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On p-rank representations

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Abstract

The p-rank of an algebraic curve X over an algebraically closed field k of characteristic $p > 0$ is the dimension of the vector space $H^1(X_{\mathrm{et}}, \mathbb{F}_p)$. We study the representations of finite subgroups $G \subset \mathrm{Aut}(X)$ induced on $H^1(X_{\mathrm{et}}, \mathbb{F}_p) \otimes k$, and obtain two main results.

First, the sum of the *nonprojective* direct summands of the representation, i.e., its *core*, is determined explicitly by local data given by the fixed point structure of the group acting on the curve. As a corollary, we derive a congruence formula for the p-rank.

Secondly, the multiplicities of the *projective* direct summands of quotient curves, i.e., their *Borne invariants*, are calculated in terms of the Borne invariants of the original curve and ramification data. In particular, this is a generalization of both Nakajima's equivariant Deuring–Shafarevich formula and a previous result of Borne in the case of free actions.

Keywords: p-rank; Galois module

1. Introduction

We fix an irreducible, smooth and complete curve X over an algebraically closed field k of positive characteristic p. The etale cohomology group $H^1(X_{\mathrm{et}}, \mathbb{F}_p)$ is a finite-

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dimensional vector space over $\mathbb{F}_p$. Its dimension, the *p-rank* of X, is a global invariant of the curve.

If we fix a finite group G of automorphisms of the curve X, then $H^1(X_{\mathrm{et}}, \mathbb{F}_p)$ becomes a finite-dimensional representation of G over $\mathbb{F}_p$. Moreover, $H^1(X_{\mathrm{et}}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k$ is a finite-dimensional representation of G over k.

First results on determining this representation up to isomorphism by local invariants of the curve and of the group action have been obtained by Shoichi Nakajima [5], under the assumption that G is a p-group, and by Niels Borne [3], under the assumption that G operates without fixed points. We continue this tradition, with no assumptions on either the group G or its action.

The local invariants (by this I essentially mean the ramification information) cannot determine the representation completely, as the example of an elliptic curve E over a field of characteristic $\neq 2$ shows. Such a curve always allows an automorphism of order 2, which stabilizes exactly 4 points (and the projection to the quotient curve is *tamely* ramified in these points). Namely, if the curve is given by the equation $y^2 = f(x)$, consider the mapping given by $(x, y) \mapsto (x, -y)$. However, the p-rank of E can be 0 *or* 1, depending on whether this curve is supersingular or not.

Accordingly, our results must be incomplete. Using the language of modular representation theory, what we do determine completely is the core of the representation (i.e., its “nonprojective” part, cf. Section 2); this is the content of Theorem 4.8. In a sense, this result is surprising, since generally the nonsemisimplicity of representations is what makes modular representation theory more difficult than representation theory in characteristic zero. Now the representation is determined completely by its core and the multiplicities of the indecomposable projective summands, which we call *Borne invariants* of the curve and introduce in Section 5. However, it is impossible to determine these by *local* invariants, as the above example shows.

The content of Theorem 5.4 is to determine explicitly, in terms of local data and the Borne invariants of X, the Borne invariants of quotient curves X/N with respect to the quotient group G/N, for any normal subgroup $N \subset G$. This gives a procedure for calculating the Borne invariants of X for those representations of G induced by quotient groups, in terms of the local invariants and the Borne invariants of the “smaller” curve X/N, and may thus be regarded as a partial solution to the problem of determining Borne invariants. In particular, if N is a p-group this approach gives all Borne invariants of X in terms of those of X/N, and if G itself is a p-group we recover Nakajima’s equivariant Deuring–Shafarevich formula.

2. Modular representation theory of finite groups

It is customary to call a (finite-dimensional) representation of a (finite) group a *modular representation* if the characteristic of the field divides the order of the group. In this situation, the notions of *simple* and *indecomposable* module no longer coincide, as would be the case in characteristic 0 by Maschke’s theorem. This makes for a richer representation theory, which we will now review.

In the following we shall fix an algebraically closed field k of characteristic $p > 0$, a finite group G, and denote by $k[G]$ the group ring of G over k. All modules under consideration will be finitely generated left $k[G]$-modules, and we identify finite-dimensional representations of G over k with such modules. All homomorphisms are assumed to be $k[G]$-linear.

2.1. Definition. A representation is *simple* (or *irreducible*) if it is nontrivial and has no proper submodules. We denote the set of isomorphism classes of simple modules by $\operatorname{Irr} G$. A representation is *indecomposable* if it is nontrivial and admits no proper direct summands. It is *projective* if the functor $\operatorname{Hom}(P, -)$ is exact.

2.2. Theorem (Krull–Schmidt). *If M is a representation, and $M \cong \bigoplus_{i=1}^{m} M_i \cong \bigoplus_{j=1}^{n} N_i$ are two decompositions with indecomposable summands, then $m = n$ and, after suitable renumbering, $M_i \cong N_i$ for all i.*

Proof. [1, Theorem 1.4.6]. □

This theorem allows us to speak of "the" indecomposable direct summands of a given module. To study modules in terms of these summands, we must introduce cores, projective covers, and loop spaces.

2.3. Definition. The (isomorphism class of the) direct sum of the nonprojective indecomposable summands of a given representation M is called *the core of M*, and will be denoted by $\operatorname{core}(M)$. If we have $M \cong \operatorname{core}(M)$, we call M itself *a core*. The (isomorphism class of the) direct sum of the projective indecomposable summands is called the *projective part of M*.

2.4. Definition.

(i) A homomorphism of modules is called *essential* if it is surjective and its restriction to every proper submodule of its domain is not surjective.
(ii) A *projective cover* of a module M consists of a projective module P and an essential map $\pi : P \to M$.

2.5. Theorem. *Any module has a projective cover, which is again finitely generated and unique up to* (*nonunique*) *isomorphism. The projective cover of a direct sum is the direct sum of the individual projective covers.*

Proof. [7, Chapter 14, Proposition 4]. □

We may thus speak of "the" projective cover $P_G(M)$ of a module.

It is known that the number of isomorphism classes of simple modules is finite [7, Chapter 18, Corollary 3]. By contrast, there are in general infinitely many isomorphism classes of indecomposable modules [1, Theorem 4.4.4]. However, the *projective* indecomposable modules are easily described by the following theorem.

2.6. Theorem. *The operation "projective cover" induces a bijection between the set* $\operatorname{Irr} G$ *of isomorphism classes of simple modules and the set of isomorphism classes of projective indecomposable modules.*

Proof. [7, Chapter 14, Corollary 1]. □

It follows from the above theorem that any module M has a decomposition

$$M \cong \operatorname{core}(M) \oplus \bigoplus_{S \in \operatorname{Irr} G} P_G(S)^{\oplus b(M,S)}$$

for unique integers $b(M,S) \geqslant 0$. To know the isomorphism class of M is to know its core and to know the value of these integers.

The core of a module is the degree zero case of a concept of "loop spaces" developed to understand modules "up to projectives." Other authors write $\Omega_G^0(M) := \operatorname{core}(M)$. We will need the degree one case:

2.7. Definition. Given a module M, its (*first*) *loop space* is

$$\Omega_G(M) := \Omega_G^1(M) := \ker\bigl(P_G(M) \to M\bigr).$$

Recursively, we define $\Omega_G^i(M) := \Omega_G(\Omega_G^{i-1}(M))$ for $i > 1$.

What follows are some technical lemmas. The reader only interested in the statements of our theorems now has the necessary notation, and may skip the rest of this subsection.

2.8. Proposition. *Given a module* M *and a simple module* S, *we have*

$$\operatorname{Hom}_G(M,S) = \operatorname{Hom}_G\bigl(P_G(M), S\bigr).$$

Proof. We apply the functor $\operatorname{Hom}_G(-,S)$ to the exact sequence

$$0 \to \Omega_G(M) \xrightarrow{i} P_G(M) \xrightarrow{\pi_M} M \to 0$$

to get the exact sequence

$$0 \to \operatorname{Hom}_G(M,S) \to \operatorname{Hom}_G\bigl(P_G(M),S\bigr) \xrightarrow{i^*} \operatorname{Hom}_G\bigl(\Omega_G(M),S\bigr).$$

The lemma follows from the equation $i^* = 0$. Assume that $i^* \neq 0$, then there exists a nonzero map $f \in \operatorname{Hom}_G(\Omega_G(M),S)$ which factors through $P_G(M)$ as $f = Fi$, for some $F \in \operatorname{Hom}_G(P_G(M),S)$. Since S is irreducible, f and F must be surjective. The map

$$P_G(M) \xrightarrow{(\pi_M,F)} M \oplus S$$

is still surjective. Thus $\ker F \xrightarrow{\pi_M} M$ is surjective. Since $F \neq 0$, i.e., $\ker F \subsetneq P(M)$, this is a contradiction to the fact that π_M is essential. □

2.9. Corollary. *If we write*

$$P_G(M) \cong \bigoplus_{S \in \operatorname{Irr} G} P_G(S)^{b(P_G(M),S)},$$

then the multiplicities $b(P_G(M), S)$ are given by $b(P_G(M), S) = \dim_k \operatorname{Hom}_G(M, S)$.

Proof. We fix $S \in \operatorname{Irr} G$ and calculate by means of the previous lemma:

$$\begin{aligned}\operatorname{Hom}_G(M, S) = \operatorname{Hom}_G(P_G(M), S) &\cong \bigoplus_{T \in \operatorname{Irr} G} \operatorname{Hom}_G\big(P_G(T), S\big)^{\oplus b(P_G(M),T)} \\ &= \bigoplus_{T \in \operatorname{Irr} G} \operatorname{Hom}_G(T, S)^{\oplus b(P_G(M),T)}.\end{aligned}$$

By Schur's lemma, the dimension of $\operatorname{Hom}_G(T, S)$ is 0 or 1, depending on whether T and S are isomorphic or not. Thus the corollary follows by counting dimensions. □

2.10. Proposition. *Given a module M, the following are equivalent*:

(i) *M is projective,*
(ii) *M is injective,*
(iii) $\operatorname{core}(M) = 0$,
(iv) *$\Omega_G(M)$ is projective, and*
(v) $\Omega_G(M) = 0$.

Furthermore, $\Omega_G(M)$ is always a core.

Proof. The equivalence of (i) and (ii) follows from [1, Propositions 1.6.2 and 3.1.2]. Clearly, (i) and (iii) are equivalent by definition. Since $\Omega_G(M) = 0$ if and only if $P_G(M) \to M$ is an isomorphism, (i) and (v) are equivalent. The equivalence of (iv) and (v) follows from the claim that $\Omega_G(M)$ is a core, which we now prove.

Assume that $P \subset \Omega_G(M) \subset P_G(M)$ is a nonzero projective submodule. Then (by the equivalence of (i) and (ii)) $P_G(M)$ decomposes as a direct sum $P_G(M) \cong P \oplus Q$, and the image of Q in M is all M. This is a contradiction to the fact that $P_G(M) \to M$ is essential; hence $\Omega_G(M)$ is a core. □

The following proposition is well known; we give a proof here since it will be a central component in the proof of our Theorem 4.8.

2.11. Proposition. *Consider an exact sequence $0 \to N \to P \to M \to 0$ of modules, where P is projective. Then there exists an isomorphism*

$$\operatorname{core}(N) \cong \Omega_G(M).$$

Furthermore, if we denote the projective part of N by Q, we have $P \cong P_G(M) \oplus Q$.

Proof. We construct the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Omega_G(M) & \longrightarrow & P_G(M) & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

The middle vertical arrow exists because P is projective; it is surjective because $P_G(M) \to M$ is essential. Let Q be the kernel of this middle arrow. Since $P_G(M)$ is projective, so is Q. By the snake lemma, the first vertical arrow is surjective, and its kernel is isomorphic to Q. Since Q is injective (Proposition 2.10), we have an isomorphism

$$N \cong \Omega_G(M) \oplus Q,$$

which proves the second claim. Since Q is projective, and $\Omega_G(M)$ is a core (Proposition 2.10), $\Omega_G(M)$ is the core of N. $\square$

2.12. Proposition. *Let p^n be the p-part of the order of G, i.e., $|G| = p^n k$ with $k \in \mathbb{N}$ and $p \nmid k$. Then the dimension of every projective module is divisible by p^n.*

Proof. [7, Exercise 16.3]. $\square$

2.13. Proposition. *Let $N \subset G$ be a normal subgroup, and consider the group $H := G/N$. There is an inclusion $\operatorname{Irr} H \subset \operatorname{Irr} G$. Given $S \in \operatorname{Irr} G$, we have*

$$P_G(S)^N \cong \begin{cases} P_H(S) & \text{if } S \in \operatorname{Irr} H, \\ 0 & \text{if } S \in \operatorname{Irr} G \setminus \operatorname{Irr} H. \end{cases}$$

Proof. [3, Lemma 2.7]. $\square$

In the last section we will need the following statement about group cohomology.

2.14. Proposition. *Let $K \subset G$ be a subgroup with $p \nmid [G : K]$. Then for any representation M of G, and for all $i \geqslant 0$, the restriction map*

$$\operatorname{Res}: H^i(G, M) \to H^i(K, M)$$

is injective. In particular, if $p \nmid |G|$, then $H^i(G, M) = 0$ for all $i > 0$.

Proof. [1, Corollary 3.6.18]. $\square$

3. The p-rank of curves

We continue to assume as given an algebraically closed field k of characteristic $p > 0$. In this article, a *curve* signifies a complete, smooth, connected, 1-dimensional variety over k. The (*absolute*) *Frobenius morphism* F of such a curve X is the (canonical) morphism which is the identity on topological spaces, and the p-power map on sections of the structure sheaf. It induces maps on the (Zariski) cohomology groups $H^i(X, \mathcal{O}_X)$. These are additive, but not k-linear maps: They are p-linear, meaning that

$$F(\lambda\xi) = \lambda^p F(\xi) \quad \text{for } \lambda \in k \text{ and } \xi \in H^i(X, \mathcal{O}_X).$$

The only nontrivial case for curves is the induced map on $H^1(X, \mathcal{O}_X)$.

For this, let us review some material on p-linear maps. There is a category of p-linear maps, with objects the pairs (V, F) consisting of a finite-dimensional vector space V and a p-linear endomorphism F of V. The morphisms in this category are the linear maps on the underlying vector spaces which commute with the given p-linear endomorphisms. Given such an object (V, F), we set $V^F := \{v \in V\colon\ Fv = v\}$, the fixed vectors of F in V, furthermore $V^s := \bigcap_{i>0} \operatorname{im} F^i$, and $V^n := \bigcup_{i>0} \ker F^i$.

The integer $h = \dim_k V^s$ is called the *stable rank of* F. The vector space V^s is often called the *semisimple* part of V.

3.1. Proposition. *In the above situation, we have*

(i) *V^F is a $\mathbb{F}_p$-vector space.*
(ii) *V^s and V^n are k-vector spaces stable under F.*
(iii) $\dim_k V^s = \dim_{\mathbb{F}_p} V^F$.
(iv) $V = V^s \oplus V^n$.
(v) *F restricted to V^s is bijective, F restricted to V^n is nilpotent.*
(vi) *$(-)^s$ is an exact functor on the category of p-linear maps.*

Proof. See [4] or [6] for (i) to (v). The last statement is clear, since we assume the maps in the category to be compatible with the respective p-linear maps F. □

On the dual vector space $V^* = \operatorname{Hom}_k(V, k)$ we can define a map C by setting $C(\psi)(v) := \psi(F(v))^{1/p}$ for $v \in V$ and $\psi \in V^*$. This map is additive and $1/p$-linear, i.e., we have $C(\lambda\psi) = \lambda^{1/p} C(\psi)$. The decomposition $V = V^s \oplus V^n$ corresponds to a decomposition of V^*, and C has the same stable rank as F. Since any $1/p$-linear map can be viewed as the dual of a p-linear map, the structure theory of the previous proposition can be translated to $1/p$-linear maps.

3.2. Definition. The *p-rank h_X of a curve X* is the stable rank of the Frobenius morphism on $H^1(X, \mathcal{O}_X)$.

It is clear that we have estimates $0 \leqslant h_X \leqslant g_X$, where $g_X = \dim_k H^1(X, \mathcal{O}_X)$ is the genus of X.

We would like to know explicitly the dual map of the Frobenius morphism on $H^1(X, \mathcal{O}_X)$. Recall that a rational function $t \in k(X)$ is called *separating* if the field extension $k(X)/k(t)$ is separable. Given a meromorphic differential $\omega = f \cdot dt$, where $f \in k(X)$ and t is separating, we may write

$$f = f_0^p + f_1^p t + \cdots + f_{p-1}^p t^{p-1}.$$

The *Cartier operator* on differentials is defined by setting

$$\mathcal{C}(\omega) := f_{p-1}\, dt = \left(\sqrt[p]{-\left(\frac{d}{dt}\right)^{p-1} f}\right) dt. \tag{3.1}$$

This is well defined and independent of the choice of t [6].

3.3. Proposition. *The dual vector space of $H^1(X, \mathcal{O}_X)$ is, by Serre duality, the vector space $H^0(X, \Omega_X)$ of holomorphic differentials. Under this identification, the Cartier operator $\mathcal{C}$ is the dual map C of the Frobenius morphism F.*

Proof. [6]. □

The geometric meaning of the p-rank is the following: There are p^{h_X} unramified Galois coverings of the curve X with Galois group $\mathbb{F}_p$ (one of which is the trivial cover), up to isomorphism of the covering curve together with the action of $\mathbb{F}_p$. More precisely, the group $\mathrm{Hom}(\pi_1^{\mathrm{et}}(X), \mathbb{F}_p)$ classifies such covers, and there are natural isomorphisms

$$\left(H^0(X, \Omega_X)^{\mathcal{C}}\right)^* \cong H^1(X, \mathcal{O}_X)^F \cong H^1(X_{\mathrm{et}}, \mathbb{F}_p) \cong \mathrm{Hom}\left(\pi_1^{\mathrm{et}}(X), \mathbb{F}_p\right),$$

compatible with the operation of automorphisms of X on the respective vector spaces. For proofs and further background, we refer to the survey in [2]. In this article, we will avoid rationality questions in representation theory by studying $H^1(X, \mathcal{O}_X)^s = H^1(X_{\mathrm{et}}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k$ instead of $H^1(X, \mathcal{O}_X)^F = H^1(X_{\mathrm{et}}, \mathbb{F}_p)$. Also, we will study the dual representation $H^0(X, \Omega_X)^s$ instead of $H^1(X, \mathcal{O}_X)^s$ to simplify computations.

4. The cores of *p*-rank representations

Consider a curve X, and a finite subgroup $G \subset \mathrm{Aut}(X)$. If X is of genus $g_X \geqslant 2$, then $\mathrm{Aut}(X)$ itself is finite, but even in that case we wish to allow ourselves the freedom of choosing a smaller group.

Given a point $x \in X$, we use the notation $v_x(-)$ for the function which assigns to a function, differential or divisor its order at x.

4.1. Proposition. *Let D be an effective divisor on a curve X. The Cartier operator $\mathcal{C}$ operates on the sheaf $\Omega_X(D)$. If D is G-invariant, then G also operates on this sheaf, and the two operations commute.*

In particular, the vector space $H^0(X, \Omega_X(D))^s$ of semisimple differentials with respect to $\mathcal{C}$ is a (finite-dimensional) representation of G.

Proof. Consider an open set $U \subset X$ and a differential $\omega \in \Omega_X(D)(U)$. For $x \in U$ choose a local parameter t at P and write

$$\omega = \left(f_0^p + f_1^p \cdot t + \cdots + f_{p-1}^p \cdot t^{p-1}\right) dt = f \cdot dt \tag{4.1}$$

as in Section 3, noting that t is separating. Setting $n = v_x(D) \geqslant 0$, the assumptions imply that $v_x(f) = v_x(\omega) \geqslant -n$. Thus the estimate

$$p \cdot v_x(f_{p-1}) + p - 1 = v_x\left(f_{p-1}^p t^{p-1}\right) \geqslant \min_i\left(v_x\left(f_i^p t^i\right)\right) = v_x(f) \geqslant -n$$

holds true. We now see that $v_x(\mathcal{C}(\omega)) = v_x(f_{p-1}) \geqslant \lceil (1-p-n)/p \rceil \geqslant -n$, where $\lceil y \rceil$ signifies the smallest integer greater than y. Therefore, we have $\mathcal{C}(\omega) \in \Omega_X(D)(U)$.

Choose $g \in G$. We have $\mathcal{C}(\omega)^g = (f_{p-1}\, dt)^g = f_{p-1}^g (dt)^g$. On the other hand, if t is separating, so is $s = t^g$, thus if we write

$$\omega^g = \left(\cdots + \left(f_{p-1}^g\right)^p \cdot s^{p-1}\right) ds,$$

we have $\mathcal{C}(\omega^g) = f_{p-1}^g\, ds = \mathcal{C}(\omega)^g$, since the definition of $\mathcal{C}$ does not depend on the choice of separating variable. □

4.2. Definition. The module $V_D := H^0(X, \Omega_X(D))^s$ of the previous proposition is the *p-rank representation* of G associated to the (G-invariant and effective, but not necessarily reduced) divisor D.

We introduce the notion D^{red} for the reduced effective divisor associated to D. The following observation will prove to be helpful:

4.3. Proposition. *If D is a G-invariant effective divisor on X, then the p-rank representation does not depend on the multiplicities of D, i.e., $V_D = V_{D^{\text{red}}}$.*

Proof. The claim is that elements of $H^0(X, \Omega_X(D))^s$ have poles of order at most one. By Proposition 3.1(iii), it is sufficient to prove this claim for differentials of the form $\omega = \mathcal{C}(\omega)$. If $v_x(\omega) = -n < 0$, then as in the proof of Proposition 4.1 we have $v_x(\mathcal{C}(\omega)) \geqslant \lceil (1-p-n)/p \rceil$. It is elementary to prove that

$$\frac{1-p-n}{p} = -n \quad \Leftrightarrow \quad n = 1,$$

so we see that $v_x(\omega) \geqslant -1$. □

In the following, we will always assume that D and $\widetilde{D}$ are effective and G-invariant reduced divisors.

4.4. Definition. We will call $\widetilde{D}$ *sufficiently large with respect to* G if it is nonempty, and contains all points of X with nontrivial stabilizer in G. (Remember that by our convention $\widetilde{D}$ is also effective, reduced and G-invariant.)

4.5. Proposition (Nakajima). *If $\widetilde{D}$ is sufficiently large with respect to G, the p-rank representation $V_{\widetilde{D}}$ is a projective $k[G]$-module.*

Proof. Let $P \subset G$ be a p-Sylow subgroup of G. By [5, Theorem 1] we know that $V_{\widetilde{D}}$ is $k[P]$-free. This is equivalent to the fact that $V_{\widetilde{D}}$ is $k[G]$-projective [1, Corollary 3.6.10]. □

We will present the core of a p-rank representation as a loop space of the following *ramification module*.

4.6. Definition. Given a G-invariant effective reduced divisor D as above, we *choose* a sufficiently large divisor $\widetilde{D} \supset D$. The *ramification module of* V_D (*with respect to* $\widetilde{D}$) is the following:

$$R_{G,D,\widetilde{D}} := \begin{cases} k[\widetilde{D} \setminus D], & \text{if } D \neq \emptyset, \\ \ker(k[\widetilde{D}] \to k, \sum \lambda_x x \mapsto \sum \lambda_x), & \text{if } D = \emptyset, \end{cases}$$

where, for any reduced effective divisor E, by $k[E] := \bigoplus_{x \in E} k \cdot x$ we denote the affine coordinate ring of the reduced subvariety of X associated to E.

The *core module of* V_D is the loop space

$$C_D := \Omega_G(R_{G,D,\widetilde{D}}).$$

4.7. Remark. We note that the module C_D does not depend on the choice of $\widetilde{D}$, since enlarging $\widetilde{D}$ corresponds to adding to $R_{G,D,\widetilde{D}}$ direct summands isomorphic to $k[G]$, and such free summands are annihilated by the loop space operator. Furthermore,

$$k[\widetilde{D} \setminus D] \cong \bigoplus_{x \in \widetilde{D} \setminus D \pmod{G}} k[G/G_x]$$

is a sum of induced representations of the trivial representation.

4.8. Theorem. *The core of the p-rank representation associated to a G-invariant effective divisor D* (*not necessarily reduced*) *is given by the following formula*:

$$\operatorname{core}(V_D) \cong C_{D^{\mathrm{red}}}.$$

Proof. By Proposition 4.3 we may assume that D is reduced. We choose $\widetilde{D} \supset D$ sufficiently large. Then $\widetilde{D} \setminus D$ is also reduced and G-invariant, and the residue map induces an exact sequence of sheaves

$$0 \to \Omega_X(D) \to \Omega_X(\widetilde{D}) \xrightarrow{\mathrm{Res}} \mathcal{O}_{\widetilde{D} \setminus D} \to 0,$$

which is invariant under the operation of G. This induces a long exact sequence

$$0 \to H^0\big(X, \Omega_X(D)\big) \to H^0\big(X, \Omega_X\big(\widetilde{D}\big)\big) \to k\big[\widetilde{D} \setminus D\big] \xrightarrow{\delta} H^1\big(X, \Omega_X(D)\big) \to 0,$$

which terminates at $H^1(X, \Omega_X(\widetilde{D})) = 0$ since $\widetilde{D} \neq \emptyset$. Clearly, we have $\ker \delta = R_{G,D,\widetilde{D}}$; hence there is an exact sequence of $k[G]$-modules

$$0 \to H^0\big(X, \Omega_X(D)\big) \to H^0\big(X, \Omega_X\big(\widetilde{D}\big)\big) \xrightarrow{\mathrm{Res}} R_{G,D,\widetilde{D}} \to 0.$$

In order to extract from this an exact sequence of semisimple parts, we define a $1/p$-linear map on $k[\widetilde{D} \setminus D] = \bigoplus_{d \in \widetilde{D} \setminus D} k \cdot d$ by letting it operate on the standard basis $\{d\}_{d \in \widetilde{D} \setminus D}$ as the identity. This induces a $1/p$-linear map on $R_{G,D,\widetilde{D}}$, compatible with the operation of G. Since we know that $\mathrm{Res}(\mathcal{C}\omega)^p = \mathrm{Res}(\omega)$ by [6], the above sequence is an exact sequence in the category of $1/p$-linear maps. Thus, by the exactness of $(-)^s$, we obtain the exact sequence

$$0 \to V_D \to V_{\widetilde{D}} \to R_{G,D,\widetilde{D}} \to 0.$$

By Proposition 4.5 the middle term is a projective module, and Proposition 2.11 gives the desired result. □

4.9. Remark. If G has no fixed points, then for $D = \emptyset$ the core of the associated p-rank representation is

$$\mathrm{core}(V_\emptyset) = \Omega_G^1(R_{G,\emptyset,\widetilde{D}}) = \Omega_G^2(k),$$

since $k[\widetilde{D}] \cong k[G]^r$ for some $r \geqslant 1$, which implies that $\mathrm{core}(R_{G,\emptyset,\widetilde{D}}) = \Omega_G^2(k)$. This particular core has been calculated by Borne in [3].

4.10. Remark. Since a projective representation is determined up to isomorphism by its composition factors [7, Chapter 14, Corollary 3 to Proposition 41], the local invariants used in Theorem 4.8 and the modular character of a p-rank representation determine such a representation up to *isomorphism*.

4.11. Corollary. *Consider a curve X and a finite group G of automorphisms of X. Let r be the number of points of X with nontrivial stabilizer in G, and let p^n be the p-part of the order of G. Then*

$$h_X \equiv 1 - r \pmod{p^n}.$$

Proof. We choose a minimal sufficiently large divisor $\widetilde{D} \supset \emptyset$, and set $R := R_{G,\emptyset,\widetilde{D}}$. Since h_X is the dimension of $V_\emptyset$ and by Theorem 4.8 this module differs from its core only by projective summands, Proposition 2.12 implies that

$$h_X \equiv \dim \Omega_G(R) \pmod{p^n}.$$

Similar reasoning applies to $\Omega_G(R)$ and R, which have dimensions adding up to the dimension of the projective module $P_G(R)$, and shows that

$$\dim \Omega_G(R) \equiv -\dim R \pmod{p^n}.$$

If G has a point with nontrivial stabilizer, then $\dim R = \dim k[\widetilde{D}] - 1 = r - 1$, and if not, then $\dim R = \dim k[\widetilde{D}] - 1 = |G| - 1 \equiv 0 - 1 = r - 1 \pmod{p^n}$; hence we can combine the above congruences to obtain the corollary. □

4.12. Remark. Akio Tamagawa has reminded me that the above corollary also follows from the Deuring–Shafarevich formula (cf. [2,5]) applied to the covering $X \to X/P$, where P is a p-Sylow subgroup of G. Note that while the Deuring–Shafarevich formula only captures wildly ramified points, the number of tamely ramified points is a multiple of p^n.

5. Borne invariants of quotient curves

In addition to the notation and conventions of the previous section, we consider a normal subgroup N of G, and the short exact sequence

$$1 \to N \to G \to H \to 1.$$

A representation of H lifts to a representation of G, and we obtain an inclusion $\operatorname{Irr} H \subset \operatorname{Irr} G$ of the set of irreducible representations.

Let $Y := X/N$ be the quotient curve, and let $\pi : X \to Y$ be the canonical projection. There is a natural induced operation of H on Y. The notation of the last section will sometimes have to be decorated by subscripts X or Y.

5.1. Definition. The *Borne invariants* $b(G, D, S)$ *of the curve* X (with respect to G and D) are the multiplicities of the projective indecomposable modules in the p-rank representation of G with respect to D. Thus, we have an isomorphism

$$V_D = \operatorname{core}(V_D) \oplus \bigoplus_{S \in \operatorname{Irr} G} P_G(S)^{\oplus b(G,D,S)}.$$

We simplify notation, setting $b(G, S) := b(G, \emptyset, S)$.

5.2. Proposition (Pink). *Let D be an N-invariant reduced effective divisor on X. There is a natural isomorphism of sheaves*

$$\pi_* \Omega_X(D)^N \cong \Omega_Y(E)$$

for an effective divisor E on Y, which commutes with the Cartier operator and the operation of G. We have

$$E^{\mathrm{red}} = \pi(D)^{\mathrm{red}} \cup \{y \in Y \mid \pi \text{ is wildly ramified over } y\}.$$

Proof. Pulling back differentials from Y to X via π induces an injective sheaf homomorphism

$$\Omega_Y \to \pi_* \Omega_X(D)^N.$$

The target of this homomorphism is a torsion-free, coherent sheaf of rank 1; hence there is a unique effective divisor E on Y such that the above homomorphism extends to an isomorphism $\Omega_Y(E) \to \pi_* \Omega_X(D)^N$. By construction of the Cartier operator, this is Cartier-equivariant.

We now proceed to determine E. If R is a local ring, we denote its completion by $\widehat{R}$. Choose $y \in Y$, we then have

$$\begin{aligned}\pi_* \Omega_X(D)^N \otimes_{\mathcal{O}_{Y,y}} \widehat{\mathcal{O}_{Y,y}} &= \Big(\bigoplus_{x \in \pi^{-1}(y)} \Omega_X(D) \otimes_{\mathcal{O}_{X,x}} \widehat{\mathcal{O}_{X,x}} \Big)^N \\ &= \big(\Omega_X(D) \otimes_{\mathcal{O}_{X,x}} \widehat{\mathcal{O}_{X,x}} \big)^{N_x} \quad \text{for any } x \in \pi^{-1}(y) \\ &= \widehat{\Omega_X(D)_x}^{N_x}.\end{aligned}$$

Choose $x \in \pi^{-1}(y)$, and denote again by x and y local parameters at x and y respectively. We have

$$\widehat{\mathcal{O}_{X,x}} = k[[x]] \quad \text{and} \quad \widehat{\mathcal{O}_{Y,y}} = k[[y]] = k[[x]]^{N_x}.$$

Setting $n := |N_x|$, we may express y as

$$y = x^n + \text{terms of higher order in } x.$$

It follows that $m := v_x(dy/dx) = n - 1$ if $p \nmid n$, and $m \geqslant n$ if $p \mid n$. Let us set $d := v_x(D)$ and write $d + m = an + b$, for integers $a, b \geqslant 0$ with $b \leqslant n - 1$. The following are equivalent:

(i) $a \geqslant 1$,
(ii) $d + m \geqslant n$,
(iii) $d \geqslant 1$ or $p \mid n$.

We now see that

$$\widehat{\Omega_X(D)_x}^{N_x} = \left(\frac{1}{x^d} k[[x]] dx \right)^{N_x} = \left(\frac{1}{x^{d+m}} k[[x]] \right)^{N_x} dy = \frac{1}{y^a} k[[y]] dy,$$

which implies that $\pi_* \Omega_X(D)^N = \Omega_Y(E)$ if we set $v_Q(E) := a$, as claimed. $\square$

5.3. Definition. Consider an irreducible representation $S \in \operatorname{Irr} G$. The restriction maps $H^1(G, S) \to H^1(G_x, S)$ combine to a global restriction map

$$r_{G,X,S} : H^1(G, S) \to \prod_{x \in X} H^1(G_x, S) \cong \bigoplus_{\{x \in X:\, G_x \neq 1\}} H^1(G_x, S).$$

Its kernel $H^1_{LT,X}(G, S) := \ker r_{G,X,S}$ consists of the *locally trivial first cohomology classes of* S (with respect to G and X). We set

$$d(G, X, S) := \dim H^1_{LT,X}(G, S).$$

5.4. Theorem. *The Borne invariants of X and $Y = X/N$ with respect to G and $H = G/N$ for $T \in \operatorname{Irr} H$ are related by the following formula*:

$$b(G, T) + d(G, X, T) = b(H, T) + d(H, Y, T).$$

Proof. This is a lengthy calculation, which we divide into several steps. We choose a sufficiently large divisor $\widetilde{D}$ on X with respect to G, and set $\widetilde{E} := \pi_*(\widetilde{D})^{\mathrm{red}}$; this is a sufficiently large divisor on Y with respect to H.

Step 1. Since $\widetilde{D}$ contains all ramified points, wild or not, Proposition 5.2 implies that

$$(V_{X,\widetilde{D}})^N = V_{Y,\widetilde{E}}.$$

In particular, since $V_{X,\widetilde{D}}$ is projective by Proposition 4.5, we may apply Proposition 2.13 to its indecomposable summands to obtain

$$b\bigl(G, \widetilde{D}, T\bigr) = b\bigl(H, \widetilde{E}, T\bigr) \quad \text{for } T \in \operatorname{Irr} H. \tag{5.1}$$

Step 2. The short exact sequence

$$0 \to V_{X,\emptyset} \to V_{X,\widetilde{D}} \to R_{G,\emptyset,\widetilde{D}} \to 0,$$

established at the end of the proof of Theorem 4.8 induces, by the second claim of Proposition 2.11, an isomorphism

$$\bigoplus_{S \in \operatorname{Irr} G} P_G(S)^{b(G,\widetilde{D},S)} \cong P_G(R_{G,\emptyset,\widetilde{D}}) \oplus \bigoplus_{S \in \operatorname{Irr} G} P_G(S)^{b(G,\emptyset,S)}.$$

In particular, using Proposition 2.8, we may apply $\operatorname{Hom}_G(-, S)$ to deduce the equation

$$b\bigl(G, \widetilde{D}, S\bigr) = \dim_k \operatorname{Hom}_G(R_{G,\emptyset,\widetilde{D}}, S) + b(G, \emptyset, S) \quad \text{for } S \in \operatorname{Irr} G. \tag{5.2}$$

Step 3. On the other hand, let us consider $S \in \operatorname{Irr} G$ and the short exact sequence

$$0 \to R_{G,\emptyset,\widetilde{D}} \to k\big[\widetilde{D}\big] \to k \to 0.$$

Applying $\operatorname{Hom}_G(-, S)$ to this sequence gives an exact sequence

$$\begin{aligned} 0 \to S^G \to \bigoplus_{x \in \widetilde{D} \pmod G} S^{G_x} \to \operatorname{Hom}_G(R_{G,\emptyset,\widetilde{D}}, S) \\ \to H^1(G, S) \xrightarrow{r_{G,X,S}} \bigoplus_{x \in \widetilde{D} \pmod G} H^1(G_x, S), \end{aligned}$$

that is, an exact sequence

$$0 \to S^G \to \bigoplus_{x \in \widetilde{D} \pmod G} S^{G_x} \to \operatorname{Hom}_G(R_{G,\emptyset,\widetilde{D}}, S) \to H^1_{LT,X}(G, S) \to 0.$$

Similar reasoning applies to $T \in \operatorname{Irr} H$, leading to the exact sequence

$$0 \to T^H \to \bigoplus_{y \in \widetilde{E} \pmod H} T^{H_y} \to \operatorname{Hom}_H(R_{H,\emptyset,\widetilde{E}}, T) \to H^1_{LT,Y}(H, T) \to 0.$$

Using the fact that the alternating sum of dimensions in an exact sequence is 0, the equality $T^G = T^H$ and, for $y = \pi(x)$, the analogous equalities $V^{G_x} = V^{H_y}$, we have

$$\dim_k \operatorname{Hom}_H(R_{H,\emptyset,\widetilde{E}}, T) - \dim_k \operatorname{Hom}_G(R_{G,\emptyset,\widetilde{D}}, T) = d(H, Y, T) - d(G, X, T). \quad (5.3)$$

Step 4. Finally, combining Eqs. (5.1) and (5.2) (for X *and* Y), and (5.3) gives the result. □

5.5. Remark. If N is a p-group, then it is known that $\operatorname{Irr} G = \operatorname{Irr} H$ [3, Remark after Definition 2.5]. Thus, in this case, the Borne invariants of Y determine *all* the Borne invariants of X. In this sense, Theorem 5.4 generalizes the equivariant Deuring–Shafarevich formula of Shoichi Nakajima [5], which is the special case of $N = G$ being a p-group.

5.6. Remark. If the operation of G on X is *tame*, that is if $p \nmid |G_x|$ for all $x \in X$, then all higher cohomology groups of the stabilizers G_x vanish by Proposition 2.14. Thus $d(G, X, S) = \dim H^1(G, S)$, which proves the conjecture that Niels Borne states in [3] after Proposition 2.4.

Under certain circumstances, the calculation of the locally trivial cohomology groups is not necessary:

5.7. Proposition. *The following estimate holds true*:

$$b(G, T) \leqslant b(H, T) \quad \textit{for all } T \in \operatorname{Irr} H.$$

Furthermore, if there is an $x \in X$ such that $p \nmid [G : G_x]$ or $p \nmid [N : N_x]$, then

$$b(G, T) = b(H, T) \quad \textit{for all } T \in \operatorname{Irr} H.$$

Proof. Given $x \in X$ and setting $y = \pi(x)$, the sequence

$$1 \to N_x \to G_x \to H_y \to 1$$

is exact. We choose $T \in \operatorname{Irr} H$ and use the abbreviations $LT_G := H^1_{LT,X}(G, T)$ and $LT_H := H^1_{LT,Y}(H, T)$. The inflation-restriction sequence of group cohomology [1, Chapter 3.4, Exercise] gives the exact sequence

$$0 \to H^1(H, T) \xrightarrow{\text{inf}} H^1(G, T) \xrightarrow{\text{res}} H^1(N, T)^H.$$

We use this to construct the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & LT_H \arrow[r] \arrow[d, "i"'] & H^1(H, T) \arrow[r, "a"] \arrow[d] & \bigoplus_y H^1(H_y, T) \arrow[d] \\
0 \arrow[r] & LT_G \arrow[r] \arrow[d] & H^1(G, T) \arrow[r, "b"] \arrow[d] & \bigoplus_x H^1(G_x, T) \arrow[d] \\
 & \operatorname{coker} i \arrow[r, "j"] \arrow[d] & H^1(N, T)^H \arrow[r, "c"] & \bigoplus_x H^1(N_x, T)^{H_y} \\
 & 0 & &
\end{tikzcd}$$

The first two rows are exact by the definition of locally trivial cohomology classes. The last two columns are exact by the inflation-restriction sequence. The injectivity of the inflation maps shows that i is injective, so the first column is exact and in particular $d(H, Y, T) \leqslant d(G, X, T)$, which implies that $b(G, T) \leqslant b(H, T)$.

Now a diagram chase shows that $LT_H = LT_G \cap H^1(H, T)$; hence the induced map j is injective and its image lies in $\ker c$. To show that $b(G, T) = b(H, T)$ in the cases mentioned in the proposition, we will show that $LT_G = LT_H$.

If $p \nmid [N : N_x]$ holds for some $x \in X$, then by Proposition 2.14 the map $H^1(N, T) \to H^1(N_x, T)$ is injective, so c is injective. Since $\operatorname{coker} i \subset \ker c = 0$, it follows that $LT_G = LT_H$.

If $p \nmid [G : G_x]$ holds for some $x \in X$, then since N is normal, $[N : N_x]$ divides $[G : G_x]$. So by the above paragraph, $LT_G = LT_H$. However, a direct analysis shows more: By

Proposition 2.14 the restriction map $H^1(G, T) \to H^1(G_x, T)$ is injective. Thus, b is injective, and $LT_G = \ker b = 0$. Since $LT_H \subset LT_G = 0$, it follows that $LT_H = LT_G = 0$. $\square$

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