CANTOR SETS IN $S^3$
WITH SIMPLY CONNECTED COMPLEMENTS

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A wild Cantor set in $S^3$ is constructed with simply connected complement. It is proved that a Cantor set $\mathcal{C} \subset S^3$ is tame if and only if every piecewise-linear, unknotted, simple loop in $S^3 \setminus \mathcal{C}$ may be engulfed. And a Cantor set $\mathcal{C} \subset S^3$ is tame if and only if $\pi_1(S^3 \setminus \mathcal{C} \setminus K)$ is finitely generated for all piecewise-linear, unknotted, simple loops $K$ in $S^3 \setminus \mathcal{C}$.

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Introduction

Let $\mathcal{C}$ be a Cantor set embedded in $S^n$. Then $\mathcal{C}$ is tame if there is a (topological) homeomorphism $h : S^n \to S^n$ such that $h(\mathcal{C})$ lies in a piecewise-linear arc. Otherwise $\mathcal{C}$ is wild. Two Cantor sets $\mathcal{C}_1, \mathcal{C}_2 \subset S^n$ are equivalent if there is a (topological) homeomorphism $h : S^n \to S^n$ such that $h(\mathcal{C}_1) = \mathcal{C}_2$. It is well known that any two Cantor sets in $S^n$, $n \leq 2$, are equivalent, and any two tame Cantor sets in $S^n$ are equivalent.

The first example of a wild Cantor set in $S^3$ was given by Antoine [1]. Next Blankinship [5] produced wild Cantor sets in $S^n$, $3 < n$. The examples above were distinguished from the tame embedding by showing the complements of the Cantor sets to be not simply connected. Kirkor [9] produced the first example of a wild Cantor set in $S^3$ with simply connected complement. Starting with yet another wild Cantor set in $S^3$ with simply connected complement, DeGryse and Osborne [7] gave examples of wild Cantor sets in $S^n$, $3 \leq n$, with simply connected complement.

The construction and proof by Kirkor is complicated. The construction by DeGryse and Osborne uses a well known shrinking argument of Bing [3].

Below a wild Cantor set $\mathcal{W}$ in $S^3$ is constructed with simply connected complement. The virtues of $\mathcal{W}$ are that both the construction and derivation of the essential properties are easy. The set $\mathcal{W}$ is distinguished from the other examples, but a very close similarity to Kirkor's example is shown.
In Section 2 it is shown that a Cantor set $C$ in $S^3$ is tame if and only if every piecewise-linear, unknotted, simple loop $K$ in $S^3 \setminus C$ lies in the interior of a piecewise-linear 3-ball in $S^3 \setminus C$, and that a Cantor set $C$ in $S^3$ is tame if and only if $\pi_1(S^3 \setminus C \setminus K)$ is finitely generated for all piecewise-linear, unknotted, simple loops $K$ in $S^3 \setminus C$.

1. The example $\mathcal{W}$

The example $\mathcal{W}$ is described by taking the iteration implied by Fig. 1. The figure shows a genus-2-handlebody $S$ containing disjoint genus-2-handlebodies $S_i$, $i = 1, 2, \ldots, 5$, in its interior. Also each $S, S_i$ has a preferred loop $W, W_i$ on its boundary which is referred to as the waist. Define a sequence $\{H_n\}_{n=0}^\infty$ of manifolds inductively as follows. Let $H_0 = S, H_1 = \bigcup S_i$. If $H_N$ is defined, then define $H_{N+1} \subset \hat{H}_N$ as follows. For each component $S'$ of $H_N$ with waist $W'$, there is a homeomorphism $h : (S, W) \to (S', W')$ such that $h(\bigcup S_i) = S' \cap H_{N+1}$. The images of the $W_i$'s will be the waists. Also choose the $H_N$'s such that the diameters of the components go to 0 as $N$ goes to $\infty$. Let $\mathcal{W} = \bigcup H_n$. Clearly $\mathcal{W}$ is a Cantor set.

![Fig. 1.](image)

The fancy linking among the $S_i$'s was chosen only to give the construction a symmetry which will be seen to simplify the argument below. Also it should be noticed that the definition of $H_{N+1}$ involved choices. The homeomorphism $h : S \to S'$ is not unique even up to isotopy. So the construction above may yield many inequivalent Cantor sets.

1.1. Theorem. $\mathcal{W}$ is wild and $\pi_1(S^3 \setminus \mathcal{W}) = 0$.

Proof. To show that $\mathcal{W}$ is wild it suffices to show that $\pi_1(\partial H_0) \to \pi_1(H_0 \setminus \mathcal{W})$ is a monomorphism. Start the argument by showing that $\pi_1(\partial H_0) \to \pi_1(H_0 \setminus \hat{H}_1)$ is a monomorphism. Let $A_i, i = 1, \ldots, 10$ be annuli properly embedded in $H_0 \setminus \hat{H}_1$ as in


Fig. 2. Observe that $H_1 \cup \bigcup A_i$ contains a spine of $H_0$. Now let $f: (D^2, \partial D^2) \to (H_0 \setminus \hat{H}_1, \partial H_0)$ be a map of a 2-disk. Using general position and innermost components of $f^{-1}(\bigcup A_i)$, one may suppose $f(D^2) \cap \bigcup A_i = \emptyset$. So $f|\partial D^2$ must be trivial on $\partial H_0$, and $\pi_1(\partial H_0) \to \pi_1(H_0 \setminus \hat{H}_1)$ is a monomorphism.

Let $S'$ be a component of $H_1$. The pair $(H_0 \setminus \hat{H}_1, \partial S')$ is homeomorphic to $(H_0 \setminus \hat{H}_1, \partial H_0)$ (see Fig. 3). Therefore $\pi_1(\partial S') \to \pi_1(H_0 \setminus \hat{H}_1)$ is a monomorphism and $\partial(H_0 \setminus \hat{H}_1)$ is incompressible in $H_0 \setminus \hat{H}_1$. Since $H_0 \setminus \mathcal{W}$ is the union of manifolds with incompressible boundary glued along their boundaries, $\pi_1(\partial H_0) \to \pi_1(H_0 \setminus \mathcal{W})$ is a monomorphism.

To see that $\pi_1(S^3 \setminus \mathcal{W}) = 0$ it should suffice to see how one loop is shrunk. See Fig. 4. The moral is that each meridional curve of a component $S'$ of some $H_N$ is homotopic in $S^3 \setminus \mathcal{W}$ to the waist of $S'$. And each waist of a component of $H_N$ is homotopic in $S^3 \setminus \mathcal{W}$ to a waist of a component of $H_{N-1}$. The waist of $H_0$ is null homotopic in $S^3 \setminus \mathcal{W}$. Using general position and induction, it is clear that $\pi_1(S^3 \setminus \mathcal{W}) = 0$. □

1.2. Remark. The Cantor set $\mathcal{M}$ constructed by Bing [3] has the property that for any distinct $p, q \in \mathcal{M}$, there exists a p.l. 2-sphere $S^2 \subset S^3 \setminus \mathcal{M}$ which separates $p$ from

Fig. 2.

Fig. 3.
The Cantor set $\mathcal{W}$ also has this property. One may see this by first showing the components of $H_1$ may be separated by 2-spheres in $S^3 \setminus H_1$.

This property is actually always true of Cantor sets $\mathcal{C}$ in $S^3$ with $\pi_1(S^3 \setminus \mathcal{C}) = 0$—just apply the Hurewicz Isomorphism Theorem and the Sphere Theorem. It should be noticed, however, that $\pi_1(S^3 \setminus \mathcal{W}) \neq 0$.

**1.3. Remark.** As mentioned before Theorem 1.1, the homeomorphism $h : S \to S'$ does not determine $H_{N+1}$ even up to isotopy. This freedom may be a source of inequivalent $\mathcal{W}$'s.

**Question.** Does the above construction yield an uncountable collection $\{\mathcal{W}\}$ of inequivalent Cantor sets with simply connected complements?

**1.4. Remark.** The choice of $H_N$ having $5^N$ components is unnecessary. Suppose only that each component of $H_{N-1}$ contains at least two components of $H_N$. Without a fancy shrinking argument, one may see that the $H_N$'s may be chosen such that the diameters of their components go to 0. So $\bigcap H_\alpha = \mathcal{W}$ is a Cantor set satisfying Theorem 1.1.

Varying the components in each $H_N$, however, does not yield inequivalent Cantor sets. That is, any Cantor set constructed as above but varying the number of components of $H_N$ in each component of $H_{N-1}$ may also be constructed by varying the number of components in any other way. See Fig. 5 for a hint, or wait for the description of Kirkor's Cantor set. This is in contrast to the Antoine Necklace construction where varying the number of components yields inequivalent Cantor sets [12].

**1.5. Remark.** If the sequence $\{H_\alpha\}$ is chosen such that each $H_\alpha$ has one component, then one gets the Fox–Artin arc [8]. Figure 6 shows the construction for the
alternating Fox-Artin arc. The author is grateful to William Eaton for pointing out such a construction for the Fox-Artin arc.

1.6. Remark. Before showing the exact connection of $\mathcal{W}$ with the example $\mathcal{X}$ of Kirkor, a quick description of $\mathcal{X}$ is in order. The building block for $\mathcal{X}$ is shown in Fig. 7. The pair $(A, J)$ is a disjoint union $J$ of two alternating Fox-Artin arcs and a neighborhood $A$ of $J \setminus \{p, q, p_x, q_x\}$. $A$ is a disjoint union of two 3-balls which are
p.l. modulo \{p, q\} (warning: the figures in Kirkor's paper are upside-down and have superfluous crossings). Define \((\{A_\alpha, J_\alpha\}_\alpha\) inductively as follows. \((A_0, J_0)\) is homeomorphic to the p.l. unknotted ball pair \((B^3, B^1)\). Suppose \((A_N, J_N)\) is defined, where \(J_N\) is a disjoint union of Fox–Artin arcs and \(A_N\) is a neighborhood of \(J_N \setminus \{\text{wild points of } J_N\} \cup J_0\). Then \((A_{N+1}, J_{N+1})\) is gotten by thinning down \(A_N\) and replacing a finite number of p.l. unknotted ball pairs in \((A_N, J_N)\) by the building block \((A, J)\). Then \(K = \bigcap A_\alpha\). See Fig. 8.

\[ |A_0 \cdot J_0| \]

\[ |A_1 \cdot J_1| \]

Fig. 8.

Now \(\mathcal{K}\) is inequivalent to \(\mathcal{W}\) because \(S^3 \setminus \mathcal{K}\) is 1-locally-connected at each point of \(\partial J_0\). The proof of Theorem 1.1 shows that \(S^3 \setminus \mathcal{W}\) is not 1-locally-connected at any point of \(\mathcal{W}\).

Considering Remark 1.5, the reader should see a similarity between Kirkor's Cantor set and \(\mathcal{W}\). The precise similarity is as follows. Let \(\frac{1}{2} \mathcal{W} = \mathcal{W} \cap B^3\), where \(B^3\) is a p.l. 3-ball in \(S^3\) such that \(\partial B^3\) intersects \(H_0\) at its waist and intersects each middle component of \(H_N\) also at its waist. So \(\partial B^3 \cap \mathcal{W}\) is one point. If \(\mathcal{W}\) is chosen correctly, then \(\mathcal{K}\) is equivalent to the disjoint union of two copies of \(\frac{1}{2} \mathcal{W}\).

1.7. Remark. \(\mathcal{W}\) is distinguished from \(\mathcal{A}\), the Cantor set of DeGryse and Osborne, by the following property which \(\mathcal{W}\) has, but \(\mathcal{A}\) does not. Given \(\varepsilon > 0\), there exists \(\delta > 0\) such that each map \(f: S^1 \to S^3 \setminus \mathcal{W}\) with image of diameter less than \(\delta\), extends to a map \(F: D^2 \to S^3\) with diameter less than \(\varepsilon\) and \(F^{-1}(\mathcal{W})\) is finite.

1.8. Remark. The p.l. unknotted simple loop \(K\) in Fig. 4 has two interesting properties. There is no 3-ball \(B^3 \subset S^3 \setminus \mathcal{C}\) such that \(K \subset B^3\), and \(\pi_1(S^3 \setminus \mathcal{C} \setminus K)\) is not finitely generated. The proofs are left to the reader. Both of these phenomena are studied in the next section.

2. Engulfing/finitely generated groups

A handlebody \(H \subset S^3\) is unknotted if \(\overline{H^c}\) is a handlebody. The following lemma is exactly as in [2], except that the handlebodies are unknotted. The lemma's proof however, follows from the same techniques.
2.1. Lemma. Let $\mathcal{C} \subset S^3$ be a Cantor set, then there is a sequence $\{H_\alpha\}$ of piecewise-linear manifolds in $S^3$ such that

1. Each $H_\alpha$ is a finite disjoint union of unknotted handlebodies,
2. $H_{\alpha+1} \subset H_\alpha$, and
3. $\bigcap H_\alpha = \mathcal{C}$.

Let $M$ be a 3-manifold and $K \subset M$. Say $K$ may be engulfed if there is a piecewise-linear 3-ball $B^3$ such that $K \subset B^3 \subset B^3 \subset M$.

2.2. Theorem. Let $\mathcal{C} \subset S^3$ be a Cantor set. Then $\mathcal{C}$ is tame if and only if every piecewise-linear, unknotted, simple loop in $S^3 \setminus \mathcal{C}$ may be engulfed.

Proof. The forward direction is easy. So suppose $S^3 \setminus \mathcal{C}$ has the engulfing property. Let $\{H_\alpha\}$ be given by Lemma 2.1. It suffices to show that for any component $H$ of any $H_\alpha$, there is a p.l. 3-ball $D^3$ such that $H \cap \mathcal{C} \subset \bar{D}^3 \subset D^3 \subset H$. Let $H$ be such a given unknotted handlebody. Let $\Gamma$ be the graph in $H^c \setminus \mathcal{C}$ which is a spine for $H^c$. Then by a theorem of Bing [4] there is a p.l., simple loop $K$ near $\Gamma$ with the property that $K$ may be engulfed if and only if $\Gamma$ may be engulfed. A careful examination of the proof shows that because $H^c$ is unknotted, $K$ may be chosen a p.l., unknotted, simple loop. Let $B^3 \subset S^3 \setminus \mathcal{C}$ be a 3-ball containing $\Gamma$. By adjusting $B^3$ one may suppose $B^3 \subset S^3 \setminus (H \cap \mathcal{C})$ and $H^c \subset B^3$, so $H \cap \mathcal{C} \subset (B^3)^c$. Let $D^3 = (B^3)^c$. □

The statement of Theorem 2.2 without the hypothesis 'unknotted' follows from the theorems in [13].

2.3. Theorem. Let $\mathcal{C} \subset S^3$ be a Cantor set. Then $\mathcal{C}$ is tame if and only if $\pi_1(S^3 \setminus \mathcal{C} \setminus K)$ is finitely generated for all piecewise-linear, unknotted, simple loops $K \subset S^3 \setminus \mathcal{C}$.

Proof. The forward direction is easy, so suppose $\pi_1(S^3 \setminus \mathcal{C} \setminus K)$ is finitely generated for all p.l. unknotted simple loops $K$. Let $K$ be given; by Theorem 2.2 it suffices to engulf $K$. Since $\pi_1(S^3 \setminus \mathcal{C} \setminus K)$ is finitely generated, a theorem of Scott [11] gives a compact, embedded 3-manifold $Q \subset S^3 \setminus \mathcal{C} \setminus K$ such that $\pi_1(Q) \to \pi_1(S^3 \setminus \mathcal{C} \setminus K)$ is an isomorphism. Since $H_1(S^3 \setminus \mathcal{C} \setminus K) = \mathbb{Z}$, $\partial Q = T$ if $Q$ is homeomorphic to a punctured knot complement. If $T$ is incompressible in $Q$, then it is clear that $K$ must be parallel to $T$, but this contradicts that $K$ is unknotted. It follows that $Q$ is a punctured solid torus and $\pi_1(S^3 \setminus \mathcal{C} \setminus K) = \mathbb{Z}$. In particular, $K$ bounds a disk $D$ in $S^3 \setminus \mathcal{C}$ and a regular neighborhood of $D$ is the desired engulfing 3-ball. □

2.4. Remark. The main interest in Theorem 2.2 is when $\pi_1(S^3 \setminus \mathcal{C}) = 0$, for if $\pi_1(S^3 \setminus \mathcal{C}) \neq 0$, then there is a p.l. unknotted simple loop in $S^3 \setminus \mathcal{C}$ which is not null homotopic in $S^3 \setminus \mathcal{C}$. Clearly such a loop may not be engulfed.
Similarly the main interest in Theorem 2.3 is when $\pi_1(S^3 \setminus \mathcal{C}) = 0$. If $\pi_1(S^3 \setminus \mathcal{C}) \neq 0$, then $\pi_1(S^3 \setminus \mathcal{C})$ is already not finitely generated ([10] or the above methods). Contrast this with examples in higher dimensions. When $n \geq 5$, Daverman [6] has constructed examples of $\mathcal{C} \subset S^n$ with $\pi_1(S^n \setminus \mathcal{C})$ non-trivial finite group.

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References