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Discrete Mathematics 308 (2008) 1319 – 1327

DISCRETE MATHEMATICS

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Minimal non-1-planar graphs

Vladimir P. Korzhik

National University of Chernivtsi, Chernivtsi 58012, Ukraine

Received 23 October 2006; received in revised form 20 March 2007; accepted 2 April 2007
Available online 7 April 2007

Abstract

A graph is 1-*planar* if it can be drawn on the plane so that each edge is crossed by no more than one other edge. A non-1-planar graph G is *minimal* if the graph $G-e$ is 1-planar for every edge e of G. We prove that there are infinitely many minimal non-1-planar graphs (MN-graphs). It is known that every 6-vertex graph is 1-planar. We show that the graph $K_7 - K_3$ is the unique 7-vertex MN-graph.

MSC: 05C10

Keywords: 1-planar graph; Minimal non-1-planar graph

1. Introduction

A graph is 1-*immersed* into a surface if it can be drawn on the surface so that each edge is crossed by no more than one other edge. A graph is 1-*planar* if it is 1-immersed into the plane (has a plane 1-immersion).

The notion of 1-immersion of a graph was introduced by Ringel [9] when trying to color the vertices and faces of a plane graph so that adjacent or incident elements receive distinct colors (here we have a 1-planar graph G whose vertex set is the union of the vertex set and the face set of the plane graph, and two vertices of G are adjacent if and only if the corresponding elements are adjacent or incident).

Little is known about 1-planar graphs. Borodin [1,2] proved that every 1-planar graph is 6-colorable. Some properties of maximal 1-planar graphs are considered in [10]. It was shown in [3] that every 1-planar graph is acyclically 20-colorable. The existence of subgraphs of bounded vertex degrees in 1-planar graphs is investigated in [7]. It was shown in [4,5] that a 1-planar graph with n vertices has at most $4n-8$ edges and that this upper bound is tight. In the paper [6] it was observed that the class of 1-planar graphs is not closed under the operation of edge deletion. Much less is known about non-1-planar graphs. The only result [8] (not counting the Borodin's result that every non-6-colorable graph is non-1-planar) is that the graph obtained from the complete graph K_6 by adding a new vertex adjacent to exactly four vertices of the K_6 is non-1-planar.

A graph G is a *minimal* non-1-planar graph (MN-graph, for short) if G is non-1-planar, but $G-e$ is 1-planar for every edge e of G. In this paper we study MN-graphs.

The problem of constructing MN-graphs is connected with the problem of determining whether a given graph is non-1-planar. By Kuratowski's theorem, planarity can be characterized by forbidding the minors K_5 and $K_{3,3}$. Here

E-mail address: korzhik@sacura.net.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.04.009

the following natural question arises: Can 1-planarity be characterized by forbidding some minors? The answer to the question is in the negative since there is a 1-planar subdivision of every graph G. Indeed, take a plane drawing of G and then for every edge with at least two crossing points, place on the edge a new 2-valent vertex between every pair of adjacent crossing points. We obtain a 1-planar subdivision of G. In other words, the set of 1-planar graphs is not closed under taking minors.

Every non-1-planar graph G contains as a subgraph some MN-graph which can be obtained by deleting edges from G trying to preserve non-1-planarity of the obtained graphs. If there were only finitely many MN-graphs, then we would have the following good characterization of non-1-planar graphs: a graph is non-1-planar if and only if it contains as a subgraph some of the MN-graphs. However, in Section 2 we show that there are infinitely many MN-graphs.

Now, when we face a problem of determining whether a given graph G is not 1-planar, we can use two approaches. We can try to show that the chromatic number of G is at least 7; then, by the Borodin's result, the graph G is not 1-planar. But it is not easy to determine the chromatic number of a graph and, besides, many non-1-planar graphs are 6-colorable. Then we can try to show that G contains as a subgraph some graph H such that we already know that H is non-1-planar. As such a graph H, it is convenient to use graphs with small number of vertices and edges, say, MN-graphs. Surprisingly enough, as far as we know, no MN-graphs were given in the literature.

Since K_6 has a plane 1-immersion, all 6-vertex graphs are 1-planar. In Section 3 we show that the graph $K_7 - K_3$ is the unique 7-vertex MN-graph.

Having no characterization of non-1-planar graphs, it is difficult to show that a graph is non-1-planar. Here the main question is: What structural properties of a graph make the graph to be non-1-planar? One of such structural properties is implemented in Section 2 when constructing infinitely many MN-graphs: a non-1-planar graph G has a subgraph Q with a unique plane 1-immersion, and the 1-immersion of Q prevents to draw the remaining part of G on the plane so that to obtain a plane 1-immersion of G.

2. An infinite family of minimal non-1-planar graphs

In this section we first give some definitions. Then we construct an infinite family of MN-graphs.

From this point on, by a 1-immersion of a graph we mean a plane 1-immersion. The notation $(v_1, v_2, \ldots, v_n)$ denotes the cycle of a graph consisting of edges (v_i, v_{i+1}), $i = 1, 2, \ldots, n$, where $v_{n+1} = v_1$.

The boundary of a face of an embedding of a graph G is a closed walk in G called the *boundary walk* of the face. The boundary walk of a face has two opposite directions. A face of an embedding of G will be designated as a cyclic sequence $[v_1, v_2, \ldots, v_m]$ of vertices (for convenience, we enclose the sequence in brackets) obtained by listing the incident boundary vertices when traversing the boundary walk of the face in some chosen direction. The sequences $[v_1, v_2, \ldots, v_m]$ and $[v_m, \ldots, v_2, v_1]$ designate the same face. By the face set of a plane embedding we mean the set of the boundary walks of all faces of the embedding.

Given a 1-immersion of a graph, delete all pairs of crossing edges. We obtain a plane embedding of a subgraph of the graph called the *spanning embedding* corresponding to the 1-immersion. The 1-immersion is uniquely determined by the face set of the spanning embedding and by pairs of crossing edges lying inside the faces.

When we say that a graph has exactly one 1-immersion, we mean that for every two 1-immersions φ and φ' of the graph, the following holds:

- If $\overline{\varphi}$ and $\overline{\varphi'}$ are spanning embeddings of φ and φ', respectively, then the embeddings are embeddings of the same graph and have the same face set.
- The same pairs of crossing edges lie inside every faces F and F' of $\overline{\varphi}$ and $\overline{\varphi'}$, respectively, such that the faces have the same boundary walk.

Considering the plane, when we say that a vertex lies *inside* (resp. *outside*) a face or a non-self-intersecting embedded cycle, we mean that the vertex lies in the interior (resp. exterior) of the face or the embedded cycle, and does not lie on the boundary walk of the face or on the embedded cycle, respectively. A vertex lies *in* a face if the vertex lies either inside the face or on the boundary walk of the face.

The following lemma gives a method of proving the existence of an MN-graph whose edge number lies in some known interval.

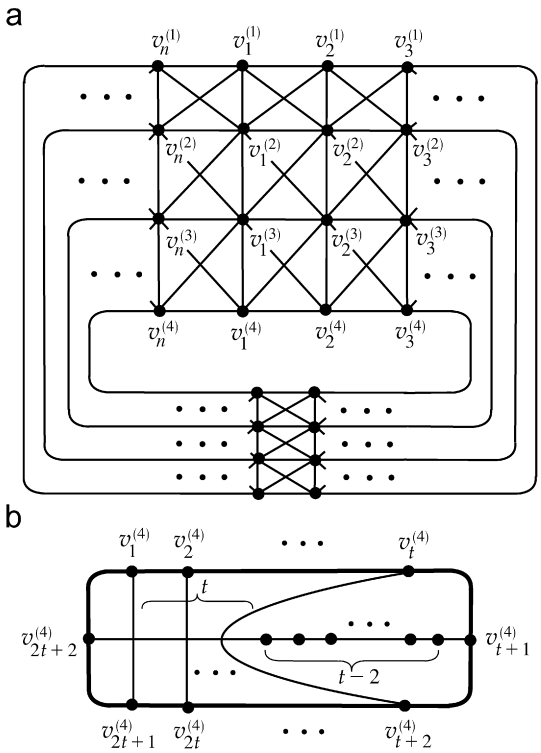

Fig. 1. The graph $G(t)$, $t \geqslant 2$.

Lemma 1. *If there is a non-1-planar graph G such that $G - e$ is 1-planar for some m edges e of G, then the graph G contains as a subgraph an MN-graph P such that $m \leqslant |E(P)| \leqslant |E(G)|$.*

Proof. Deleting successively some edges from G such that the obtained graphs are non-1-planar, we will finally obtain an MN-graph P which is a subgraph of G. Suppose that there is a subset $\widetilde{E} \subseteq E(G)$ with $|\widetilde{E}| = m$ such that $G - e$ is 1-planar for all $e \in \widetilde{E}$. Clearly, $\widetilde{E} \subseteq E(P)$, whence the lemma follows. $\square$

In this section, for every $t \geqslant 2$ we construct a non-1-planar graph $G(t)$ with $|E(G(t))| = 28t + 25$, such that the graph $G(t) - e$ is 1-planar for some $2t - 1$ edges e of $G(t)$.

Theorem 1. *There are infinitely many MN-graphs.*

Proof. The graphs $G(t)$ having been constructed for every $t \geqslant 2$, then, by Lemma 1, for every $t \geqslant 2$ there is an MN-graph $R(t)$ with $2t - 1 \leqslant |E(R(t))| \leqslant 28t + 25$. Now, for every infinite sequence $2 \leqslant t_1 < t_2 < \cdots < t_i < t_{i+1} < \cdots$ of integers such that $28t_i + 25 < 2t_{i+1} - 1$ for $i = 1, 2, \ldots$, we obtain an infinite sequence $R(t_1), R(t_2), \ldots, R(t_i), R(t_{i+1}) \ldots$ of different MN-graphs such that $2t_i - 1 \leqslant |E(R(t_i))| \leqslant 28t_i + 25 < 2t_{i+1} - 1 \leqslant |E(R(t_{i+1}))|$ for $i = 1, 2, \ldots$. $\square$

The non-1-planar graph $G(t)$, $t \geqslant 2$, has a subgraph $Q(t)$ which has exactly one 1-immersion, and the 1-immersion of $Q(t)$ prevents to draw the remaining part of $G(t)$ on the plane so that to obtain a 1-immersion of $G(t)$.

Fig. 1(a) shows a 1-immersion of $Q(t)$ (to avoid clattering the figure and the text we denote $n = 2t + 2$). The spanning embedding has $3n$ quadrangular faces and the 1-immersion of $Q(t)$ is obtained if we insert a pair of crossing diagonals into each of the faces. The *side edges* join 5-valent vertices. We will show (Theorem 2) that $Q(t)$ has exactly one 1-immersion. The graph $G(t)$ is obtained from $Q(t)$ if we add t new edges $(v_i^{(4)}, v_{n-i}^{(4)})$, $i = 1, 2, \ldots, t$, and join the vertices $v_{t+1}^{(4)}$ and $v_n^{(4)}$ by a path consisting of $t - 1$ edges (see Fig. 1(b) where all the edges and the path are depicted

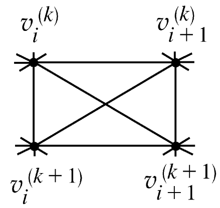

Fig. 2. A subgraph K_4 of the graph $Q(t)$.

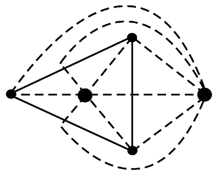

Fig. 3. Connecting two vertices by 6 disjoint paths.

inside the 2-cell bounded by the cycle $(v_1^{(4)}, v_2^{(4)}, \ldots, v_n^{(4)})$). Clearly, $G(t) - e$ is 1-planar for each of the $2t-1$ new edges e shown inside the cycle.

Now we show that $G(t)$ is non-1-planar. Indeed, if $G(t)$ is 1-planar, then every 1-immersion of $G(t)$ is obtained if we take the unique 1-immersion of $Q(t)$ and place the new t edges and the new path in such a way as to obtain a 1-immersion of $G(t)$. Since the new edges and the new path do not cross the crossing diagonals of the 1-immersion of $Q(t)$, there must be a 1-immersion of $G(t)$ in which the new edges and path are placed inside the 2-cell with the boundary cycle $(v_1^{(4)}, v_2^{(4)}, \ldots, v_n^{(4)})$ such that every edge is crossed by no more than one other edge, but it is impossible since the path can be crossed by at most $t-1$ new edges (see Fig. 1(b)).

Before proving Theorem 2, we need Lemmas 2–5.

Two paths connecting vertices v and w of a graph are *disjoint* if v and w are the only common vertices of the paths.

By inspecting Fig. 1(a), we obtain the following observation.

Observation 1. *The vertex sets of the subgraphs K_4 of $Q(t)$ are exactly all the sets*

$$V_i^{(k)} = \{v_i^{(k)}, v_{i+1}^{(k)}, v_{i+1}^{(k+1)}, v_i^{(k+1)}\}$$

for all $i = 1, 2, \ldots, n$, and $k = 1, 2, 3$ (here $v_{n+1}^{(k)} = v_1^{(k)}$ for $k = 1, 2, 3$), see Fig. 2. For every $V_i^{(k)}$, either all vertices of the $V_i^{(k)}$ are 8-valent or exactly two vertices are 5-valent and the other two vertices are 8-valent.

Lemma 2. *In every 1-immersion of $Q(t)$, for every embedded 3-cycle of $Q(t)$, either there is no 8-valent vertex of $Q(t)$ inside the cycle or there is no 8-valent vertex of $Q(t)$ outside the cycle.*

Proof. If there is an embedded 3-cycle such that one 8-valent vertex lies inside the cycle and one other 8-valent vertex lies outside the cycle, then the two vertices can be connected by at most 6 disjoint paths (see Fig. 3). Now, to prove the lemma it suffices to show that in $Q(t)$ every two 8-valent vertices v and w are connected by 7 disjoint paths. Fig. 4 shows these 7 paths. In the figure, the vertices v and w are encircled and the edges of the paths are depicted in thick lines. Recall that in Fig. 1(a) we have $n = 2t + 2 \geqslant 6$. □

Lemma 3. *In every 1-immersion of $Q(t)$, two edges of every subgraph K_4 of $Q(t)$ intersect.*

Proof. Suppose to the contrary that given a 1-immersion of $Q(t)$, there is a subgraph K_4 of $Q(t)$ such that no two edges of the subgraph intersect. Then in the 1-immersion the subgraph K_4 is triangularly embedded in the plane (see Fig. 5(a)). By Lemma 2, all 8-valent vertices of $Q(t)$ lie in one face of the triangular embedding, say, $[y, v, w]$. Then

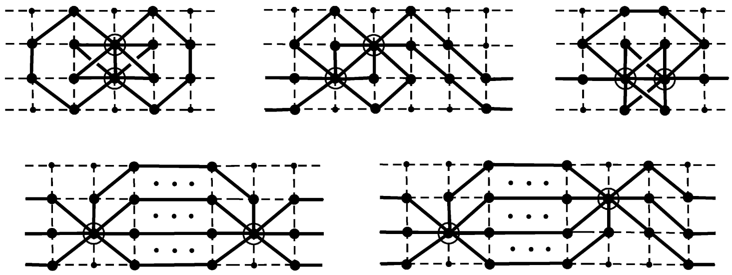

Fig. 4. Seven disjoint paths joining two 8-valent vertices of $Q(t)$.

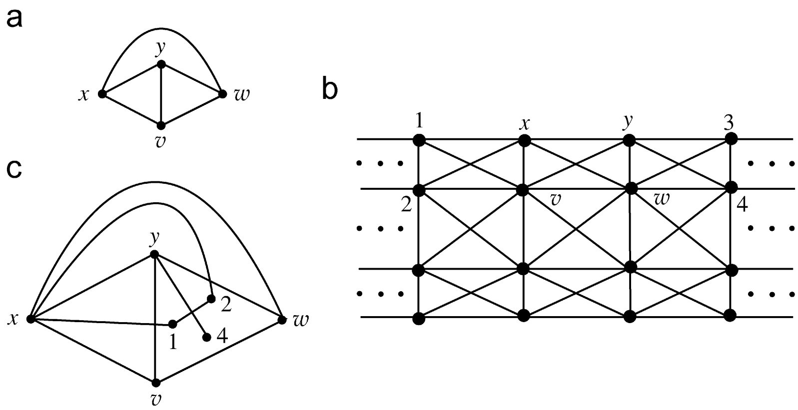

Fig. 5. A triangular embedding of a subgraph K_4 of $Q(t)$.

the vertex x is 5-valent and, by Observation 1, one other vertex of the K_4, say, the vertex y, is 5-valent also. Without loss of generality, let the vertices of the K_4 be located in $Q(t)$ as shown in Fig. 5(b). We have the following:

(A) The vertex x is adjacent to the 8-valent vertex 2 lying inside the face $[y, v, w]$ in Fig. 5(a), hence the edge $(x, 2)$ crosses some edge of the 3-cycle (y, v, w).

Every 5-valent vertex z different from the vertices 1, x, y, and 3 is adjacent to three 8-valent vertices different from v and w, hence, taking (A) into account, we obtain that the vertex z lies inside the face $[y, v, w]$. Now we have that each of the vertices 1 and 3 is adjacent to three vertices lying inside the face $[y, v, w]$: two 8-valent vertices (different from v and w) and one 5-valent vertex. Hence, taking (A) into account, the vertices 1 and 3 must lie inside the face $[y, v, w]$ also. We obtain that all vertices of $Q(t)$, except the 5-valent vertex x, lie in the face $[y, v, w]$.

Now consider the 3-cycle $(x, 1, 2)$. The edges $(x, 1)$ and $(x, 2)$ cross distinct edges of the cycle (y, v, w) and, as a result, one of the vertices y,v, and w lies inside the cycle. Since 8-valent vertices are not separated by an embedded 3-cycle, the 5-valent vertex y lies inside the cycle $(x, 1, 2)$ (see Fig. 5(c)) and no 8-valent vertex lies inside the cycle $(x, 1, 2)$. The vertex y is adjacent to the 8-valent vertex 4, hence, the edge $(y, 4)$ crosses the edge $(1, 2)$. The vertex y is adjacent to the vertex 3. Since the edges of the cycle $(x, 1, 2)$ are crossed by the edges (y, v), (y, w), and $(y, 4)$, respectively, the vertex 3 lies inside the cycle. But then the vertex 3 is not adjacent to any 8-valent vertex, a contradiction. □

It follows from Lemma 3 that in every 1-immersion of $Q(t)$, for every subgraph K_4 of $Q(t)$, some 4-cycle of the K_4 is quadrangularly embedded in the plane and two crossing diagonals are inserted in one of the two faces of the embedding. The 4-gonal face in which the crossing diagonals are inserted is called the 4-*cross* of the subgraph K_4 of $Q(t)$ in the 1-immersion of $Q(t)$.

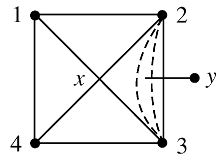

Fig. 6. The interior of the 4-cross of a subgraph K_4 containing some vertices of $Q(t)$.

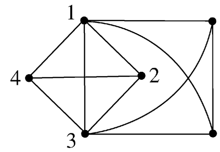

Fig. 7. An intersected edge of a 4-cross is an Ω-edge.

Lemma 4. *In every* 1-*immersion of* $Q(t)$, *for every subgraph* K_4 *of* $Q(t)$, *there are no vertices of* $Q(t)$ *inside the* 4-*cross of the* K_4.

Proof. First we show that $Q(t)$ is 5-connected. Suppose, for a contradiction, that there is a cutset W of $Q(t)$ such that $|W| < 5$. It was shown in the proof of Lemma 2 that every two 8-valent vertices of $Q(t)$ are connected by 7 disjoint paths. Hence, one of the connected components of $Q(t) - W$ does not contain 8-valent vertices, that is, the connected component is a path on $m \geqslant 1$ vertices such that all vertices of the path are 5-valent in $Q(t)$. It is easy to see that if $m < 5$ (resp. $m \geqslant 5$), then in $Q(t)$ the vertices of the path are adjacent to $m + 4 \geqslant 5$ (resp. to at least m) vertices of W, a contradiction. Hence, $Q(t)$ is 5-connected.

Now, suppose, for a contradiction, that in some 1-immersion of $Q(t)$, for some subgraph K_4 of $Q(t)$, there are vertices of $Q(t)$ inside the 4-cross of the K_4, say, inside the cycle $(x, 2, 3)$ (see Fig. 6, where $[1, 2, 3, 4]$ is the 4-cross of the K_4 and x is the crossing point). Then there can be at most one edge crossing the edge $(2, 3)$ and joining a vertex inside the cycle $(x, 2, 3)$ with some vertex y outside the 4-cross. Hence, the set $\{2, 3, y\}$ contains a cutset of $Q(t)$, a contradiction, since $Q(t)$ is 5-connected. □

An edge that a subgraph K_4 of $Q(t)$ shares with another subgraph K_4 of $Q(t)$ is called an *Ω-edge*.

Lemma 5. *In every* 1-*immersion of* $Q(t)$, *for every subgraph* K_4 *of* $Q(t)$, *the two intersected edges of the subgraph are not Ω-edges*.

Proof. Suppose that in some 1-immersion of $Q(t)$, for some subgraph K_4 of $Q(t)$, in the 4-cross $[1, 2, 3, 4]$ of the K_4 the intersected edge $(1, 3)$ is an Ω-edge (see Fig. 7). Then one of the vertices 2 and 4 lie inside the 4-cross of another subgraph K_4 of $Q(t)$, contrary to Lemma 4. □

Lemma 6. *In every* 1-*immersion of* $Q(t)$, *for* $i = 1, 2, \ldots, n$, *and* $k = 1, 2, 3$, *the* 4-*cross of the subgraph* K_4 *of* $Q(t)$ *with vertex set* $V_i^{(k)}$ *is*

$$[v_i^{(k)}, v_{i+1}^{(k)}, v_{i+1}^{(k+1)}, v_i^{(k+1)}]. \tag{1}$$

Proof. Consider an arbitrary 1-immersion of $Q(t)$.

If all vertices of $V_i^{(k)}$ are 8-valent, then the subgraph K_4 with vertex set $V_i^{(k)}$ has exactly four Ω-edges, and the Ω-edges form a 4-cycle

$$(v_i^{(k)}, v_{i+1}^{(k)}, v_{i+1}^{(k+1)}, v_i^{(k+1)}). \tag{2}$$

By Lemma 5, the boundary walk of the 4-cross of the K_4 is a 4-cycle consisting of all four Ω-edges. But the K_4 has only one such cycle, namely, the cycle (2). Hence, the 4-cross of the K_4 with vertex set $V_i^{(k)}$ is (1).

If two vertices of $V_i^{(k)}$ are 5-valent, the subgraph K_4 with vertex set $V_i^{(k)}$ has exactly three Ω-edges, and the Ω-edges together with the side edge of the K_4 form the 4-cycle (2). The crossing diagonals in the 4-cross of the K_4 are nonadjacent edges of the K_4. The only edge of the K_4 not adjacent to the side edge is an Ω-edge. Hence, by Lemma 5, the boundary walk of the 4-cross of the K_4 is a 4-cycle consisting of all three Ω-edges and the side edge. But the K_4 has only one such 4-cycle, namely, the cycle (2). Hence, the 4-cross of the K_4 with vertex set $V_i^{(k)}$ is (1). □

Theorem 2. *The graph $Q(t)$, $t \geqslant 2$, has exactly one* 1*-immersion.*

Proof. It follows from Lemma 4, that given an arbitrary 1-immersion φ of $Q(t)$, the 4-cross of a subgraph K_4 of $Q(t)$ is a face of the spanning embedding $\overline{\varphi}$ corresponding to the 1-immersion. Then, by Lemma 6, we obtain that (1) is a face of $\overline{\varphi}$ for $i = 1, 2, \ldots, n$, and $k = 1, 2, 3$, and the 4-gonal faces of $\overline{\varphi}$ are exactly the 4-gonal faces of the spanning embedding $\overline{\psi}$ corresponding to the 1-immersion ψ of $Q(t)$ given in Fig. 1(a). Each of the remaining faces of $\overline{\varphi}$ is bounded by a closed walk of $Q(t)$ consisting of side edges only (in $\overline{\varphi}$ every nonside edge belongs to two 4-gonal faces). But the side edges of $Q(t)$ form two disjoint cycles $(v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)})$ and $(v_1^{(4)}, v_2^{(4)}, \ldots, v_n^{(4)})$. Hence, the remaining faces of $\overline{\varphi}$ are determined uniquely, they are the same as in $\overline{\psi}$,namely, $[v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}]$ and $[v_1^{(4)}, v_2^{(4)}, \ldots, v_n^{(4)}]$, and, in φ and ψ, no crossing edges are inserted in the two n-gonal faces. Hence, $Q(t)$ has exactly one 1-immersion. □

3. The 7-vertex minimal non-1-planar graph

In this section we show that the graph $K_7 - K_3$ is the unique 7-vertex MN-graph.

Lemma 7. *The graph $K_7 - K_3$ is not* 1*-planar.*

Proof. The graph K_5 has exactly one (up to isomorphism) 1-immersion shown in Fig. 8. Given a 1-immersion of $K_6 - K_2$, deleting a 4-valent vertex we obtain the 1-immersion of K_5. It is easy to see that there are only three ways to place the sixth (4-valent) vertex in Fig. 8 to obtain non-isomorphic 1-immersions of $K_6 - K_2$; these three non-isomorphic 1-immersions of $K_6 - K_2$ are shown in Fig. 9 where the two non-adjacent vertices are encircled.

If there is a 1-immersion of $K_7 - K_3$, then deleting a 4-valent vertex in the 1-immersion we obtain one of the three 1-immersions of $K_6 - K_2$ shown in Fig. 9. But the reader can check that there is no way to place the seventh (4-valent) vertex in each of the 1-immersions shown in Fig. 9 to obtain a 1-immersion of $K_7 - K_3$. □

For a graph H with at most 7 vertices, denote by $K_7 - H$ the graph obtained from K_7 by deleting all edges of a subgraph H of K_7. Clearly, for a subgraph H' of H, if $K_7 - H'$ is 1-planar, then $K_7 - H$ is 1-planar too.

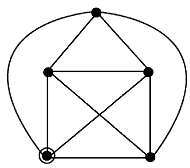

Fig. 8. The 1-immersion of K_5.

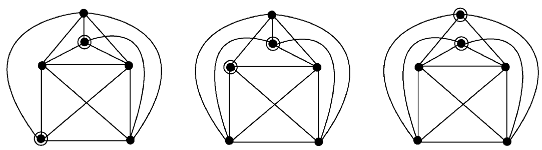

Fig. 9. Three non-isomorphic 1-immersions of $K_6 - K_2$.

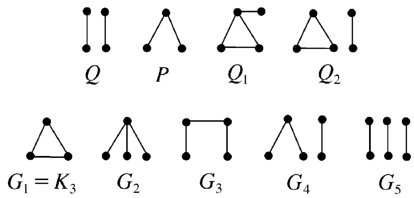

Fig. 10. Graphs with small number of edges.

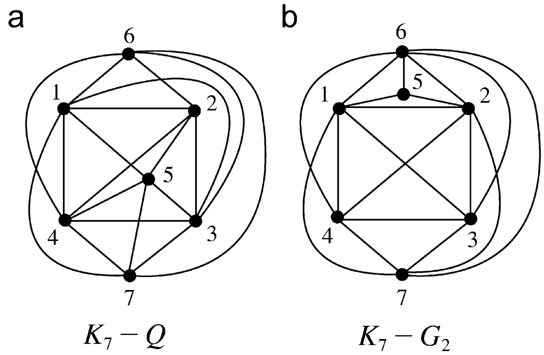

Fig. 11. Plane 1-immersions of $K_7 - Q$ and $K_7 - G_2$.

Theorem 3. *The graph $K_7 - K_3$ is the unique 7-vertex MN-graph.*

Proof. First we show that $K_7 - K_3$ is an MN-graph. Because of the symmetry of the graph, it suffices to show that the graphs $K_7 - Q_1$ and $K_7 - Q_2$ are 1-planar (see Fig. 10). Fig. 11(a) shows a 1-immersion of $K_7 - Q$ (two non-adjacent edges (2, 7) and (5, 6) are missing). Since the graph Q is a subgraph of Q_1 and Q_2, the graphs $K_7 - Q_1$ and $K_7 - Q_2$ are 1-planar.

Now we show that $K_7 - K_3$ is a unique MN-graph. Since the graph P (see Fig. 10) is a subgraph of K_3, the graphs K_7, K_7 minus one edge, and $K_7 - P$ are non-1-planar and are not MN-graphs (that is, all 7-vertex graphs with 21, 20, and 19 edges are non-1-planar and are not MN-graphs). Every 7-vertex graph with 18 edges is obtained from K_7 by deleting three edges. These three edges induce one of the five graphs $G_1, G_2, \dots, G_5$ shown in Fig. 10. A 1-immersion of $K_7 - G_2$ is shown in Fig. 11(b) (the edges (5, 3), (5, 4) and (5, 7) are missing). The graph Q is a subgraph of G_3, G_4, and G_5, hence the graphs $K_7 - G_i$, $i = 3, 4, 5$, are 1-planar. We obtain that $K_7 - K_3$ is the unique non-1-planar graph with 18 edges. It follows that all 7-vertex graphs with 17 edges are 1-planar. The proof is complete. $\square$

One can wonder what other MN-graphs with more vertices can look like. It is not difficult to show that the 8-vertex graph obtained from $K_7 - K_3$ if we place a new 2-valent vertex on an edge joining two 6-valent vertices is an MN-graph.

Acknowledgment

The author thanks the referees for many helpful comments and suggestions.

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