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On Holomorphic Maps Which Commute with Hyperbolic Automorphisms*

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INTRODUCTION

Let Δ_n be the unit ball of $\mathbf{C}^n$ and let γ be a hyperbolic automorphism of Δ_n. In this work we study the class of holomorphic mappings $f \in \mathrm{Hol}(\Delta_n, \Delta_n)$, from Δ_n into itself, which commute with γ (with respect to the usual composition of mappings).

In the one-dimensional case, it is well known (see [6]) that if $f \in \mathrm{Hol}(\Delta, \Delta)$ commutes with a hyperbolic automorphism γ of Δ, then f is either the identity map or it is a hyperbolic automorphism of Δ with the same fixed points of γ (for a more recent exposition of this and related results, see, *e.g.*, [1]). Still in the one-dimensional case Behan and Shields [3, 11] proved that, except for the case of two hyperbolic automorphisms of Δ, two non-trivial commuting holomorphic maps belonging to $\mathrm{Hol}(\Delta, \Delta)$ have the same fixed point in Δ or the same "Wolff point" in $\partial\Delta$.

If the dimension n of the space is strictly greater than one, then the problem of characterizing the holomorphic maps which commute with a given hyperbolic automorphism of Δ_n is still open and in this paper we give some contribution at this regard.

Suppose that $f \in \mathrm{Hol}(\Delta_n, \Delta_n)$ commutes with a given hyperbolic automorphism γ of Δ_n.

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We first prove that the two fixed points p_1 and p_2 ($\in \partial\Delta_n$) of γ are "fixed points" for f as well (Corollary 1.6). Since we can suppose, up to conjugation in Aut Δ_n, that the fixed points of γ are e_1 and $-e_1$, where $e_1 = (1, 0, ..., 0)$, then the finiteness of

$$\liminf_{z \to e_1} \frac{1 - \|f(z)\|}{1 - \|z\|}$$

follows (as well as the finiteness of the same lim inf at $-e_1$). This implies, via the Julia–Wolff–Carathéodory theorem, that, among others, the functions

(i) $(1 - f_1(z))/(1 - z_1)$,
(ii) $Q_{e_1}(f(z))/(1 - z_1)^{1/2}$,
(iii) $\langle df_z e_1^{\perp}, e_1 \rangle/(1 - z_1)^{1/2}$,

defined in Theorem 1.5 have restricted K-limit at e_1 (see Definition 1.4).

At this point we assume a "regularity condition" on f, that is, we assume that the K-limit (and not only the restricted K-limit) of function (i) exists at e_1. With this hypothesis we prove the main result of the paper, *i.e.*, that f_1 is a function depending only on one complex variable, and we can find an explicit formula for f_1 (Theorem 2.2 and Theorem 2.4). We then show that the assumption of analogous "regularity conditions" on (ii) at e_1 does not make any sense.

Finally, after having given (under conjugation in Aut Δ_n) a special form to the hyperbolic automorphism γ of Δ_n, we show that the existence of the K-limit of function (iii), for $z \to e_1$, brings to the same conclusions on f as in Theorem 2.4.

For a statement of the Wolff theorem, for a definition of the "Wolff point," and for other preliminaries and notations we refer the reader to, *e.g.*, [10].

1. THE GENERAL CASE

Let us denote by $SU(n, 1)$ the special unitary group with respect to the standard Hermitian form of signature $(n, 1)$, *i.e.*,

$$SU(n, 1) = \{g \in SL(n+1, \mathbf{C}) : g^* J g = J\},$$

where $J = \left(\begin{smallmatrix} I_n & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and I_n is the $n \times n$ identity matrix. Let us write any $g \in SU(n, 1)$, as customary, in the form of a complex $(n+1) \times (n+1)$ matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, with $D \in \mathbf{C}$ and A, B, C matrices of type $n \times n$, $n \times 1$ and $1 \times n$, respectively.

It is well known that there exists a surjective homomorphism Ψ: $SU(n, 1) \to \operatorname{Aut} \Delta_n$ mapping $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SU(n, 1)$ to $\Psi_g \in \operatorname{Aut} \Delta_n$ defined by

$$\Psi_g(z) = (Az + B)(Cz + D)^{-1},$$

for all $z \in \Delta_n$. The kernel of Ψ is given by the center of $SU(n, 1)$, *i.e.*, by the subgroup

$$\{e^{2i\pi k/(n+1)} I_{n+1}, k = 0, ..., n\}$$

(for a proof see, *e.g.*, [5, 10]).

The proof of the following theorem can be found, *e.g.*, in [1].

THEOREM 1.1. *Each element γ of the group* $\operatorname{Aut} \Delta_n$ *can be extended holomorphically to an open neighborhood of $\bar{\Delta}_n$ and, if $\gamma \neq \mathrm{id}_{\Delta_n}$, then either γ has at least one fixed point in Δ_n, or it has no fixed points in Δ_n and it has one or two fixed points in $\partial \Delta_n$.*

DEFINITION 1.1. In the case in which γ has some fixed point in Δ_n, then it is called *elliptic*; if γ has no fixed points in Δ_n and only one fixed point in $\partial \Delta_n$, then it is called *parabolic*; if γ has no fixed points in Δ_n and two fixed points in $\partial \Delta_n$, then it is called *hyperbolic*.

As we already noticed in the Introduction, in the case $n = 1$, the set of all holomorphic maps of the unit disc Δ of $\mathbf{C}$ into itself which commute with a given hyperbolic automorphism was studied in 1941 by M. H. Heins who proved the following

THEOREM 1.2. *Let γ be a hyperbolic automorphism of Δ and let $f \in \mathrm{Hol}(\Delta, \Delta)$ be such that $f \circ \gamma = \gamma \circ f$. Then either $f = \mathrm{id}_\Delta$ or f is a hyperbolic automorphism of Δ with the same fixed points of γ.*

A proof of this theorem can be found in [6]: the proof relies upon the existence result for the derivative of f at its Wolff point.

From now on γ will be a hyperbolic element of $\operatorname{Aut} \Delta_n$. Since $\operatorname{Aut} \Delta_n$ acts doubly transitively on $\partial \Delta_n$, we can find a suitable element φ in $\operatorname{Aut} \Delta_n$ such that the fixed points of $\varphi \gamma \varphi^{-1}$ in $\partial \Delta_n$ are e_1 and $-e_1$, where e_j denotes the j-th element of the standard basis of $\mathbf{C}^n$, $j = 1, ..., n$. If γ is a hyperbolic element in $\operatorname{Aut} \Delta_n$ such that its fixed points in $\partial \Delta_n$ are e_1 and $-e_1$, then the elements of $SU(n, 1)$ which represent γ have the form

$$\begin{pmatrix} e^{i\theta} \cosh t_0 & 0 & e^{i\theta} \sinh t_0 \\ 0 & A_1 & 0 \\ e^{i\theta} \sinh t_0 & 0 & e^{i\theta} \cosh t_0 \end{pmatrix},$$

where $t_0 \in \mathbf{R} \backslash \{0\}$, $A_1 \in U(n-1)$, and $\det A_1 = e^{-2i\theta}$.

In fact e_1 and $-e_1$ are the fixed points of γ in $\partial\Delta_n$ if, and only if, e_1+e_{n+1} and e_1-e_{n+1} are the isotropic eigenvectors in $\mathbf{C}^{n+1}$ of any of the matrices in $\Psi^{-1}(\gamma)$. In what follows, we will choose any element g of the $n+1$ elements of $\Psi^{-1}(\gamma)$. All that we will say is independent of the choice made. By conjugating this chosen element g with a suitable element in $SU(n-1)\subset SU(n,1)$ we can suppose that A_1 is a diagonal matrix. This implies that if $z=(z_1, ..., z_n)\in\Delta_n$, then

$$\gamma(z)=\frac{(\cosh t_0 z_1+\sinh t_0,\, e^{i\theta_2}z_2, ..., e^{i\theta_n}z_n)}{\sinh t_0 z_1+\cosh t_0}. \tag{1.1}$$

If γ is any hyperbolic automorphism of Δ_n, then the search for all the solutions $f\in \mathrm{Hol}(\Delta_n, \Delta_n)$ of equation $f\circ\gamma=\gamma\circ f$ can, clearly, be made up to conjugation by elements of $\mathrm{Aut}\,\Delta_n$. Therefore we can suppose that γ has the form (1.1). Our first results concern the form of the first component of f, when restricted to the unit disc $\Delta\times\{0\}\subset\Delta_n$. The fact that f and γ commute implies the following

PROPOSITION 1.3. *Let $\gamma\in\mathrm{Aut}\,\Delta_n$ be as in* (1.1) *and let $f=(f_1, ..., f_n)\in \mathrm{Hol}(\Delta_n, \Delta_n)$. If $f\circ\gamma=\gamma\circ f$, then there exists $t_1\in\mathbf{R}$ such that*

$$f_1(z_1, 0, ..., 0)=\frac{\cosh t_1 z_1+\sinh t_1}{\sinh t_1 z_1+\cosh t_1}. \tag{1.2}$$

Proof. Let us consider the holomorphic maps $\tilde{f}$ and $\tilde{\gamma}$ from Δ into Δ defined by $\tilde{f}(\zeta)=f_1(\zeta, 0, ..., 0)$ and $\tilde{\gamma}(\zeta)=\gamma_1(\zeta, 0, ..., 0)$. It is easy to see that the map $\tilde{\gamma}$ is a holomorphic automorphism of Δ and that its fixed points are 1 and -1. Since $\gamma_1(z)$ depends only on z_1 and since $\gamma_j(z_1, 0, ..., 0)=0$ for all $2\leqslant j\leqslant n$, then $\tilde{\gamma}$ and $\tilde{f}$ commute.

By Theorem 1.2, there exists $t_1\in\mathbf{R}$ such that for all $\zeta\in\Delta$,

$$\tilde{f}(\zeta)=\frac{\cosh t_1\zeta+\sinh t_1}{\sinh t_1\zeta+\cosh t_1}$$

and the proposition is proved. ∎

The explicit form of $\tilde{f}$ we have found allows us to prove that

$$\liminf_{\zeta\to 1}\frac{1-|\tilde{f}(\zeta)|}{1-|\zeta|}<+\infty \qquad\text{and}\qquad \liminf_{\zeta\to -1}\frac{1-|\tilde{f}(\zeta)|}{1-|\zeta|}<+\infty. \tag{1.3}$$

In fact, if $t_1=0$, then the lim inf is equal to 1; if $t_1\neq 0$, then we can perform a direct computation, taking the limit on the real segment $(-1, 1)$.

Let now $\|\cdot\|$ denote the norm associated to the standard Hermitian product $\langle\cdot, \cdot\rangle$ on $\mathbf{C}^n$. We will use inequalities (1.3) to study the function f.

With the aim of applying the Julia–Wolff–Carathéodory theorem for $n > 1$, we will prove

PROPOSITION 1.4. *Let $f \in \mathrm{Hol}(\Delta_n, \Delta_n)$ be such that*

$$f_1(z_1, 0, ..., 0) = \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1}.$$

Then

$$\liminf_{z \to e_1} \frac{1 - \|f(z)\|}{1 - \|z\|} < +\infty \qquad \textit{and} \qquad \liminf_{z \to -e_1} \frac{1 - \|f(z)\|}{1 - \|z\|} < +\infty.$$

Proof. Obviously we have

$$\frac{1 - \|f(z)\|}{1 - \|z\|} \leqslant \frac{1 - |f_1(z)|}{1 - \|z\|}.$$

Then we get

$$\liminf_{z \to e_1} \frac{1 - |f_1(z)|}{1 - \|z\|} \leqslant \liminf_{z_1 \to 1} \frac{1 - |f_1(z_1, 0, ..., 0)|}{1 - |z_1|} = \liminf_{z_1 \to 1} \frac{1 - |\tilde{f}(z_1)|}{1 - |z_1|} < +\infty.$$

The finiteness of the same lim inf at $-e_1$ can be proved analogously. ∎

To state the Julia–Wolff–Carathéodory theorem we will recall some notations concerning curves in Δ_n (see, *e.g.*, [10]). Let $x \in \partial\Delta_n$; a x-curve is a curve σ: $[a, b) \to \Delta_n$ such that $\lim_{t \to b^-} \sigma(t) = x$. We denote by σ_x the projection of σ into the complex line $\mathbf{C}x$ through 0 and x, *i.e.*, we set $\sigma_x(t) = \langle \sigma(t), x \rangle x$.

DEFINITION 1.2. Let σ be a x-curve; we say that σ is *special* if

$$\lim_{t \to b^-} \frac{\|\sigma(t) - \sigma_x(t)\|^2}{1 - \|\sigma_x(t)\|^2} = 0.$$

DEFINITION 1.3. Let σ be a special x-curve; then σ is said to be *restricted* if there exists $A > 0$ such that

$$\frac{\|\sigma_x(t) - x\|}{1 - \|\sigma_x(t)\|} \leqslant A \qquad \forall t \in [a, b).$$

The Korányi regions take the place of the Stolz regions in the definition of the "non-tangential limits" in dimension greater than 1.

The *Korányi region* $K(x, M)$ of vertex $x \in \partial \Delta_n$ and amplitude $M > 0$ is given by (see, *e.g.*, [10])

$$K(x, M) = \left\{ z \in \Delta_n : \frac{|1 - \langle z, x \rangle|}{1 - \|z\|} < M \right\}.$$

The Korányi region $K(x, M)$ is empty if $M \leqslant 1$ and, for any x in the boundary of Δ_n, the regions $K(x, M)$ "fill" Δ_n as M approaches $+\infty$.

DEFINITION 1.4. Let $f\colon \Delta_n \to \mathbf{C}$ be a function. We shall say that f has *K-limit* λ at $x \in \partial \Delta_n$ (possibly $\lambda = \infty$) if $f(z) \to \lambda$ as $z \to x$ within $K(x, M)$ for any $M > 1$. We shall say that f has *restricted K-limit* λ at x if $f(\sigma(t)) \to \lambda$ as $t \to b^-$ for any restricted x-curve σ. We can now state precisely the following classical result (see, *e.g.*, [10, 1]).

THEOREM 1.5 (Julia–Wolff–Carathéodory). *Let $f \in \mathrm{Hol}(\Delta_n, \Delta_n)$ be such that, for $x \in \partial \Delta_n$*

$$\liminf_{z \to x} \frac{1 - \|f(z)\|}{1 - \|z\|} = c < +\infty.$$

Then f has K-limit $y \in \partial \Delta_n$ at x and the following functions are bounded on any Korányi region:

(i) $(1 - \langle f(z), x \rangle)/(1 - \langle z, x \rangle)$,

(ii) $Q_y(f(z))/(1 - \langle z, x \rangle)^{1/2}$,

(iii) $\langle df_z x^\perp, y \rangle/(1 - \langle z, x \rangle)^{1/2}$,

where $Q_y(z) = z - \langle z, y \rangle y$ is the orthogonal projection on the orthogonal complement of $\mathbf{C}y$ and $x^\perp$ is any vector in $\mathbf{C}^n$ orthogonal to x. Moreover the functions (ii) *and* (iii) *have restricted K-limit* 0 *at x and the function* (i) *has restricted K-limit c at x.*

By Proposition 1.4, the Julia–Wolff–Carathéodory theorem yields the following result, which guarantees that the fixed points of γ are "fixed points" for f.

COROLLARY 1.6. *Let γ be a hyperbolic automorphism of Δ_n, let $p_1, p_2 \in \partial \Delta_n$ be the fixed points of γ in $\bar{\Delta}_n$, and let $f \in \mathrm{Hol}(\Delta_n, \Delta_n)$. If $f \circ \gamma = \gamma \circ f$, then $K - \lim_{z \to p_1} f(z) = p_1$ and $K - \lim_{z \to p_2} f(z) = p_2$.*

Proof. Let $\varphi \in \mathrm{Aut}\, \Delta_n$ be such that $\varphi(e_1) = p_1$, $\varphi(-e_1) = p_2$, and $\check{\gamma} = \varphi^{-1} \circ \gamma \circ \varphi$ has the form (1.1). Set $\check{f} = \varphi^{-1} \circ f \circ \varphi$. Then $\check{f}$ commutes with $\check{\gamma}$. Since φ sends Korányi regions with vertex at p_1 (p_2) in Korányi regions with vertex at e_1 $(-e_1)$, then we can restrict ourselves to the case in which γ has the form (1.1).

By Proposition 1.3 there exists $t_1 \in \mathbf{R}$ such that

$$f_1(z_1, 0, ..., 0) = \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1}.$$

Proposition 1.4 together with Theorem 1.5 implies that f admits K-limit y at e_1. The above form of f_1 yields that $f_1(z_1, 0, ..., 0)$ approaches to 1 when z_1 approaches to 1. Hence $f(z_1, 0, ..., 0) \to e_1$ when $z_1 \to 1$ (because f maps Δ_n into itself) and therefore $y = e_1$. The same argument applied to the point $-e_1$ implies that $K-\lim_{z \to -e_1} f(z) = -e_1$. ▮

We will now obtain the final results of this section, which completely describe the behaviour of f on the disc $\Delta \times \{0\}$.

Proposition 1.7. *Let γ be the hyperbolic automorphism of Δ_n given by* (1.1) *and let $f \in \mathrm{Hol}(\Delta_n, \Delta_n)$ be such that $f \circ \gamma = \gamma \circ f$. Then $f_2(z_1, 0, \cdots, 0) = \cdots = f_n(z_1, 0, ..., 0) = 0$ for all $z_1 \in \Delta$.*

Proof. Fix $z_1 \in \Delta$, set $z = (z_1, 0, ..., 0)$, and define

$$\sigma(t) = \left(\frac{\cosh tz_1 + \sinh t}{\sinh tz_1 + \cosh t}, 0, ..., 0 \right).$$

The curve σ is a restricted e_1-curve when $t \to +\infty$. In fact $\sigma = \sigma_{e_1}$ and therefore σ is trivially special; the fact that σ is restricted follows from an easy computation.

We consider now the function (ii) in Theorem 1.5. By Propositions 1.3 and 1.4 we obtain that

$$\lim_{t \to +\infty} \frac{\|(f_2(\sigma(t)), ..., f_n(\sigma(t)))\|}{(1 - |\sigma_1(t)|)^{1/2}} = 0, \tag{1.4}$$

since $\sigma = \sigma_1$ and σ is restricted.

By the definition of σ, (1.4) is equivalent to the fact that

$$\lim_{t \to +\infty} \|(f_2(\sigma(t)), ..., f_n(\sigma(t)))\| \left(1 - \left| \frac{\cosh tz_1 + \sinh t}{\sinh tz_1 + \cosh t} \right| \right)^{-1/2} = 0. \tag{1.5}$$

Now, the curve σ was chosen in such a way that it contains all the points $\gamma^m(z)$ for $m \in \mathbf{N}$: in fact $\sigma(mt_0) = \gamma^m(z)$, as it can be seen by the definition of σ and the form of γ (see (1.1)). Hence, the fact that f and γ commute implies that

$$\begin{aligned}(f_2(\sigma(mt_0)), ..., f_n(\sigma(mt_0)))\\ = A_1^m(f_2(z), ..., f_n(z))(\sinh mt_0 z_1 + \cosh mt_0)^{-1}.\end{aligned}$$

Since $A_1 \in U(n-1)$, the last equation implies that

$$\begin{aligned}\|(f_2(\sigma(mt_0)), ..., f_n(\sigma(mt_0)))\| \\ = \|(f_2(z), ..., f_n(z))\| |\sinh mt_0 z_1 + \cosh mt_0|^{-1}.\end{aligned} \tag{1.6}$$

By considering the argument of the limit in (1.5) at the point $t = mt_0$ and by calling in (1.6), we obtain that

$$\lim_{m \to +\infty} \frac{\|(f_2(z), ..., f_n(z))\|}{|\sinh mt_0 z_1 + \cosh mt_0|} \left(1 - \left|\frac{\cosh mt_0 z_1 + \sinh mt_0}{\sinh mt_0 z_1 + \cosh mt_0}\right|\right)^{-1/2} = 0. \tag{1.7}$$

Squaring the argument of the limit in (1.7) and multiplying it by

$$\left(1 + \left|\frac{\cosh mt_0 z_1 + \sinh mt_0}{\sinh mt_0 z_1 + \cosh mt_0}\right|\right)^{-1},$$

which is strictly less than 1, we obtain that

$$\lim_{m \to +\infty} \frac{\|(f_2(z), ..., f_n(z))\|^2}{|\sinh mt_0 z_1 + \cosh mt_0|^2} \left(1 - \left|\frac{\cosh mt_0 z_1 + \sinh mt_0}{\sinh mt_0 z_1 + \cosh mt_0}\right|^2\right)^{-1} = 0. \tag{1.8}$$

This equality is equivalent to

$$\begin{aligned}\lim_{m \to +\infty} \|(f_2(z), ..., f_n(z))\|^2 \, (|\sinh mt_0 z_1 + \cosh mt_0|^2 \\ - |\cosh mt_0 z_1 + \sinh mt_0|^2)^{-1} = 0.\end{aligned}$$

Straightforward computations yield now that

$$\lim_{m \to +\infty} \|(f_2(z), ..., f_n(z))\|^2 \, (1 - |z_1|^2)^{-1} = 0,$$

and hence $f_2(z_1, 0, ..., 0) = \cdots = f_n(z_1, 0, ..., 0) = 0$ for all $z_1 \in \Delta$ and the proposition is proved. ∎

Before passing to the general case, we want to study the situation in which two holomorphic automorphisms of Δ_n, one of which is hyperbolic, commute. The result that we find generalizes to dimension $n > 1$, a well known result on commuting automorphisms (see [6, and 2]).

PROPOSITION 1.8. *Let γ be a hyperbolic automorphism of Δ_n, and let f be an automorphism of Δ_n. If γ and f commute, then either f is hyperbolic and it* *has the same fixed points of* γ *or it is elliptic and its fixed points set has positive dimension and contains the fixed point set of* γ.

Proof. Let l_1 and $l_2 \in SU(n, 1)$ be such that $\Psi_{l_1} = \gamma$ and $\Psi_{l_2} = f$. As before, the statement of the proposition is invariant by inner conjugation in Aut Δ_n. Therefore, by conjugating both l_1 and l_2 by a same element in $SU(n, 1)$ we can suppose that $l_1 = \left(\begin{smallmatrix} U & 0 \\ 0 & V \end{smallmatrix}\right)$, where U is a diagonal $(n-1) \times (n-1)$ unitary matrix and where $V = e^{i\theta}\left(\begin{smallmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{smallmatrix}\right)$, with $t \neq 0$. (Here we choose the fixed points of γ to be e_n and $-e_n$ only for technical reasons.)

The form of l_2 will now be

$$l_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with A, B, C, D, respectively, $(n-1) \times (n-1)$, $(n-1) \times 2$, $2 \times (n-1)$, 2×2 complex matrices. The fact that f and γ commute is equivalent to $l_1 l_2 = e^{2im\pi/(n+1)} l_2 l_1$ for a suitable $m \in \{0, ..., n\}$.

This last equation implies in particular that $UB = e^{i\theta} e^{2im\pi/(n+1)}$ $B\left(\begin{smallmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{smallmatrix}\right)$. Setting $B = (B_1, B_2)$ for B_1, B_2 vectors of $\mathbf{C}^{n-1}$ and letting $U_1 = e^{-i\theta} e^{-2mi\pi/(n+1)} U$, we obtain

$$\begin{pmatrix} U_1 & 0 \\ 0 & U_1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} \cosh tI & \sinh tI \\ \sinh tI & \cosh tI \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Thus $\left(\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right)$ belongs to Ker M, where $M = \left(\begin{smallmatrix} U_1 - \cosh tI & -\sinh tI \\ -\sinh tI & -\cosh tI + U_1 \end{smallmatrix}\right)$.

Since U_1 is a diagonal unitary matrix, say $U_1 = \operatorname{diag}[e^{i\theta_1}, ..., e^{i\theta_{n-1}}]$, an easy inductive procedure shows that $\det M = ((e^{i\theta_1} - \cosh t)^2 - \sinh^2 t) \cdots ((e^{i\theta_{n-1}} - \cosh t)^2 - \sinh^2 t) \neq 0$. Hence $B_1 = B_2 = 0$, whence $A \in U(n-1)$ and $D \in U(1, 1)$. In the remaining one-dimensional case a direct inspection proves that $D = e^{i\alpha}\left(\begin{smallmatrix} \cosh \tau & \sinh \tau \\ \sinh \tau & \cosh \tau \end{smallmatrix}\right)$. If $\tau \neq 0$, then f is hyperbolic and its fixed point set is equal to the fixed point set of γ, otherwise f is elliptic and its fixed point set has positive dimension and contains both the fixed points of γ. ∎

2. WHAT "REGULARITY" CAN ADD

Let $f \in \operatorname{Hol}(\Delta_n, \Delta_n)$ be a map which commutes with the holomorphic automorphism γ defined by (1.1). We will pass now to the investigation of the behaviour of f outside the disc $\Delta \times \{0\}$, in the case in which f is a holomorphic self map of Δ_n. We will consider the case in which the map f has a "sort of regularity" at the boundary and will deduce some consequences on the form of f.

Notice that, in the one-dimensional case, if τ is the Wolff point of $f\colon \Delta \to \Delta$, then

$$K-\lim_{z\to\tau}\frac{\tau-f(z)}{\tau-z}=d_f(\tau,\tau)\leqslant 1,$$

where $d_f(\tau,\tau)$ denotes the dilatation coefficient of f at τ (see, *e.g.*, [1]). In the multidimensional case this is no more true because in this case the statement of the Julia–Wolff–Carathéodory theorem involves the restricted K-limit instead of the K-limit.

Given any $z\in\Delta_n$, we want to introduce curves which contain all points of the form $\{\gamma^m(z)\}$ for $m\in\mathbf{N}$. By taking the limit along these curves we will be able to understand the behaviour of f at any point $z\in\Delta_n$. To do this, fix $z\in\Delta_n$ and define the curve $\sigma\colon[0,+\infty)\to\Delta_n$ by

$$\sigma(t)=\frac{(\cosh tz_1+\sinh t,\, e^{i\theta_2 t/t_0}z_2,\,\ldots,\, e^{i\theta_n t/t_0}z_n)}{\sinh tz_1+\cosh t}. \tag{2.1}$$

First of all notice that $\sigma(mt_0)=\gamma^m(z)$ for all $m\in\mathbf{N}$.

Since we want to use these curves to compute K-limits, we have to prove that, for a fixed $z\in\Delta_n$, σ lies in a suitable Korányi region with vertex at e_1.

PROPOSITION 2.1. *There exists $M>1$ such that $\sigma(t)\in K(e_1,M)$ for all $t\geqslant 0$.*

Proof. Consider the ratio $|1-\sigma_1(t)|/(1-\|\sigma(t)\|)$. It is evident that it is bounded on $[0,+\infty)$ iff $|1-\sigma_1(t)|/(1-\|\sigma(t)\|^2)$ is. If we compute this last ratio, we obtain

$$\frac{|1-\sigma_1(t)|}{1-\|\sigma(t)\|^2}=\frac{|1-z_1|}{1-\|z\|^2}(\cosh t-\sinh t)\,|\cosh t+\sinh tz_1|$$

$$=e^{-t}\frac{|1-z_1|}{1-\|z\|^2}\,|\cosh t+\sinh tz_1|$$

$$\frac{|1-z_1|}{1-\|z\|^2}(e^{-t}\cosh t+e^{-t}\sinh t\,|z_1|)\leqslant\frac{|1-z_1|}{1-\|z\|^2}(1+|z_1|)=M,$$

because $\cosh t\leqslant e^t$ and $\sinh t\leqslant e^t$ for all $t\geqslant 0$. ∎

If $f\in\mathrm{Hol}(\Delta_n,\Delta_n)$ commutes with the hyperbolic automorphism γ given by (1.1), then, by Propositions 1.3 and 1.4, Theorem 1.5 and Corollary 1.6, both the restricted K-limit of $(1-f_1(z))/(1-z_1)$ at e_1 and the restricted K-limit of $(1+f_1(z))/(1+z_1)$ at $-e_1$ do exist. If we now suppose that (not only the restricted K-limit of $(1-f_1(z))/(1-z_1)$ exists and is finite at e_1, but also) the K-limit of $(1-f_1(z))/(1-z_1)$ exists and is finite at e_1, we can prove the following

THEOREM 2.2. *Let γ be the hyperbolic automorphism of Δ_n given by* (1.1) *and let $f \in \mathrm{Hol}(\Delta_n, \Delta_n)$ be such that*

(a) *f commutes with γ,*

(b) *there exists $K-\lim_{z\to e_1}((1-f_1(z))/(1-z_1)) = c \in \mathbf{C}$.*

Then there exists $t_1 \in \mathbf{R}$ such that, for all $z=(z_1, ..., z_n) \in \Delta_n$,

$$f_1(z) = \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1}.$$

In particular, f_1 does not depend on $z_2, ..., z_n$.

Proof. Taking γ or γ^{-1} we can always suppose that e_1 is the Wolff point of γ (that is, we can suppose that $t_0 > 0$ in (1.1)). By Proposition 1.3, there exists $t_1 \in \mathbf{R}$ such that

$$f_1(z_1, 0, ..., 0) = \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1}.$$

Corollary 1.6 gives that the K-limit of f at e_1 is equal to e_1 and this implies that the function $(1+f_1(z))/(1+z_1)$ has K-limit 1 at e_1. Then condition (b) yields that

$$K-\lim_{z\to e_1} \frac{1-f_1(z)}{1-z_1}\cdot\frac{1+z_1}{1+f_1(z)} = c.$$

Fix $z \in \Delta_n$ and define σ as in (2.1). Proposition 2.1 implies that

$$\lim_{t\to+\infty} \frac{1-f_1(\sigma(t))}{1-\sigma_1(t)}\cdot\frac{1+\sigma_1(t)}{1+f_1(\sigma(t))} = c.$$

Consider this last limit restricted to the sequence $\{mt_0\}$ for $m \in \mathbf{N}$. Since $\sigma(mt_0) = \gamma^m(z)$, we have the equality

$$\frac{1-f_1(\sigma(mt_0))}{1-\sigma_1(mt_0)}\cdot\frac{1+\sigma_1(mt_0)}{1+f_1(\sigma(mt_0))} = \frac{1-f_1(\gamma^m(z))}{1-\gamma_1^m(z)}\cdot\frac{1+\gamma_1^m(z)}{1+f_1(\gamma^m(z))}.$$

Using the fact that f and γ commute we obtain

$$\frac{1-f_1(\sigma(mt_0))}{1-\sigma_1(mt_0)}\cdot\frac{1+\sigma_1(mt_0)}{1+f_1(\sigma(mt_0))} = \frac{1-\gamma_1^m(f(z))}{1-\gamma_1^m(z)}\cdot\frac{1+\gamma_1^m(z)}{1+\gamma_1^m(f(z))}.$$

A direct computation, performed taking into account the form of γ, gives

$$\frac{1-\gamma_1^m(f(z))}{1-\gamma_1^m(z)}\cdot\frac{1+\gamma_1^m(z)}{1+\gamma_1^m(f(z))}=\frac{1-f_1(z)}{1-z_1}\cdot\frac{1+z_1}{1+f_1(z)}.$$

Therefore

$$\frac{1-f_1(z)}{1-z_1}\cdot\frac{1+z_1}{1+f_1(z)}=\lim_{m\to+\infty}\frac{1-f_1(\sigma(mt_0))}{1-\sigma_1(mt_0)}\cdot\frac{1+\sigma_1(mt_0)}{1+f_1(\sigma(mt_0))}=c,$$

and hence we obtain that $f_1(z)$ does not depend on $z_2, \ldots, z_n$ and the theorem is proved. ∎

Notice that, for any $f\in\mathrm{Hol}(\Delta_n, \Delta_n)$ such that $f_1(z)=(\cosh t_1 z_1 + \sinh t_1)/(\sinh t_1 z_1+\cosh t_1)$, then the K-limit of $(1-f_1(z))/(1-z_1)$ at e_1 exists. In fact, as f_1 depends only on z_1, the K-limit at e_1 becomes a K-limit in one-variable at 1 and in this case we can apply the fact that the function extends holomorphically to an open neighborhood of the closed disc Δ in $\mathbf{C}$ to obtain the existence of the K-limit at 1.

We will now get rid of the particular form (1.1) of the hyperbolic automorphism γ of Δ_n, to give a more general statement of Theorem 2.2. Let γ be a hyperbolic automorphism of Δ_n and let $p_1, p_2\in\partial\Delta_n$ be its fixed points. Let $\varphi\in\mathrm{Aut}\,\Delta_n$ be such that $\varphi(e_1)=p_1$ and $\varphi(-e_1)=p_2$. We can choose φ so that $\varphi^{-1}\circ\gamma\circ\varphi$ has the form (1.1). Let $f\in\mathrm{Hol}(\Delta_n, \Delta_n)$ and define $\check{f}=\varphi^{-1}\circ f\circ\varphi$ and $\check{\gamma}=\varphi^{-1}\circ\gamma\circ\varphi$. Obviously $\check{\gamma}$ commutes with $\check{f}$ iff γ commutes with f. The following lemma holds

LEMMA 2.3. *Let γ, f, φ, $\check{f}$, $\check{\gamma}$ be as above and suppose that f commutes with γ. Then the two following facts are equivalent*:

$$\text{(i)}\quad K\text{-}\lim_{z\to p_1}\frac{1-\langle f(z), p_1\rangle}{1-\langle z, p_1\rangle}\ \textit{exists and belongs to}\ \mathbf{C}$$

$$\text{(ii)}\quad K\text{-}\lim_{z\to e_1}\frac{1-\check{f}_1(z)}{1-z_1}\ \textit{exists and belongs to}\ \mathbf{C}.$$

Moreover, if the two limits exist, then they are equal.

Proof. Let us denote by v the standard Hermitian form of signature $(n, 1)$ on $\mathbf{C}^{n+1}$ and, if $a\in\mathbf{C}^n$, let us denote by a^* the vector in $\mathbf{C}^{n+1}$ given by $\binom{a}{1}$. Obviously,

$$\frac{1-\langle f(z), p_1\rangle}{1-\langle z, p_1\rangle}=\frac{v(f^*(z), p_1^*)}{v(z^*, p_1^*)}\quad\text{and}\quad\frac{1-f_1(z)}{1-z_1}=\frac{v(f^*(z), e_1^*)}{v(z^*, e_1^*)}.$$

Let $\chi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(n,1)$ be such that $\Psi_\chi = \varphi^{-1}$. Using the definition of $\check{f}$ and the fact that φ maps Korányi regions with vertex at e_1 in Korányi regions with vertex at p_1 we obtain that

$$K-\lim_{z\to e_1} \frac{1-\check{f}_1(z)}{1-z_1} = K-\lim_{\varphi(z)\to p_1} \frac{1-\langle \varphi^{-1}f(\varphi(z)), e_1\rangle}{1-\langle z, e_1\rangle}.$$

If we set $\varphi(z)=\zeta$, then the above limit is equal to

$$K-\lim_{\zeta\to p_1} \frac{1-\langle \varphi^{-1}f(\zeta), e_1\rangle}{1-\langle \varphi^{-1}(\zeta), e_1\rangle} = K-\lim_{\zeta\to p_1} \frac{v((\varphi^{-1}f(\zeta))^*, e_1^*)}{v((\varphi^{-1}(\zeta))^*, e_1^*)}. \tag{2.2}$$

A direct inspection shows that, being $\Psi_\chi = \varphi^{-1}$,

$$(\varphi^{-1}f(\zeta))^* = \chi(f(\zeta)^*)/(Cf(\zeta)+D) \quad \text{and} \quad (\varphi^{-1}(\zeta))^* = \chi(\zeta^*)/(C\zeta+D).$$

Then the K-limit in (2.2) is equal to

$$\begin{aligned} &K-\lim_{\zeta\to p_1} \frac{v(\chi(f(\zeta)^*)/(Cf(\zeta)+D), e_1^*)}{v(\chi(\zeta^*)/(C\zeta+D), e_1^*)} \\ &\quad = K-\lim_{\zeta\to p_1} \frac{C\zeta+D}{Cf(\zeta)+D} \cdot K-\lim_{\zeta\to p_1} \frac{v(\chi(f(\zeta)^*), e_1^*)}{v(\chi(\zeta^*), e_1^*)}. \end{aligned}$$

Corollary 1.6 implies that $K-\lim_{z\to p_1} f(z) = p_1$; then

$$K-\lim_{\zeta\to p_1} \frac{C\zeta+D}{Cf(\zeta)+D} = 1.$$

Hence

$$K-\lim_{z\to e_1} \frac{1-f_1(z)}{1-z_1} = K-\lim_{\zeta\to p_1} \frac{v(\chi(f(\zeta)^*), e_1^*)}{v(\chi(\zeta^*), e_1^*)}.$$

Using the fact that $\chi \in SU(n,1)$, we obtain that

$$K-\lim_{\zeta\to p_1} \frac{v(\chi(f(\zeta)^*), e_1^*)}{v(\chi(\zeta^*), e_1^*)} = K-\lim_{\zeta\to p_1} \frac{v(f(\zeta)^*, \chi^{-1}(e_1^*))}{v(\zeta^*, \chi^{-1}(e_1^*))}.$$

Now, since $\Psi_\chi = \varphi^{-1}$ and $\varphi(e_1) = p_1$, we obtain that $\Psi_{\chi^{-1}}(e_1) = p_1$. If $\chi^{-1} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$, then a direct inspection proves that $\chi^{-1}(e_1^*) = p_1^*(C_1e_1 + D_1)$. Therefore we get

$$K-\lim_{z\to e_1}\frac{1-\check{f}_1(z)}{1-z_1}=K-\lim_{\zeta\to p_1}\frac{v(f(\zeta)^*,p_1^*)}{v(\zeta^*,p_1^*)}\cdot\frac{\overline{C_1e_1+D_1}}{C_1e_1+D_1}$$
$$=K-\lim_{\zeta\to p_1}\frac{v(f(\zeta)^*,p_1^*)}{v(\zeta^*,p_1^*)}.$$

By definition, $v(\zeta^*, p_1^*) = -1 + \langle \zeta, p_1 \rangle$ and $v(f(\zeta)^*, p_1) = -1 + \langle f(\zeta), p_1 \rangle$. If follows that

$$K-\lim_{\zeta\to e_1}\frac{1-\langle f(\zeta),p_1\rangle}{1-\langle \zeta,p_1\rangle}$$

does exist if and only if

$$K-\lim_{z\to e_1}\frac{1-f_1(z)}{1-z_1}$$

does exist and that, if they exist, then they are equal. ∎

As a consequence of the above lemma we can state Theorem 2.2 in an "invariant version."

THEOREM 2.4. *Let γ be a hyperbolic automorphism of Δ_n and let p_1, p_2 be the fixed points of γ in $\partial\Delta_n$. Let $f \in \mathrm{Hol}(\Delta_n, \Delta_n)$ be such that*

(a) *f commutes with γ,*

(b) *there exists $K-\lim_{z\to p_1}((1-\langle f(z), p_1\rangle)/(1-\langle z, p_1\rangle)) = c \in \mathbf{C}$.*

Then there exists $t_1 \in \mathbf{R}$ and $\varphi \in \mathrm{Aut}\,\Delta_n$ such that

$$\langle \varphi^{-1}\circ f\circ\varphi(z), e_1\rangle = \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1}.$$

In particular, $\langle \varphi^{-1}\circ f\circ\varphi(z), e_1\rangle$ does not depend on $z_2, ..., z_n$.

By assuming a "certain regularity" on a map $f \in \mathrm{Hol}(\Delta_n, \Delta_n)$ which commutes with a hyperbolic automorphism of Δ_n, we have obtained a very precise and surprising information on the map f itself. In particular we have obtained that one of the components of f is always, up to conjugation in Aut Δ_n, a function of one complex variable. This "regularity condition" we have assumed is the existence of K-limits (instead of the existence of restricted K-limits) for function (i) in Theorem 1.5.

Now we will prove that "assuming regularity" on function (ii) in Theorem 1.5 is meaningless: namely we will prove that for γ itself (which obviously commutes with γ) it is not true that

$$K - \lim_{z \to e_1} Q_{e_1}(\gamma(z))/(1-z_1)^{-1/2} = 0$$

(here Q_{e_1} is as usual the projection on the orthogonal complement of $\mathbf{C}e_1$).
In fact we have

$$\begin{aligned}&\|Q_{e_1}(\gamma(z))\|^2\,|1-z_1|^{-1}\\&\qquad=(|z_2|^2+\cdots+|z_n|^2)\,|\sinh t_0 z_1+\cosh t_0|^{-2}\,|1-z_1|^{-1}.\end{aligned}$$

Since $|\sinh t_0 z_1+\cosh t_0|\leqslant\cosh t_0+|z_1|\sinh t_0\leqslant\cosh t_0+\sinh t_0=e^{t_0}$, then

$$\|Q_{e_1}(\gamma(z))\|^2\,|1-z_1|^{-1}\geqslant e^{-2t_0}(|z_2|^2+\cdots+|z_n|^2)\,|1-z_1|^{-1}. \tag{2.4}$$

Take $a, z_1\in(0,1)$ and set $z_2=a\sqrt{1-z_1^2}$. To prove that the point $(z_1, z_2, 0, ..., 0)$ belongs to $K(e_1, 2(1-a^2)^{-1})$, we evaluate $|1-z_1|(1-\|z\|)^{-1}$. Since z_1 and a are real, we find

$$\begin{aligned}|1-z_1|(1-\|z\|)^{-1}&\leqslant 2(1-z_1)(1-\|z\|^2)^{-1}\\&=2|1-z_1|(1-(z_1^2+a^2(1-z_1^2)))^{-1}\\&=2(1-z_1)(1-z_1^2)^{-1}\,(1-a^2)^{-1}\\&=2(1+z_1)^{-1}\,(1-a^2)^{-1}\leqslant 2(1-a^2)^{-1}.\end{aligned}$$

Therefore, fixed $a\in(0,1)$, the points of the form $(z_1, a\sqrt{1-z_1^2}, 0, ..., 0)$ belong to $K(e_1, 2(1-a^2)^{-1})$ for all $z_1\in(0,1)$. If we now compute the limit of $e^{-2t_0}(|z_2|^2+\cdots+|z_n|^2)\,|1-z_1|^{-1}$ on the points of the form $(z_1, a\sqrt{1-z_1^2}, 0, ..., 0)$, with $z_1\to 1$, we obtain

$$e^{-2t_0}a^2(1-z_1^2)(1-z_1)^{-1}=a^2e^{-2t_0}(1+z_1),$$

whose limit for $z_1\to 1$ is equal to $2a^2e^{-2t_0}\neq 0$. Comparing this result with inequality (2.4), we contradict the fact that the K-limit at e_1 of the function $Q_{e_1}(\gamma(z))/(1-z_1)^{-1/2}$ is equal to 0.

We will now conclude this paper by proving that a statement similar to the one in Theorem 2.2 holds true also in the case in which we have the existence of the K-limit (and not only of the restricted K-limit) for the function $\langle df_z e_1^{\perp}, e_1\rangle/(1-z_1)^{1/2}$ when $z\to e_1$ (here $e_1^{\perp}$ is any vector in $\mathbf{C}^n$ orthogonal to e_1). To be more precise we can state the following

THEOREM 2.5. *Let γ be as in (1.1) and let $f \in \mathrm{Hol}(\Delta_n, \Delta_n)$ be such that $f \circ \gamma = \gamma \circ f$. If $e_1^{\perp}$ denotes any vector in $\mathbf{C}^n$ orthogonal to e_1 and if*

$$K - \lim_{z \to e_1} \frac{\langle df_z e_1^{\perp}, e_1 \rangle}{(1 - z_1)^{1/2}} = 0,$$

then f_1 does not depend on $z_2, ..., z_n$ and therefore

$$f_1(z) = \frac{\cosh t_1 z_1 + \sinh t_1}{\sinh t_1 z_1 + \cosh t_1},$$

for a suitable $t_1 \in \mathbf{R}$.

Proof. Taking γ or γ^{-1} we can always suppose that e_1 is the Wolff point of γ (that is, we can suppose that $t_0 > 0$ in (1.1)).

If we fix $z \in \Delta_n$ and define σ as in (2.1), then we have

$$f(\sigma(mt_0)) = f(\gamma^m(z)) = \gamma^m(f(z)).$$

Therefore

$$f_1(\gamma^m(z)) = \frac{\cosh mt_0 \, f_1(z) + \sinh mt_0}{\sinh mt_0 \, f_1(z) + \cosh mt_0},$$

and by differentiating both members of the last equality with respect to z_j (for $j \geqslant 2$) we obtain

$$\frac{\partial f_1}{\partial z_j}(\gamma^m(z)) \frac{e^{i\theta_j m}}{\sinh mt_0 z_1 + \cosh mt_0} = \frac{\partial f_1}{\partial z_j}(z) \, (\sinh mt_0 \, f_1(z) + \cosh mt_0)^{-2},$$

that is,

$$\frac{\partial f_1}{\partial z_j}(\gamma^m(z)) = \frac{e^{-i\theta_j m}(\sinh mt_0 z_1 + \cosh mt_0)}{(\sinh mt_0 \, f_1(z) + \cosh mt_0)^2} \frac{\partial f_1}{\partial z_j}(z). \tag{2.3}$$

The fact that the K-limit of $\langle df_z e_1^{\perp}, e_1 \rangle / (1 - z_1)^{1/2}$ at e_1 is equal to 0 implies obviously that

$$K - \lim_{z \to e_1} (\langle df_z e_1^{\perp}, e_1 \rangle)^2 \, (1 - z_1)^{-1} = 0.$$

By Proposition 2.1, the curve σ is contained in a suitable Korányi region, and then we can compute the limit of $(\langle df_z e_1^{\perp}, e_1 \rangle)^2 \, (1 - z_1)^{-1}$ on the sequence $\{\sigma(mt_0)\}$ and obtain 0. Fix now $j \in \{2, ..., n\}$ and choose $e_1^{\perp} = e_j$. Then

$$\lim_{m \to +\infty} \left(\frac{\partial f_1}{\partial z_j} (\gamma^m(z)) \right)^2 (1 - \gamma_1^m(z))^{-1} = 0.$$

Formula (2.3) implies that

$$\lim_{m \to +\infty} \frac{e^{-2i\theta_j m}(\sinh mt_0 z_1 + \cosh mt_0)^3}{(\sinh mt_0 f_1(z) + \cosh mt_0)^4 (\cosh mt_0 - \sinh mt_0)(1 - z_1)} \left(\frac{\partial f_1}{\partial z_j} (z) \right)^2 = 0.$$

Taking the modulus we get

$$\lim_{m \to +\infty} \frac{e^{mt_0} \, |\sinh mt_0 z_1 + \cosh mt_0|^3}{|\sinh mt_0 f_1(z) + \cosh mt_0|^4 \, |1 - z_1|} \left| \frac{\partial f_1}{\partial z_j}(z) \right|^2 = 0.$$

Now, since the limit (for $m \to +\infty$) of the function

$$\frac{e^{mt_0} \, |\sinh mt_0 z_1 + \cosh mt_0|^3}{|\sinh mt_0 f_1(z) + \cosh mt_0|^4}$$

is equal to $|1 + z_1|^3 \, |f_1(z) + 1|^{-4}$, we have

$$\lim_{m \to +\infty} \left| \frac{\partial f_1}{\partial z_j} (z) \right|^2 |1 - z_1|^{-1} = 0$$

and therefore $(\partial f_1 / \partial z_j)(z) = 0$, for all $j \geqslant 2$. Taking into account the results of Proposition 1.3, we obtain the assertion. ∎

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