Available online at www.sciencedirect.com
ScienceDirect
Discrete Applied Mathematics 156 (2008) 2429–2435
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Communication

Minimum cost homomorphisms to semicomplete multipartite digraphs

Gregory Gutin[a,b,*], Arash Rafiey[c], Anders Yeo[a]

[a] *Department of Computer Science, Royal Holloway University of London, Egham, Surrey TW20 0EX, UK*
[b] *Department of Computer Science, University of Haifa, Israel*
[c] *School of Computing, Simon Fraser University, Burnaby, BC, V5A 1S6 Canada*

Received 12 May 2006; received in revised form 14 August 2007; accepted 29 August 2007
Available online 20 February 2008

Communicated by F.S. Roberts

Abstract

For digraphs D and H, a mapping $f : V(D) \to V(H)$ is a *homomorphism of* D *to* H if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. For a fixed directed or undirected graph H and an input graph D, the problem of verifying whether there exists a homomorphism of D to H has been studied in a large number of papers. We study an optimization version of this decision problem. Our optimization problem is motivated by a real-world problem in defence logistics and was introduced recently by the authors and M. Tso.

Suppose we are given a pair of digraphs D, H and a cost $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$. The cost of a homomorphism f of D to H is $\sum_{u\in V(D)} c_{f(u)}(u)$. Let H be a fixed digraph. The minimum cost homomorphism problem for H, MinHOMP(H), is stated as follows: For input digraph D and costs $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$, verify whether there is a homomorphism of D to H and, if it does exist, find such a homomorphism of minimum cost. In our previous paper we obtained a dichotomy classification of the time complexity of MinHOMP(H)when H is a semicomplete digraph. In this paper we extend the classification to semicomplete k-partite digraphs, $k \geq 3$, and obtain such a classification for bipartite tournaments.

Keywords: Minimum cost homomorphism; Semicomplete multipartite digraphs

1. Introduction

In our terminology and notation, we follow [1,5]. In this paper, directed (undirected) graphs have no parallel arcs (edges) or loops. The vertex (arc) set of a digraph G is denoted by $V(G)$ $(A(G))$. The vertex (edge) set of an undirected graph G is denoted by $V(G)$ $(E(G))$. A digraph D obtained from a complete k-partite (undirected) graph G by replacing every edge xy of G with arc xy, arc yx, or both xy and yx, is called a *semicomplete k-partite digraph* (or *semicomplete multipartite digraph* when k is immaterial). The *partite sets* of D are the partite sets of G. A semicomplete k-partite digraph D is *semicomplete* if each partite set of D consists of a unique vertex. A *k-partite*

* Corresponding author at: Department of Computer Science, Royal Holloway University of London, Egham, Surrey TW20 0EX, UK.
E-mail addresses: gutin@cs.rhul.ac.uk (G. Gutin), arafieyh@cs.sfu.ca (A. Rafiey), anders@cs.rhul.ac.uk (A. Yeo).

0166-218X/$ - see front matter
doi:10.1016/j.dam.2007.09.023

tournament is a semicomplete k-partite digraph with no directed cycle of length 2. Semicomplete k-partite digraphs and its subclasses mentioned above are well-studied in graph theory and algorithms, see, e.g., [1].

For introductions to homomorphisms in directed and undirected graphs, see [1,12,14]. For digraphs D and H, a mapping $f : V(D) \to V(H)$ is a *homomorphism of D to H* if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. A homomorphism f of D to H is also called an *H-coloring* of D, and $f(x)$ is called a *color* of x for every $x \in V(D)$. We denote the set of all homomorphisms from D to H by $\mathrm{HOM}(D, H)$.

For a fixed digraph H, the *homomorphism problem* $\mathrm{HOMP}(H)$ is to verify whether, for an input digraph D, there is a homomorphism of D to H (i.e., whether $\mathrm{HOM}(D, H) \neq \emptyset$). The problem $\mathrm{HOMP}(H)$ has been studied for several families of directed and undirected graphs H, see, e.g., [12,14]. The well-known result of Hell and Nešetřil [13] asserts that $\mathrm{HOMP}(H)$ for undirected graphs is polynomial time solvable if H is bipartite and it is NP-complete, otherwise.

Such a dichotomy classification for all digraphs is unknown and only partial classifications have been obtained; see [14]. For example, Bang-Jensen, Hell and MacGillivray [3] showed that $\mathrm{HOMP}(H)$ when H is a semicomplete digraph is polynomial time solvable if H has at most one cycle and $\mathrm{HOMP}(H)$ is NP-complete, otherwise. Bang-Jensen and Hell [2] proved that if a bipartite tournament H is a core, then $\mathrm{HOMP}(H)$ is polynomial time solvable when H has at most one cycle and $\mathrm{HOMP}(H)$ is NP-hard when H has at least two cycles. (A digraph H is a *core* if H does not contain no proper subdigraph H' such that there are both an H'-coloring of H and an H-coloring of H'.)

The authors of [9] introduced an optimization problem on H-colorings for undirected graphs H, $\mathrm{MinHOMP}(H)$(defined below). The problem is motivated by a problem in defence logistics (see [9]) and can be viewed (see [9]) as an important special case of the valued constraint satisfaction problem recently introduced in [4]. In our previous paper [7], we obtained a dichotomy classification for the time complexity of $\mathrm{MinHOMP}(H)$when H is a semicomplete digraph. In this paper, we extend that classification to obtain a dichotomy classification for semicomplete k-partite digraphs H, $k \geq 3$. We also obtain a classification of the complexity of $\mathrm{MinHOMP}(H)$when H is a bipartite tournament. The case of arbitrary semicomplete bipartite digraphs is significantly more complicated and was recently solved in [8]. Another difficult solved case is that of undirected graphs (or, equivalently, of symmetric digraphs) [6]. A digraph D is symmetric if $xy \in A(D)$ implies $yx \in A(D)$. The general case of arbitrary digraphs remains a very hard and interesting open problem.

Suppose we are given a pair of digraphs D, H and a real cost $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$. The *cost* of a homomorphism f of D to H is $\sum_{u \in V(D)} c_{f(u)}(u)$. For a fixed digraph H, the *minimum cost homomorphism problem* $\mathrm{MinHOMP}(H)$is formulated as follows. For an input digraph D and costs $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$, verify whether $\mathrm{HOM}(D, H) \neq \emptyset$ and, if $\mathrm{HOM}(D, H) \neq \emptyset$, find a homomorphism in $\mathrm{HOM}(D, H)$ of minimum cost.

For a digraph G, if $xy \in A(G)$, we say that x *dominates* y and y is *dominated* by x (denoted by $x \to y$). The *out-degree* $d_G^+(x)$ (*in-degree* $d_G^-(x)$) of a vertex x in G is the number of vertices dominated by x (that dominate x). For sets $X, Y \subset V(G)$, $X \to Y$ means that $x \to y$ for each $x \in X$, $y \in Y$, but no vertex of Y dominates a vertex in X. A set $X \subseteq V(G)$ is *independent* if no vertex in X dominates a vertex in X. A *k-cycle*, denoted by $\vec{C}_k$, is a directed simple cycle with k vertices. A digraph H is an *extension* of a digraph D if H can be obtained from D by replacing every vertex x of D with a set S_x of independent vertices such that if $xy \in A(D)$ then $uv \in A(H)$ for each $u \in S_x$, $v \in S_y$.

The *underlying graph* $U(G)$ of a digraph G is the undirected graph obtained from G by disregarding all orientations and deleting one edge in each pair of parallel edges. A digraph G is *connected* if $U(G)$ is connected. The *components* of G are the subdigraphs of G induced by the vertices of components of $U(G)$. A digraph G is *strongly connected* if there is a path from x to y for every ordered pair of vertices $x, y \in V(G)$. A *strong component* of G is a maximal induced strongly connected subdigraph of G. A digraph G' is the *converse* of a digraph G if G' is obtained from G by reversing orientations of all arcs.

The rest of the paper is organized as follows. In Section 2, we give all polynomial time solvable cases of $\mathrm{MinHOMP}(H)$when H is a semicomplete k-partite digraph, $k \geq 3$, or a bipartite tournament. Section 3 is devoted to a full dichotomy classification of the time complexity of $\mathrm{MinHOMP}(H)$when H is a semicomplete k-partite digraph, $k \geq 3$. A classification of the same problem for H being a bipartite tournament is proved in Section 4.

2. Polynomial time solvable cases

In this section, we will apply the following theorem which was proved in [7] using a powerful result from [4].

Theorem 2.1. *Let H be a digraph and let there exist an ordering $\pi(1), \pi(2), \dots, \pi(p)$ of the vertices of H satisfying the following Min–Max property: For any pair $\pi(i)\pi(k), \pi(j)\pi(s)$ of arcs in H, we have $\pi(\min\{i, j\})\pi(\min\{k, s\}) \in A(H)$ and $\pi(\max\{i, j\})\pi(\max\{k, s\}) \in A(H)$. Then* MinHOMP($H$) *is polynomial time solvable.*

The Min–Max property is closely related to a property of digraphs that has long been of interest [11]. We say that a digraph H has the *X-underbar property* if its vertices can be ordered $\pi(1), \pi(2), \dots, \pi(p)$ so that $\pi(i)\pi(r), \pi(j)\pi(s) \in A(H)$ implies that $\pi(\min\{i, j\})\pi(\min\{r, s\}) \in A(H)$. It is interesting that the X-underbar property is sufficient to ensure that the list homomorphism problem for H has a polynomial solution [14].

Let TT_p denote the acyclic tournament on $p \geq 1$ vertices. Let $p \geq 3$ and let TT_p^- be a digraph obtained from TT_p by deleting the arc from the vertex of in-degree zero to the vertex of out-degree zero. In [7], we proved the following result for TT_p using Theorem 2.1. Thus, our proof is only for TT_p^-.

Lemma 2.2. *The problems* MinHOMP(TT_p) *and* MinHOMP(TT_p^-) *are polynomial time solvable for $p \geq 1$ and $p \geq 3$, respectively.*

Proof. Let $V(TT_p^-) = \{\pi(1), \pi(2), \dots, \pi(p)\}$ and let $A(TT_p^-) = \{\pi(i)\pi(j) : 1 \leq i < j \leq p,\ j - i < p - 1\}$. Let $\pi(i)\pi(k)$ and $\pi(j)\pi(s)$ be distinct arcs in TT_p^-. Observe that $\pi(\min\{i, j\})\pi(\min\{k, s\}) \neq \pi(1)\pi(p)$ and $\pi(\max\{i, j\})\pi(\max\{k, s\}) \neq \pi(1)\pi(p)$. Thus, $\pi(\min\{i, j\})\pi(\min\{k, s\})$ and $\pi(\max\{i, j\})\pi(\max\{k, s\})$ are arcs in TT_p^-. Therefore, MinHOMP(TT_p^-) is polynomial time solvable by Theorem 2.1. $\diamond$

Lemma 2.3. *Suppose that* MinHOMP(H) *is polynomial time solvable. Then, for each extension H' of H,* MinHOMP(H') *is also polynomial time solvable.*

Proof. Recall that we can obtain H' from H by replacing every vertex $i \in V(H)$ with a set S_i of independent vertices. Consider an H'-coloring h' of an input digraph D. We can reduce h' into an H-coloring of D as follows: if $h'(u) \in S_i$, then $h(u) = i$.

Let $u \in V(D)$. Assign $\min\{c_j(u) : j \in S_i\}$ to be a new cost $c_i(u)$ for each $i \in V(H)$. Observe that we can find an optimal H-coloring h of D with the new costs in polynomial time and transform h into an optimal H'-coloring of D with the original costs using the obvious inverse of the reduction described above. $\diamond$

In [7], we proved that MinHOMP(H) is polynomial time solvable when $H = \vec{C}_p$, $p \geq 2$. Combining this results with Lemmas 2.2 and 2.3, we immediately obtain the following:

Theorem 2.4. *If H is an extension of TT_p $(p \geq 1)$, $\vec{C}_p$ $(p \geq 2)$ or $TT_p^-(p \geq 3)$, then* MinHOMP(H) *is polynomial time solvable.*

It seems that it is not possible to prove the following result by a straightforward application of Theorem 2.1. Nevertheless, our proof uses Theorem 2.1 in a somewhat indirect way via Lemma 2.2.

Theorem 2.5. *Let H be an acyclic bipartite tournament. Then* MinHOMP(H) *is polynomial time solvable.*

Proof. Let V_1, V_2 be the partite sets of H, let D be an input digraph and let $c_i(x)$ be the costs, $i \in V(H),\ x \in V(D)$. Observe that if D is not bipartite, then HOM$(D, H) = \emptyset$, so we may assume that D is bipartite. We can check whether D is bipartite in polynomial time. Let U_1, U_2 be the partite sets of D.

To prove that we can find a minimum cost H-coloring of D in polynomial time, it suffices to show that we can find a minimum cost H-coloring f of D such that $f(U_1) \subseteq V_1$ and $f(U_2) \subseteq V_2$. Indeed, if D is connected, to find a minimum cost H-coloring of D we can choose from a minimum cost H-coloring f with $f(U_1) \subseteq V_1$ and $f(U_2) \subseteq V_2$ and a minimum cost H-coloring h of D with $h(U_1) \subseteq V_2$ and $h(U_2) \subseteq V_1$. If D is not connected, we can find a minimum cost H-coloring of each component of D separately.

To force $f(U_1) \subseteq V_1$ and $f(U_2) \subseteq V_2$ for each H-coloring f, it suffices to modify the costs such that it is too expensive to assign any color from V_j to a vertex in U_{3-j}, $j = 1, 2$. Let $M = |V(D)| \cdot \max\{c_i(x) : i \in V(H),\ x \in V(D)\} + 1$ and replace $c_i(x)$ by $c_i(x) + M$ for each pair $x \in U_j$, $i \in V_{3-j}$, $j = 1, 2$.

Observe that the vertices of H can be ordered $i_1, i_2, \dots, i_p$ such that i_k is an arbitrary vertex of in-degree zero in $H - \{i_1, i_2, \dots, i_{k-1}\}$ for every $k \in \{1, 2, \dots, p\}$. Thus, H is a subdigraph of TT_p with vertices $i_1, i_2, \dots, i_p$

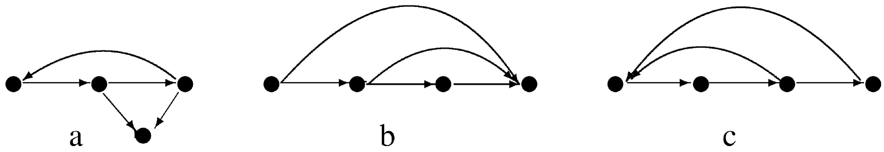

Fig. 1. Graphs used in Lemmas 3.3–3.5.

($i_s i_t \in A(TT_p)$ if and only if $s < t$). Observe that

$$\{f \in \mathrm{HOM}(D, H) : f(U_j) \subseteq V_j, j = 1, 2\} = \{f \in \mathrm{HOM}(D, TT_p) : f(U_j) \subseteq V_j, j = 1, 2\}.$$

Thus, to solve MinHOMP(H) with the modified costs it suffices to solve MinHOMP(TT_p) with the same costs (this will solve MinHOMP(H) under the assumption $f(U_j) \subseteq V_j$). We can solve the latter in polynomial time by Lemma 2.2. $\diamond$

3. Classification for semicomplete k-partite digraphs, $k \geq 3$

The following lemma allows us to prove that MinHOMP(H)is NP-hard when MinHOMP(H') is NP-hard for an induced subdigraph H' of H.

Lemma 3.1 (*[7]*). *Let H' be an induced subdigraph of a digraph H. If* MinHOMP(H') *is NP-hard, then* MinHOMP(H) *is also NP-hard.*

The following lemma is the NP-hardness part of the main result in [7].

Lemma 3.2. *Let H be a semicomplete digraph containing a cycle and let $H \notin \{\vec{C}_2, \vec{C}_3\}$. Then* MinHOMP($H$) *is NP-hard.*

The following lemma was proved in [10]. The digraph H_1 from the lemma is depicted in Fig. 1(a).

Lemma 3.3. *Let H_1 be a digraph obtained from $\vec{C}_3$ by adding an extra vertex dominated by two vertices of the cycle and let H be H_1 or its converse. Then* HOMP(H) *is NP-complete.*

We need two more lemmas for our classification. The digraph H' from the next lemma is depicted in Fig. 1(b).

Lemma 3.4. *Let H' be given by $V(H') = \{1, 2, 3, 4\}$, $A(H') = \{12, 23, 34, 14, 24\}$ and let H be H' or its converse. Then* MinHOMP(H) *is NP-hard.*

Proof. Let $H = H'$. We reduce the maximum independence set problem (MISP) to MinHOMP(H). Let G be an arbitrary graph without isolated vertices. We construct a digraph D from G as follows: every vertex of G belongs to D and, for each pair x, y of adjacent vertices of G, we add to D new vertices u_{xy} and v_{xy} together with arcs $u_{xy}x, u_{xy}v_{xy}, v_{xy}y$. (No edge of G is in D.) Let n be the number of vertices in D. Let x, y be an adjacent pair of vertices in G. We set $c_3(x) = c_3(y) = c_i(u_{xy}) = c_i(v_{xy}) = 1$ for $i = 1, 2, 3$, $c_4(x) = c_4(y) = n + 1$ and $c_j(x) = c_j(y) = c_4(u_{xy}) = c_4(v_{xy}) = n^2 + n + 1$ for $j = 1, 2$.

Consider a mapping $f : V(D) \to V(H)$ such that $f(z) = 4$ for each $z \in V(G)$ and $f(u_{xy}) = 1$, $f(v_{xy}) = 2$ for each pair x, y of adjacent vertices of G. Observe that f is an H-coloring of D of cost smaller than $n^2 + n + 1$.

Consider now a minimum cost H-coloring h of D. Let x, y be a pair of adjacent vertices in G. Due to the values of the costs, h can assign x, y only colors 3 and 4 and u_{xy}, v_{xy} only colors 1,2 or 3. The coloring can assign u_{xy} either 1 or 2 as otherwise v_{xy} must be assigned color 4. If u_{xy} is assigned 1, then v_{xy}, y, x must be assigned 2, (3 or 4) and 4, respectively. If u_{xy} is assigned 2, then v_{xy}, y, x must be assigned 3,4 and (3 or 4), respectively. In both cases, only one of the vertices x and y can receive color 3. Since h is optimal, the maximum number of vertices in D that it inherited from G must be assigned color 3. This number is the maximum number of independent vertices in G. Since MISP is NP-hard, so is MinHOMP(H). $\diamond$

The digraph H from the next lemma is depicted in Fig. 1(c).

Lemma 3.5. *Let H be given by $V(H) = \{1, 2, 3, 4\}$, $A(H) = \{12, 23, 31, 34, 41\}$. Then* MinHOMP($H$) *is NP-hard.*

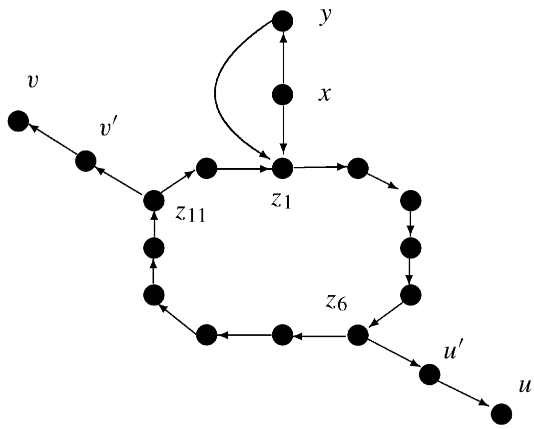

Fig. 2. Gadget for Lemma 3.5.

Proof. We will reduce the maximum independent set problem to MinHOMP(H). However before we do this we consider a digraph $D^{\text{gadget}}(u, v)$ defined as follows (see Fig. 2): $V(D^{\text{gadget}}(u, v)) = \{x, y, u', u, v', v, z_1, z_2, \ldots, z_{12}\}$ and

$$A(D^{\text{gadget}}(u, v)) = \{xy, xz_1, yz_1, z_6u', u'u, z_{11}v', v'v, z_1z_2, z_2z_3, z_3z_4, \ldots, z_{11}z_{12}, z_{12}z_1\}.$$

Observe that in any homomorphism f of $D^{\text{gadget}}(u, v)$ to H we must have $f(z_1) = 1$ since vertices x, y, z_1 can only map to 3,4,1, respectively. This implies that $(f(z_1), f(z_2), \ldots, f(z_{12}))$ has to coincide with one of the following two sequences:

$$(1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3) \quad \text{or} \quad (1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4).$$

If the first sequence is the actual one, then we have $f(z_6) = 3$, $f(u') \in \{1, 4\}$, $f(u) \in \{1, 2\}$, $f(z_{11}) = 2$, $f(v') = 3$ and $f(v) \in \{1, 4\}$. If the second sequence is the actual one, then we have a symmetrical situation $f(z_6) = 2$, $f(u') = 3$, $f(u) \in \{1, 4\}$, $f(z_{11}) = 3$, $f(v') \in \{1, 4\}$ and $f(v) \in \{1, 2\}$. Notice that we cannot assign color 2 to both u and v in a homomorphism.

Let G be a graph. Construct a digraph D as follows. Start with $V(D) = V(G)$ and, for each edge $uv \in E(G)$, add a distinct copy of $D^{\text{gadget}}(u, v)$ to D (notice that u and v are not copied but shared among gadgets). Note that the vertices in $V(G)$ form an independent set in D and that $|V(D)| = |V(G)| + 16|E(G)|$.

Let all costs $c_i(t) = 1$ for $t \in V(D)$ apart from $c_j(p) = 2$ for all $p \in V(G)$ and $j \in \{1, 4\}$. Clearly, a minimum cost H-coloring h of D must aim at assigning as many vertices of $V(G)$ in D a color different from 1 and 4. However, if pq is an edge in G, by the arguments above, h cannot assign color 2 to both p and q; h can assign color 2 to either p or q (or neither). Thus, a minimum cost homomorphism of D to H corresponds to a maximum independent set in G and vice versa (the vertices of a maximum independent set are assigned color 2 and all other vertices in $V(G)$ are assigned color 1). ◇

Theorem 3.6. *Let H be a semicomplete k-partite digraph, $k \geq 3$. If H is an extension of TT_k, $\vec{C}_3$ or TT^{-}_{k+1}, then* MinHOMP(H) *is polynomial time solvable. Otherwise,* MinHOMP(H) *is NP-hard.*

Proof. Since H is a semicomplete k-partite digraph, $k \geq 3$, if H has a cycle, then there can be three possibilities for the length of a shortest cycle in H: 2,3 or 4. Thus, we consider four cases: the above three cases and the case when H is acyclic.

Case 1: H has a 2-cycle C. Let i, j be vertices of C. The vertices i, j together with a vertex from a partite set different from those where i, j belong to form a semicomplete digraph with a 2-cycle. Thus, by Lemmas 3.1 and 3.2, MinHOMP(H)is NP-hard.

Case 2: A shortest cycle C of H has three vertices ($C = ijli$). If H has at least four partite sets, then MinHOMP(H) can be shown to be NP-hard similarly to Case 1. Assume that H has three partite sets and that MinHOMP(H) is not NP-hard. Let V_1, V_2 and V_3 be partite sets of H such that $i \in V_1$, $j \in V_2$ and $l \in V_3$. Consider a vertex $s \in V_1$ outside C. If s is dominated by j and l or dominates j and l, then MinHOMP(H) is NP-hard by

Lemmas 3.1 and 3.3, a contradiction. If $j \to s \to l$, then MinHOMP(H) is NP-hard by Lemmas 3.1 and 3.5, a contradiction. Thus, $l \to s \to j$. Similar arguments show that $l \to V_1 \to j$. Consider $p \in V_2$. Similar arguments show that $p \to V_1 \to j$ and moreover $V_3 \to V_1 \to j$. Again, similarly we can prove that $V_3 \to V_1 \to V_2$, i.e., H is an extension of $\vec{C}_3$.

Case 3: A shortest cycle C of H has four vertices ($C = ijsti$). Since C is a shortest cycle, i, s belong to the same partite set, say V_1, and j, t belong to the same partite set, say V_2. Since H is not bipartite, there is a vertex l belonging to a partite set different from V_1 and V_2. Since H has no cycle of length 2 or 3, either l dominates $V(C)$ or $V(C)$ dominates l. Consider the first case ($l \to V(C)$) as the second one can be tackled similarly. Let H' be the subdigraph of H induced by the vertices l, i, j, s. Observe now that MinHOMP(H) is NP-hard by Lemmas 3.1 and 3.4.

Case 4: H has no cycle. Assume that MinHOMP(H) is not NP-hard, but H is not an extension of an acyclic tournament. The last assumption implies that there is a pair of nonadjacent vertices i, j and a distinct vertex l such that $i \to l \to j$. Let s be a vertex belonging to a partite set different from the partite sets which i and l belong to. Without loss of generality, assume that at least two vertices in the set $\{i, j, l\}$ dominate s. If all three vertices dominate s, then by Lemmas 3.1 and 3.4, MinHOMP(H) is NP-hard, a contradiction. Since H is acyclic, we conclude that $\{i, l\} \to s \to j$. Let V_1 be the partite set of i and j. Similar arguments show that for each vertex $t \in V(H) - V_1$, $i \to t \to j$. By considering a vertex $p \in V_1 - \{i, j\}$ and using arguments similar to the ones applied above, we can show that either $p \to (V(H) - V_1)$ or $(V(H) - V_1) \to p$. This implies that we can partition V_1 into V_1' and V_1'' such that $V_1' \to (V(H) - V_1) \to V_2''$. This structure of H implies that there is no pair a, b of nonadjacent vertices in $V(H) - V_1$ such that $a \to c \to b$ for some vertex $c \in V(H)$ since otherwise the problem is NP-hard by Lemmas 3.1 and 3.4. Thus, the subdigraph $H - V_1$ is an extension of an acyclic tournament and, therefore, H is an extension of TT_{k+1}^{-}. ◇

4. Classification for bipartite tournaments

The following lemma can be proved similarly to Lemma 3.3.

Lemma 4.1. *Let H_1 be given by $V(H_1) = \{1, 2, 3, 4, 5\}$, $A(H_1) = \{12, 23, 34, 41, 15, 35\}$ and let H be H_1 or its converse. Then* MinHOMP(H) *is NP-hard.*

Now we can obtain a dichotomy classification for MinHOMP(H) when H is a bipartite tournament.

Theorem 4.2. *Let H be a bipartite tournament. If H is acyclic or an extension of a* 4*-cycle, then* MinHOMP(H) *is polynomial time solvable. Otherwise,* MinHOMP(H) *is NP-hard.*

Proof. If H is an acyclic bipartite tournament or an extension of a 4-cycle, then MinHOMP(H) is polynomial time solvable by Theorems 2.5 and 2.4. We may thus assume that H has a cycle C, but H is not an extension of a cycle. We have to prove that MinHOMP(H) is NP-hard.

Let C be a shortest cycle of H. Since H is a bipartite tournament, we have $|V(C)| = 4$. Thus, we may assume, without loss of generality, that $C = i_1i_2i_3i_4i_1$, where i_1, i_3 belong to a partite set V_1 of H and i_2, i_4 belong to the other partite set V_2 of H.

We may assume that any vertex in V_1 dominates either i_2 or i_4 and is dominated by the other vertex in $\{i_2, i_4\}$, as otherwise we are done by Lemmas 4.1 and 3.1. Analogously any vertex in V_2 dominates exactly one of the vertices in $\{i_1, i_3\}$. Therefore, we may partition the vertices in H into the following four sets.

$$J_1 = \{j_1 \in V_1 : i_4 \to j_1 \to i_2\} \qquad J_2 = \{j_2 \in V_2 : i_1 \to j_2 \to i_3\}$$
$$J_3 = \{j_3 \in V_1 : i_2 \to j_3 \to i_4\} \qquad J_4 = \{j_4 \in V_2 : i_3 \to j_4 \to i_1\}$$

If $q_2q_1 \in A(H)$, where $q_j \in J_j$ for $j = 1, 2$ then we are done by Lemmas 4.1 and 3.1 (consider the cycle $q_1i_2i_3i_4q_1$ and the vertex q_2 which dominates both q_1 and i_3). Thus, $J_1 \to J_2$ and analogously we obtain that $J_2 \to J_3 \to J_4 \to J_1$, so H is an extension of a cycle, a contradiction. ◇

Acknowledgements

We are very grateful to both referees for several valuable suggestions and to Martin Green, Pavol Hell and Eun Jung Kim for useful discussions of the paper. The research of Gutin and Rafiey was supported in part by the IST Programme of the European Community, under the PASCAL Network of Excellence, IST-2002-506778.

References

[1] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer-Verlag, London, 2000.
[2] J. Bang-Jensen, P. Hell, The effect of two cycles on the complexity of colourings by directed graphs, Discrete Appl. Math. 26 (1990) 1–23.
[3] J. Bang-Jensen, P. Hell, G. MacGillivray, The complexity of colouring by semicomplete digraphs, SIAM J. Discrete Math. 1 (1988) 281–298.
[4] D. Cohen, M. Cooper, P. Jeavons, A. Krokhin, A maximal tractable class of soft constraints, J. Artificial Intelligence Res. 22 (2004) 1–22.
[5] M. Garey, D. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman, 1979.
[6] G. Gutin, P. Hell, A. Rafiey, A. Yeo, Dichotomy for minimum cost graph homomorphisms, European J. Combin. (in press).
[7] G. Gutin, A. Rafiey, A. Yeo, Minimum cost and list homomorphisms to semicomplete digraphs, Discrete Appl. Math. 154 (2006) 890–897.
[8] G. Gutin, A. Rafiey, A. Yeo, Minimum cost homomorphisms to semicomplete bipartite digraphs (submitted for publication).
[9] G. Gutin, A. Rafiey, A. Yeo, M. Tso, Level of repair analysis and minimum cost homomorphisms of graphs, Discrete Appl. Math. 154 (2006) 881–889.
[10] W. Gutjahr, Graph colourings, Ph.D. Thesis, Free University Berlin, 1991.
[11] W. Gutjahr, E. Welzl, G. Woeginger, Polynomial graph-colorings, Discrete Appl. Math. 35 (1992) 29–45.
[12] P. Hell, Algorithmic aspects of graph homomorphisms, in: Survey in Combinatorics 2003, in: London Math. Soc. Lecture Note Series, vol. 307, Cambridge U. Press, 2003, pp. 239–276.
[13] P. Hell, J. Nešetřil, On the complexity of H-colouring, J. Combin. Theory B 48 (1990) 92–110.
[14] P. Hell, J. Nešetřil, Graphs and Homomorphisms, Oxford U. Press, Oxford, 2004.