On Holomorphic Maps Which Commute with Hyperbolic Automorphisms*

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INTRODUCTION

Let $A_n$ be the unit ball of $\mathbb{C}^n$ and let $\gamma$ be a hyperbolic automorphism of $A_n$. In this work we study the class of holomorphic mappings $f \in \text{Hol}(A_n, A_n)$, from $A_n$ into itself, which commute with $\gamma$ (with respect to the usual composition of mappings).

In the one-dimensional case, it is well known (see [6]) that if $f \in \text{Hol}(A, A)$ commutes with a hyperbolic automorphism $\gamma$ of $A$, then $f$ is either the identity map or it is a hyperbolic automorphism of $A$ with the same fixed points of $\gamma$ (for a more recent exposition of this and related results, see, e.g., [1]). Still in the one-dimensional case Behan and Shields [3, 11] proved that, except for the case of two hyperbolic automorphisms of $A$, two non-trivial commuting holomorphic maps belonging to $\text{Hol}(A, A)$ have the same fixed point in $A$ or the same “Wolff point” in $\partial A$.

If the dimension $n$ of the space is strictly greater than one, then the problem of characterizing the holomorphic maps which commute with a given hyperbolic automorphism of $A_n$ is still open and in this paper we give some contribution at this regard.

Suppose that $f \in \text{Hol}(A_n, A_n)$ commutes with a given hyperbolic automorphism $\gamma$ of $A_n$.

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We first prove that the two fixed points \( p_1 \) and \( p_2 \) (\( \gamma \neq 2 \)) of \( \gamma \) are "fixed points" for \( f \) as well (Corollary 1.6). Since we can suppose, up to conjugation in \( \text{Aut} \ A_n \), that the fixed points of \( \gamma \) are \( e_1 \) and \( -e_1 \), where \( e_1 = (1, 0, \ldots, 0) \), then the finiteness of

\[
\lim \inf_{x \to e_1} \frac{1 - \|f(z)\|}{1 - |z|}
\]

follows (as well as the finiteness of the same \( \lim \inf \) at \( -e_1 \)). This implies, via the Julia-Wolff-Carathéodory theorem, that, among others, the functions

(i) \( (1 - f_1(z))/(1 - z_1) \),

(ii) \( Q_{e_1}(f(z))/(1 - z_1)^{1/2} \),

(iii) \( \langle dfz, e_1^t, e_1 \rangle/(1 - z_1)^{1/2} \),

defined in Theorem 1.5 have restricted \( K \)-limit at \( e_1 \) (see Definition 1.4).

At this point we assume a "regularity condition" on \( f \), that is, we assume that the \( K \)-limit (and not only the restricted \( K \)-limit) of function (i) exists at \( e_1 \). With this hypothesis we prove the main result of the paper, i.e., that \( f_1 \) is a function depending only on one complex variable, and we can find an explicit formula for \( f_1 \) (Theorem 2.2 and Theorem 2.4). We then show that the assumption of analogous "regularity conditions" on (ii) at \( e_1 \) does not make any sense.

Finally, after having given (under conjugation in \( \text{Aut} \ A_n \)) a special form to the hyperbolic automorphism \( \gamma \) of \( A_n \), we show that the existence of the \( K \)-limit of function (iii), for \( z \to e_1 \), brings to the same conclusions on \( f \) as in Theorem 2.4.

For a statement of the Wolff theorem, for a definition of the "Wolff point," and for other preliminaries and notations we refer the reader to, e.g., [10].

1. THE GENERAL CASE

Let us denote by \( SU(n, 1) \) the special unitary group with respect to the standard Hermitian form of signature \( (n, 1) \), i.e.,

\[
SU(n, 1) = \{ g \in SL(n + 1, \mathbb{C}) : g^*Jg = J \},
\]

where \( J = (I_n^t \ 0) \), and \( I_n \) is the \( n \times n \) identity matrix. Let us write any \( g \in SU(n, 1) \), as customary, in the form of a complex \( (n + 1) \times (n + 1) \) matrix \( (A \ B) \), with \( D \in \mathbb{C} \) and \( A, B, C \) matrices of type \( n \times n, n \times 1 \) and \( 1 \times n \), respectively.
It is well known that there exists a surjective homomorphism \( \Psi : SU(n, 1) \to \text{Aut}_n \) mapping \( g = (A, B) \in SU(n, 1) \) to \( \Psi_g \in \text{Aut}_n \) defined by

\[
\Psi_g(z) = (Az + B)(Cz + D)^{-1},
\]

for all \( z \in A_n \). The kernel of \( \Psi \) is given by the center of \( SU(n, 1) \), i.e., by the subgroup

\[
\{ e^{2\pi i k (n+1)} I_{n+1}, k = 0, ..., n \}
\]

(for a proof see, e.g., [5, 10]).

The proof of the following theorem can be found, e.g., in [1].

**Theorem 1.** Each element \( \gamma \) of the group \( \text{Aut}_n \) can be extended holomorphically to an open neighborhood of \( \partial A_n \) and, if \( \gamma \neq \text{id}_{A_n} \), then either \( \gamma \) has at least one fixed point in \( A_n \), or it has no fixed points in \( A_n \) and it has one or two fixed points in \( \partial A_n \).

**Definition 1.1.** In the case in which \( \gamma \) has some fixed point in \( A_n \), then it is called *elliptic*; if \( \gamma \) has no fixed points in \( A_n \) and only one fixed point in \( \partial A_n \), then it is called *parabolic*; if \( \gamma \) has no fixed points in \( A_n \) and two fixed points in \( \partial A_n \), then it is called *hyperbolic*.

As we already noticed in the Introduction, in the case \( n = 1 \), the set of all holomorphic maps of the unit disc \( \Delta \) of \( \mathbb{C} \) into itself which commute with a given hyperbolic automorphism was studied in 1941 by M. H. Heins who proved the following

**Theorem 1.2.** Let \( \gamma \) be a hyperbolic automorphism of \( \Delta \) and let \( f \in \text{Hol}(\Delta, \Delta) \) be such that \( f \cdot \gamma = \gamma \cdot f \). Then either \( f = \text{id}_\Delta \) or \( f \) is a hyperbolic automorphism of \( \Delta \) with the same fixed points of \( \gamma \).

A proof of this theorem can be found in [6]: the proof relies upon the existence result for the derivative of \( f \) at its Wolff point.

From now on \( \gamma \) will be a hyperbolic element of \( \text{Aut}_n \). Since \( \text{Aut}_n \) acts doubly transitively on \( \partial A_n \), we can find a suitable element \( \varphi \) in \( \text{Aut}_n \) such that the fixed points of \( \varphi \gamma \varphi^{-1} \) in \( \partial A_n \) are \( e_1 \) and \( -e_1 \), where \( e_j \) denotes the \( j \)-th element of the standard basis of \( \mathbb{C}^n \), \( j = 1, ..., n \). If \( \gamma \) is a hyperbolic element in \( \text{Aut}_n \) such that its fixed points in \( \partial A_n \) are \( e_1 \) and \( -e_1 \), then the elements of \( SU(n, 1) \) which represent \( \gamma \) have the form

\[
\begin{pmatrix}
e^{\mu} \cosh t_0 & e^{\mu} \sinh t_0 \\
0 & A_1 \\
e^{\mu} \sinh t_0 & e^{\mu} \cosh t_0
\end{pmatrix},
\]

where \( t_0 \in \mathbb{R} \setminus \{0\} \), \( A_1 \in U(n-1) \), and \( \det A_1 = e^{-2\mu} \).
In fact $e_1$ and $-e_1$ are the fixed points of $\gamma$ in $\partial A_n$ if, and only if, $e_1 + e_{n+1}$ and $-e_1 - e_{n+1}$ are the isotropic eigenvectors in $\mathbb{C}^{n+1}$ of any of the matrices in $\Psi^{-1}(\gamma)$. In what follows, we will choose any element $g$ of the $n+1$ elements of $\Psi^{-1}(\gamma)$. All that we will say is independent of the choice made. By conjugating this chosen element $g$ with a suitable element in $SU(n-1) \subset SU(n,1)$ we can suppose that $A_1$ is a diagonal matrix. This implies that if $z = (z_1, ..., z_n) \in A_n$, then

$$\gamma(z) = \frac{\cosh t_0 z_1 + \sinh t_0, e^{i\theta} z_2, ..., e^{i\theta} z_n}{\sinh t_0 z_1 + \cosh t_0}$$

(1.1)

If $\gamma$ is any hyperbolic automorphism of $A_n$, then the search for all the solutions $f \in \text{Hol}(A_n, A_n)$ of equation $f \circ \gamma = \gamma \circ f$ can, clearly, be made up to conjugation by elements of Aut $A_n$. Therefore we can suppose that $\gamma$ has the form (1.1). Our first results concern the form of the first component of $f$, when restricted to the unit disc $D \times \{0\} \subset A_n$. The fact that $f$ and $\gamma$ commute implies the following

**Proposition 1.3.** Let $\gamma \in \text{Aut} A_n$ be as in (1.1) and let $f = (f_1, ..., f_n) \in \text{Hol}(A_n, A_n)$. If $f \circ \gamma = \gamma \circ f$, then there exists $t_1 \in \mathbb{R}$ such that

$$f_1(z_1, 0, ..., 0) = \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1}$$

(1.2)

*Proof.* Let us consider the holomorphic maps $\tilde{f}$ and $\tilde{\gamma}$ from $A$ into $A$ defined by $\tilde{f}(\zeta) = f_1(\zeta, 0, ..., 0)$ and $\tilde{\gamma}(\zeta) = \gamma_1(\zeta, 0, ..., 0)$. It is easy to see that the map $\tilde{\gamma}$ is a holomorphic automorphism of $A$ and that its fixed points are 1 and $-1$. Since $\gamma_j(z)$ depends only on $z_1$ and since $\gamma_j(z_1, 0, ..., 0) = 0$ for all $2 \leq j \leq n$, then $\tilde{\gamma}$ and $\tilde{f}$ commute.

By Theorem 1.2, there exists $t_1 \in \mathbb{R}$ such that for all $\zeta \in A$,

$$\tilde{\gamma}(\zeta) = \frac{\cosh t_1 \zeta + \sinh t_1}{\sinh t_1 \zeta + \cosh t_1}$$

and the proposition is proved. $\blacksquare$

The explicit form of $\tilde{f}$ we have found allows us to prove that

$$\liminf_{\zeta \to 1} \frac{1 - |\tilde{\gamma}(\zeta)|}{1 - |\zeta|} < +\infty$$

and

$$\liminf_{\zeta \to -1} \frac{1 - |\tilde{\gamma}(\zeta)|}{1 - |\zeta|} < +\infty.$$  

(1.3)

In fact, if $t_1 = 0$, then the limit is equal to 1; if $t_1 \neq 0$, then we can perform a direct computation, taking the limit on the real segment $(-1, 1)$.

Let now $\|\cdot\|$ denote the norm associated to the standard Hermitian product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$. We will use inequalities (1.3) to study the function $f$.  

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With the aim of applying the Julia–Wolff–Carathéodory theorem for \( n > 1 \), we will prove

**Proposition 1.4.** Let \( f \in \text{Hol}(\mathbb{D}_n, \mathbb{D}_n) \) be such that

\[
f_1(z_1, 0, ..., 0) = \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1}.
\]

Then

\[
\liminf_{z \to e_1} \frac{1 - \|f(z)\|}{1 - \|z\|} < +\infty \quad \text{and} \quad \liminf_{z \to -e_1} \frac{1 - \|f(z)\|}{1 - \|z\|} < +\infty.
\]

**Proof.** Obviously we have

\[
\frac{1 - \|f(z)\|}{1 - \|z\|} \leq \frac{1 - \|f(z_1)\|}{1 - \|z_1\|}.
\]

Then we get

\[
\liminf_{z \to e_1} \frac{1 - \|f_1(z)\|}{1 - \|z\|} = \liminf_{z_1 \to 1} \frac{1 - \|f_1(z_1)\|}{1 - \|z_1\|} = \liminf_{z_1 \to 1} \frac{1 - \|f_1(z_1)\|}{1 - \|z_1\|} < +\infty.
\]

The finiteness of the same \( \liminf \) at \(-e_1\) can be proved analogously.

To state the Julia–Wolff–Carathéodory theorem we will recall some notations concerning curves in \( \mathbb{D}_n \) (see, e.g., [10]). Let \( x \in \mathbb{D}_n \); a \( x \)-curve is a curve \( \gamma: [a, b] \to \mathbb{D}_n \) such that \( \lim_{t \to b^{-}} \gamma(t) = x \). We denote by \( \gamma_x \) the projection of \( \gamma \) into the complex line \( \mathbb{C} \times \{x\} \) through 0 and \( x \), i.e., we set \( \gamma_x(t) = \langle \gamma(t), x \rangle \).

**Definition 1.2.** Let \( \gamma \) be a \( x \)-curve; we say that \( \gamma \) is special if

\[
\lim_{t \to b^{-}} \frac{\|\gamma(t) - \gamma_x(t)\|^2}{1 - \|\gamma_x(t)\|^2} = 0.
\]

**Definition 1.3.** Let \( \gamma \) be a special \( x \)-curve; then \( \gamma \) is said to be restricted if there exists \( A > 0 \) such that

\[
\frac{\|\sigma_x(t) - x\|}{1 - \|\sigma_x(t)\|} \leq A \quad \forall t \in [a, b).
\]

The Korányi regions take the place of the Stolz regions in the definition of the “non-tangential limits” in dimension greater than 1.
The Korányi region $K(x, M)$ of vertex $x \in \partial A_n$ and amplitude $M > 0$ is given by (see, e.g., [10])

$$K(x, M) = \left\{ z \in A_n : \frac{|\langle z, x \rangle|}{1 - \|z\|} < M \right\}.$$ 

The Korányi region $K(x, M)$ is empty if $M \leq 1$ and, for any $x$ in the boundary of $A_n$, the regions $K(x, M)$ “fill” $A_n$ as $M$ approaches $+\infty$.

**Definition 1.4.** Let $f : A_n \to \mathbb{C}$ be a function. We shall say that $f$ has K-limit $\gamma$ at $x \# 2n$ (possibly $\gamma = \infty$) if $f(z) \to \gamma$ as $z \to x$ within $K(x, M)$ for any $M > 1$. We shall say that $f$ has restricted K-limit $\gamma$ at $x$ if $f(\sigma(t)) \to \gamma$ as $t \to b^-$ for any restricted $x$-curve $\sigma$. We can now state precisely the following classical result (see, e.g., [10, 1]).

**Theorem 1.5 (Julia–Wolff–Carathéodory).** Let $f \in \text{Hol}(A_n, A_n)$ be such that, for $x \# 2n$,

$$\liminf_{z \to x} \frac{1 - \|f(z)\|}{1 - \|z\|} = c < +\infty.$$ 

Then $f$ has K-limit $\gamma \in \partial A_n$ at $x$ and the following functions are bounded on any Korányi region:

(i) $(1 - \langle f(z), x \rangle)/(1 - \langle z, x \rangle)$,

(ii) $Q_\gamma(f(z))/(1 - \langle z, x \rangle)^{1/2}$,

(iii) $\langle df_z(x), y \rangle/(1 - \langle z, x \rangle)^{1/2}$,

where $Q_\gamma(z) = z - \langle z, y \rangle y$ is the orthogonal projection on the orthogonal complement of $C_\gamma$ and $x^\perp$ is any vector in $C^\perp$ orthogonal to $x$. Moreover the functions (ii) and (iii) have restricted K-limit 0 at $x$ and the function (i) has restricted K-limit $c$ at $x$.

By Proposition 1.4, the Julia–Wolff–Carathéodory theorem yields the following result, which guarantees that the fixed points of $\gamma$ are “fixed points” for $f$.

**Corollary 1.6.** Let $\gamma$ be a hyperbolic automorphism of $A_n$, let $p_1, p_2 \in \partial A_n$ be the fixed points of $\gamma$ in $A_n$, and let $f \in \text{Hol}(A_n, A_n)$. If $f \circ \gamma = \gamma \circ f$, then $K - \lim_{z \to p_1} f(z) = p_1$ and $K - \lim_{z \to p_2} f(z) = p_2$.

**Proof.** Let $\varphi \in \text{Aut} A_n$ be such that $\varphi(e_1) = p_1$, $\varphi(-e_1) = p_2$, and $\tilde{\gamma} = \varphi^{-1} \circ \gamma \circ \varphi$ has the form (1.1). Set $\tilde{f} = \varphi^{-1} \circ f \circ \varphi$. Then $\tilde{f}$ commutes with $\tilde{\gamma}$. Since $\varphi$ sends Korányi regions with vertex at $p_1$ ($p_2$) in Korányi regions...
with vertex at $e_1 (-e_1)$, then we can restrict ourselves to the case in which $\gamma$ has the form (1.1).

By Proposition 1.3 there exists $t_1 \in \mathbb{R}$ such that

$$f_1(z_1, 0, \ldots, 0) = \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1}.$$

Proposition 1.4 together with Theorem 1.5 implies that $f$ admits $K$-limit $y$ at $e_1$. The above form of $f_1$ yields that $f_1(z_1, 0, \ldots, 0)$ approaches 1 when $z_1$ approaches to 1. Hence $f(z_1, 0, \ldots, 0) \to e_1$ when $z_1 \to 1$ (because $f$ maps $\Delta_n$ into itself) and therefore $y = e_1$. The same argument applied to the point $-e_1$ implies that $K \lim_{z \to -e_1} f(z) = -e_1$.

We will now obtain the final results of this section, which completely describe the behaviour of $f$ on the disc $\Delta \times \{0\}$.

**Proposition 1.7.** Let $\gamma$ be the hyperbolic automorphism of $\Delta_n$ given by (1.1) and let $f \in \text{Hol}(\Delta_n, \Delta_n)$ be such that $f \circ \gamma = \gamma \circ f$. Then $f_2(z_1, 0, \ldots, 0) = \cdots = f_n(z_1, 0, \ldots, 0) = 0$ for all $z_1 \in \Delta$.

**Proof.** Fix $z_1 \in \Delta$, set $z = (z_1, 0, \ldots, 0)$, and define

$$\sigma(t) = \left( \frac{\cosh t z_1 + \sinh t}{\sinh t z_1 + \cosh t}, 0, \ldots, 0 \right).$$

The curve $\sigma$ is a restricted $e_1$-curve when $t \to +\infty$. In fact $\sigma = \sigma_e$ and therefore $\sigma$ is trivially special; the fact that $\sigma$ is restricted follows from an easy computation.

We consider now the function (ii) in Theorem 1.5. By Propositions 1.3 and 1.4 we obtain that

$$\lim_{t \to +\infty} \frac{\|f_2(\sigma(t)), \ldots, f_n(\sigma(t))\|}{(1 - |\sigma(t)|)^{1/2}} = 0, \quad (1.4)$$

since $\sigma = \sigma_e$ and $\sigma$ is restricted.

By the definition of $\sigma$, (1.4) is equivalent to the fact that

$$\lim_{t \to +\infty} \|f_2(\sigma(t)), \ldots, f_n(\sigma(t))\| \left( 1 - \frac{\cosh t z_1 + \sinh t}{\sinh t z_1 + \cosh t} \right)^{-1/2} = 0. \quad (1.5)$$

Now, the curve $\sigma$ was chosen in such a way that it contains all the points $\gamma_m(z)$ for $m \in \mathbb{N}$ in fact $\sigma(\gamma_m) = \gamma_m(z)$, as it can be seen by the definition of $\sigma$ and the form of $\gamma$ (see (1.1)). Hence, the fact that $f$ and $\gamma$ commute implies that

$$(f_2(\sigma(\gamma_m)), \ldots, f_n(\sigma(\gamma_m)))$$

$$= A^m_1(f_2(z), \ldots, f_n(z)) (\sinh \gamma_m z_1 + \cosh \gamma_m)^{-1}.$$
Since $A_1 \in U(n-1)$, the last equation implies that
\begin{align*}
\|(f_2(\sigma(mt_0)), \ldots, f_n(\sigma(mt_0)))\| = \|(f_2(z), \ldots, f_n(z))\||\sinh mt_0 z_1 + \cosh mt_0|^{-1}.
\end{align*}
(1.6)

By considering the argument of the limit in (1.5) at the point $t = mt_0$ and by calling in (1.6), we obtain that
\begin{align*}
\lim_{m \to +\infty} \frac{\|(f_2(z), \ldots, f_n(z))\|}{|\sinh mt_0 z_1 + \cosh mt_0|} \left(1 - \frac{\cosh mt_0 z_1 + \sinh mt_0}{\sinh mt_0 z_1 + \cosh mt_0}\right)^{-1/2} = 0.
\end{align*}
(1.7)

Squaring the argument of the limit in (1.7) and multiplying it by
\begin{align*}
\left(1 + \frac{\cosh mt_0 z_1 + \sinh mt_0}{\sinh mt_0 z_1 + \cosh mt_0}\right)^{-1},
\end{align*}
which is strictly less than 1, we obtain that
\begin{align*}
\lim_{m \to +\infty} \frac{\|(f_2(z), \ldots, f_n(z))\|^2}{|\sinh mt_0 z_1 + \cosh mt_0|^2} \left(1 - \frac{\cosh mt_0 z_1 + \sinh mt_0}{\sinh mt_0 z_1 + \cosh mt_0}\right)^{-1} = 0.
\end{align*}
(1.8)

This equality is equivalent to
\begin{align*}
\lim_{m \to +\infty} \frac{\|(f_2(z), \ldots, f_n(z))\|^2 (|\sinh mt_0 z_1 + \cosh mt_0|^2 - |\cosh mt_0 z_1 + \sinh mt_0|^2)^{-1} = 0.
\end{align*}

Straightforward computations yield now that
\begin{align*}
\lim_{m \to +\infty} \frac{\|(f_2(z), \ldots, f_n(z))\|^2 (1 - |z_1|^2)^{-1} = 0,
\end{align*}
and hence $f_2(z_1, 0, \ldots, 0) = \cdots = f_n(z_1, 0, \ldots, 0) = 0$ for all $z_1 \in A$ and the proposition is proved. \[\square\]

Before passing to the general case, we want to study the situation in which two holomorphic automorphisms of $A_n$, one of which is hyperbolic, commute. The result that we find generalizes to dimension $n > 1$, a well-known result on commuting automorphisms (see [6, and 2]).

**Proposition 1.8.** Let $\gamma$ be a hyperbolic automorphism of $A_n$, and let $f$ be an automorphism of $A_n$. If $\gamma$ and $f$ commute, then either $f$ is hyperbolic and it
has the same fixed points of $\gamma$ or it is elliptic and its fixed points set has positive dimension and contains the fixed point set of $\gamma$.

**Proof.** Let $l_1$ and $l_2 \in SU(n, 1)$ be such that $\Psi_{l_1} = \gamma$ and $\Psi_{l_2} = f$. As before, the statement of the proposition is invariant by inner conjugation in $\text{Aut} \ A_n$. Therefore, by conjugating both $l_1$ and $l_2$ by a same element in $SU(n, 1)$ we can suppose that $l_1 = (U \ 0 \ \bar{0})$, where $U$ is a diagonal $(n-1) \times (n-1)$ unitary matrix and where $V = e^{\theta (\sinh \tau \cosh \tau)}$, with $\tau \neq 0$. (Here we choose the fixed points of $\gamma$ to be $e_n$ and $-e_n$ only for technical reasons.)

The form of $l_2$ will now be

$$l_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $A, B, C, D$, respectively, $(n-1) \times (n-1)$, $(n-1) \times 2$, $2 \times (n-1)$, $2 \times 2$ complex matrices. The fact that $f$ and $\gamma$ commute is equivalent to

$$l_1 l_2 = e^{2im\pi/(n+1)} l_2 l_1$$

for a suitable $m \in \{0, ..., n\}$.

This last equation implies in particular that $UB = e^{i\theta} e^{-2im\pi/(n+1)} B$. Setting $B = (B_1, B_2)$ for $B_1, B_2$ vectors of $\mathbb{C}^{n-1}$ and letting $U_1 = e^{-i\theta} e^{-2im\pi/(n+1)} U$, we obtain

$$\begin{pmatrix} U_1 & 0 \\ 0 & U_1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} \cosh t I & \sinh t I \\ \sinh t I & \cosh t I \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$ 

Thus $(B_1, B_2)$ belongs to $\text{Ker} M$, where $M = \begin{pmatrix} U_1 & -\cosh t I \\ -\sinh t I & -\cosh t I \end{pmatrix}$. Since $U_1$ is a diagonal unitary matrix, say $U_1 = \text{diag}[e^{i\theta_1}, ..., e^{i\theta_n}]$, an easy inductive procedure shows that $\det M = ((e^{i\theta_1} - \cosh t)^2 - \sinh^2 t) \cdots ((e^{i\theta_{n-1}} - \cosh t)^2 - \sinh^2 t) \neq 0$. Hence $B_1 = B_2 = 0$, whence $A \in U(n-1)$ and $D \in U(1, 1)$. In the remaining one-dimensional case a direct inspection proves that $D = e^{i\theta (\sinh \tau \cosh \tau)}$. If $\tau \neq 0$, then $f$ is hyperbolic and its fixed point set is equal to the fixed point set of $\gamma$, otherwise $f$ is elliptic and its fixed point set has positive dimension and contains both the fixed points of $\gamma$.

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**2. WHAT “REGULARITY” CAN ADD**

Let $f \in \text{Hol}(A_n, A_n)$ be a map which commutes with the holomorphic automorphism $\gamma$ defined by (1.1). We will pass now to the investigation of the behaviour of $f$ outside the disc $A \times \{0\}$, in the case in which $f$ is a holomorphic self map of $A_n$. We will consider the case in which the map $f$ has a “sort of regularity” at the boundary and will deduce some consequences on the form of $f$. 

Notice that, in the one-dimensional case, if $\tau$ is the Wolff point of $f: \mathbb{D} \to \mathbb{D}$, then

$$K \lim_{z \to \tau} \frac{\tau - f(z)}{\tau - z} = d_f(\tau, \tau) \leq 1,$$

where $d_f(\tau, \tau)$ denotes the dilatation coefficient of $f$ at $\tau$ (see, e.g., [1]). In the multidimensional case this is no more true because in this case the statement of the Julia–Wolff–Carathéodory theorem involves the restricted $K$-limit instead of the $K$-limit.

Given any $z \in \mathbb{D}_n$, we want to introduce curves which contain all points of the form $\gamma_m(z)$ for $m \in \mathbb{N}$. By taking the limit along these curves we will be able to understand the behaviour of $f$ at any point $z \in \mathbb{D}_n$. To do this, fix $z \in \mathbb{D}_n$ and define the curve $\sigma: [0, +\infty) \to \mathbb{D}_n$ by

$$\sigma(t) = \left( \cosh tz_1 + \sinh t, e^{i\theta_2/n}z_2, \ldots, e^{i\theta_n/n}z_n \right),$$

First of all notice that $\sigma(mt_0) = \gamma^m(z)$ for all $m \in \mathbb{N}$.

Since we want to use these curves to compute $K$-limits, we have to prove that, for a fixed $z \in \mathbb{D}_n$, $\sigma$ lies in a suitable Korányi region with vertex at $e_1$.

**Proposition 2.1.** There exists $M > 1$ such that $\sigma(t) \in K(e_1, M)$ for all $t \geq 0$.

**Proof.** Consider the ratio $|1 - \sigma(t)|/(1 - \|\sigma(t)\|)$. It is evident that it is bounded on $[0, +\infty)$ if $|1 - \sigma(t)|/(1 - \|\sigma(t)\|^2)$ is. If we compute this last ratio, we obtain

$$\frac{|1 - \sigma(t)|}{1 - \|\sigma(t)\|^2} = \frac{|1 - z_1|}{1 - \|z\|^2} (\cosh t - \sinh t) |\cosh t + \sinh tz_1|$$

$$= e^{-t} \frac{|1 - z_1|}{1 - \|z\|^2} (\cosh t + \sinh tz_1)$$

$$\frac{1 - |z_1|}{1 - \|z\|^2} (e^{-t} \cosh t + e^{-t} \sinh t |z_1|) \leq \frac{1 - |z_1|}{1 - \|z\|^2} (1 + |z_1|) = M,$$

because $\cosh t \leq e^t$ and $\sinh t \leq e^t$ for all $t \geq 0$.

If $f \in \text{Hol}(\mathbb{D}_n, \mathbb{D}_n)$ commutes with the hyperbolic automorphism $\gamma$ given by (1.1), then, by Propositions 1.3 and 1.4, Theorem 1.5 and Corollary 1.6, both the restricted $K$-limit of $(1 - f(z))/(1 - z)$ at $e_1$ and the restricted $K$-limit of $(1 + f(z))/(1 + z)$ at $-e_1$ do exist. If we now suppose that (not only the restricted $K$-limit of $(1 - f(z))/(1 - z_1)$ exists and is finite at $e_1$, then...
but also the $K$-limit of $(1 - f(z))/(1 - z)$ exists and is finite at $e_1$, we can prove the following

**Theorem 2.2.** Let $\gamma$ be the hyperbolic automorphism of $\mathbb{A}_n$ given by (1.1) and let $f \in \text{Hol}(\mathbb{A}_n, \mathbb{A}_n)$ be such that

(a) $f$ commutes with $\gamma$,

(b) there exists $K = \lim_{z \to e_1} (1 - f(z))/(1 - z) = c \in \mathbb{C}$.

Then there exists $t \in \mathbb{R}$ such that, for all $z = (z_1, \ldots, z_n) \in \mathbb{A}_n$,

$$f(z) = \frac{\cosh t z_1 + \sinh t}{\sinh t z_1 + \cosh t}.$$ 

In particular, $f_1$ does not depend on $z_2, \ldots, z_n$.

**Proof.** Taking $\gamma$ or $\gamma^{-1}$ we can always suppose that $e_1$ is the Wolff point of $\gamma$ (that is, we can suppose that $t_0 > 0$ in (1.1)). By Proposition 1.3, there exists $t \in \mathbb{R}$ such that

$$f_1(z_1, 0, \ldots, 0) = \frac{\cosh t z_1 + \sinh t}{\sinh t z_1 + \cosh t}.$$ 

Corollary 1.6 gives that the $K$-limit of $f$ at $e_1$ is equal to $e_1$ and this implies that the function $(1 + f_1(z))/(1 + z)$ has $K$-limit 1 at $e_1$. Then condition (b) yields that

$$K = \lim_{z \to e_1} \frac{1 - f(z)}{1 - z} \cdot \frac{1 + z_1}{1 + f_1(z)} = c.$$ 

Fix $z \in \mathbb{A}_n$ and define $\sigma$ as in (2.1). Proposition 2.1 implies that

$$\lim_{t \to +\infty} \frac{1 - f_1(\sigma(t))}{1 - \sigma(t)} \cdot \frac{1 + \sigma_1(t)}{1 + f_1(\sigma(t))} = c.$$ 

Consider this last limit restricted to the sequence $\{mt_0\}$ for $m \in \mathbb{N}$. Since $\sigma(mt_0) = \gamma^m(z)$, we have the equality

$$\frac{1 - f_1(\sigma(mt_0))}{1 - \sigma_1(mt_0)} \cdot \frac{1 + \sigma_1(mt_0)}{1 + f_1(\sigma(mt_0))} = \frac{1 - f_1(\gamma^m(z))}{1 - \gamma_1^m(z)} \cdot \frac{1 + \gamma_1^m(z)}{1 + f_1(\gamma^m(z))}.$$ 

Using the fact that $f$ and $\gamma$ commute we obtain

$$\frac{1 - f_1(\sigma(mt_0))}{1 - \sigma_1(mt_0)} \cdot \frac{1 + \sigma_1(mt_0)}{1 + f_1(\sigma(mt_0))} = \frac{1 - \gamma_1^m(f(z))}{1 - \gamma_1^m(z)} \cdot \frac{1 + \gamma_1^m(z)}{1 + f_1(\gamma^m(z))}.$$
A direct computation, performed taking into account the form of \( \gamma \), gives

\[
\frac{1 - \gamma^m(f(z))}{1 + \gamma^m(f(z))} = \frac{1 - f(z)}{1 + f(z)}.
\]

Therefore

\[
\frac{1 - f(z)}{1 + f(z)} = \lim_{m \to +\infty} \frac{1 - f_1(s(m t_0))}{1 + f_1(s(m t_0))} = c,
\]

and hence we obtain that \( f_1(z) \) does not depend on \( z_2, \ldots, z_n \) and the theorem is proved.

Notice that, for any \( f \in \text{Hol}(A_n, A_n) \) such that \( f_1(z) = \frac{\cosh t_1 z_1}{\sinh t_1} \), then the \( K \)-limit of \( (1 - f(z)) / (1 - z_1) \) at \( e_1 \) exists. In fact, as \( f_1 \) depends only on \( z_1 \), the \( K \)-limit at \( e_1 \) becomes a \( K \)-limit in one-variable at 1 and in this case we can apply the fact that the function extends holomorphically to an open neighborhood of the closed disc \( A \) in \( C \) to obtain the existence of the \( K \)-limit at 1.

We will now get rid of the particular form (1.1) of the hyperbolic automorphism \( \gamma \) of \( A_n \) to give a more general statement of Theorem 2.2.

Let \( \gamma \) be a hyperbolic automorphism of \( A_n \) and let \( p_1, p_2 \in \partial A_n \) be its fixed points. Let \( \varphi \in \text{Aut} A_n \) be such that \( \varphi(e_1) = p_1 \) and \( \varphi(-e_1) = p_2 \). We can choose \( \varphi \) so that \( \varphi^{-1} \gamma \varphi \) has the form (1.1). Let \( f \in \text{Hol}(A_n, A_n) \) and define \( \tilde{f} = \varphi^{-1} f \varphi \) and \( \tilde{\gamma} = \varphi^{-1} \gamma \varphi \). Obviously \( \tilde{\gamma} \) commutes with \( \tilde{f} \) iff \( \gamma \) commutes with \( f \). The following lemma holds.

**Lemma 2.3.** Let \( \gamma, f, \varphi, \tilde{f}, \tilde{\gamma} \) be as above and suppose that \( f \) commutes with \( \gamma \). Then the two following facts are equivalent:

(i) \( K \)-limit \( \lim_{z \to \partial A_n} \frac{1 - \langle f(z), p_1 \rangle}{1 - \langle z, p_1 \rangle} \) exists and belongs to \( C \).

(ii) \( K \)-limit \( \lim_{z \to e_1} \frac{1 - f_1(z)}{1 - z_1} \) exists and belongs to \( C \).

Moreover, if the two limits exist, then they are equal.

**Proof.** Let us denote by \( \nu \) the standard Hermitian form of signature \((n, 1)\) on \( C^{n+1} \) and, if \( a \in C^n \), let us denote by \( a^* \) the vector in \( C^{n+1} \) given by \( \langle f, a \rangle \). Obviously,

\[
\frac{1 - \langle f(z), p_1 \rangle}{1 - \langle z, p_1 \rangle} = \frac{\nu(f^*(z), p_1^*)}{\nu(z^*, p_1^*)} \quad \text{and} \quad \frac{1 - f_1(z)}{1 - z_1} = \frac{\langle f^*(z), e_1^* \rangle}{\langle z^*, e_1^* \rangle}.
\]
Let $\chi = (\xi, \eta) \in SU(n, 1)$ be such that $\Psi_\chi = \varphi^{-1}$. Using the definition of $\hat{f}$ and the fact that $\varphi$ maps Korányi regions with vertex at $e_1$ in Korányi regions with vertex at $p_1$ we obtain that

$$K - \lim_{z \rightarrow e_1} \frac{1 - \hat{f}(z)}{1 - z_1} = K - \lim_{\varphi(z) \rightarrow p_1} \frac{1 - 1 \langle \varphi^{-1} f(\varphi(z), e_1) \rangle}{1 - 1 \langle z, e_1 \rangle}.$$ 

If we set $\varphi(z) = \zeta$, then the above limit is equal to

$$K - \lim_{\zeta \rightarrow p_1} \frac{1 - 1 \langle \varphi^{-1} f(\zeta), e_1 \rangle}{1 - 1 \langle \zeta, e_1 \rangle} = K - \lim_{\zeta \rightarrow p_1} \frac{v((\varphi^{-1} f(\zeta))^*, e_1^\#)}{v((\varphi^{-1} f(\zeta))^*, e_1^\#)}. \quad (2.2)$$

A direct inspection shows that, being $\Psi_\chi = \varphi^{-1}$, 

$$\varphi^{-1} f(\zeta) = \frac{\chi(\zeta^*)}{(C\chi(\zeta) + D)} \quad \text{and} \quad \varphi^{-1} (\zeta^*) = \frac{\chi(\zeta^*)}{(C\chi + D)}.$$ 

Then the $K$-limit in (2.2) is equal to

$$K - \lim_{\zeta \rightarrow p_1} \frac{v(\chi(\zeta^*), e_1^\#)}{v(\chi(\zeta^*)/(C\chi + D), e_1^\#)} = K - \lim_{\zeta \rightarrow p_1} \frac{C\chi + D}{C\chi(\zeta) + D} \cdot \lim_{\zeta \rightarrow p_1} \frac{v(\varphi^{-1} f(\zeta)^*, e_1^\#)}{v(\varphi^{-1} f(\zeta)^*, e_1^\#)}.$$ 

Corollary 1.6 implies that $K - \lim_{z \rightarrow p_1} f(z) = p_1$; then

$$K - \lim_{\zeta \rightarrow p_1} \frac{C\chi + D}{C\chi(\zeta) + D} = 1.$$ 

Hence

$$K - \lim_{z \rightarrow e_1} \frac{1 - \hat{f}(z)}{1 - z_1} = K - \lim_{\zeta \rightarrow p_1} \frac{v(\chi(\zeta^*), e_1^\#)}{v(\chi(\zeta^*)/(C\chi + D), e_1^\#)}.$$ 

Using the fact that $\chi \in SU(n, 1)$, we obtain that

$$K - \lim_{\zeta \rightarrow p_1} \frac{v(\chi(\zeta^*), e_1^\#)}{v(\chi(\zeta^*)/(C\chi + D), e_1^\#)} = K - \lim_{\zeta \rightarrow p_1} \frac{v(\chi(\zeta^*), \varphi^{-1}(e_1^\#))}{v(\chi(\zeta^*)/(C\chi + D), \varphi^{-1}(e_1^\#))}.$$ 

Now, since $\Psi_\chi = \varphi^{-1}$ and $\varphi(e_1) = p_1$, we obtain that $\Psi_\chi^{-1}(e_1) = p_1$. If $\chi^{-1}(\xi, \eta) = (\xi', \eta')$, then a direct inspection proves that $\chi^{-1} e_1 = p_1 (C_1 e_1 + D_1)$. Therefore we get
By definition, \( v(\zeta^*, p_1^*) = -1 + \langle \zeta, p_1 \rangle \) and \( v(f(\zeta)^*, p_1) = -1 + \langle f(\zeta), p_1 \rangle \). If follows that

\[
K - \lim_{\zeta \to \zeta_1} \frac{1 - f(\zeta)}{1 - z_1}
\]
does exist if and only if

\[
K - \lim_{z_1 \to z_1} \frac{1 - f(z)}{1 - z_1}
\]
does exist and that, if they exist, then they are equal. }

As a consequence of the above lemma we can state Theorem 2.2 in an “invariant version.”

**Theorem 2.4.** Let \( \gamma \) be a hyperbolic automorphism of \( A_n \) and let \( p, p_2 \) be the fixed points of \( \gamma \) in \( \partial A_n \). Let \( \varphi \in \text{Hol}(A_n, A_n) \) be such that

1. \( \varphi \) commutes with \( \gamma \); 
2. there exists \( K - \lim_{z \to z_1} ((1 - \langle f(z), p_1 \rangle)/(1 - \langle z, p_1 \rangle)) = c \in \mathbb{C} \).

Then there exists \( t_1 \in \mathbb{R} \) and \( \varphi \in \text{Aut} A_n \) such that

\[
\langle \varphi^{-1} f \varphi(z), e_1 \rangle = \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1}
\]

In particular, \( \langle \varphi^{-1} f \varphi(z), e_1 \rangle \) does not depend on \( z_2, \ldots, z_n \).

By assuming a “certain regularity” on a map \( f \in \text{Hol}(A_n, A_n) \) which commutes with a hyperbolic automorphism of \( A_n \), we have obtained a very precise and surprising information on the map \( f \) itself. In particular we have obtained that one of the components of \( f \) is always, up to conjugation in \( \text{Aut} A_n \), a function of one complex variable. This “regularity condition” we have assumed is the existence of \( K \)-limits (instead of the existence of restricted \( K \)-limits) for function (i) in Theorem 1.5.
Now we will prove that “assuming regularity” on function (ii) in Theorem 1.5 is meaningless: namely we will prove that for \( z \) itself (which obviously commutes with \( \gamma \)) it is not true that

\[
K - \lim_{z \to \gamma(z)} Q_\gamma(z)/(1 - z_1)^{-1/2} = 0
\]

(here \( Q_\gamma \) is as usual the projection on the orthogonal complement of \( C \).

In fact we have

\[
\|Q_\gamma(z)\|^2 |1 - z_1|^{-1} = (|z_1|^2 + \cdots + |z_n|^2) |\sinh t_0 z_1 + \cosh t_0|^{-2} |1 - z_1|^{-1}.
\]

Since \( |\sinh t_0 z_1 + \cosh t_0| \leq \cosh t_0 + |z_1| \sinh t_0 \leq \cosh t_0 + \sinh t_0 = e^0 \), then

\[
\|Q_\gamma(z)\|^2 |1 - z_1|^{-1} \geq e^{-2\sigma} (|z_1|^2 + \cdots + |z_n|^2) |1 - z_1|^{-1}. \tag{2.4}
\]

Take \( a, z_1 \in (0, 1) \) and set \( z_2 = a \sqrt{1 - z_1^2} \). To prove that the point \((z_1, z_2, 0, \ldots, 0)\) belongs to \( K(e_1, 2(1 - a^2))^{-1} \), we evaluate \(|1 - z_1/(1 - |z|)|^{-1}\) on the points of the form \((z_1, a \sqrt{1 - z_1^2}, 0, \ldots, 0)\), with \( z_1 \to 1 \), we obtain

\[
e^{-2\sigma} a^2 (1 - z_1^2)/(1 - z_1) = a^2 e^{-2\sigma} (1 + z_1),
\]

whose limit for \( z_1 \to 1 \) is equal to \( 2a^2 e^{-2\sigma} \neq 0 \). Comparing this result with inequality (2.4), we contradict the fact that the \( K \)-limit at \( e_1 \) of the function \( Q_\gamma(z) \) is equal to 0.

We will now conclude this paper by proving that a statement similar to the one in Theorem 2.2 holds true also in the case in which we have the existence of the \( K \)-limit (and not only of the restricted \( K \)-limit) for the function \( df \) when \( z \to e_1 \) (here \( e_i \) is any vector in \( C^s \) orthogonal to \( e_i \)). To be more precise we can state the following
THEOREM 2.5. Let $\gamma$ be as in (1.1) and let $f \in \text{Hol}(A_n, A_n)$ be such that $f \circ \gamma = \gamma \circ f$. If $e_1^\perp$ denotes any vector in $C^n$ orthogonal to $e_1$ and if

$$K - \lim_{z \to e_1} \frac{\langle df_z e_1^\perp, e_1 \rangle}{(1-z_1)^{1/2}} = 0,$$

then $f_1$ does not depend on $z_2, \ldots, z_n$ and therefore

$$f_1(z) = \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 - \cosh t_1},$$

for a suitable $t_1 \in \mathbb{R}$.

**Proof.** Taking $\gamma$ or $\gamma^{-1}$ we can always suppose that $e_1$ is the Wolff point of $\gamma$ (that is, we can suppose that $t_0 > 0$ in (1.1)).

If we fix $z \in A_n$ and define $\sigma$ as in (2.1), then we have

$$f(\sigma(mt_0)) = f(\gamma^m(z)) = \gamma^m(f(z)).$$

Therefore

$$f_1(\gamma^m(z)) = \frac{\cosh mt_0 f_1(z) + \sinh mt_0}{\sinh mt_0 f_1(z) + \cosh mt_0},$$

and by differentiating both members of the last equality with respect to $z_j$ (for $j \geq 2$) we obtain

$$\frac{\partial f_1}{\partial z_j}(\gamma^m(z)) \frac{e^{\theta m}}{\sinh mt_0 z_1 + \cosh mt_0} = \frac{\partial f_1}{\partial z_j}(z) \frac{(\sinh mt_0 f_1(z) + \cosh mt_0)^{-2}}{2 \sinh mt_0 f_1(z) + \cosh mt_0},$$

that is,

$$\frac{\partial f_1}{\partial z_j}(\gamma^m(z)) = \frac{e^{-\theta m}(\sinh mt_0 z_1 + \cosh mt_0)}{(\sinh mt_0 f_1(z) + \cosh mt_0)^2} \frac{\partial f_1}{\partial z_j}(z). \quad (2.3)$$

The fact that the $K$-limit of $\langle df_z e_1^\perp, e_1 \rangle/((1-z_1)^{1/2}$ at $e_1$ is equal to 0 implies obviously that

$$K - \lim_{z \to e_1} \frac{\langle df_z e_1^\perp, e_1 \rangle}{(1-z_1)^{1/2}} = 0.$$

By Proposition 2.1, the curve $\sigma$ is contained in a suitable Korányi region, and then we can compute the limit of $\langle df_z e_1^\perp, e_1 \rangle^2 (1-z_1)^{-1}$ on the
sequence \( \{ \sigma(mt_0) \} \) and obtain 0. Fix now \( j \in \{ 2, \ldots, n \} \) and choose \( \epsilon_j = \epsilon_j^2 = 0. \) Then

\[
\lim_{m \to +\infty} \left( \frac{\partial f_j}{\partial z_j}(\gamma_m(z)) \right)^2 (1 - \gamma_m(z))^{-1} = 0.
\]

Formula (2.3) implies that

\[
\lim_{m \to +\infty} \frac{e^{-2m\epsilon_j/\epsilon_j}}{(\sinh mt_0 f_1(z) + \cosh mt_0)^3 (\cosh mt_0 - \sinh mt_0 |1 - z_1|)} \left( \frac{\partial f_j}{\partial z_j}(z) \right)^2 = 0.
\]

Taking the modulus we get

\[
\lim_{m \to +\infty} \frac{e^{m\epsilon_j/\epsilon_j} |\sinh mt_0 z_1 + \cosh mt_0|}{|\sinh mt_0 f_1(z) + \cosh mt_0|} \left| \frac{\partial f_j}{\partial z_j}(z) \right|^2 = 0.
\]

Now, since the limit (for \( m \to +\infty \)) of the function

\[
\frac{e^{m\epsilon_j/\epsilon_j} |\sinh mt_0 z_1 + \cosh mt_0|}{|\sinh mt_0 f_1(z) + \cosh mt_0|}
\]

is equal to \( |1 + z_1|^{1/3} |f_1(z) + 1|^{-4} \), we have

\[
\lim_{m \to +\infty} \left| \frac{\partial f_j}{\partial z_j}(z) \right|^2 |1 - z_1|^{-1} = 0
\]

and therefore \( (\partial f_j/\partial z_j)(z) = 0 \), for all \( j \geq 2 \). Taking into account the results of Proposition 1.3, we obtain the assertion.

REFERENCES