation of $\frac{\partial^2 f}{\partial x_a^2}(x_1, x_2)$ utilizing Δx_a.

THEOREM 6. *Let G and G^ϵ be discrete functions defined on K, satisfying $\|G - G^\epsilon\|_\infty \le \epsilon$. Then, for $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$,*

$$\left|\mathbf{D}^2_{x_a}(J_\delta G)\left(x_1^{(i)}, x_2^{(j)}\right) - \mathbf{D}^2_{x_a}(J_\delta G^\epsilon)\left(x_1^{(i)}, x_2^{(j)}\right)\right| \le C\frac{\epsilon}{|\delta|^2_{-\infty}}, \qquad a = 1, 2.$$

THEOREM 7. *If $g \in C^0(I)$, and G and G^ϵ are discrete functions defined on K satisfying $\|G - G^\epsilon\|_\infty \le \epsilon$, then for all $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$,*

$$\left|\mathbf{D}_{x_a}(J_\delta g)\left(x_1^{(i)}, x_2^{(j)}\right) - \frac{\partial}{\partial x_a}(J_\delta g)\left(x_1^{(i)}, x_2^{(j)}\right)\right| \le C\frac{(\epsilon + \Delta x)}{|\delta|_{-\infty}} + C_\delta(\Delta x)^2,$$

and

$$\left|\mathbf{D}^2_{x_a}(J_\delta g)\left(x_1^{(i)}, x_2^{(j)}\right) - \frac{\partial^2}{\partial x_a^2}(J_\delta g)\left(x_1^{(i)}, x_2^{(j)}\right)\right| \le C\frac{(\epsilon + \Delta x)}{|\delta|^2_{-\infty}} + C_\delta(\Delta x)^2,$$

where $a = 1, 2$.

2.3. Implementation

Computation of $J_\delta f$ throughout the domain $I = [0,1] \times [0,1]$, requires the extension of f to a slightly larger rectangle $I'_\delta = [-p_1\delta_1, 1+p_1\delta_1] \times [-p_2\delta_2, 1+p_2\delta_2]$ and since $J_\delta f = J_{\delta_2}(J_{\delta_1} f(x_1, x_2))$, only one-dimensional extensions are needed.

As indicated previously, the parameter $\delta = (\delta_1, \delta_2)$ plays a crucial role in the regularization procedure. The discrete δ-mollification of $G = \{g_{ij} : 1 \le i \le m,\ 1 \le j \le n\}$,

$$J_\delta G(x) = \sum_{j=1}^{n} \int_{s_2^{(j-1)}}^{s_2^{(j)}} \rho_{\delta_2, x_2}(x_2 - s_2) \left(\sum_{i=1}^{m} \int_{s_1^{(i-1)}}^{s_1^{(i)}} \rho_{\delta_1, x_1}(x_1 - s_1) g_{ij}\, ds_1 \right) ds_2,$$

is reduced to a double "mollification sweep" of several one-dimensional functions. First, for each fixed j, the discrete δ-mollification of the one-dimensional data set $\{g_{ij} : 1 \le i \le m\}$ is evaluated and then, for each fixed x_1, another discrete δ-mollification with respect to x_2 of the previously mollified data (the one-dimensional data set $\{\sum_{i=1}^{m} \int_{s_1^{(i-1)}}^{s_1^{(i)}} \rho_{\delta_1, x_1}(x_1 - s_1) g_{ij}\, ds_1 : 1 \le j \le n\}$) is computed. Thus, the two-dimensional automatic parameter selection is reduced to a sequence of one-dimensional ones.

3. SOURCE TERM AND TEMPERATURE IDENTIFICATION IN 2-D IHCP

3.1. Description of the Problem

Find $u(x, y, t)$, $\nabla u(x, y, t)$, and $f(x, y, t)$ throughout the domain $[0, x_{\max}] \times [0,1] \times [0,1]$ of the (x, y, t) plane, from measured approximations of $\alpha(y, t)$, $\beta(y, t)$, and $\gamma(y, t)$ satisfying

$$u_t = \nabla(a(x, y, t)\nabla u(x, y, t)) + f(x, y, t), \qquad 0 < x < x_{\max}, \quad 0 < y < 1, \quad 0 < t < 1,$$

$$\begin{aligned} u(0, y, t) &= \alpha^\varepsilon(y, t), && 0 \le y \le 1, \quad 0 \le t \le 1, \\ u_x(0, y, t) &= \beta^\varepsilon(y, t), && 0 \le y \le 1, \quad 0 \le t \le 1, \\ f(0, y, t) &= \gamma^\varepsilon(y, t), && 0 \le y \le 1, \quad 0 \le t \le 1, \end{aligned}$$

where $a(x, y, t)$ is given and $\alpha(y, t)$, $\beta(y, t)$, and $\gamma(y, t)$ are measured. The known data functions α^ε, β^ε, and γ^ε for α, β, and γ, respectively, are discrete noisy functions defined on the discrete set $\{(y_j, t_k) : y_j = jl, t_k = kn : 1 \le j \le N,\ 1 \le k \le T\}$ with $l = 1/N$ and $n = 1/T$. Moreover, $\|\alpha^\varepsilon - \alpha\| \le \varepsilon$, $\|\beta^\varepsilon - \beta\| \le \varepsilon$, and $\|\gamma^\varepsilon - \gamma\| \le \varepsilon$, where ε is a positive tolerance.

3.2. Regularized Problem

The regularized problem, based on mollification, is formulated as follows. Determine $u(x,y,t)$, $\nabla v(x,y,t)$, and $f(x,y,t) \in [0,x_{\max}] \times [0,1] \times [0,1]$ such that

$$v_t = \nabla(a(x,y,t)\nabla v(x,y,t)) + f(x,y,t), \qquad 0 < x < x_{\max}, \quad 0 < y < 1, \quad 0 < t < 1,$$

$$\begin{aligned} v(0,y,t) &= J_{\delta_0}\alpha(y,t), && 0 \le y \le 1, \quad 0 \le t \le 1, \\ v_x(0,y,t) &= J_{\delta_0^*}\beta(y,t), && 0 \le y \le 1, \quad 0 \le t \le 1, \\ f(0,y,t) &= J_{\hat{\delta}_0}\gamma(y,t), && 0 \le y \le 1, \quad 0 \le t \le 1. \end{aligned}$$

Note: δ-mollificatons of α^ε, β^ε, and γ^ε are taken with respect to y and t using $\delta_0 = (\delta_1,\delta_2)$, $\delta_0^* = (\delta_3, \delta_4)$, and $\hat{\delta}_0 = (\delta_5, \delta_6)$, respectively. Applying the mollification method, the space marching scheme to compute $f(x,y,t)$ in $[0,x_{\max}] \times [0,1] \times [0,1]$ and $v(x,y,t)$, $v_x(x,y,t)$, $v_{xx}(x,y,t)$, $v_y(x,y,t)$, $v_{yy}(x,y,t)$, and $v_t(x,t)$ throughout $[0,x_{\max}] \times [0,1] \times [0,1]$ is described in the next section.

3.3. The Marching Scheme

Let M, N, and T be positive integers such that $h = \Delta x = 1/M$, $l = \Delta y = 1/N$, and $k = \Delta t = 1/T$. We define $x_i = ih, i = 0,\dots,i_{\max}$ (with $x_{\max} = h\ i_{\max}$); $y_j = jl$, $j = 0,1,\dots,N$; $t_n = nk$, $n = 0,1,\dots,T$; $q(ih,jl,nk) = a(x_i,y_j,t_k)\ v_x(ih,jl,nk)$, and introduce the discrete functions

$$\begin{aligned} R_{i,j}^n &: \text{the discrete computed approximation of } v(ih,jl,nk), \\ W_{i,j}^n &: \text{the discrete computed approximation of } v_t(ih,jl,nk), \\ Q_{i,j}^n &: \text{the discrete computed approximation of } v_x(ih,jl,nk), \\ U_{i,j}^n &: \text{the discrete computed approximation of } v_{xx}(ih,jl,nk), \\ O_{i,j}^n &: \text{the discrete computed approximation of } v_{xt}(ih,jl,nk), \\ S_{i,j}^n &: \text{the discrete computed approximation of } v_{xy}(ih,jl,nk), \\ P_{i,j}^n &: \text{the discrete computed approximation of } v_y(ih,jl,nk), \\ Z_{i,j}^n &: \text{the discrete computed approximation of } v_{yy}(ih,jl,nk), \\ F_{i,j}^n &: \text{the discrete computed approximation of } f(ih,jl,nk). \end{aligned}$$

The space marching scheme is defined as follows.

STEP 1. Select $\delta_0, \delta_0^*, \hat{\delta}_0$.

STEP 2. Perform mollification of $\alpha^\varepsilon, \beta^\varepsilon$, and γ^ε.
Set

$$\begin{aligned} R_{0,j}^n &= J_{\delta_0}\alpha^\varepsilon(jl,nk), \\ Q_{0,j}^n &= J_{\delta_0^*}\beta^\varepsilon(jl,nk), \\ F_{0,j}^n &= J_{\hat{\delta}_0}\gamma^\varepsilon(jl,nk). \end{aligned}$$

STEP 3. Perform mollified differentiation in time of $J_{\delta_0}\alpha^\varepsilon(jl,nk)$ and $J_{\delta_0^*}\beta^\varepsilon(jl,nk)$.
Set

$$\begin{aligned} W_{0,j}^n &= \mathbf{D}_t\left(J_{\delta_0}\alpha^\varepsilon\right)(jl,nk), \\ O_{0,j}^n &= \mathbf{D}_t\left(J_{\delta_0^*}\beta^\varepsilon\right)(jl,nk). \end{aligned}$$

STEP 4. Perform mollified differentiation in y-space of

$$P_{0,j}^n = \mathbf{D}_y \left(J_{\delta_0}\alpha^\varepsilon\right)(jl, nk),$$
$$Z_{0,j}^n = \mathbf{D}_{yy}^2 \left(J_{\delta_0}\alpha^\varepsilon\right)(jl, nk),$$
$$S_{0,j}^n = \mathbf{D}_y \left(J_{\delta_0^*}\beta^\varepsilon\right)(jl, nk).$$

STEP 5. Initialize $i = 0$. Do while $i \le i_{\max} - 1$.

a : $R_{i+1,j}^n = R_{i,j}^n + \dfrac{h}{a(x_i, y_j, t_n)} Q_{i,j}^n$.

b : $Q_{i+1,j}^n = Q_{i,j}^n + \left(W_{i,j}^n - \left(a_y\left(x_i, a\left(x_i, y_j, t_n\right) Z_{i,j}^n + F_{i,j}^n\right)\right) h, y_j, t_n\right) P_{i,j}^n$
$\quad + a(x_i, y_j, t_n) Z_{i,j}^n + F_{i,j}^n))h$.

c : $P_{i+1,j}^n = P_{i,j}^n + \dfrac{h}{a(x_i, y_j, t_n)} \left(S_{i,j}^n - \dfrac{a_y(x_i, y_j, t_n)}{a(x_i, y_j, t_n)} Q_{i,j}^n\right)$.

d : $W_{i+1,j}^n = W_{i,j}^n + \dfrac{h}{a(x_i, y_j, t_n)} O_{i,j}^n$.

e : Select $\delta_{i+1}, \delta_{i+1}^*$.

f : Perform mollified differentiation in x-space of $R_{i+1,j}^n$.
Set: $U_{i+1,j}^n = \mathbf{D}_x^2 \left(J_{\delta_{i+1}} R_{i+1,j}^n\right)$.

g : Perform mollified differentiation in y-space of $Q_{i+1,j}^n$ and $P_{i+1,j}^n$.
Set: $S_{i+1,j}^n = \mathbf{D}_y \left(J_{\delta_{i+1}^*} Q_{i+1,j}^n\right)$, $Z_{i+1,j}^n = \mathbf{D}_y \left(J_{\delta_{i+1}} P_{i+1,j}^n\right)$.

h : Perform mollified differentiation in y-space of $Q_{i+1,j}^n$.
Set: $O_{i+1,j}^n = \mathbf{D}_t \left(J_{\delta_{i+1}^*} Q_{i+1,j}^n\right)$.

i : $F_{i+1,j}^n = W_{i+1,j}^n - a\left(x_{i+1}, y_j, t_n\right)\left(U_{i+1,j}^n + Z_{i+1,j}^n\right) - \dfrac{a_x(x_{i+1}, y_j, t_n)}{a(x_{i+1}, y_j, t_n)} Q_{i+1,j}^n$
$\quad - a_y(x_{i+1}, y_j, t_n) P_{i+1,j}^n$.

j : Set $i = i + 1$.

4. STABILITY OF THE SCHEME

In this section, we prove a stability estimate for the numerical variables $R_{i,j}^n, Q_{i,j}^n, P_{i,j}^n, W_{i,j}^n$, and $F_{i,j}^n$.

We start with some necessary definitions. If $A_{i,j}^n$ is a discrete function, we denote

$$|A_i| = \max_{j,n} \left|A_{i,j}^n\right| \qquad \text{and} \qquad \|A\|_\infty = \max_j |A_j|.$$

Without loss of generality, throughout this section and the next, we assume $|\delta|_{-\infty} = \min(\delta_i, \delta_i^*, \hat{\delta}_i) \le 1$, where $i = 0, 1, \dots, M$.

ASSUMPTION 4.1. *For all* $(x, y, t) \in I_{\max} = [0, x_{\max}] \times [0, 1] \times [0, 1]$, *we further assume*

1. $u(x, y, t) \in C^2(I_{\max})$,
2. $a(x, y, t) \in C^1(I_{\max})$,
3. $f(x, y, t) \in C(I_{\max})$,
4. $a(x, y, t) \neq 0, (x, y, t) \in I_{\max}$.

THEOREM 8. *If Assumption 4.1 holds, then there exist constants* C_0 *and* C_1, *such that*

$$\max\{|R_i|, |Q_i|, |W_i|, |P_i|, |F_i|\} \le C_0 \exp(C_1) \max\{|R_0|, |Q_0|, |W_0|, |P_0|, |F_0|\}.$$

PROOF. Applying Theorem 4, there exists a constant C, independent of δ, such that

$$|O_i| \le \frac{C}{|\delta|_{-\infty}} |Q_i|, \tag{4.1}$$

$$|S_i| \le \frac{C}{|\delta|_{-\infty}} |Q_i|, \tag{4.2}$$

$$|Z_i| \le \frac{C}{|\delta|_{-\infty}} |P_i|, \tag{4.3}$$

and

$$|U_i| \le \frac{C}{|\delta|^2_{-\infty}} |R_i|. \tag{4.4}$$

According to Assumption 4.1, there exist positive constants ξ and η, such that

$$\min_{x \in I_{\max}, y \in I, t \in I} |a(x,y,t)| \ge \xi \tag{4.5}$$

and

$$\max_{x \in I_{\max}, y \in I, t \in I} \{|a(x,y,t)|, |a_x(x,y,t)|, |a_y(x,y,t)|\} \le \eta. \tag{4.6}$$

Thus,

$$\begin{aligned}
\left|R^n_{i+1,j}\right| &= \left| R^n_{i,j} + \frac{h}{a(x_i,y_j,t_n)} Q^n_{i,j} \right| \\
&\le \left|R^n_{i,j}\right| + \frac{h}{|a(x_i,y_j,t_n)|} \left|Q^n_{i,j}\right| \\
&\le |R_i| + \frac{h}{\xi} |Q_i|.
\end{aligned}$$

Using inequalities (4.1) and (4.2), we have

$$\begin{aligned}
\left|P^n_{i+1,j}\right| &= \left| P^n_{i,j} + \frac{h}{a(x_i,y_j,t_n)} \left(S^n_{i,j} - \frac{a_y(x_i,y_j,t_n)}{a(x_i,y_j,t_n)} Q^n_{i,j} \right) \right| \\
&\le \left|P^n_{i,j}\right| + \frac{h}{|a(x_i,y_j,t_n)|} \left(\left|S^n_{i,j}\right| + \frac{|a_y(x_i,y_j,t_n)|}{|a(x_i,y_j,t_n)|} \left|Q^n_{i,j}\right| \right) \\
&\le |P_i| + \frac{h}{|a(x_i,y_j,t_n)|} \left(|S_i| + \frac{|a_y(x_i,y_j,t_n)|}{|a(x_i,y_j,t_n)|} |Q_i| \right) \\
&\le |P_i| + \frac{h}{|a(x_i,y_j,t_n)|} \left(\frac{C}{|\delta|_{-\infty}} |Q_i| + \frac{|a_y(x_i,y_j,t_n)|}{|a(x_i,y_j,t_n)|} |Q_i| \right) \\
&\le |P_i| + \frac{h}{\xi} \left(\frac{C}{|\delta|_{-\infty}} |Q_i| + \frac{\eta}{\xi} |Q_i| \right)
\end{aligned}$$

and

$$\begin{aligned}
\left|W^n_{i+1,j}\right| &= \left| W^n_{i,j} + \frac{h}{a(x_i,y_j,t_n)} O^n_{i,j} \right| \\
&\le \left|W^n_{i,j}\right| + \frac{h}{|a(x_i,y_j,t_n)|} \left|O^n_{i,j}\right| \\
&\le |W_i| + \frac{h}{|a(x_i,y_j,t_n)|} |O_i| \\
&\le |W_i| + h \frac{C}{|a(x_i,y_j,t_n)| \, |\delta|_{-\infty}} |Q_i| \\
&\le |W_i| + h \frac{C}{\xi |\delta|_{-\infty}} |Q_i|.
\end{aligned}$$

By Section 3.3(i),

$$F_{i,j}^n = W_{i,j}^n - a(x_i,y_j,t_n)\left(U_{i,j}^n + Z_{i,j}^n\right) - \frac{a_x(x_{i+1},y_j,t_n)}{a(x_{i+1},y_j,t_n)}Q_{i,j}^n - a_y(x_i,y_j,t_n)P_{i,j}^n,$$

and applying (4.3), (4.4), and (4.6),

$$\begin{aligned}
\left|F_{i,j}^n\right| &= \left|W_{i,j}^n - a(x_i,y_j,t_n)\left(U_{i,j}^n + Z_{i,j}^n\right) - \frac{a_x(x_{i+1},y_j,t_n)}{a(x_{i+1},y_j,t_n)}Q_{i,j}^n - a_y(x_i,y_j,t_n)P_{i,j}^n\right| \\
&\le \left|W_{i,j}^n\right| + |a(x_i,y_j,t_n)|\left(\left|U_{i,j}^n\right| + \left|Z_{i,j}^n\right|\right) + \left|\frac{a_x(x_{i+1},y_j,t_n)}{a(x_{i+1},y_j,t_n)}\right|\left|Q_{i,j}^n\right| \\
&\quad + |a_y(x_i,y_j,t_n)|\left|P_{i,j}^n\right| \\
&\le |W_i| + |a(x_i,y_j,t_n)|(|U_i| + |Z_i|) + \left|\frac{a_x(x_{i+1},y_j,t_n)}{a(x_{i+1},y_j,t_n)}\right||Q_i| \\
&\quad + |a_y(x_i,y_j,t_n)|\,|P_i| \\
&\le |W_i| + \eta\left(|U_i| + |Z_i| + \frac{1}{\xi}|Q_i| + |P_i|\right) \\
&\le |W_i| + \eta\left(\frac{C}{|\delta|_{-\infty}^2}|R_i| + \frac{C}{|\delta|_{-\infty}}|P_i| + \frac{1}{\xi}|Q_i| + |P_i|\right) \\
&= |W_i| + \eta\left(\frac{C}{|\delta|_{-\infty}^2}|R_i| + \frac{1}{\xi}|Q_i| + \left(1 + \frac{C}{|\delta|_{-\infty}}\right)|P_i|\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
Q_{i+1,j}^n &= Q_{i,j}^n + \left(W_{i,j}^n - \left(a_y\left(x_i,y_j,t_n\right)P_{i,j}^n + a\left(x_i,y_j,t_n\right)Z_{i,j}^n + F_{i,j}^n\right)\right)h \\
&= Q_{i,j}^n + \left(W_{i,j}^n - \left(a_y\left(x_i,y_j,t_n\right)P_{i,j}^n + a\left(x_i,y_j,t_n\right)Z_{i,j}^n + W_{i,j}^n\right.\right. \\
&\quad - a(x_i,y_j,t_n)\left(U_{i,j}^n + Z_{i,j}^n\right) - \frac{a_x(x_{i+1},y_j,t_n)}{a(x_{i+1},y_j,t_n)}Q_{i,j}^n \\
&\quad \left.\left.- a_y\ (x_i,y_j,t_n)P_{i,j}^n\right)\right)h \\
&= Q_{i,j}^n + \left(a(x_i,y_j,t_n)U_{i,j}^n + \frac{a_x(x_{i+1},y_j,t_n)}{a(x_{i+1},y_j,t_n)}Q_{i,j}^n\right)h,
\end{aligned}$$

and it follows that

$$\begin{aligned}
\left|Q_{i+1,j}^n\right| &\le \left|Q_{i,j}^n\right| + \left(|a(x_i,y_j,t_n)|\,\left|U_{i,j}^n\right| + \left|\frac{a_x(x_{i+1},y_j,t_n)}{a(x_{i+1},y_j,t_n)}\right|\left|Q_{i,j}^n\right|\right)h \\
&\le |Q_i| + \left(|a(x_i,y_j,t_n)|\,|U_i| + \left|\frac{a_x(x_{i+1},y_j,t_n)}{a(x_{i+1},y_j,t_n)}\right||Q_i|\right)h \\
&\le |Q_i| + \left(\eta|U_i| + \frac{\eta}{\xi}|Q_i|\right)h \\
&\le |Q_i| + \left(\eta\frac{C}{|\delta|_{-\infty}^2}|R_i| + \frac{\eta}{\xi}|Q_i|\right)h.
\end{aligned}$$

Let $A_i = \max\{|R_i|, |Q_i|, |W_i|, |P_i|\}$. Then,

$$\begin{aligned}
\left|R_{i+1,j}^n\right| &\le \left(1 + h\frac{1}{\xi}\right)A_i, \\
\left|P_{i+1,j}^n\right| &\le \left(1 + h\left(\frac{C}{\xi|\delta|_{-\infty}} + \frac{\eta}{\xi^2}\right)\right)A_i, \\
\left|W_{i+1,j}^n\right| &\le \left(1 + h\frac{C}{\xi|\delta|_{-\infty}}\right)A_i, \\
\left|Q_{i+1,j}^n\right| &\le \left(1 + h\left(\frac{\eta}{|\delta|_{-\infty}^2}C + \frac{\eta}{\xi}\right)\right)A_i,
\end{aligned}$$

and denoting

$$M_\delta = \max\left\{\frac{1}{\xi}, \frac{C}{\xi|\delta|_{-\infty}} + \frac{\eta}{\xi^2}, \frac{C}{\xi|\delta|_{-\infty}}, \frac{\eta}{|\delta|^2_{-\infty}}C + \frac{\eta}{\xi}\right\},$$

we obtain

$$\begin{aligned} A_i &\le (1 + hM_\delta)A_{i-1}, \\ A_i &\le (1 + hM_\delta)^i A_0 \\ &\le \exp(M_\delta)A_0. \end{aligned}$$

Since

$$\begin{aligned} \left|F^n_{i,j}\right| &\le |W_i| + \eta\left(\frac{C}{|\delta|^2_{-\infty}}|R_i| + \frac{1}{\xi}|Q_i| + \left(1 + \frac{C}{|\delta|_{-\infty}}\right)|P_i|\right) \\ &\le \left(1 + \eta\left(1 + \frac{1}{\xi} + \frac{C}{|\delta|^2_{-\infty}} + \frac{C}{|\delta|_{-\infty}}\right)\right)|A_i|, \end{aligned}$$

we have

$$\left|F^n_{i,j}\right| \le \exp(M_\delta)\left(1 + \eta\left(1 + \frac{1}{\xi} + \frac{C}{|\delta|^2_{-\infty}} + \frac{C}{|\delta|_{-\infty}}\right)\right)A.$$

Combining the above expressions,

$$\max\{|F_i|, |A_i|\} \le \left(1 + \eta\left(1 + \frac{1}{\xi} + \frac{C}{|\delta|^2_{-\infty}} + \frac{C}{|\delta|_{-\infty}}\right)\right)\exp(M_\delta)\max\{|F_0|, |A_0|\}.$$

4.1. Error Analysis

We define the discrete error functions

$$\begin{aligned} \Delta R^n_{i,j} &= R^n_{i,j} - v(ih, jl, nk), \\ \Delta Q^n_{i,j} &= Q^n_{i,j} - q(ih, jl, nk), \\ \Delta P^n_{i,j} &= P^n_{i,j} - v_y(ih, jl, nk), \\ \Delta W^n_{i,j} &= W^n_{i,j} - v_t(ih, jl, nk), \\ \Delta F^n_{i,j} &= F^n_{i,j} - f(ih, jl, nk), \end{aligned}$$

and denote

$$\Delta_i = \max\{|\Delta R_i|, |\Delta Q_i|, |\Delta P_i|, |\Delta W_i|, |\Delta F_i|\}.$$

LEMMA 1. *There exists a constant C, independent of δ, ϵ, h, l, and k, such that $\Delta_0 \le (C/|\delta|_{-\infty})(\epsilon + l + k)$.*

PROOF. Applying Theorems 2 and 3, we have

$$\left|\Delta R^n_{0,j}\right| = \left|\left(J_{\delta_0}\alpha^\epsilon\right)(jl, nk) - \alpha(jl, nk)\right| \le C(\epsilon + l + k)$$

and

$$\left|\Delta P^n_{0,j}\right| = \left|\mathbf{D}_y\left(J_{\delta_0^*}\beta^\epsilon\right)(jl, nk) - \frac{\partial}{\partial y}\beta(jl, nk)\right| \le \frac{C}{|\delta|_{-\infty}}(\epsilon + l).$$

Similarly,

$$\left|\Delta W^n_{0,j}\right| = \left|\mathbf{D}_t\left(J_{\delta_0}\alpha^\epsilon\right)(jl, nk) - \frac{\partial}{\partial t}\alpha(jl, nk)\right| \le \frac{C}{|\delta|_{-\infty}}(\epsilon + k).$$

Finally,

$$\left|\Delta F^n_{0,j}\right| = \left|J_{\hat\delta_0}(\gamma^\epsilon)(jl, nk) - \gamma(jl, nk)\right| \le C(\epsilon + l + k).$$

Combining the above inequalities,

$$\Delta_0 \le \frac{C}{|\delta|_{-\infty}}(\epsilon + l + k).$$

THEOREM 9. *If Assumption 4.1 holds, then*

$$\|\Delta\|_\infty \le M_\delta \exp(M_\delta)\Delta_0 + M_\delta(l+k+h) + 2M_\delta(h+l),$$

where

$$M_\delta = \max\left\{2+\eta\left(2+\frac{2C}{|\delta|_{-\infty}}+\frac{C}{|\delta|_{-\infty}^2}+\frac{1}{\xi}\right), \frac{1}{\xi}\left(\frac{C}{|\delta|_{-\infty}}+\frac{\eta}{\xi}+C_\delta\right), 2\eta C_\delta, \frac{1}{\xi}\right\}.$$

PROOF. We observe that the mollified solution $v(x,y,t)$ satisfies

$$\begin{aligned}
v((i+1)h,jl,nk) &= v(ih,jl,nk) + \frac{h}{a(ih,jl,nk)}q(ih,jl,nk) + O\left(h^2\right),\\
q((i+1)h,jl,nk) &= q(ih,jl,nk) + h\left(v_t(ih,jl,nk)\right.\\
&\quad - \left(a_y(x_i,y_j,t_k)\frac{\partial}{\partial y}J_{\delta_i}v(ih,jl,nk)\right.\\
&\quad + a(x_i,y_j,t_k)\frac{\partial^2}{\partial y^2}J_{\delta_i}v(ih,jl,nk)\\
&\quad \left.\left. + f(ih,jl,nk)\right)\right) + O\left(h^2\right),\\
v_y((i+1)h,jl,nk) &= v_y(ih,jl,nk) + \frac{h}{a(x_i,y_j,t_k)}\left(q_y(ih,jl,nk)\right.\\
&\quad \left. -\frac{a_y(x_i,y_j,t_k)}{a(x_i,y_j,t_k)}q(ih,jl,nk)\right) + O\left(h^2\right),\\
v_t((i+1)h,jl,nk) &= v_t(ih,jl,nk) + \frac{h}{a(x_i,y_j,t_k)}q_t(ih,jl,nk) + O\left(h^2\right).
\end{aligned}$$

We compare the equalities above with those from the marching scheme. Let C_{δ_i}, $C_{\delta_i^*}$, and $C_{\hat{\delta}_i}$ represent the upper bounds, in magnitude, of the higher-order derivatives of the convolution kernels corresponding to the radii of mollification δ_i, δ_i^*, and $\hat{\delta}_i$, respectively, where $i = 0,1,\dots,M$. Define $C_\delta = \max_i\{C_{\delta_i}, C_{\delta_i^*}, C_{\hat{\delta}_i}\}$. We neglect the effect of δ-mollification on the already mollified solution q and its derivatives q_t, q_y, and f. The error estimates for the numerical variables $R^n_{i,j}, Q^n_{i,j}$, $W^n_{i,j}, P^n_{i,j}$, and $F^n_{i,j}$ are as follows:

$$\begin{aligned}
\Delta R^n_{i+1,j} &= \Delta R^n_{i,j} + \left(R^n_{i+1,j} - R^n_{i,j}\right) - \left(v((i+1)h,jl,nk) - v((ih,jl,nk)\right)\\
&= \Delta R^n_{i,j} + h\frac{Q^n_{i,j}}{a(x_i,y_j,t_n)} - \frac{h}{a(ih,jl,nk)}q(ih,jl,nk) + O\left(h^2\right)\\
&= \Delta R^n_{i,j} + \frac{h}{a(ih,jl,nk)}\left(Q^n_{i,j} - q(ih,jl,nk)\right) + O\left(h^2\right)\\
&= \Delta R^n_{i,j} + \frac{h}{a(ih,jl,nk)}\Delta Q^n_{i,j} + O\left(h^2\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|\Delta R^n_{i+1,j}\right| &\le |\Delta R_i| + \frac{h}{|a(ih,jl,nk)|}|\Delta Q_i| + O\left(h^2\right)\\
&\le |\Delta R_i| + \frac{h}{\xi}|\Delta Q_i| + O\left(h^2\right).
\end{aligned} \tag{4.7}$$

Also,

$$\begin{aligned}
\Delta Q_{i+1,j}^n &= \Delta Q_{i,j}^n + \left(Q_{i+1,j}^n - Q_{i,j}^n\right) - \left(q((i+1)h, jl, nk) - q(ih, jl, nk)\right) \\
&= \Delta Q_{i,j}^n + h\left(W_{i,j}^n - \left(a_y(x_i, y_j, t_n)P_{i,j}^n + a(x_i, y_j, t_n)Z_{i,j}^n + F_{i,j}^n\right)\right) \\
&\quad - h\left(v_t(ih, jl, nk) - \left(a_y(x_i, y_j, t_k)\frac{\partial}{\partial y}J_{\delta_i}v(ih, jl, nk)\right.\right. \\
&\quad \left.\left. +\ a(x_i, y_j, t_k)\frac{\partial^2}{\partial y^2}J_{\delta_i}v(ih, jl, nk) + f(ih, jl, nk)\right)\right) + O\left(h^2\right) \\
&= \Delta Q_{i,j}^n + h\left(\Delta W_{i,j}^n - (a_y(x_i, y_j, t_k)\Delta P_{i,j}^n + \Delta F_{i,j}^n\right. \\
&\quad \left. + a(x_i, y_j, t_k)\left(Z_{i,j}^n - \frac{\partial^2}{\partial y^2}J_{\delta_i}v(ih, jl, nk)\right)\right) + O\left(h^2\right).
\end{aligned}$$

Notice that

$$\begin{aligned}
\left|Z_{i,j}^n - \frac{\partial^2}{\partial y^2}J_\delta v(ih, jl, nk)\right| &= \left|\mathbf{D}_y\left(J_{\delta_i}P_{i,j}^n\right) - \frac{\partial}{\partial y}J_{\delta_i}v_y(ih, jl, nk)\right| \\
&\le \frac{C}{|\delta|_{-\infty}}\left|\Delta P_{i,j}^n\right| + \frac{C}{|\delta|_{-\infty}}l + C_\delta l^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|\Delta Q_{i+1,j}^n\right| &\le \left|\Delta Q_{i,j}^n\right| + h\left(\left|\Delta W_{i,j}^n\right| + |a_y(x_i, y_j, t_k)|\left|\Delta P_{i,j}^n\right| + \left|\Delta F_{i,j}^n\right|\right. \\
&\quad \left. + |a(x_i, y_j, t_k)|\left|Z_{i,j}^n - \frac{\partial^2}{\partial y^2}J_{\delta_i}v(ih, jl, nk)\right|\right) + O\left(h^2\right) \\
&\le |\Delta Q_i| + h(|\Delta W_i| + |a_y(x_i, y_j, t_k)|\,|\Delta P_i| + |\Delta F_i| \\
&\quad + |a(x_i, y_j, t_k)|\left(\frac{C}{|\delta|_{-\infty}}|\Delta P_i| + \frac{C}{|\delta|_{-\infty}}l + C_\delta l^2\right) + O\left(h^2\right) \\
&\le |\Delta Q_i| + h\left(|\Delta W_i| + \eta|\Delta P_i| + |\Delta F_i|\right. \\
&\quad \left. + \eta\left(\frac{C}{|\delta|_{-\infty}}|\Delta P_i| + \frac{C}{|\delta|_{-\infty}}l + C_\delta l^2\right)\right) + O\left(h^2\right)
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
\Delta P_{i+1,j}^n &= \Delta P_{i,j}^n + \left(P_{i+1,j}^n - P_{i,j}^n\right) - \left(v_y((i+1)h, jl, nk) - v_y(ih, jl, nk)\right) \\
&= \Delta P_{i,j}^n + \frac{h}{a(x_i, y_j, t_n)}\left(S_{i,j}^n - \frac{a_y(x_i, y_j, t_n)}{a(x_i, y_j, t_n)}Q_{i,j}^n\right) - \frac{h}{a(x_i, y_j, t_k)} \\
&\quad \times\left(q_y(ih, jl, nk) - \frac{a_y(x_i, y_j, t_k)}{a(x_i, y_j, t_k)}q(ih, jl, nk)\right) + O\left(h^2\right) \\
&= \Delta P_{i,j}^n + \frac{h}{a(x_i, y_j, t_n)}\left(\left(S_{i,j}^n - q_y(ih, jl, nk)\right)\right. \\
&\quad \left. - \frac{a_y(x_i, y_j, t_n)}{a(x_i, y_j, t_n)}\left(Q_{i,j}^n - q(ih, jl, nk)\right)\right) + O\left(h^2\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|\Delta P_{i+1,j}^n\right| &\le \left|\Delta P_{i,j}^n\right| + \frac{h}{|a(x_i, y_j, t_n)|}\left(\left|S_{i,j}^n - q_y(ih, jl, nk)\right|\right. \\
&\quad \left. + \frac{|a_y(x_i, y_j, t_n)|}{|a(x_i, y_j, t_n)|}\left|\Delta Q_{i,j}^n\right|\right) + O\left(h^2\right) \\
&\le |\Delta P_i| + \frac{h}{\xi}\left(\left|S_{i,j}^n - q_y(ih, jl, nk)\right| + \frac{\eta}{\xi}|\Delta Q_i|\right) + O\left(h^2\right).
\end{aligned}$$

Since

$$\begin{aligned}\left|S_{i,j}^n - q_y(ih,jl,nk)\right| &= \left|\mathbf{D}_y\left(J_{\delta_{i+1}^*}Q_{i+1,j}^n\right) - q_y(ih,jl,nk)\right| \\ &\le \frac{C}{|\delta|_{-\infty}}\left|\Delta Q_{i,j}^n\right| + \frac{C}{|\delta|_{-\infty}}l + C_\delta l^2 \\ &\le \frac{C}{|\delta|_{-\infty}}|\Delta Q_i| + \frac{C}{|\delta|_{-\infty}}l + C_\delta l^2,\end{aligned}$$

we obtain

$$\left|\Delta P_{i+1,j}^n\right| \le |\Delta P_i| + \frac{h}{\xi}\left(\left(\frac{C}{|\delta|_{-\infty}} + \frac{\eta}{\xi}\right)|\Delta Q_i| + \frac{C}{|\delta|_{-\infty}}l + C_\delta l^2\right)) + O\left(h^2\right). \tag{4.9}$$

Also,

$$\begin{aligned}\Delta W_{i+1,j}^n &= \Delta W_{i,j}^n + \left(W_{i+1,j}^n - W_{i,j}^n\right) - \left(v_t((i+1)h,jl,nk) - v_t(ih,jl,nk)\right) \\ &= \Delta W_{i,j}^n + \frac{h}{a(x_i,y_j,t_n)}O_{i,j}^n - \frac{h}{a(x_i,y_j,t_k)}q_t(ih,jl,nk) + O\left(h^2\right) \\ &= \Delta W_{i,j}^n + \frac{h}{a(x_i,y_j,t_n)}\left(O_{i,j}^n - q_t(ih,jl,nk)\right) + O\left(h^2\right) \\ &= \Delta W_{i,j}^n + \frac{h}{a(x_i,y_j,t_n)}\left(\mathbf{D}_t\left(J_{\delta_i^*}Q_{i,j}^n\right) - q_t(ih,jl,nk)\right) + O\left(h^2\right)\end{aligned}$$

and

$$\begin{aligned}\left|\Delta W_{i+1,j}^n\right| &\le \left|\Delta W_{i,j}^n\right| + \frac{h}{|a(x_i,y_j,t_n)|}\left|\mathbf{D}_t\left(J_{\delta_i^*}Q_{i,j}^n\right) - q_t(ih,jl,nk)\right| + O\left(h^2\right), \\ \left|\Delta W_{i+1,j}^n\right| &\le \left|\Delta W_{i,j}^n\right| + \frac{h}{|a(x_i,y_j,t_n)|}\left|\mathbf{D}_t\left(J_{\delta_i^*}Q_{i,j}^n\right) - q_t(ih,jl,nk)\right| + O\left(h^2\right).\end{aligned}$$

Moreover,

$$\begin{aligned}\left|\mathbf{D}_t\left(J_{\delta_i^*}Q_{i,j}^n\right) - q_t(ih,jl,nk)\right| &\le \frac{C}{|\delta|_{-\infty}}\left|\Delta Q_{i,j}^n\right| + \frac{C}{|\delta|_{-\infty}}k + C_\delta k^2 \\ &\le \frac{C}{|\delta|_{-\infty}}|\Delta Q_i| + \frac{C}{|\delta|_{-\infty}}k + C_\delta k^2,\end{aligned}$$

$$\left|\Delta W_{i+1,j}^n\right| \le |\Delta W_i| + \frac{h}{\xi}\left(\frac{C}{|\delta|_{-\infty}}|\Delta Q_i| + \frac{C}{|\delta|_{-\infty}}k + C_\delta k^2\right) + O\left(h^2\right). \tag{4.10}$$

Thus,

$$\begin{aligned}\Delta F_{i,j}^n &= F_{i,j}^n - f(ih,jl,nk) \\ &= W_{i,j}^n - a(x_i,y_j,t_n)\left(U_{i,j}^n + Z_{i,j}^n\right) - \frac{a_x(x_{i+1},y_j,t_n)}{a(x_{i+1},y_j,t_n)}Q_{i,j}^n - a_y(x_i,y_j,t_n)P_{i,j}^n \\ &\quad - \left(v_t(ih,jl,nk) - a(x_i,y_j,t_n)(v_{xx}(ih,jl,nk) + v_{yy}(ih,jl,nk))\right. \\ &\quad \left. - \frac{a_x(x_{i+1},y_j,t_n)}{a(x_{i+1},y_j,t_n)}q(ih,jl,nk) - a_y(x_i,y_j,t_n)v_y(ih,jl,nk)\right) \\ &= \left(W_{i,j}^n - v_t(ih,jl,nk)\right) - a(x_i,y_j,t_n)\left(\left(U_{i,j}^n - v_{xx}(ih,jl,nk)\right)\right. \\ &\quad \left. + \left(Z_{i,j}^n - v_{yy}(ih,jl,nk)\right)\right) - \frac{a_x(x_{i+1},y_j,t_n)}{a(x_{i+1},y_j,t_n)}\left(Q_{i,j}^n - q(ih,jl,nk)\right) \\ &\quad - a_y(x_i,y_j,t_n)\left(P_{i,j}^n - v_y(ih,jl,nk)\right) \\ &= \Delta W_{i,j}^n - a(x_i,y_j,t_n)\left(\left(U_{i,j}^n - v_{xx}(ih,jl,nk)\right) + \left(Z_{i,j}^n - v_{yy}(ih,jl,nk)\right)\right) \\ &\quad - \frac{a_x(x_{i+1},y_j,t_n)}{a(x_{i+1},y_j,t_n)}\Delta Q_{i,j}^n - a_y(x_i,y_j,t_n)\Delta P_{i,j}^n\end{aligned}$$

and it follows that

$$\begin{aligned}\left|\Delta F_{i,j}^n\right| &\le \left|\Delta W_{i,j}^n\right| + |a(x_i,y_j,t_n)|\left(\left|U_{i,j}^n - v_{xx}(ih,jl,nk)\right| + \left|Z_{i,j}^n - v_{yy}(ih,jl,nk)\right|\right) \\ &\quad + \frac{|a_x(x_{i+1},y_j,t_n)|}{|a(x_{i+1},y_j,t_n)|}\left|\Delta Q_{i,j}^n\right| + |a_y(x_i,y_j,t_n)|\left|\Delta P_{i,j}^n\right| \\ &\le |\Delta W_i| + \eta\left(\left|U_{i,j}^n - v_{xx}(ih,jl,nk)\right| + \left|Z_{i,j}^n - v_{yy}(ih,jl,nk)\right|\right) \\ &\quad + \frac{\eta}{\xi}|\Delta Q_i| + \eta|\Delta P_i|.\end{aligned}$$

Since

$$\begin{aligned}\left|U_{i,j}^n - v_{xx}(ih,jl,nk)\right| &= \left|\mathbf{D}_x^2\left(J_{\delta_{i+1}}R_{i,j}^n\right) - v_{xx}(ih,jl,nk)\right| \\ &\le \frac{C}{|\delta|_{-\infty}^2}\left|\Delta R_{i,j}^n\right| + \frac{C}{|\delta|_{-\infty}^2}h + C_\delta h^2 \\ &\le \frac{C}{|\delta|_{-\infty}^2}|\Delta R_i| + \frac{C}{|\delta|_{-\infty}^2}h + C_\delta h^2\end{aligned}$$

and

$$\begin{aligned}\left|Z_{i,j}^n - v_{yy}(ih,jl,nk)\right| &= \left|\mathbf{D}_y\left(J_{\delta_i}P_{i,j}^n\right) - \frac{\partial}{\partial y}J_{\delta_i}v_y(ih,jl,nk)\right| \\ &\le \frac{C}{|\delta|_{-\infty}}\left|\Delta P_{i,j}^n\right| + \frac{C}{|\delta|_{-\infty}}l + C_\delta l^2 \\ &\le \frac{C}{|\delta|_{-\infty}}|\Delta P_i| + \frac{C}{|\delta|_{-\infty}}l + C_\delta l^2,\end{aligned}$$

we have

$$\begin{aligned}|\Delta F_i| \le |\Delta W_i| + \eta\Bigg(&\frac{C}{|\delta|_{-\infty}^2}|\Delta R_i| + \frac{C}{|\delta|_{-\infty}}|\Delta P_i| + \frac{C}{|\delta|_{-\infty}^2}h + \frac{C}{|\delta|_{-\infty}}l \\ &+ C_\delta h^2 + C_\delta l^2\Bigg) + \frac{\eta}{\xi}|\Delta Q_i| + \eta|\Delta P_i|.\end{aligned} \tag{4.11}$$

Substituting these into (4.8), we obtain

$$\begin{aligned}\left|\Delta Q_{i+1,j}^n\right| \le |\Delta Q_i| + h\Bigg(&2|\Delta W_i| + \eta\left(2 + \frac{2C}{|\delta|_{-\infty}}\right)|\Delta P_i| + \eta\frac{C}{|\delta|_{-\infty}^2}|\Delta R_i| \\ &+ \frac{\eta}{\xi}|\Delta Q_i| + \eta\left(\frac{C}{|\delta|_{-\infty}^2}h + \frac{2C}{|\delta|_{-\infty}}l + C_\delta h^2 + 2C_\delta l^2\right)\Bigg) + O\left(h^2\right).\end{aligned}$$

Let $\hat{\Delta}_i = \max\{|\Delta Q_i|, |\Delta W_i|, |\Delta P_i|, |\Delta R_i|\}$. Then, using (4.11),

$$|\Delta F_i| \le \left(1 + \eta\left(\frac{C}{|\delta|_{-\infty}^2} + \frac{C}{|\delta|_{-\infty}} + \frac{1}{\xi} + 1\right)\right)\hat{\Delta}_i + \eta\left(\frac{C}{|\delta|_{-\infty}^2}h + \frac{C}{|\delta|_{-\infty}}l + C_\delta h^2 + C_\delta l^2\right) \tag{4.12}$$

and

$$\begin{aligned}\left|\Delta Q_{i+1,j}^n\right| \le &\left(1 + h\left(2 + \eta\left(2 + \frac{2C}{|\delta|_{-\infty}} + \frac{C}{|\delta|_{-\infty}^2} + \frac{1}{\xi}\right)\right)\right)\hat{\Delta}_i \\ &+ h\eta\left(\frac{C}{|\delta|_{-\infty}^2}h + \frac{2C}{|\delta|_{-\infty}}l + C_\delta h^2 + 2C_\delta l^2\right) + O\left(h^2\right).\end{aligned}$$

From (4.7), (4.9), and (4.10),

$$\begin{aligned}\left|\Delta R_{i+1,j}^n\right| &\le \left(1 + h\frac{1}{\xi}\right)\hat{\Delta}_i + O\left(h^2\right), \\ \left|\Delta P_{i+1,j}^n\right| &\le \left(1 + h\frac{1}{\xi}\left(\frac{C}{|\delta|_{-\infty}} + \frac{\eta}{\xi}\right)\right)\hat{\Delta}_i + h\frac{1}{\xi}\left(\frac{C}{|\delta|_{-\infty}}l + C_\delta l^2\right) + O\left(h^2\right), \\ \left|\Delta W_{i+1,j}^n\right| &\le \left(1 + h\frac{1}{\xi}\frac{C}{|\delta|_{-\infty}}\right)\hat{\Delta}_i + h\frac{1}{\xi}\left(\frac{C}{|\delta|_{-\infty}}k + C_\delta k^2\right) + O\left(h^2\right),\end{aligned}$$

and if we define

$$M_\delta = \max\left\{2+\eta\left(2+\frac{2C}{|\delta|_{-\infty}}+\frac{C}{|\delta|_{-\infty}^2}+\frac{1}{\xi}\right), \frac{1}{\xi}\left(\frac{C}{|\delta|_{-\infty}}+\frac{\eta}{\xi}+C_\delta\right), 2\eta C_\delta, \frac{1}{\xi}\right\},$$

we can write

$$\hat{\Delta}_{i+1} \le (1+M_\delta h)\hat{\Delta}_i + M_\delta h(l+k+h) + O\left(h^2\right).$$

Again, by calculating i iterations,

$$\begin{aligned}
\hat{\Delta}_i &\le (1+M_\delta h)^i \hat{\Delta}_i + \sum_{k=0}^{i-1}(1+M_\delta h)^k M_\delta h(l+k+h) + O\left(h^2\right) \\
&= (1+M_\delta h)^i \hat{\Delta}_0 + \frac{(1+M_\delta h)^i - 1}{1+M_\delta h - 1} M_\delta h(l+k+h) + O\left(h^2\right) \\
&\le \exp(M_\delta)\left(\hat{\Delta}_0 + l + k + h\right).
\end{aligned}$$

Applying this to (4.12),

$$|\Delta F_i| \le M_\delta \exp(M_\delta)\left(\hat{\Delta}_0 + l + k + h\right) + 2M_\delta(h+l).$$

Since $\Delta_i = \max\{\hat{\Delta}_i, |F_i|\}$, there exist constants

$$C_1 = M_\delta \exp(M_\delta)$$

and

$$C_2 = M_\delta(l+k+h) + 2M_\delta(h+l),$$

such that

$$\|\Delta\|_\infty \le C_1\Delta_0 + C_2.$$

COROLLARY 1. *If the hypotheses of the above theorem hold, then for fixed* δ, $\lim_{\epsilon,h,k,l\to 0}\|\Delta\|_\infty = 0$.

PROOF. Since $\Delta_0 \le (C/|\delta|_{-\infty})(\epsilon + l + k)$ and $\|\Delta\|_\infty \le C_1\Delta_0 + C_2$, with $C_1 = M_\delta\exp(M_\delta)$ and $C_2 = M_\delta(l+k+h) + 2M_\delta(h+l)$ independent of ϵ, l, k, or h, for fixed δ, $\|\Delta\|_\infty$ approaches zero when ϵ, l, k, and h tend to zero.

4.2. Numerical Results

The algorithm of Section 3, with parameters $x_{\max} = 0.2$, $\Delta x = \Delta y = \Delta t = 1/64,\ 1/128$, $p = 3$, $\varepsilon = 0.000,\ 0.005$, and 0.010, has been applied to approximately solve the following problem.

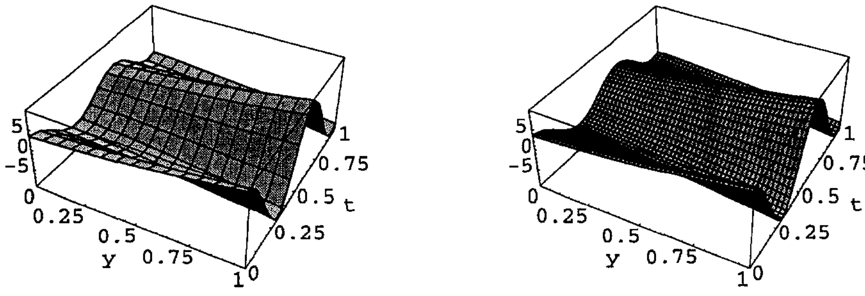
Identify $u(x,y,t), \nabla u(x,y,t)$, and $f(x,y,t)$ satisfying

$$\begin{aligned}
u_t &= \Delta u(x,y,t) + f(x,y,t), && 0<x<x_{\max}, \quad 0<y<1, \quad 0<t<1 \\
u(0,y,t) &= e^{1+y}\cos(10t), && 0\le y\le 1, \quad 0\le t\le 1, \\
u_x(0,y,t) &= e^{1+y}\cos(10t), && 0\le y\le 1, \quad 0\le t\le 1, \\
f(0,y,t) &= -2e^{y+1}(\cos(10t)+5\sin(10t)), && 0\le y\le 1, \quad 0\le t\le 1.
\end{aligned}$$

The exact solution functions are $u(x,y,t) = e^{x+y+1}\cos(10t)$ and $f(x,y,t) = -2e^{x+y+1}(\cos(10t) + 5\sin(10t))$, respectively.

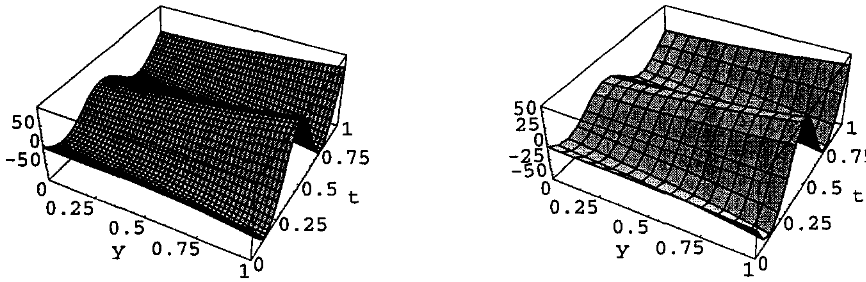
Table 1. Relative l_2 error norms at $x = 0.2$.

M	N	T	ϵ	f	u	u_t	u_x	u_{xx}	u_{yy}
64	64	64	0.000	0.1469	0.0654	0.1054	0.1061	0.1537	0.1581
64	64	64	0.005	0.1600	0.0773	0.1038	0.1189	0.1973	0.1878
64	64	64	0.010	0.1578	0.0731	0.1005	0.1215	0.1859	0.1894
128	128	128	0.000	0.1421	0.0478	0.0972	0.1060	0.1567	0.1574
128	128	128	0.005	0.1642	0.0765	0.1062	0.1267	0.1974	0.1932
128	128	128	0.010	0.1553	0.0634	0.1035	0.1291	0.1895	0.1804



(a) (b)

Figure 1. Exact and computed temperatures at $x = 0.2$, with parameters $M = N = T = 128$, $\epsilon = 0.005$.



(a) (b)

Figure 2. Exact and computed source terms at $x = 0.2$, with parameters $M = N = T = 128$, $\epsilon = 0.005$.

For these examples, Table 1 and Figures 1 and 2 illustrate, respectively, the quantitative and qualitative behavior of the method. We observe that continuous dependency with respect to errors in the data (at $x = 0$) has been restored.

REFERENCES

1. Z. Yi and D.A. Murio, Source term identification in the 1-D IHCP, *Computers Math. Applic.* (to appear).
2. C. Coles and D.A. Murio, Simultaneous space diffusivity and source term reconstruction in 2D IHCP, *Computers Math. Applic.* **42** (12), 1549–1564, (2001).
3. C.E. Mejía and D.A. Murio, Mollified hyperbolic method for coefficient identification problems, *Computers Math. Applic.* **26** (5), 1–12, (1993).
4. D.A. Murio, *Mollification and Space Marching, Inverse Engineering Handbook*, Chapter 4, (Edited by K. Woodbury), pp. 219–326, CRC Press, Boca Ratón, FL, (2002).
5. J.R. Cannon and P. Du Chateau, Inverse problems for an unknown source in the heat equation, *Journal of Mathematical Analysis and Applications* **75**, 465–485, (1980).
6. R. Ewing and T. Lin, Parameter identification problems in single-phase and two-phase flow, In *International Series of Numerical Mathematics, Volume 91*, pp. 85–108, Birkhäuser Verlag, Basel, (1989).
7. A. Nanda and P. Das, Determination of the source term in the heat conduction equation, *Inverse Problems* **12**, 325–339, (1996).

8. V. Isakov, *Inverse Source Problems*, American Mathematical Society, Providence, RI, (1990).
9. D.A. Murio, C.E. Mejía and S. Zhan, Discrete mollification and automatic numerical differentiation, *Computers Math. Applic.* **35** (5), 1–16, (1998).