Global Stability for Two-Species Lotka–Volterra Systems with Delay

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In this paper, a two-species delayed Lotka–Volterra system without delayed intraspecific competitions is considered. It is proved that the system is globally stable for all off-diagonal delays \( \tau_{12}, \tau_{21} \geq 0 \), if and only if the interaction matrix of the system satisfies Condition WDD.

We study here the global stability of the two-species Lotka–Volterra system with discrete delay,

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t) \left[ r_2 + a_{11}x_1(t) + a_{12}x_2(t - \tau_{12}) \right], \\
\dot{x}_2(t) &= x_2(t) \left[ r_2 + a_{21}x_1(t - \tau_{21}) + a_{22}x_2(t) \right],
\end{align*}
\]

with initial conditions

\[
\begin{align*}
x_i(t) &= \phi_i(t) \geq 0, \ t \in [-\tau_0, 0]; \quad \phi_i(0) > 0, \ i = 1, 2.
\end{align*}
\]

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Here \(x_i\) represents the density of species \(i\), and \(r_i\) the reproduction rate, \(\tau_{ij} \geq 0\) \((i \neq j; \ i, j = 1, 2)\) the constant time lag, and \(\tau_0 = \max(\tau_{ij})\). \(a_{ij}\) \((i, j = 1, 2)\) is constant and \(\phi_i(t)\) \((i = 1, 2)\) continuous on \([-\tau_0, 0]\).

In the sequel, system (1) is supposed to have a unique positive equilibrium \(x^* = (x_1^*, x_2^*)\). For system (1), sufficient conditions are given to ensure the global stability of \(x^*,\) for example, [1]. It can be checked that all the known sufficient conditions to ensure the global stability of \(x^*\) are stronger than the following weakly diagonally dominant condition.

**Condition (WDD).** \(a_{11} < 0, \ a_{22} < 0, \ -a_{12}a_{21} \leq a_{11}a_{22}, \) and \(a_{12}a_{21} < a_{11}a_{22}\) if \(a_{12}a_{21} < 0\).

In fact, it will be proved that in some sense, **Condition (WDD)** is necessary for global stability of the system. The main result of this note is as follows.

**Theorem.** System (1) is globally stable for all \(\tau_{12}, \tau_{21} \geq 0\) if and only if **Condition (WDD)** holds.

**Proof.** Sufficiency. Since the system has a positive equilibrium \(x^* = (x_1^*, x_2^*)\), by using the transformation

\[
\tilde{x}_1 = x_1 - x_1^*, \quad \tilde{x}_2 = x_2 - x_2^*,
\]

it becomes

\[
\begin{align*}
\dot{\tilde{x}}_1(t) &= (x_1^* + x_1(t))[a_{11}x_1(t) + a_{12}x_2(t - \tau_{12})], \\
\dot{\tilde{x}}_2(t) &= (x_2^* + x_2(t))[a_{21}x_1(t - \tau_{21}) + a_{22}x_2(t)],
\end{align*}
\]

where we used \(x_i(t)\) instead of \(\tilde{x}_i(t), \ i = 1, 2.\) Following [4], we consider the Liapunov functional \(V: C(-\tau, 0)^2 \to R\) by

\[
V(\phi) = \sum_{i=1}^{2} c_i \left( \phi_i(0) - x_i^* \ln \frac{\phi_i(0) + x_i^*}{x_i^*} \right) + \sum_{i,j=1, i \neq j}^{2} d_{ij} \int_{-\tau_{ij}}^{0} \phi_j(\theta)d\theta,
\]

where \(c_1 = -2a_{11}a_{21}, \ c_2 = -2a_{11}a_{22}, \ d_{12} = a_{11}^2a_{21}^2, \) and \(d_{21} = a_{11}^2a_{22}^2.\) Since the case \(a_{12}a_{21} = 0\) is easy, we consider the case \(a_{12}a_{21} \neq 0.\) Now by **Condition (WDD)**,

\[
\dot{V}(\phi) \leq -a_{12}^2(a_{11}\phi_1(0) + a_{12}\phi_2(-\tau_{12}))^2 \\
- a_{11}^2(a_{21}\phi_1(-\tau_{21}) + a_{22}\phi_2(0))^2.
\]

Hence,

\[
E = \{\phi = (\phi_1, \phi_2): \dot{V}(\phi) = 0\} \\
= \{\phi = (\phi_1, \phi_2): a_{11}\phi_1(0) + a_{12}\phi_2(-\tau_{12}) = 0, \ a_{21}\phi_1(-\tau_{21}) + a_{22}\phi_2(0) = 0\}.
\]
To show the global stability, we just need to prove that the LaSalle’s invariant set $M$ contained in $E$ has only the trivial solution [2]. Now suppose that $x(t) = (x_1(t), x_2(t))$ is any solution in $M$, then it must satisfy

$$a_{11}x_1(t) + a_{12}x_2(t - \tau_{12}) = 0, \quad a_{21}x_1(t - \tau_{21}) + a_{22}x_2(t) = 0. \quad (4)$$

Clearly, system (3) and the equalities in (4) lead to

$$\dot{x}_i = 0 \quad i = 1, 2.$$ 

Hence, $x_i = 0$ for $i = 1, 2$.

Therefore, $x^*$ is globally stable for system (1).

Necessity. The characteristic equation of system (1) at $x^*$ takes the form

$$(z^2 + pz + q) + re^{-\tau} = 0, \quad (5)$$

where $p = a_{11}x_1^* + a_{22}x_2^*$, $q = a_{11}a_{22}x_1^*x_2^*$, $r = -a_{12}a_{21}x_1^*x_2^*$, and $\tau = \tau_{12} + \tau_{21}$.

When $\tau = 0$, (5) becomes

$$z^2 + pz + (q + r) = 0. \quad (6)$$

The uniqueness of the positive equilibrium $x^*$ implies $q + r \neq 0$. Since system (1) with $\tau_{12} = \tau_{21} = 0$ is globally stable and $q + r \neq 0$, the eigenvalues of (6) have negative real parts, namely

$$p > 0, \quad \text{and} \quad q + r > 0. \quad (7)$$

Here we used the fact that $p = 0$ is the sufficient and necessary condition for (1) with $\tau_{12} = \tau_{21} = 0$ to be integrable [3].

In the case of $a_{12}a_{21} \geq 0$, (8) is clearly identical to Condition (WDD).

In the case of $a_{12}a_{21} < 0$, we will show that if Condition (WDD) fails, then there is a $\tau_0$ such that for $\tau \neq \tau_0$, system (1) can possess a periodic solution. Clearly, if Condition (WDD) does not hold, (7) together with $a_{12}a_{21} < 0$ implies

$$r^2 > q^2.$$ 

Substituting $z = x + iy$ into (5), we have

$$(x^2 + ixy - y^2 + px + ipy + q) + re^{-\tau}(\cos \tau y - i \sin \tau y) = 0. \quad (8)$$

By separating the real and imaginary parts of (8), we obtain

$$x^2 - y^2 + px + q + re^{-\tau} \cos \tau y = 0,$$

$$2xy + py - re^{-\tau} \sin \tau y = 0. \quad (9)$$
Letting $x = 0$, (9) leads to

\[ y^2 - q = r \cos \tau y, \]

(10)

\[ 2py = r \sin \tau y. \]

(11)

From (10) and (11), we obtain

\[ (y^2 - q)^2 + p^2 y^2 = r^2. \]

Letting $y$ be a positive solution of the above equation and substituting it into (11), we can get $\tau_0$ such that at $\tau_0$, (5) has an eigenvalue $iy$. Furthermore, at $\tau_0$,

\[ \dot{x} = \left( y^2 + r^2 - q^2 \right) / \left( (p + q\tau - y^2\tau)^2 + (2y + py\tau)^2 \right) > 0. \]

By the Hopf bifurcation theorem [2], system (1) has a periodic solution near $\tau_0$.

This completes the proof of the theorem.

Remark. This theorem can serve as a partial extension of the main result of [4].

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