Stability of nonautonomous differential equations in Hilbert spaces

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Abstract

We introduce a large class of nonautonomous linear differential equations \( v' = A(t)v \) in Hilbert spaces, for which the asymptotic stability of the zero solution, with all Lyapunov exponents of the linear equation negative, persists in \( v' = A(t)v + f(t, v) \) under sufficiently small perturbations \( f \). This class of equations, which we call Lyapunov regular, is introduced here inspired in the classical regularity theory of Lyapunov developed for finite-dimensional spaces, that is nowadays apparently overlooked in the theory of differential equations. Our study is based on a detailed analysis of the Lyapunov exponents. Essentially, the equation \( v' = A(t)v \) is Lyapunov regular if for every \( k \) the limit of \( \Gamma(t)^{1/t} \) as \( t \to \infty \) exists, where \( \Gamma(t) \) is any \( k \)-volume defined by solutions \( v_1(t), \ldots, v_k(t) \). We note that the class of Lyapunov regular linear equations is much larger than the class of uniformly asymptotically stable equations.

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1. Introduction

Let us first consider the finite-dimensional setting by which our work was inspired. We are interested in the study of the persistence of the asymptotic stability of the zero

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solution of a nonautonomous linear differential equation

$$x' = A(t)x,$$  \hspace{1cm} (1)

under “perturbations” $f$ of the original equation

$$x' = A(t)x + f(t, x).$$ \hspace{1cm} (2)

We recall that there are examples, going back to Perron, showing that an arbitrarily small perturbation (2) of an asymptotically stable nonautonomous linear equation (1) may be unstable, and in fact may be exponentially unstable in some directions, even if all Lyapunov exponents of the linear equation (1) are negative. It is of course possible to provide additional assumptions of general nature under which the stability persists. This is the case, for example, with the assumption of uniform asymptotic stability for the linear equation, although this requirement is dramatically restrictive for a nonautonomous system. Incidentally, this assumption is analogous to the restrictive requirement of existence of an exponential dichotomy for the evolution operator of a nonautonomous equation in the case when there exist simultaneously positive and negative Lyapunov exponents (we refer to [4] for a related discussion). It is thus desirable to look for general assumptions that are substantially weaker than uniform asymptotic stability, under which one can still establish the persistence of stability of the zero solution in (2), when the perturbation $f$ is sufficiently small. This is the case of the so-called notion of regularity introduced by Lyapunov in his doctoral thesis [8] (the expression is his own), which unfortunately seems nowadays apparently overlooked in the theory of differential equations (either related to stability or otherwise).

We now briefly recall the classical notion of Lyapunov regularity, or regularity for short, in the finite-dimensional setting. We first introduce the Lyapunov exponent associated to the linear differential equation (1) in $\mathbb{R}^n$. We assume that $A(t)$ depends continuously on $t$, and that all solutions of (1) are global. The Lyapunov exponent $\lambda: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is defined by

$$\lambda(x_0) = \limsup_{t \to +\infty} \frac{1}{t} \log \|x(t)\|,$$ \hspace{1cm} (3)

where $x(t)$ denotes the solution of (1) with $x(0) = x_0$. To introduce the notion of regularity we also need to consider the adjoint equation

$$y' = -A(t)^*y,$$ \hspace{1cm} (4)

where $A(t)^*$ denotes the transpose of $A(t)$. The associated Lyapunov exponent $\mu: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is defined by

$$\mu(y_0) = \limsup_{t \to +\infty} \frac{1}{t} \log \|y(t)\|,$$
where \( y(t) \) denotes the solution of (4) with \( y(0) = y_0 \). It follows from the abstract theory of Lyapunov exponents (see the book by Barreira and Pesin [2, Section 1.2]) that the function \( \lambda \) can take at most \( n \) values on \( \mathbb{R}^n \setminus \{0\} \), say \(-\infty \leq \lambda_1 < \cdots < \lambda_p \) for some integer \( p \leq n \). Furthermore, for each \( i = 1, \ldots, p \) the set

\[ E_i = \{ x \in \mathbb{R}^n : \lambda(x) \leq \lambda_i \} \]

is a linear space. We consider the values \( \lambda'_1 \leq \cdots \leq \lambda'_n \) of the Lyapunov exponent \( \lambda \) on \( \mathbb{R}^n \setminus \{0\} \) counted with multiplicities, obtained by repeating each value \( \lambda_i \) a number of times equal to \( \dim E_i - \dim E_{i-1} \) (with \( E_0 = \{0\} \)). In a similar manner, we can consider the values \( \mu'_1 \geq \cdots \geq \mu'_n \) of the Lyapunov exponent \( \mu \) on \( \mathbb{R}^n \setminus \{0\} \) counted with multiplicities. We say that the linear equation (1) is \textit{Lyapunov regular} if

\[ \lambda'_i + \mu'_i = 0 \quad \text{for} \quad i = 1, \ldots, n. \quad (5) \]

It is well known that if all values of the Lyapunov exponent are \textit{negative} then the zero solution of (1) is asymptotically stable. However, there may still exist arbitrarily small perturbations \( f(t, x) \) with \( f(t, 0) = 0 \), such that the zero solution of (2) is not asymptotically stable. An explicit example in \( \mathbb{R}^2 \) is the equation \((u', v') = A(t)(u, v)\), with the diagonal matrix

\[ A(t) = \begin{pmatrix} -15 - 14(\sin \log t + \cos \log t) & 0 \\ 0 & -15 + 14(\sin \log t + \cos \log t) \end{pmatrix} \]

and the perturbation \( f(t, (u, v)) = (0, u^4) \). In this example, one can show that the Lyapunov exponent \( \lambda \) in (3) is constant and equal to \(-1\), but there exists a solution \((u(t), v(t))\) of the perturbed system (2), i.e., of the equation \((u', v') = A(t)(u, v) + (0, u^4)\), with

\[ \lim \sup_{t \to +\infty} \frac{1}{t} \log \| (u(t), v(t)) \| > 0 \]

(we refer to [2] for full details about the example). In other words, assuming that all values of the Lyapunov exponent \( \lambda \) are negative is not sufficient to guarantee that the asymptotic stability of the linear equation (1) persists under sufficiently small perturbations. On the other hand, if (1) is \textit{Lyapunov regular}, then for any sufficiently small perturbation \( f(t, x) \) with \( f(t, 0) = 0 \) for every \( t \geq 0 \), the zero solution of the perturbed equation (2) is asymptotically stable (see Theorem 12 below).

It should be noted that while Lyapunov regularity requires much from the structure of the original linear equation, it is substantially weaker than the requirement of uniform asymptotic stability (note that a priori Lyapunov regularity also requires much from the structure of the associated adjoint equation, although there are alternative characterizations of regularity that do not use the adjoint equation; we refer to [2] for
full details). More precisely, consider the evolution operator $U(t,s)$ associated to (1), satisfying $x(t) = U(t,s)x(s)$ for each $t \geq s$, where $x(t)$ is a solution of (1). When the linear system (1) is Lyapunov regular and all values of the Lyapunov exponent $\lambda$ are negative one can show that for every $\beta > 0$ there exist positive constants $c$ and $\alpha$, such that

$$\|U(t,s)\| \leq ce^{-\alpha(t-s) + \beta s} \text{ for every } t \geq s.$$ 

However, in general one cannot take $\beta = 0$, and thus the system need not be uniformly asymptotically stable. In particular,

$$\|x(t)\| \leq ce^{\beta s}e^{-\alpha(t-s)}\|x(s)\|,$$

(6)

where the constant $ce^{\beta s}$ deteriorates exponentially along the orbit of a solution. This means that the “size” of the neighborhood at time $s$ where the exponential stability of the zero solution is guaranteed may decay with exponential rate, although small when compared to the Lyapunov exponents by choosing $\beta$ sufficiently small.

It is possible and relevant to describe counterparts of the above theory and the related stability results for dynamical systems in infinite-dimensional spaces (the reader can see, for example, the related discussion in the book by Hale et al. [6, Section 7.5]; the book presents a detailed discussion of the state-of-the-art of the geometric theory of dynamical systems in infinite-dimensional spaces). Our main goals in this paper are:

1. to introduce a version of Lyapunov regularity in Hilbert spaces, mimicking as much as possible the classical theory, and which in the finite-dimensional setting reduces to the classical notion introduced by Lyapunov;
2. to establish the persistence of the asymptotic stability of the zero solution under sufficiently small perturbations of Lyapunov regular nonautonomous linear differential equations, in the infinite-dimensional setting of Hilbert spaces.

We also describe the important geometric consequences of regularity related to the existence of exponential growth rates of norms, angles, and volumes along the solutions.

While the notion of Lyapunov regularity makes considerable demands on the linear system, it turns out that within the context of ergodic theory it is typical under fairly general assumptions. Here, we formulate only one of the major results in this direction, which in fact is one of the fundamental pieces at the basis of the so-called smooth ergodic theory or Pesin theory (see [2]). Recall that a finite measure $\nu$ in $\mathbb{R}^n$ is invariant under the flow $\{\varphi_t\}_{t \in \mathbb{R}}$ if

$$\nu(\varphi_t(A)) = \nu(A) \text{ for every measurable set } A \subset \mathbb{R}^n \text{ and } t \in \mathbb{R}.$$ 

The following statement is a particular version of the celebrated multiplicative ergodic theorem of Oseledeets in [11]. It is a simple consequence of the general theory, as described for example in [2].
Theorem 1 (see [11]). Consider a differential equation \( x' = F(x) \) in \( \mathbb{R}^n \) with \( F \) of class \( C^1 \), and assume that it generates a flow \( \{ \varphi_t \}_{t \in \mathbb{R}} \) which preserves a finite measure \( \nu \) with compact support in \( \mathbb{R}^n \). Then for \( \nu \)-almost every \( x \in \mathbb{R}^n \) the linear variational equation

\[
y' = A_x(t)y \quad \text{with} \quad A_x(t) = d\varphi_t \cdot F
\]

is Lyapunov regular.

We refer to [2] for a detailed exposition of the multiplicative ergodic theorem. We remark that since the general solution of Eq. (7) is given by \( y(t) = (d_x \varphi_t) y_0 \), with \( y_0 \in \mathbb{R}^n \), the Lyapunov exponent \( \lambda \) in (3) associated to (7) coincides with the “usual” Lyapunov exponent associated to each solution \( \varphi_t(x) \) of \( x' = F(x) \) along a direction \( y_0 \), i.e.,

\[
\lambda(x, y_0) = \limsup_{t \to +\infty} \frac{1}{t} \log \| (d_x \varphi_t) y_0 \|.
\]

We can apply Theorem 1 for example to any Hamiltonian equation and the associated invariant Liouville–Lebesgue measure. More generally, any flow defined by a differentiable vector field with zero divergence preserves Lebesgue measure. This happens in particular with the geodesic flow on the unit tangent bundle of a smooth manifold.

Theorem 1 and its related versions should be considered strong motivations to study Lyapunov regular systems, in view of the ubiquity of these systems at least in the measurable category. Furthermore, and this is another motivation for our study, there exist several related results in the infinite-dimensional setting. Namely, it turns out that the notion of Lyapunov regularity in a finite-dimensional space has several important geometric consequences, related to the existence of exponential growth rates of norms, angles, and volumes (see the discussions in Sections 2.3 and 5.3 for details). Ruelle [12] was the first to obtain related “geometric” results in Hilbert spaces (see Section 2.4 for a related discussion). Later on Mañé [9] considered transformations in Banach spaces under some compactness assumptions (including the case of differentiable maps with compact derivative at each point). The results of Mañé were extended by Thieullen in [13] for a class of transformations satisfying a certain asymptotic compactness. In view of the regularity theory in finite-dimensional spaces one should ask, and this is another motivation for our study, whether the above “geometric” results in the infinite-dimensional setting have behind them an analogous (infinite-dimensional) regularity theory, which additionally reduces to the classical theory when applied to the finite-dimensional setting. We shall show that this is indeed the case (see Section 2.4). Note that the answer to this question largely depends on finding an appropriate generalization of the notion of Lyapunov regularity for infinite-dimensional spaces.

One may of course try other approaches to our stability problem. In this respect we should mention the work of Lillo [7]. Although of very different nature, it is related to our problem and we would like to highlight the crucial differences in order to
emphasize the novelty of our work. His approach is motivated by the fact that the Lyapunov exponents are not upper semi-continuous with respect to $A: t \mapsto A(t)$ in the space of bounded continuous functions with the norm

$$
\|A\| = \sup \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(t)| : t \geq 0 \right\},
$$

where $a_{ij}(t)$ are the entries of the matrix $A(t)$. In particular, if all Lyapunov exponents for the system $x' = A(t)x$ are negative, it may happen that there exist arbitrarily small perturbations $B(t)$ of $A(t)$, with respect to the above norm, with at least one nonnegative Lyapunov exponent. This shows that the stability of a nonautonomous linear equation may not persist under perturbations even at the linear level. The approach of Lillo intends to replace the Lyapunov exponents by another invariant, that he calls major characteristic exponent, which is upper-semicontinuous in the matrix of coefficients with respect to the above norm. This is the number

$$
\lambda_A = \limsup_{t \to +\infty} \sup_{x \geq 0, \ x_0 \in \mathbb{R}^n} \frac{1}{t - s} \log \frac{\|x(t, s, x_0)\|}{\|x_0\|},
$$

where $x(t, s, x_0)$ is the solution of (1) with $x(s) = x_0$. The upper semi-continuity of $A \mapsto \lambda_A$ yields that sufficiently small perturbations $B(t)$ of a function $A(t)$ with $\lambda_A < 0$ have also $\lambda_B < 0$ (in particular, all Lyapunov exponents of the system $x' = B(t)x$ are negative). He also discusses how this property can be used to study nonlinear perturbations as in (2) assuming that $A(t)$ is almost periodic with $\lambda_A < 0$, and $f(t, x)$ is almost periodic in $t$ uniformly with respect to $x$ in some neighborhood of zero (essentially he uses the assumption $\lambda_A < 0$ to show that $A(t)$ can be reduced to an upper triangular matrix $C(t)$, with arbitrarily small entries above the diagonal, and exponentially contracting entries in the diagonal; he then applies work of Mitropolski in [10] to obtain a stable almost periodic solution for the perturbed equation).

We now compare this approach to our work. Note that it follows readily from (8) that for every $\varepsilon > 0$ there exists $c > 0$, such that

$$
\|x(t, s, x_0)\| \leq ce^{-(\lambda_A + \varepsilon)(t-s)}\|x(s)\| \text{ for every } t \geq s \text{ and every } x_0 \in \mathbb{R}^n.
$$

When $\lambda_A < 0$ this inequality is the same as (6) provided that $\varepsilon$ is sufficiently small so that $\lambda_A + \varepsilon < 0$, but now with the crucial difference that $\beta = 0$. We emphasize that, instead we also consider the case of a sufficiently small $\beta > 0$ (thus, we may have no “stable” exponential dichotomy for our linear system). Furthermore, we consider arbitrary continuous functions (in particular, they need not be almost periodic), and the boundedness of $A(t)$ is here replaced by the more general subexponential growth condition in (9). Another advantage of our approach is that the above smallness condition on $\beta$ can be formulated based solely on the Lyapunov exponents (since the same happens with the above notion of regularity) and requires no further invariants. In addition, we
also consider the infinite-dimensional case and we can show that regular systems are typical in the context of ergodic theory (see Section 2.4).

We should also discuss why we work with Hilbert spaces instead of Banach spaces. We believe that we can proceed with a formal generalization and effect an analogous approach in the case of Banach spaces, namely for operators $A(t)$ in Banach spaces with a Schauder basis. This is the case for example of the spaces $L^p[0, 1]$. We note that a Banach space with a Schauder basis must be separable, although not all separable Banach spaces have a Schauder basis, as shown by Enflo in [5]. Our approach in the case of Hilbert spaces starts by considering finite-dimensional subspaces. To effect a generalization for Banach spaces we need to study the adjoint equation in the dual space, and consider in parallel finite-dimensional objects for the Banach space and for its dual (starting with the subspaces and the associated differential equations), instead of only finite-dimensional objects for the original space. Due to this additional technical complication, we believe that the writing would substantially hide the main principles of our approach, while this does not happen in the case of Hilbert spaces. Another difficulty is that several norm estimates in the proofs strongly use the fact that we are in a Hilbert space. It seems to us that in the case of Banach spaces it may not be possible to establish such strong estimates. Furthermore, one of the crucial aspects of the classical concept of regularity is the subexponential asymptotic behavior of angles between solutions (see Section 5.3). In the case of Banach spaces we can consider norms of projections instead of angles, but at present there is not even a related finite-dimensional theory at our disposal. In conclusion, we consider it a challenge to effect an analogous approach to the one in this paper in the case of Banach spaces. The above discussion stresses the main points to start dealing with.

The structure of the paper is the following. In Section 2, we introduce the notion of Lyapunov regularity in Hilbert spaces, mimicking as much as possible the classical theory described above for the finite-dimensional setting. We also give examples of regular and nonregular systems, and show that from the point of view of ergodic theory the regular systems are typical. In Section 3, we show that it is sufficient to consider operators $A(t)$ which are upper triangular with respect to some fixed orthonormal basis. Our results concerning the stability under perturbations of Lyapunov regular nonautonomous equations are established in Section 4. The proofs are inspired by the corresponding proofs in [2] in the finite-dimensional case, but require several nontrivial modifications. Alternative characterizations of our notion of Lyapunov regularity are given in Section 5. In particular, we give a geometric characterization in terms of the existence of exponential growth rates of volumes defined by solutions of the linear equation.

2. Lyapunov regularity in Hilbert spaces

2.1. The notion of regularity

We introduce here the concept of Lyapunov regularity in a separable Hilbert space by closely imitating the corresponding classical notion introduced by Lyapunov for
finite-dimensional spaces (see the introduction for the definition; we refer to [2] for full details on the classical notion).

Let $H$ be a separable real Hilbert space (we can also consider complex Hilbert spaces with minor changes). We denote by $\mathcal{B}(H)$ the space of bounded linear operators on $H$. Let $A: (0, +\infty) \to \mathcal{B}(H)$ be a continuous function, such that

$$\limsup_{t \to +\infty} \frac{1}{t} \log^+ \|A(t)\| = 0,$$

where $\log^+ x = \max\{0, \log x\}$ and $\|A(t)\|$ denotes the operator norm. Consider the initial value problem

$$v' = A(t)v, \quad v(0) = v_0$$

with $v_0 \in H$. Under the above assumptions, one can easily show, for example using Gronwall's lemma, that (10) has a unique solution $v(t)$ and that this solution is global for positive time. We define the Lyapunov exponent $\lambda: H \to \mathbb{R} \cup \{-\infty\}$ for (10) by

$$\lambda(v_0) = \limsup_{t \to +\infty} \frac{1}{t} \log \|v(t)\|$$

(with the convention that $\log 0 = -\infty$). We also fix an increasing sequence of subspaces $H_1 \subset H_2 \subset \cdots$ of dimension $\dim H_n = n$ for each $n \in \mathbb{N}$, and with union equal to $H$. It follows from the abstract theory of Lyapunov exponents (see [2, Section 1.2]) that for each $n \in \mathbb{N}$ the function $\lambda$ restricted to $H_n \setminus \{0\}$ can take at most $n$ values, say

$$-\infty \leq \lambda_{1,n} < \cdots < \lambda_{p,n,n}$$

for some integer $p_n \leq n$. Furthermore, for each $i = 1, \ldots, p_n$ the set

$$E_{i,n} = \{v \in H_n : \lambda(v) \leq \lambda_{i,n}\}$$

is a linear subspace of $H_n$. We can also consider the values

$$\lambda'_{1,n} \leq \cdots \leq \lambda'_{p,n,n}$$

of the Lyapunov exponent $\lambda$ on $H_n \setminus \{0\}$ counted with multiplicities, obtained by repeating each value $\lambda_{i,n}$ a number of times equal to the difference $\dim E_{i,n} - \dim E_{i-1,n}$ (with $E_{0,n} = \{0\}$).

We now consider the initial value problem for the adjoint equation

$$w' = -A(t)^*w, \quad w(0) = w_0$$
with \( w_0 \in H \), where \( A(t)^* \) denotes the transpose of the operator \( A(t) \). We define the Lyapunov exponent \( \mu : H \to \mathbb{R} \cup \{ -\infty \} \) for (14) by

\[
\mu(w_0) = \limsup_{t \to +\infty} \frac{1}{t} \log \| w(t) \|.
\]

Again it follows from the abstract theory of Lyapunov exponents that for each \( n \in \mathbb{N} \) the function \( \mu \) restricted to \( H_n \setminus \{ 0 \} \) can take at most \( n \) values, say

\[
-\infty \leq \mu_{q_n,n} < \cdots < \mu_{1,n}
\]

for some integer \( q_n \leq n \). (15)

Furthermore, for each \( i = 1, \ldots, q_n \) the set

\[
F_{i,n} = \{ w \in H_n : \mu(w) \leq \mu_{i,n} \}
\]

is a linear subspace of \( H_n \). Similarly, we consider the values

\[
\mu_{1,n}' \geq \cdots \geq \mu_{n,n}'
\]

of the Lyapunov exponent \( \mu \) on \( H_n \setminus \{ 0 \} \) counted with multiplicities, obtained by repeating each value \( \mu_{i,n} \) a number of times equal to the difference \( \dim F_{i,n} - \dim F_{i+1,n} \) (with \( F_{n+1,n} = \{ 0 \} \)).

According to the above discussion, each of the Lyapunov exponents \( \lambda \) and \( \mu \) takes at most a countable number of values. Let \( \lambda_i \) and \( \mu_i \) for \( i \in \mathbb{N} \) be respectively the values of \( \lambda \) and \( \mu \) on \( H \setminus \{ 0 \} \) counted with multiplicities.

We recall that two bases \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_n \) of \( H_n \) are said to be dual if \( \langle v_i, w_j \rangle = \delta_{ij} \) for every \( i \) and \( j \), where \( \delta_{ij} \) is the Kronecker symbol. Mimicking the abstract theory of Lyapunov exponents in finite-dimensional spaces, we introduce the regularity coefficient of \( \lambda \) and \( \mu \),

\[
\gamma(\lambda, \mu) = \sup \{ \gamma_n(\lambda, \mu) : n \in \mathbb{N} \},
\]

where

\[
\gamma_n(\lambda, \mu) = \min \max \{ \lambda(v_i) + \mu(w_i) : 1 \leq i \leq n \}
\]

with the minimum taken over all dual bases \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_n \) of the space \( H_n \). It follows from Theorem 1.2.6 in [2] applied to the Lyapunov exponents \( \lambda \) and \( \mu \) restricted to the finite-dimensional space \( H_n \) that \( \gamma_n(\lambda, \mu) \geq 0 \) for each \( n \in \mathbb{N} \), and thus \( \gamma(\lambda, \mu) \geq 0 \). We say that the equation in (10) is Lyapunov regular or simply regular if \( \gamma(\lambda, \mu) = 0 \). Note that \( \gamma(\lambda, \mu) = 0 \) if and only if \( \gamma_n(\lambda, \mu) = 0 \) for every \( n \in \mathbb{N} \).

We refer to Section 5 for several alternative characterizations of Lyapunov regularity. We note that in the finite-dimensional case our notion coincides with the classical
notion introduced by Lyapunov (see also the discussions in the introduction and in Section 5). When there exists \( \delta > 0 \), such that

\[-\infty \leq \lambda_1 \leq \lambda_2 \leq \cdots < -\delta \quad \text{and} \quad \mu_1 \geq \mu_2 \geq \cdots > \delta, \tag{19}\]

the Lyapunov regularity of the equation in (10) can be shown to imply that (see Theorem 19 below)

\[ \lambda_i + \mu_i = 0 \quad \text{for every} \quad i \in \mathbb{N}. \tag{20}\]

Property (20) can be seen as a justification of our version of regularity in Hilbert spaces (compare with (5) in the introduction). We emphasize that our stability results in Section 4 never require the condition (19).

Although Lyapunov regularity is a strong requirement, at least in certain natural contexts a “typical” nonautonomous linear differential equation is regular (see the discussions in the introduction and in Section 2.4).

### 2.2. Regular and nonregular equations

We present here examples of regular and nonregular equations \( v' = A(t)v \). We also motivate some of the geometric consequences of regularity.

We fix an orthonormal basis of \( H \) by vectors \( u_1, u_2, \ldots \) (recall that \( H \) is a separable Hilbert space), such that \( H_n = \text{span}\{u_1, \ldots, u_n\} \) for each \( n \), i.e., the first \( n \) elements of the basis generate \( H_n \). We will show in Section 3 (see Theorem 7) that it is always possible to reduce the case of a general function \( A(t) \) to that when \( A(t) \) is upper triangular for each \( t \geq 0 \), with respect to the basis \( u_1, u_2, \ldots \) of \( H \). This means that \( \langle A(t)u_i, u_j \rangle = 0 \) for each \( t \geq 0 \) whenever \( i < j \). As such, in view of the simplicity of the discussion, it is reasonable to consider here only the upper triangular case. Set

\[ \underline{\lambda}_i = \lim \inf_{t \to +\infty} \frac{1}{t} \int_0^t \langle A(s)u_i, u_i \rangle \, ds \quad \text{and} \quad \overline{\lambda}_i = \lim \sup_{t \to +\infty} \frac{1}{t} \int_0^t \langle A(s)u_i, u_i \rangle \, ds. \]

We first show how to use these numbers to obtain good estimates for the regularity coefficient \( \gamma(\lambda, \mu) \) in the upper triangular case. This will allow us to establish the regularity (or the nonregularity) of the equation. The following is established in Theorem 23 below (see (82)).

**Proposition 2.** If \( A(t) \) is upper triangular for each \( t \geq 0 \), then

\[ \sup_{n \geq 1} \frac{1}{n^2} \sum_{i=1}^n (\overline{\lambda}_i - \underline{\lambda}_i) \leq \gamma(\lambda, \mu) \leq \sum_{i=1}^{\infty} (\overline{\lambda}_i - \underline{\lambda}_i). \]

We note that Proposition 2 and any of its consequences described below are used nowhere in the paper other than in this section and in Sections 2.3 and 2.4, and thus
there is no danger of circular reasoning. Proposition 2 immediately yields the following criterion for regularity (or for nonregularity).

**Proposition 3.** Assume that \( A(t) \) is upper triangular for each \( t \geq 0 \). Then \( \gamma(\lambda, \mu) = 0 \) if and only if \( \bar{z}_i = \overline{z}_i \) for every \( i \in \mathbb{N} \).

A simple consequence of Proposition 3 is that any equation \( v' = Av \) with a continuous (upper triangular) operator \( A \) in \( H \) (independent of \( t \)) is Lyapunov regular; moreover, \( \bar{z}_i = \overline{z}_i = \langle Au_i, u_i \rangle \) for each \( i \).

We now present a statement which is a rewriting of Lemma 1.3.5 in [2] for the finite-dimensional system \( v'_n = A_n(t)v_n \), where \( A_n(t) = A(t)|H_n \) is the restriction of the operator \( A(t) \) to \( H_n \); note that due to the upper triangular property the space \( H_n \) is invariant under solutions of \( v' = A(t)v \).

**Proposition 4** (Barreira and Pesin [2]). Assume that \( A(t) \) is upper triangular for each \( t \geq 0 \). If \( \bar{z}_i := \overline{z}_i \) for \( i = 1, \ldots, n \), then \( \gamma_n(\lambda, \mu) = 0 \). Furthermore

1. the numbers \( \bar{z}_1, \ldots, \bar{z}_n \) are the values of the Lyapunov exponent \( \lambda \) on \( H_n \setminus \{0\} \) counted with multiplicities;
2. the numbers \( -\bar{z}_1, \ldots, -\bar{z}_n \) are the values of the Lyapunov exponent \( \mu \) on \( H_n \setminus \{0\} \) counted with multiplicities.

Note that the values \( \bar{z}_1, \ldots, \bar{z}_n \) need not be ordered in Proposition 4. When \( \gamma(\lambda, \mu) = 0 \) it follows from Proposition 3 that \( \bar{z}_i = \overline{z}_i \) for each \( i \in \mathbb{N} \), and thus, as an immediate consequence of Proposition 4, we have the following properties:

1. the numbers \( \bar{z}_i \) for \( i \in \mathbb{N} \) are the values of the Lyapunov exponent \( \lambda \) on \( H \setminus \{0\} \) counted with multiplicities;
2. the numbers \( -\bar{z}_i \) for \( i \in \mathbb{N} \) are the values of the Lyapunov exponent \( \mu \) on \( H \setminus \{0\} \) counted with multiplicities.

We refer to Sections 5.2 and 5.3 for criteria of regularity in the general case, that is, when \( A(t) \) is not necessarily upper triangular. Unfortunately, these criteria are not as easy to apply as the criterion in Proposition 3.

**2.3. Geometric consequences of regularity**

We describe here several important geometric consequences of Lyapunov regularity when \( A(t) \) is upper triangular. We refer to Section 5.3 for a detailed discussion in the case of an arbitrary function \( A(t) \); similar results hold in this general case although the approach is somewhat more complicated.

For each \( n \in \mathbb{N} \) consider a continuous function \( a_n \colon [0, +\infty) \to \mathbb{R} \), such that

\[
\limsup_{t \to +\infty} \frac{1}{t} \log^+ |a_n(t)| = 0.
\]
We define a (diagonal) operator $A(t)$ by $A(t)u_n = a_n(t)u_n$ for each $n \in \mathbb{N}$. Note that
\[ \|A(t)\| = \sup\{|a_n(t)| : n \in \mathbb{N}\}, \]
and thus the condition (9) may fail but one can verify that for this simpler class of operators all the arguments apply under the weaker assumption (21). The solution of the initial value problem (10) with $v_0 = \sum_{i=1}^{n} c_i u_i$ is given by
\[ v(t) = \sum_{i=1}^{n} c_i \exp \left( \int_{0}^{t} a_i(s) \, ds \right) u_i. \]
One can easily verify that
\[ \limsup_{t \to +\infty} \frac{1}{t} \log \|v(t)\| = \max \{\overline{c}_i : c_i \neq 0 \text{ and } i = 1, \ldots, n\}. \]
By Proposition 3, if the system is regular, then the limit
\[ \lim_{t \to +\infty} \frac{1}{t} \log \|v(t)\| = \max \{c_i : c_i \neq 0 \text{ and } i = 1, \ldots, n\} \]
exists. This is one of the important geometric consequences of regularity, i.e., the existence of the exponential growth rate of norms along a solution. Another consequence concerns the exponential growth rate of volumes. To explain this, let $v_1(t), \ldots, v_n(t)$ be the solutions of the linear equation $v' = A(t)v$, with $A(t)$ diagonal as above for each $t$, such that $v_i(0) = u_i$ for $i = 1, \ldots, n$. Then the $n$-volume $\Gamma_n(t)$ defined by the vectors $v_1(t), \ldots, v_n(t)$ is given by
\[ \Gamma_n(t) = \prod_{i=1}^{n} \exp \left( \int_{0}^{t} a_i(s) \, ds \right). \]
By Proposition 3, if the system is regular, then the limit
\[ \lim_{t \to +\infty} \frac{1}{t} \log \Gamma_n(t) = \sum_{i=1}^{n} \lim_{t \to +\infty} \frac{1}{t} \int_{0}^{t} a_i(s) \, ds = \sum_{i=1}^{n} \overline{c}_i = \sum_{i=1}^{n} \dot{\lambda}(v_i) \]
exists.

2.4. Ubiquity of Lyapunov regularity

We show here that in the context of ergodic theory our notion of Lyapunov regularity, as introduced in Section 2.1, is rather common. Let $\{\phi_t\}_{t \geq 0}$ be a measurable semiflow
in $H$. We assume that for each $x \in H$ and $t \geq 0$ there exists the Fréchet derivative $d_x \phi_t$, and that $(x, t) \mapsto d_x \phi_t$ is measurable. We say that a finite measure $\nu$ in $H$ is invariant under the semiflow if

$$
\nu(\phi_{-t}(A)) = \nu(A) \text{ for every measurable set } A \subset H \text{ and every } t \geq 0.
$$

The following statement is a continuous time version of results formulated by Ruelle in [12] (see Corollaries 2.2 and 2.3 in that paper; see also his related discussion in Section 7 in the case of semiflows). The proof can be obtained by carefully modifying the approach in [12]. The necessary modifications are analogous to the modifications that are needed to obtain the continuous time version of the multiplicative ergodic theorem from the corresponding discrete time version in the classical finite-dimensional setting (see also [11]). We refer to [1] for details. Recall that a (linear) cocycle over the semiflow $\{\phi_t\}_{t \geq 0}$ is a function $A : H \times [0, +\infty) \to B(H)$, such that

1. $A(x, 0) = \text{Id}$ for every $x \in H$;
2. for every $x \in H$ and $t, s \geq 0$,

$$
A(x, t + s) = A(\phi_s x, t) A(x, s).
$$

**Theorem 5.** Assume that the semiflow $\{\phi_t\}_{t \geq 0}$ preserves a finite measure $\nu$ in $H$, and that $A$ is a cocycle over the semiflow, such that $A(x, t)$ is a sum of a unitary operator with a compact operator for each $x \in H$ and $t \geq 0$. If

$$
\int_H \log^+ \sup \{\|A(x, t)\| : t \in [0, 1]\} \, d\nu(x) < \infty,
$$

then for $\nu$-almost every $x \in H$ the limit

$$
\lim_{t \to +\infty} (A(x, t)^* A(x, t))^{1/(2t)}
$$

exists in norm.

Consider now a differential equation $u' = F(u)$ in the Hilbert space $H$ with $F$ Fréchet differentiable, and assume that it generates a semiflow $\{\phi_t\}_{t \in \mathbb{R}}$ as above which preserves a finite measure $\nu$ in $H$. Consider also a fixed orthonormal basis $u_1, u_2, \ldots$ of the space $H$, as in Section 2.2, such that $H_n = \text{span}\{u_1, \ldots, u_n\}$ for each $n$.

**Theorem 6.** Assume that

1. the cocycle $A(x, t) = d_x \phi_t$ satisfies the conditions of Theorem 5;
2. $A(x, t)$ is upper triangular for each $x \in H$ and $t \geq 0$ with respect to the orthonormal basis $u_1, u_2, \ldots$ of $H$. 

Then the linear variational equation

\[ v' = A_x(t)v \quad \text{with} \quad A_x(t) = d_{\phi, x} F \]

is Lyapunov regular for \( v \)-almost every point \( x \in H \).

**Proof.** Note that since \( \mathcal{A}(x, t) \) is upper triangular, the space \( H_n \) is invariant under \( \mathcal{A}(x, t) \) for each \( n \in \mathbb{N} \), and we can consider the (finite-dimensional) cocycle \( \mathcal{A}_n(x, t) = \mathcal{A}(x, t)|H_n \). Since \( \|\mathcal{A}_n(x, t)\| \leq \|\mathcal{A}(x, t)\| \), each cocycle \( \mathcal{A}_n(x, t) \) satisfies the conditions of Theorem 5, and thus, for every \( x \in H \) in a full \( v \)-measure set \( \Lambda_n \), the limit

\[ \lim_{t \to +\infty} \frac{1}{t} \int_0^t \operatorname{tr}[\mathcal{A}_n(x, s)] ds \]

exists in norm. Consider the full \( v \)-measure set \( \Lambda = \bigcap_{n=1}^{\infty} \Lambda_n \). We denote by \( [\mathcal{A}(x, t)]_n \) the \( n \times n \) matrix obtained from the first \( n \) “rows” and \( n \) “columns” of \( \mathcal{A}(x, t) \) (or equivalently of \( \mathcal{A}_n(x, t) \)) with respect to the above fixed basis, that is, the matrix with entries \( \langle u_i, \mathcal{A}(x, t)u_j \rangle \) for \( i, j = 1, \ldots, n \). Since \( [\mathcal{A}(x, t)]_n \) is upper triangular one can easily verify that

\[ [\mathcal{A}(x, t)^*\mathcal{A}(x, t)]_n = [\mathcal{A}(x, t)^*]_n [\mathcal{A}(x, t)]_n = [\mathcal{A}(x, t)]_n^* [\mathcal{A}(x, t)]_n \]

and thus

\[ \det[\mathcal{A}(x, t)^*\mathcal{A}(x, t)]_n = (\det[\mathcal{A}(x, t)]_n)^2. \]

Consider a point \( x \in \Lambda \) (i.e., a point for which (23) holds for every \( n \in \mathbb{N} \)). Noticing that \( \mathcal{A}_n(x, t) \) is a monodromy operator for \( v' = (A_x(t)|H_n)v \), we have the well-known identity

\[ \det[\mathcal{A}(x, t)]_n = \det[\mathcal{A}(x, 0)]_n \exp \int_0^t \operatorname{tr}[\mathcal{A}_x(s)]_n ds. \]

We can thus conclude that the limits

\[ \lim_{t \to +\infty} \frac{1}{t} \int_0^t \operatorname{tr}[\mathcal{A}_x(s)]_n ds = \lim_{t \to +\infty} \frac{1}{t} \log |\det[\mathcal{A}(x, t)]_n| \]

\[ = \lim_{t \to +\infty} \frac{1}{2t} \log \det[\mathcal{A}(x, t)^*\mathcal{A}(x, t)]_n \]

\[ = \log \lim_{t \to +\infty} \left( \det[\mathcal{A}(x, t)^*\mathcal{A}(x, t)]_n \right)^{1/(2t)} \]

\[ = \log \lim_{t \to +\infty} \left( |\mathcal{A}(x, t)^*\mathcal{A}(x, t)|_n \right)^{1/(2t)} \]

\[ = \log \det \lim_{t \to +\infty} \left( |\mathcal{A}(x, t)^*\mathcal{A}(x, t)|_n \right)^{1/(2t)} \]
exist. Note that the existence of the last limit is equivalent to the existence of the limit in (23). We have used here the fact that the determinant is a continuous function of the entries of a matrix, and the identity \((\det C)^z = \det(C^z)\) for a positive semi-definite symmetric matrix \(C\) and \(z > 0\) (not necessarily an integer). To conclude the proof note that the limit
\[
\lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle A(s)u_n, u_n \rangle \, ds = \lim_{t \to +\infty} \frac{1}{t} \int_0^t (\text{tr}[A_x(s)]_n - \text{tr}[A_x(s)]_{n-1}) \, ds
\]
exists. The desired result follows now immediately from Proposition 3. \(\square\)

Theorem 6 indicates that in the context of ergodic theory the notion of Lyapunov regularity in Hilbert spaces is very common. More precisely, under the standard integrability assumption in (22), for a measurable flow preserving a finite measure, almost all orbits have a Lyapunov regular linear variational equations. Using our stability results (see Section 4), or appropriate generalizations when zero is not a solution of the perturbed system (2), one could show that for almost all orbits (or more precisely for almost all initial conditions) of a perturbed system, such that the associated linear variational equation has only negative Lyapunov exponents, any sufficiently small perturbation does not destroy the asymptotic stability of the original orbit in the perturbed equation (i.e., the orbit originating the linear variational equation).

We now describe how to reduce the study of an arbitrary cocycle to the study of an upper triangular cocycle. We consider an extension of the semiflow \(\{\phi_t\}_{t \geq 0}\) in \(H\),
\[
\Phi_t: H \times U_H \to H \times U_H,
\]
where \(U_H\) is the group of unitary operators in \(H\). Given \(t \geq 0\) and \((x, U) \in H \times U_H\), we apply the Gram–Schmidt orthogonalization procedure to the “columns” of \((d_x \phi_t)U\) (with respect to the basis \(u_1, u_2, \ldots\)) and write
\[
(d_x \phi_t)U = R_t(x, U)T_t(x, U), \tag{24}
\]
where \(R_t(x, U) \in U_H\) and \(T_t(x, U)\) is upper triangular (with positive entries in the diagonal). Note that the operators \(R_t(x, U)\) and \(T_t(x, U)\) are uniquely defined. The new semiflow in \(H \times U_H\) is defined by
\[
\Phi_t(x, U) = (\phi_t x, R_t(x, U)).
\]
Consider now the projection \(\pi: (x, U) \mapsto U\). By (24), we have
\[
T_t(x, U) = ((\pi \circ \Phi_t)(x, U))^{-1}(d_x \phi_t)\pi(x, U).
\]
Therefore,
\[
T_{t+s}(x, U) = T_t(\Phi_s(x, U))T_s(x, U)
\]
and $B((x,U),t) = T_t(x,U)$ is an upper triangular cocycle over the semiflow $\{\Phi_t\}_{t \geq 0}$ in $H \times U_H$. The regularity of a point $x \in H$ with respect to the cocycle $d_x \Phi_t$ can be expressed in terms of the regularity of (any of) the points $(x,U) \in H \times U_H$, with $x$ as above, with respect to the upper triangular cocycle $B$ in the extended space $H \times U_H$. This is due to the fact that, in view of (24), the values of the corresponding Lyapunov exponents,

$$(x, y) \mapsto \limsup_{t \to +\infty} \frac{1}{t} \log \| (d_x \Phi_t) y \|$$

for $y \in H$, and

$$( (x, U), (y, V) ) \mapsto \limsup_{t \to +\infty} \frac{1}{t} \log \| B((x, U), t)(y, V) \|$$

for $(y, V) \in H \times T_H U_H$, are equal respectively at the points $x$ and $(x, U)$ (since $\| R_t(x, U) \| = \| U \| = 1$ for every $t \geq 0$ and $(x, U) \in H \times U_H$).

Note that if the semiflow $\{\Phi_t\}_{t \geq 0}$ preserves a finite measure $\nu$ in $H \times U_H$, such that

$$\tau(B \times U_H) = \nu(B)$$

for every measurable set $B \subseteq H$, then the set of regular points for $d_x \Phi_t$ has full $\nu$-measure if and only if the set of regular points for the upper triangular cocycle $B$ has full $\tau$-measure. For example, if the semiflow $\Phi_t$ possesses a compact invariant set $\Lambda \subseteq H \times U_H$, then there exists a finite measure $\nu$ supported on $\Lambda$ which satisfies (25) (see the next paragraph), and we conclude from Theorem 6 that the set of regular points for $d_x \Phi_t$ has full $\nu$-measure.

Concerning the existence of finite invariant measures on compact invariant sets (recall that a set $\Lambda$ is invariant under the semiflow $\{\varphi_t\}_{t \geq 0}$ if $\varphi_{-t}\Lambda = \Lambda$ for each $t \geq 0$) we have the following well-known statement: a semiflow on a metric space $H$, such that $\varphi_t$ is continuous for each $t \geq 0$ possesses at least one invariant probability measure on each compact invariant set $\Lambda \subseteq H$. This is a simple consequence of the fact that the space of probability measures on a compact metric space is compact for the weak convergence of measures, together with an averaging argument along orbits (namely, for an arbitrary probability measure $\nu$, any weak limit of the sequence of measures $\frac{1}{n} \int_0^n \varphi_s^* \nu \, ds$, with $(\varphi_s^* \nu)(A) = \nu(\varphi_{-s} A)$, is invariant under the semiflow).

3. Upper triangular reduction

The following result shows that we can always assume, without loss of generality, that the operator $A(t)$ in (10) is “upper triangular” for every $t$ with respect to the fixed
orthonormal basis \( u_1, u_2, \ldots \) of \( H \) considered in Section 2.2. Note that the basis is independent of \( t \).

**Theorem 7.** Given a continuous function \( A: [0, +\infty) \to \mathcal{B}(H) \) satisfying \( (9) \), there is a continuous function \( B: [0, +\infty) \to \mathcal{B}(H) \), such that

1. \( B(t) \) is upper triangular for each \( t \geq 0 \), i.e., \( \langle B(t)u_i, u_j \rangle = 0 \) for each \( t \geq 0 \) whenever \( i < j \), and for each \( n \in \mathbb{N} \),

\[
\limsup_{t \to +\infty} \frac{1}{t} \log^+ \| B(t) |H_n| \| = 0; \tag{26}
\]

2. the initial value problem \( (10) \) is equivalent to

\[
x' = B(t)x, \quad x(0) = v_0, \tag{27}
\]

with the solutions \( v(t) \) and \( x(t) \) related by \( v(t) = U(t)x(t) \) for some Fréchet differentiable function \( U: [0, +\infty) \to \mathcal{B}(H) \) with \( U(t) \) unitary for each \( t \).

**Proof.** We continue to denote by \( v(t) \) the solution of the initial value problem \( (10) \). We first establish an auxiliary statement.

**Lemma 1.** There are continuous operator functions \( B: [0, +\infty) \to \mathcal{B}(H) \) and \( U: [0, +\infty) \to \mathcal{B}(H) \) with \( U(0) = \text{Id} \) and \( U(t) \) unitary for each \( t \) such that

1. \( B(t) \) is upper triangular for each \( t \geq 0 \), and \( |\langle B(t)u_i, u_j \rangle| \leq 2\| A(t) \| \) for each \( t \geq 0 \) and every \( i \) and \( j \);
2. \( t \mapsto U(t) \) is Fréchet differentiable, and setting \( x(t) = U(t)^{-1}v(t) \) for each \( t \geq 0 \), we have \( x'(t) = B(t)x(t) \).

**Proof of the lemma.** We construct the operator \( U(t) \) by applying the Gram–Schmidt orthogonalization procedure to the vectors \( v_1(t), v_2(t), \ldots \), where \( v_i(t) \) is the solution of \( (10) \) with \( v_0 = u_i \) for each \( i \geq 1 \) (where \( u_1, u_2, \ldots \) is the fixed orthonormal basis of \( H \)). In this manner, we obtain functions \( u_1(t), u_2(t), \ldots \), such that

1. \( \langle u_i(t), u_j(t) \rangle = \delta_{ij} \) for each \( i \) and \( j \),
2. each function \( u_k(t) \) is a linear combination of \( v_1(t), \ldots, v_k(t) \).

Note that each \( v_k(t) \) is also a linear combination of \( u_1(t), \ldots, u_k(t) \), and thus

\[
\langle v_i(t), u_j(t) \rangle = 0 \text{ for } i < j. \tag{28}
\]

Given \( t \geq 0 \) we define the linear operator \( U(t): H \to H \), such that \( U(t)u_i = u_i(t) \) for each \( i \). Clearly, the operator \( U(t) \) is unitary for each \( t \), and \( t \mapsto U(t) \) is Fréchet differentiable with \( U'(t)u_i = u'_i(t) \) for each \( i \). Set now \( x(t) = U(t)^{-1}v(t) \). We obtain

\[
v'(t) = U'(t)x(t) + U(t)x'(t) = A(t)v(t) = A(t)U(t)x(t) \tag{29}
\]
and thus \( x'(t) = B(t)x(t) \), where

\[
B(t) = U(t)^{-1}A(t)U(t) - U(t)^{-1}U'(t).
\]  (30)

Clearly, \( B : \mathbb{R} \to \mathcal{B}(H) \) is a continuous function.

Given \( t \geq 0 \), let now \( V(t) \) be the operator, such that \( V(t)u_i = v_i(t) \) for each \( i \), and set \( X(t) = U(t)^{-1}V(t) \). Since \( U(t) \) is unitary, by (28) we obtain

\[
0 = \langle v_i(t), u_j(t) \rangle = \langle V(t)u_i, U(t)u_j \rangle = \langle X(t)u_i, u_j \rangle \text{ for } i < j.
\]  (31)

Therefore \( X(t) \) is upper triangular, and taking derivatives in (31) we conclude that the same happens with \( X'(t) \). Proceeding in a similar way to that in (29) with \( V(t) = U(t)X(t) \) we obtain

\[
X'(t) = B(t)X(t) \text{ for } t \geq 0.
\]  (32)

Thus, \( B(t) = X'(t)X(t)^{-1} \) and it easily follows that \( B(t) \) is upper triangular.

It remains to establish the bound in the first statement. Since \( U(t) \) is unitary, by (30) we have

\[
B(t) + B(t)^* = U(t)^*(A(t) + A(t)^*)U(t) - (U(t)^*U'(t) + U'(t)^*U(t))
\]

\[
= U(t)^*(A(t) + A(t)^*)U(t) - \frac{d}{dt}(U(t)^*U(t))
\]

\[
= U(t)^*(A(t) + A(t)^*)U(t).
\]  (33)

Write for each \( i, j \in \mathbb{N} \) and \( t \geq 0 \),

\[
b_{ij}(t) = \langle B(t)u_i, u_j \rangle \quad \text{and} \quad \tilde{a}_{ij}(t) = \langle A(t)u_i(t), u_j(t) \rangle.
\]

Since \( B(t) \) is upper triangular, it follows from (33) that

\[
b_{ii}(t) = \tilde{a}_{ii}(t) \quad \text{and} \quad b_{ij}(t) = \tilde{a}_{ij}(t) + \tilde{a}_{ji}(t)
\]  (34)

whenever \( i \neq j \). Since \( U(t) \) is unitary, the vectors \( u_1(t) = U(t)u_1, u_2(t) = U(t)u_2, \ldots \) form an orthonormal basis of \( H \), and thus

\[
\|A(t)\| \geq \|A(t)u_i(t)\| = \left\| \sum_{j=1}^{\infty} \langle Au_i(t), u_j(t) \rangle u_j(t) \right\|
\]

\[
= \left( \sum_{j=1}^{\infty} \tilde{a}_{ji}(t)^2 \right)^{1/2} \geq |\tilde{a}_{ij}(t)|
\]
for every $i$ and $j$. It follows from (34) that $|b_{ij}(t)| \leq 2\|A(t)\|$ for every $i$ and $j$. This completes the proof. □

We emphasize that the function $B(t)$ in Lemma 1 does not depend on the particular solution $v(t)$ of (10). It follows from the bound $|b_{ij}(t)| \leq 2\|A(t)\|$ that given $v = \sum_{i=1}^{n} a_i u_i \in H_n$ with $\|v\| = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} = 1$,

$$\|B(t)v\|^2 = \left\| \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \langle B(t)u_i, u_j \rangle u_j \right\|^2 \leq \sum_{j=1}^{n} \left( \sum_{i=j}^{n} a_i b_{ij}(t) \right)^2 \leq \sum_{j=1}^{n} \left( \sum_{i=j}^{n} a_i^2 \sum_{i=j}^{n} b_{ij}(t)^2 \right) \leq \sum_{j=1}^{n} \sum_{i=j}^{n} b_{ij}(t)^2 \leq 4n^2\|A(t)\|^2.$$

Therefore, $\|B(t)\|_{H_n} \leq 2n\|A(t)\|$, and the property (26) follows immediately from (9). For the last statement in the theorem it remains to observe that $U(t)$ is invertible for each $t$, and that $v(0) = x(0) = v_0$ since $U(0) = \text{Id}$. This establishes the theorem. □

The advantage of the upper triangular systems is that we can consider finite-dimensional systems in $H_n = \text{span}\{u_1, \ldots, u_n\}$ given by

$$y_n' = B_n(t)y_n, \quad \text{with } B_n(t) = B(t)|_{H_n} \quad \text{and} \quad y_n(0) = v_0|_{H_n},$$

since for each $n$ the space $H_n$ is invariant under solutions of (27). We obtain the solution of (27) in the form $y(t) = \lim_{n \to \infty} y_n(t)$. The condition (26) ensures that for each $n \in \mathbb{N}$ the initial value problem in (36) has a unique and global solution. In this manner our initial value problem (10) becomes in essence a finite-dimensional problem.

4. Stability of nonautonomous equations in Hilbert spaces

4.1. Setup

Here we consider nonlinear perturbations $v' = A(t)v + f(t, v)$ of the linear equation $v' = A(t)v$, and study the persistence of the stability of solutions under sufficiently small perturbations. Without loss of generality, we may assume that the operator $A(t)$ is upper triangular for every $t$ with respect to the fixed orthonormal basis $u_1, u_2, \ldots$ of $H$ considered in Section 3 (see Theorem 7).
Consider the initial value problem

\[ v' = A(t)v + f(t, v), \quad v(0) = v_0 \]  \hspace{1cm} (37)

with \( v_0 \in H \). We also consider the conditions:

C1. \( A: [0, +\infty) \to B(H) \) is a continuous function satisfying (9) and

\[ \langle A(t)u_i, u_j \rangle = 0 \text{ for every } i < j \text{ and every } t \geq 0; \]  \hspace{1cm} (38)

C2. \( f: [0, +\infty) \times H \to H \) is a continuous function satisfying \( f(t, 0) = 0 \) for all \( t \geq 0 \), and there exists constants \( c, r > 0 \), such that

\[ \| f(t, u) - f(t, v) \| \leq c\| u - v \| (\| u \| r + \| v \| r) \]

for every \( t \geq 0 \), and \( u, v \in H \);

C3. \( |\langle v_0, u_n \rangle| < \| v_0 \|/a_n \) for every \( n \geq 0 \), and

\[ |\langle f(t, u) - f(t, v), u_n \rangle| \leq \frac{1}{a_n} \| u - v \| (\| u \| r + \| v \| r) \]  \hspace{1cm} (39)

for every \( t \geq 0 \), \( u, v \in H \), and \( n \geq 0 \), for some positive increasing sequence \( (a_n)_n \) that diverges sufficiently fast.

Under the conditions C1–C2, it can easily be shown that the perturbed equation in (37) has a unique solution \( v(t) \). We note that \( v(t) \equiv 0 \) is always a solution of (37).

A description of the required speed of \( a_n \) in (39) is given at the end of this section. We remark that the condition (39) corresponds to the requirement that the perturbation is sufficiently small (with respect to some basis). It should be noted that when the perturbation is finite-dimensional, that is, when there exists \( n \in \mathbb{N} \), such that \( f(t, v) \in H_n \) for every \( t \geq 0 \) and \( v \in H \), then the requirement (39) is not needed, since in this case \( \langle f(t, u) - f(t, v), u_m \rangle = 0 \) for every \( m > n \). On the other hand, we emphasize that the perturbations that we consider need not be finite-dimensional.

Consider now the condition

\[ r \sup\{\lambda_i : i \in \mathbb{N}\} + \gamma(\lambda, \mu) < 0. \]  \hspace{1cm} (40)

Since \( \gamma(\lambda, \mu) \geq 0 \) (see Section 2), this implies that

\[ \sup\{\lambda_i : i \in \mathbb{N}\} < 0. \]  \hspace{1cm} (41)

This property ensures the asymptotic stability of the linear equation in (10). We recall from the introduction that the asymptotic stability of (10) is not sufficient to ensure the
stability of the zero solution of (37). In fact, there exist examples for which a small perturbation $f$ makes zero an exponentially unstable solution (an explicit example is given in the introduction).

4.2. Stability results

We can now formulate our main results on the persistence of stability of the zero solution of (10) under perturbations. It should be emphasized that the results deal with equations in which the operators $A(t)$ are bounded for every $t$. This has some drawbacks, since stability questions arise naturally in nonautonomous partial differential equations in which the operators $A(t)$ may be unbounded.

**Theorem 8.** If conditions C1–C3 and (40) hold, then for any positive sequence $(a_n)_n$ diverging sufficiently fast, given $\varepsilon > 0$ sufficiently small there exists a constant $a > 0$, such that any solution of Eq. (37) with $\|v_0\|$ sufficiently small is global and satisfies

$$\|v(t)\| \leq ae^{(\sup|\lambda_i|: i \in \mathbb{N}|+\varepsilon)t}\|v_0\| \text{ for every } t \geq 0.$$  \hspace{1cm} (42)

Note that $\sup|\lambda_i| + \varepsilon < 0$ for every sufficiently small $\varepsilon > 0$. The proof of Theorem 8 and of the remaining results in this section are given in Sections 4.4 and 4.5. The following is an immediate corollary of Theorem 8 for regular equations.

**Theorem 9.** If conditions C1–C3 and (41) hold, and the equation in (10) is Lyapunov regular, then for any positive sequence $(a_n)_n$ diverging sufficiently fast, given $\varepsilon > 0$ sufficiently small there exists a constant $a > 0$, such that any solution of Eq. (37) with $\|v_0\|$ sufficiently small is global and satisfies (42).

Theorem 8 establishes the persistence of stability of the zero solution allowing a certain degree of nonregularity for the equation in (10), that is, it may happen that $\gamma(\lambda, \mu) > 0$. We note that by (40) a higher order $r$ of the perturbation $f$ allows a larger regularity coefficient. When $\gamma(\lambda, \mu) > 0$ the angles between distinct solutions may vary with exponential speed, essentially related to $\gamma(\lambda, \mu)$, although this speed is small when compared to the values of the Lyapunov exponent, that is, to $\inf{\{ |\lambda_i| : i \in \mathbb{N} \}}$. This strongly contrasts to what happens in Theorem 9 in which case the regularity assumption forces the angles between distinct solutions to vary at most with subexponential speed. We refer to Section 5 for a detailed discussion.

We now formulate an abstract stability result which will be obtained as a consequence of the proof of Theorem 8. It is somewhat more explicit about the required speed of $a_n$ in (39). Let $X(t)$ be (upper triangular) monodromy operators for the equation $v' = A(t)v$. These are operators such that the solution with $v(0) = v_0$ is given by $v(t) = X(t)X(0)^{-1}v_0$.

**Theorem 10.** Assume that conditions C1–C3 hold, and that there exist constants $\alpha < 0$ and $\beta > 0$, with $r\alpha + \beta < 0$, and a positive sequence $(c_n)_n$ with $\sum_{k=1}^{\infty} c_k/a_k < \infty$.
such that for every $n \in \mathbb{N}$ and $t \geq s \geq 0$,
\[ \|X(t)X(s)^{-1}\|_{H_n} \leq c_ne^{\beta(t-s)} \]  
(43)

Then there exists a constant $a > 0$, such that any solution of Eq. (37) with $\|v_0\|$ sufficiently small is global and satisfies
\[ \|v(t)\| \leq ae^{2t} \|v_0\| \text{ for every } t \geq 0. \]  
(44)

Note that Theorem 10 tells us that the required speed for the sequence $(a_n)_n$ is related to norm estimates for the monodromy operators in finite-dimensional spaces (we can set for example $a_n = c_n n^{1+\tau}$ with $\tau > 0$).

The following is another consequence of the proof of Theorem 8. It has the advantage of not mentioning the spaces $H_n$, although at the expense of requiring more from the monodromy operators.

**Theorem 11.** Assume that conditions C1–C2 hold, and that there exist $\alpha < 0$ and $\beta > 0$, with $\alpha + \beta < 0$, and $C > 0$ such that $\|X(t)X(s)^{-1}\| \leq Ce^{\alpha(t-s)+\beta s}$ for every $t \geq s \geq 0$. Then there exists a constant $a > 0$, such that any solution of Eq. (37) with $\|v_0\|$ sufficiently small is global and satisfies (44).

We also consider the finite-dimensional case. For simplicity we consider the space $H = \mathbb{R}^n$ with the standard inner product. In this case we can obtain the following stronger statement, where $M(\mathbb{R}^n)$ is the set of $n \times n$ matrices with real entries.

**Theorem 12** (Barreira et al. [2, Theorem 1.4.3]). Assume that
1. $A: [0, +\infty) \to M(\mathbb{R}^n)$ is a continuous function satisfying (9),
2. $f: [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function satisfying $f(t, 0) = 0$ for all $t \geq 0$, and there exist constants $c, r > 0$, such that for every $t \geq 0$, and $u, v \in \mathbb{R}^n$,
\[ \|f(t, u) - f(t, v)\| \leq c\|u - v\| (\|u\|^r + \|v\|^r), \]
3. $r \sup\{\lambda_i' : i = 1, \ldots, n\} + \gamma_n(\lambda, \mu) < 0.$

Then the solution $v(t) \equiv 0$ of the perturbed equation (37) is exponentially stable.

We shall reobtain Theorem 12 as a consequence of the infinite-dimensional version in Theorem 8.

4.3. Smallness of the perturbation

We now describe the required speed of the sequence $(a_n)_n$ in (39). For each fixed $n \in \mathbb{N}$, we consider dual bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ of $H_n$, such that
\[ \max\{\lambda(v_i) + \mu(w_i) : i = 1, \ldots, n\} = \gamma_n(\lambda, \mu) \]  
(45)
(this is always possible since the minimum in (18) attains at most a finite number of values). It follows easily from the definition of the Lyapunov exponents that given \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) there exists a constant \( D_{\varepsilon,n} > 0 \), such that

\[
\| v_i(t) \| \leq D_{\varepsilon,n} e^{(\lambda(v_i)_{+,+}+\varepsilon) t} \quad \text{and} \quad \| w_i(t) \| \leq D_{\varepsilon,n} e^{(\mu(w_i)_{+,+}+\varepsilon) t}
\]

(46)

for every \( t \geq 0 \) and \( i = 1, \ldots, n \), where \( v_i(t) \) is the solution of (10) with \( v_0 = v_i \), and \( w_i(t) \) is the solution of (14) with \( w_0 = w_i \) for each \( i \). We assume that the sequence \((a_n)n\) diverges sufficiently fast so that

\[
d := \sum_{k=1}^{\infty} \frac{k^2 D_{\varepsilon,k}^2}{a_k} < \infty
\]

(47)

for some choice of dual bases \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_n \) of \( H_n \) satisfying (45), and of \( \varepsilon > 0 \) satisfying

\[
r(\sup \{ \lambda_i : i \in \mathbb{N} \} + \varepsilon) + \gamma(\lambda, \mu) + 2\varepsilon < 0.
\]

(48)

Note that in view of (40) any sufficiently small \( \varepsilon > 0 \) satisfies (48).

In the particular case of a regular equation, the constants \( D_{\varepsilon,n} \) in (46) can be made somewhat more explicit. We recall the numbers \( \lambda_{i,n} \) and \( p_n \) in (11), and the numbers \( \mu_{i,n} \) and \( q_n \) in (15).

**Proposition 13.** When the equation in (10) is Lyapunov regular, we have \( D_{\varepsilon,n} \leq \max \{ c_{\varepsilon,n}, d_{\varepsilon,n} \} \) with

\[
c_{\varepsilon,n} = \sup_{1 \leq i \leq p_n} \sup_{t \geq 0} \left\{ \frac{\| v(t) \|}{e^{(\lambda_{i,n}+\varepsilon) t}} : v(0) \in E_{i,n} \right\},
\]

\[
d_{\varepsilon,n} = \sup_{1 \leq i \leq q_n} \sup_{t \geq 0} \left\{ \frac{\| w(t) \|}{e^{(\mu_{i,n}+\varepsilon) t}} : w(0) \in F_{i,n} \right\},
\]

where \( v(t) \) is a solution of (10) and \( w(t) \) is a solution of (14).

The proof of Proposition 13 is given in Section 5.2, as a consequence of an alternative characterization of regularity.

In the case of a “uniform” behavior of the Lyapunov exponents, we can be more explicit about the smallness condition on the perturbation \( f \). Namely, assume that for each \( \varepsilon > 0 \) there exists \( C = C(\varepsilon) > 0 \), such that

\[
\| v(t) \| \leq C e^{(\lambda(v)+\varepsilon) t} \| v(0) \| \quad \text{and} \quad \| w(t) \| \leq C e^{(\mu(w)+\varepsilon) t} \| w(0) \|
\]

(49)
for every $t \geq 0$ and every $v(0) \in H$, where $v(t)$ is a solution of (10) and $w(t)$ is a solution of (14). The following is a version of Theorem 8 in this particular case.

**Theorem 14.** Assume that conditions C1–C3, (40), and (49) hold. If $\sum_{k=1}^{\infty} k^2/a_k < \infty$, then given $\varepsilon > 0$ sufficiently small there exists a constant $a > 0$, such that any solution of Eq. (37) with $\|v_0\|$ sufficiently small is global and satisfies (42).

**Proof.** This is an immediate consequence of Theorem 8 and of the above description of the required speed of $(a_n)_n$ in (39): set $D_{\varepsilon,n} = C$ in (47). $\square$

Alternatively, Theorem 14 can be obtained combining Theorem 10 with the norm estimates for the monodromy operators obtained in Theorem 15 below.

### 4.4. Norm estimates for the monodromy operators

Here, we establish crucial estimates for the proofs of the stability results. We use the same notation as in the proof of Lemma 1. Consider the upper triangular monodromy operator $X(t) = U(t)^{-1}V(t)$ constructed in the proof of the lemma. In the following result we obtain bounds on the norm of $X(t)X(s)^{-1}$ restricted to each finite-dimensional space $H_n$ by combining information about the solutions of the equations $v' = A(t)v$ and $w' = -A(t)^*w$ through the study of the Lyapunov exponents $\lambda$ and $\mu$. For each $n \in \mathbb{N}$, we fix dual bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ of $H_n$ satisfying (45) and (46). We recall that $\lambda_{n,n} = \lambda_{p_n,n}$ (see (11) and (13)) is the top value of the Lyapunov exponent $\lambda$ (for Eq. (10)) on $H_n \setminus \{0\}$.

**Theorem 15.** For every $n \in \mathbb{N}$, $\varepsilon > 0$, and $t \geq s \geq 0$ we have

$$\|X(t)X(s)^{-1}|H_n\| \leq n^2 D_{\varepsilon,n}^2 \exp\left(\lambda_{n,n}^+\varepsilon(t-s) + \left(\lambda_n^+(\lambda_n^+ - 2) + 2\varepsilon\right)s\right).$$

**Proof.** Consider the operator $Y(t) = [X(t)^{-1}]^*$ for each $t$. Taking derivatives in the identity $X(t)X(t)^{-1} = X(t)Y(t)^* = \text{Id}$ we obtain

$$X'(t)X(t)^{-1} + X(t)Y'(t)^* = 0.$$

It follows from (32) that

$$X(t)Y'(t)^* = -B(t)X(t)X(t)^{-1} = -B(t).$$

Therefore,

$$Y'(t)^* = -X(t)^{-1}B(t) = -Y(t)^*B(t)$$

and hence,

$$Y'(t) = -B(t)^*Y(t). \quad (50)$$
By (32), the function \( x_i(t) = X(t)v_i \) is a solution of \( x' = B(t)x \) for each \( i = 1, \ldots, n \). Similarly, by (50), the function \( y_i(t) = Y(t)w_i \) is a solution of \( y' = -B(t)^*y \) for each \( i = 1, \ldots, n \). Note that

\[
x_i(t) = U(t)^{-1}v_i(t) \quad \text{and} \quad y_i(t) = U(t)^{-1}w_i(t),
\]

(51)

where \( w_i(t) = [V(t)^{-1}]^*w_i \) for each \( i \). Using (30) we obtain

\[
w'_i(t) = U'(t)y_i(t) + U(t)y'_i(t)
\]

\[
= [U'(t)U(t)^{-1} - U(t)B(t)^*U(t)^{-1}]w_i(t)
\]

\[
= [-A(t)^* + U'(t)U(t)^{-1} + U(t)U'(t)^*]w_i(t)
\]

\[
= \left[-A(t)^* + \frac{d}{dt}(U(t)U(t)^*)\right]w_i(t) = -A(t)^*w_i(t).
\]

Therefore, \( w_i(t) \) is the solution of (14) with \( w_0 = w_i \) for each \( i \).

Since \( U(t) \) is unitary, it follows from (46) and (51) that

\[
\|x_i(t)\| \leq D_{e,n}e^{(\lambda(v_i)+\epsilon)t} \quad \text{and} \quad \|y_i(t)\| \leq D_{e,n}e^{(\mu(w_i)+\epsilon)t}
\]

for every \( t \geq 0 \) and \( i = 1, \ldots, n \). Given \( i \) and \( j \) such that \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \) we consider the number

\[
a_{ij} = \langle X(t)X(s)^{-1}u_i, u_j \rangle.
\]

Since \( X(t) \) is upper triangular for every \( t \geq 0 \), we have \( a_{ij} = 0 \) for \( i < j \). We now consider the case when \( i \geq j \). Observe that

\[
X(t)X(s)^{-1} = X(t)Y(s)^*
\]

for any \( t \geq s \geq 0 \). Since each operator \( X(t) \) leaves invariant the space \( H_n \), and \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_n \) are dual bases, we obtain

\[
a_{ij} = \langle Y(s)^*u_i, X(t)^*u_j \rangle = \sum_{k=1}^n \langle Y(s)^*u_i, w_k \rangle \langle v_k, X(t)^*u_j \rangle
\]

\[
= \sum_{k=1}^n \langle u_i, Y(s)w_k \rangle \langle X(t)v_k, u_j \rangle
\]

\[
= \sum_{k=1}^n \langle u_i, y_k(s) \rangle \langle x_k(t), u_j \rangle
\]
and thus, using (45),

\[ |a_{ij}| \leq \sum_{k=1}^{n} \|y_k(s)\| \cdot \|x_k(t)\| \]

\[ \leq \sum_{k=1}^{n} D_{e,n}^2 e^{(\lambda(v_k)+\varepsilon)t+(\mu(w_k)+\varepsilon)s} \]

\[ = \sum_{k=1}^{n} D_{e,n}^2 e^{(\lambda(v_k)+\varepsilon)(t-s)+(\lambda(v_k)+\mu(w_k)+2\varepsilon)s} \]

\[ \leq n D_{e,n}^2 e^{(\gamma_n'(\lambda,\mu)+2\varepsilon)s}. \]

We can now proceed in a similar manner to that in the proof of Theorem 7 (see (35)) to conclude that given \( v = \sum_{i=1}^{n} \alpha_i u_i \in H_n \) with \( \|v\| = 1 \),

\[ \|X(t)X(s)^{-1}v\|^2 = \left\| \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \langle X(t)X(s)^{-1}u_i, u_j \rangle u_j \right\|^2 \]

\[ = \sum_{j=1}^{n} \left( \sum_{i=j}^{n} \alpha_i a_{ij} \right)^2 \]

\[ \leq \sum_{j=1}^{n} \left( \sum_{i=j}^{n} \alpha_i^2 \sum_{i=j}^{n} a_{ij}^2 \right) \leq \sum_{j=1}^{n} \sum_{i=j}^{n} a_{ij}^2. \]

Therefore,

\[ \|X(t)X(s)^{-1}v\| \leq n^2 D_{e,n}^2 e^{(\gamma_n'(\lambda,\mu)+2\varepsilon)s}. \]

This establishes the desired inequality. \( \square \)

Note that in Theorem 15 the operators \( A(t) \) need not be upper triangular. When the operators \( X(t) \) are diagonal we can somewhat improve the statement in Theorem 15.

**Theorem 16.** Assume that the operator \( X(t) \) is diagonal for every \( t \geq 0 \). Then for every \( n \in \mathbb{N} \), \( \varepsilon > 0 \), and \( t \geq s \geq 0 \) we have

\[ \|X(t)X(s)^{-1}\|_{H_n} \leq D_{e,n}^2 e^{(\gamma_n(\lambda,\mu)+2\varepsilon)s}, \]

where \( \gamma_n = \max\{\lambda(u_i) + \mu(u_i) : i = 1, \ldots, n\} \).
Proof. We use the same notation as in the proof of Theorem 15. Let now \( v = \sum_{i=1}^{n} \alpha_i u_i \in H_n \) with \( \| v \| = 1 \). Using the fact that the operators \( Y(s)^* \) and \( X(t)^* \) are diagonal, we obtain
\[
\| X(t)X(s)^{-1} v \| = \left\| \sum_{i=1}^{n} \alpha_i (X(t)X(s)^{-1} u_i, u_i) u_i \right\|
\]
\[
= \left( \sum_{i=1}^{n} \alpha_i^2 (Y(s)^* u_i, X(t)^* u_i)^2 \right)^{1/2}
\]
\[
\leq \max_{1 \leq i \leq n} |\langle Y(s)^* u_i, X(t)^* u_i \rangle| \leq \max_{1 \leq i \leq n} |\langle Y(s)^* u_i, u_i \rangle \langle u_i, X(t)^* u_i \rangle|.
\]
Therefore,
\[
\| X(t)X(s)^{-1} |H_n \| \leq \max_{1 \leq i \leq n} |\langle u_i, Y(s)u_i \rangle \langle X(t)u_i, u_i \rangle| \leq \max_{1 \leq i \leq n} (\| Y(s)u_i \| \cdot \| X(t)u_i \|)
\]
\[
\leq D_{e,n}^2 \max_{1 \leq i \leq n} e^{(\mu(u_i)+\varepsilon)s+(\lambda(u_i)+\varepsilon)t}
\]
\[
\leq D_{e,n}^2 \max_{1 \leq i \leq n} e^{(\lambda(u_i)+\varepsilon)(t-s)+(\lambda(u_i)+\mu(u_i)+2\varepsilon)s}
\]
\[
\leq D_{e,n}^2 e^{(\lambda_{n,n}'+\varepsilon)(t-s)+(\tau_n+2\varepsilon)s}.
\]
This completes the proof. □

Since the basis \( u_1, \ldots, u_n \) of \( H_n \) is dual to itself, it follows from Lemma 2 in the proof of Theorem 19 below that \( \lambda(u_i) + \mu(u_i) \geq 0 \) for \( i = 1, \ldots, n \). Therefore, the number \( \tau_n \) in the statement of Theorem 16 satisfies \( \tau_n \geq 0 \).

4.5. Proofs of the stability results

We use the same notation as in the proof of Lemma 1, but now applied to the case when \( A(t) \) is upper triangular for every \( t \). In this case we can take \( U(t) = 1d \) for every \( t \), and thus we can consider the monodromy operators \( X(t) = V(t) \); we shall always make this choice.

Proof of Theorem 8. We denote by \( v(t) \) the solution of the initial value problem (37). This problem is equivalent to the integral equation
\[
v(t) = X(t)v_0 + \int_{0}^{t} X(t)X(s)^{-1} f(s, v(s)) \, ds.
\]
Consider the operator
\[(Tv)(t) = X(t)v_0 + \int_0^t X(t)X(s)^{-1} f(s, v(s)) \, ds\]
on the space
\[B_\delta = \{ v: [0, \infty) \to H \text{ continuous} : \|v(t)\| \leq \delta e^{\alpha t} \text{ for every } t \geq 0 \},\]
where \(\delta > 0\) (to be chosen later), and \(\alpha = \sup\{\lambda_i : i \in \mathbb{N}\} + \epsilon\) for some \(\epsilon > 0\), such that \(\alpha < 0\) (recall that (41) is a consequence of (40)). We introduce the norm on \(B_\delta\) given by
\[\|v\| = \sup\{\|v(t)\|e^{-\alpha t} : t \geq 0\}.\]
One can easily verify that \(B_\delta\) becomes a complete metric space with respect to the induced distance. Observe now that by Theorem 15, for every \(n \in \mathbb{N}\), \(\epsilon > 0\), and \(t \geq s \geq 0\),
\[\|X(t)X(s) - 1\|_{H_n} \leq n^2 D_{t, n}^2 e^{(\gamma_n + \epsilon)(t-s) + (\gamma_n + 2\epsilon)s} \leq n^2 D_{t, n}^2 e^{2(t-s) + \beta s},\] (53)
where \(\beta = \gamma(\lambda, \mu) + 2\epsilon\). Let \(v_1, v_2 \in B_\delta\). Since \(X(t)\) is upper triangular for every \(t\), using (53) and condition C3 we obtain
\[\|X(t)X(s)^{-1}(f(s, v_1(s)) - f(s, v_2(s)))\|\]
\[= \left\| X(t)X(s)^{-1} \sum_{k=1}^\infty \langle f(s, v_1(s)) - f(s, v_2(s)), u_k \rangle u_k \right\|\]
\[\leq \sum_{k=1}^\infty |(f(s, v_1(s)) - f(s, v_2(s)), u_k)| \cdot \|X(t)X(s)^{-1}\|_{H_k}\]
\[\leq \sum_{k=1}^\infty \frac{1}{a_k} \|v_1(s) - v_2(s)\|(\|v_1(s)\|^r + \|v_2(s)\|^r) k^2 D_{s, k}^2 e^{2(t-s) + \beta s}\]
\[\leq \sum_{k=1}^\infty \frac{k^2 D_{s, k}^2}{a_k} \|v_1(s) - v_2(s)\|(\|v_1(s)\|^r + \|v_2(s)\|^r) e^{2t + (r\alpha + \beta)s}\]
\[\leq \sum_{k=1}^\infty \frac{2\delta' k^2 D_{s, k}^2}{a_k} \|v_1(s) - v_2(s)\| e^{2t + (r\alpha + \beta)s}.\] (54)
That is,
\[ \|X(t)X(s)^{-1} (f(s, v_1(s)) - f(s, v_2(s)))\| \leq 2d \delta^s \|v_1 - v_2\| e^{2t(r \alpha + \beta)s}, \] (55)
where \(d\) is the constant in (47). We assume that \(d < \infty\) for some \(\varepsilon > 0\), such that (see (48))
\[ r \alpha + \beta = r(\sup \{\lambda_i : i \in \mathbb{N}\} + \varepsilon) + 2 \varepsilon < 0, \]
which is always possible due to (40). The assumption \(d < \infty\) corresponds to require that the sequence \((a_n)_n\) diverges sufficiently fast. Therefore,
\[ \|T v_1(t) - T v_2(t)\| \leq 2d \delta^s \|v_1 - v_2\| e^{2t} \int_0^t e^{(r \alpha + \beta)s} ds \leq 2d \kappa \delta^s \|v_1 - v_2\| e^{2t}, \]
where \(\kappa = \int_0^\infty e^{(r \alpha + \beta)s} ds\). Hence,
\[ \|T v_1 - T v_2\| \leq 2d \kappa \delta^s \|v_1 - v_2\|. \] (56)
Choose now \(\delta \in (0, 1)\) such that \(\theta := 2d \kappa \delta^s < 1\). For each \(v_0 \in H\) satisfying condition C3 we obtain in a similar manner, using (53) with \(s = 0\), that
\[ \|X(t)v_0\| \leq \lim_{n \to \infty} \sum_{k=1}^n \|\langle v_0, u_k \rangle\| \cdot \|X(t)|H_k\| \leq \sum_{k=1}^\infty k^2 D_{e,k}^2 \frac{a_k}{a_k} e^{2t} \|v_0\| = d e^{2t} \|v_0\|. \] (57)
Note that \(X(t)v_0 = (T0)(t)\). Therefore, for each \(v \in B_\delta\), setting \(v_1 = v \in B_\delta\) and \(v_2 = 0\) in (56), we obtain
\[ \|(Tv)(t)e^{-2t} \| \leq \|X(t)v_0\| + \|Tv - T0\| \leq d \|v_0\| + 0 \delta < \delta \]
provided that \(v_0\) is chosen sufficiently small. Therefore, \(T(B_\delta) \subset B_\delta\), and the operator \(T\) is a contraction on the complete metric space \(B_\delta\). Hence, there exists a unique function \(v \in B_\delta\) which solves (52). It remains to establish the stability of the zero solution. For this, set \(u(t) = X(t)v_0\) and observe that the solution \(v(t)\) can be obtained by
\[ v(t) = \lim_{n \to \infty} (T^n 0)(t) = \sum_{n=0}^\infty (J^n u)(t), \]
where
\[(Ju)(t) = \int_0^t X(t)X(s)^{-1} f(s, u(s)) \, ds.\]

It follows from (56) that \(\|Ju\| \leq \theta \|u\|\). Hence, using (57),
\[\|v\| \leq \sum_{n=0}^{\infty} \|J^n u\| \leq \sum_{n=0}^{\infty} \theta^n \|u\| = \frac{\|u\|}{1 - \theta} \leq \frac{\|v_0\|}{1 - \theta}.
\]

Therefore,
\[\|v(t)\| \leq \frac{d}{1 - \theta} e^{\alpha t} \text{ for every } t \geq 0.
\]

This concludes the proof of the theorem. □

**Proof of Theorem 10.** We can repeat almost verbatim the proof of Theorem 8, replacing the inequality (53) by the condition (43), and the inequalities (55) (see also (54)) and (57), respectively, by
\[\|X(t)X(s)^{-1}(f(s, v_1(s)) - f(s, v_2(s)))\| \leq 2\eta \delta^r \|v_1 - v_2\| e^{\alpha t + (r\alpha + \beta)s},\]
where \(\eta = \sum_{k=1}^{\infty} c_k/a_k < \infty\), and
\[\|X(t)v_0\| \leq \eta e^{\alpha t} \|v_0\| \text{ for each } v_0 \in H \text{ satisfying condition C3}.
\]

That is, we obtain similar inequalities to those in (55) and (57), with \(d\) replaced by \(\eta\). It then follows from the proof of Theorem 8 (see (58)) that choosing \(\delta \in (0, 1)\), such that \(\theta := 2\eta \delta^r \int_0^{\infty} e^{(r\alpha + \beta)s} ds < 1\), any solution \(v(t)\) of Eq. (37) with \(\|v_0\|\) sufficiently small satisfies the estimate (44) with \(a = \eta/(1 - \theta)\). □

**Proof of Theorem 11.** As in the proof of Theorem 10 we can repeat almost verbatim the proof of Theorem 8, replacing the inequalities (55) and (57), respectively, by
\[\|X(t)X(s)^{-1}(f(s, v_1(s)) - f(s, v_2(s)))\|
\leq \|X(t)X(s)^{-1}\| \cdot \|f(s, v_1(s)) - f(s, v_2(s))\|
\leq Ce^{\alpha(t-s) + \beta s} \|v_1(s) - v_2(s)\| (\|v_1(s)\|^r + \|v_2(s)\|^r)
\leq Cc \|v_1 - v_2\| (\|v_1\|^r + \|v_2\|^r) e^{\alpha t + (r\alpha + \beta)s}
\leq 2Cc \delta^r \|v_1 - v_2\| e^{\alpha t + (r\alpha + \beta)s},\]
and \( \|X(t)v_0\| \leq \|X(t)\| \cdot \|v_0\| \leq Ce^{\alpha t} \|v_0\| \). We can now proceed in a similar manner to that in the proof of Theorem 8 to obtain the desired result. \( \square \)

**Proof of Theorem 12.** Note that condition C2 is explicitly stated as an hypothesis in the theorem. Furthermore, since in the proof of Theorem 8 the series are now replaced by finite sums, we do not need (38) or condition C3, and thus in particular any sequence \((a_n)_n\) controlling the smallness of the perturbation. In addition, the third hypothesis in the theorem is equivalent to (40). The statement is thus an immediate consequence of Theorem 8. \( \square \)

## 5. Characterizations of Lyapunov regularity

### 5.1. Regularity coefficient and Perron coefficient

We use the same notation as in Section 2. In particular, we consider the values

\[
\lambda_{1,n} \leq \cdots \leq \lambda'_{n,n} \quad \text{and} \quad \mu_{1,n} \geq \cdots \geq \mu'_{n,n}
\]

respectively of the Lyapunov exponents \( \lambda \) and \( \mu \) on \( H_n \setminus \{0\} \) counted with multiplicities (see (13) and (17)). Mimicking once more the abstract theory of Lyapunov exponents in finite-dimensional spaces, we introduce the **Perron coefficient** of \( \lambda \) and \( \mu \),

\[
\pi(\lambda, \mu) = \sup\{\lambda_i + \mu_i : i \in \mathbb{N}\}.
\]

We also consider for each \( n \in \mathbb{N} \) the number

\[
\pi_n(\lambda, \mu) = \max\{\lambda'_{i,n} + \mu'_{i,n} : i = 1, \ldots, n\}.
\]

In the abstract theory of Lyapunov exponents in finite-dimensional spaces the numbers \( \gamma_n(\lambda, \mu) \) (see (18)) and \( \pi_n(\lambda, \mu) \) are called, respectively, the **regularity coefficient** and the **Perron coefficient** of \( \lambda \) and \( \mu \).

The following theorem establishes some relations between the regularity coefficients and the Perron coefficients.

**Theorem 17.** For each \( n \in \mathbb{N} \),

\[
0 \leq \pi_n(\lambda, \mu) \leq \gamma_n(\lambda, \mu) \leq n \pi_n(\lambda, \mu).
\]

In addition, if there exists \( \delta > 0 \) such that (19) holds, i.e.,

\[
-\infty \leq \lambda_1 \leq \lambda_2 \leq \cdots < -\delta \quad \text{and} \quad \mu_1 \geq \mu_2 \geq \cdots \geq \delta,
\]

then

\[
0 \leq \pi_n(\lambda, \mu) - \gamma_n(\lambda, \mu) \leq \frac{\delta}{n}
\]

and

\[
\lambda_1 \leq \lambda_2 \leq \cdots < \frac{\delta}{n} \quad \text{and} \quad \mu_1 \geq \mu_2 \geq \cdots \geq \delta + \frac{\delta}{n}.
\]
then

1.

\[ 0 \leq \pi(\lambda, \mu) = \lim_{n \to \infty} \pi_n(\lambda, \mu) \leq \lim_{n \to \infty} \gamma_n(\lambda, \mu) \leq \gamma(\lambda, \mu); \]  

(61)

2. for any increasing sequence of subspaces \( H'_1 \subset H'_2 \subset \cdots \) with union equal to \( H \),

\[ \gamma(\lambda, \mu) = \sup \{ \gamma'_n(\lambda, \mu) : n \in \mathbb{N} \}, \]

where

\[ \gamma'_n(\lambda, \mu) = \min \max \{ \lambda(v'_i) + \mu(w'_i) : 1 \leq i \leq \dim H'_n \}, \]

and \( m = \dim H'_n \), with the minimum taken over all dual bases \( v'_1, \ldots, v'_m \) and \( w'_1, \ldots, w'_m \) of the space \( H'_n \).

**Proof.** The first statement follows from Theorem 1.2.6 in [2] applied to the Lyapunov exponents \( \lambda \) and \( \mu \) restricted to the finite-dimensional space \( H_n \).

To show that the sequence \((\pi_n)_n\) with \( \pi_n = \pi_n(\lambda, \mu) \) is convergent, note that by the monotonicity in (19), given \( \varepsilon > 0 \) one can choose \( k \in \mathbb{N} \) such that

\[ \lambda_i \in (a - \varepsilon, a) \quad \text{and} \quad \mu_i \in (b, b + \varepsilon) \quad \text{for every} \quad i \geq k, \]

(62)

where \( a = \sup_i \lambda_i \) and \( b = \inf_i \mu_i \). In particular,

\[ a + b - \varepsilon < \lambda_k + \mu_k < a + b + \varepsilon. \]

(63)

Furthermore, the numbers \( \lambda_i \) and \( \mu_i \) are obtained, respectively, from collecting the numbers \( \lambda'_{j,n} \) and \( \mu'_{j,n} \). More precisely, for each \( i \in \mathbb{N} \) there exist integers \( n, p, q \in \mathbb{N} \), with \( p \leq n \) and \( q \leq n \), such that \( \lambda_i = \lambda'_{p,n} \) and \( \mu_i = \mu'_{q,n} \). We have \( i \geq p \) and \( i \geq q \), and these inequalities may be strict. However, since the sequence \( H_n \) is increasing, for a given integer \( k \), if \( n \) is sufficiently large, then all numbers in \( \lambda_1 \leq \cdots \leq \lambda_k \) and \( \mu_1 \geq \cdots \geq \mu_k \) must occur respectively in the two finite sequences in (59) (otherwise they would never occur as values of the Lyapunov exponents \( \lambda \) and \( \mu \)). But due to the monotonicity of the sequences (see (19) and (59)), we conclude that \( \lambda'_{i,n} = \lambda_i \) and \( \mu'_{i,n} = \mu_i \) for every \( i \leq k \) (and every sufficiently large \( n \)). Therefore, in view of (62),

\[ \max \{ c_k, a + b - \varepsilon \} \leq \pi_n \leq \max \{ c_k, a + b + \varepsilon \}, \]

where \( c_k = \max \{ \lambda_i + \mu_i : 1 \leq i \leq k \} \). By (63), we conclude that

\[ c_k - 2\varepsilon \leq \max \{ c_k, \lambda_k + \mu_k - 2\varepsilon \} \leq \pi_n \leq \max \{ c_k, \lambda_k + \mu_k + 2\varepsilon \} \leq c_k + 2\varepsilon. \]
Letting \( k \to \infty \) we obtain \( n \to \infty \), and the arbitrariness of \( \varepsilon \) in the above inequalities implies that the sequence \( (\gamma_n)_n \) is convergent, with limit \( \pi(\lambda, \mu) \). We now show that the sequence \( (\gamma_n)_n \) with \( \gamma_n = \gamma_n(\lambda, \mu) \) is convergent. For each \( n, m \in \mathbb{N} \) we have

\[
\gamma_{n+m} = \min \max \{ \lambda(v_i) + \mu(w_i) : 1 \leq i \leq n + m \} \\
\leq \min \max \{ \lambda(v'_i) + \mu(w'_i) : 1 \leq i \leq n + m \},
\]

where the first minimum is taken over all dual bases \( v_1, \ldots, v_{n+m} \) and \( w_1, \ldots, w_{n+m} \) of the space \( H_{n+m} \), and the second minimum is taken over all dual bases \( v'_1, \ldots, v'_{n+m} \) and \( w'_1, \ldots, w'_{n+m} \) of the space \( H_{n+m} \), such that

\[
\langle v'_1, \ldots, v'_n \rangle = \langle w'_1, \ldots, w'_n \rangle = H_n,
\]
i.e., the first \( n \) elements of each basis generate \( H_n \). In a similar manner to that for the sequence \( (\pi_n)_n \), it follows from the monotonicity in \((19)\) that given \( \varepsilon > 0 \), if \( n \) is sufficiently large, then for each \( m \in \mathbb{N} \),

\[
\lambda(v_{n+i}) \in (a - \varepsilon, a) \quad \text{and} \quad \mu(w_{n+i}) \in (b, b + \varepsilon) \quad \text{for every} \quad i \leq m.
\]

It follows from \((64)\) that \( \gamma_{n+m} \leq \max\{\gamma_n, a + b + \varepsilon\} \). Finally, note that for each \( n \) sufficiently large there exists \( 1 \leq i \leq n \), such that \( \lambda(v_i) \in (a - \varepsilon, a) \) and \( \mu(w_i) \in (b, b + \varepsilon) \). Therefore, for this \( i \) we have \( \lambda(v_i) + \mu(w_i) > a + b - \varepsilon \), and hence,

\[
\gamma_{n+m} \leq \max\{\gamma_n, a + b + \varepsilon\} \leq \max\{\gamma_n, \lambda(v_i) + \mu(w_i) + 2\varepsilon\} \leq \gamma_n + 2\varepsilon.
\]

Letting \( m \to \infty \) and then \( n \to \infty \), we conclude from the arbitrariness of \( \varepsilon \) that \( \limsup_{n \to \infty} \gamma_n \leq \liminf_{n \to \infty} \gamma_n \). The inequalities in \((61)\) follow now immediately from \((60)\) taking limits when \( n \to \infty \).

The independence of the definition of \( \gamma(\lambda, \mu) \) with respect to the choice of subspaces \( H_n \) (and their dimension) follows readily from the convergence of the sequence \( (\gamma_n)_n \) together with the observation that if \( H'_n \) is an increasing sequence of subspaces with union equal to \( H \), then for each \( n \in \mathbb{N} \) there exist \( m, \ell \in \mathbb{N} \), such that \( H'_n \subset H_m \subset H'_\ell \). \( \square \)

### 5.2. Characterizations of regularity

We recall that a basis \( v_1, \ldots, v_n \) of the space \( H_n \) is normal for the filtration by subspaces

\[
E_1 \subset E_2 \subset \cdots \subset E_p = H_n
\]

if for each \( i = 1, \ldots, p \) there exists a basis of \( E_i \) composed of vectors in \( \{v_1, \ldots, v_n\} \). When \( v_1, \ldots, v_n \) is a normal basis for the filtration of subspaces \( E_{i,n} \) with \( i = 1, \ldots, p_n \)
(see (12)) we also say that it is normal for the Lyapunov exponent \( \lambda \) (or simply normal when it is clear from the context to which exponent we are referring to). We shall refer to dual bases \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_n \) which are normal respectively for the Lyapunov exponents \( \lambda \) and \( \mu \), i.e., respectively for the filtration by subspaces
\[
E_{1,n} \subset \cdots \subset E_{p_n,n} = H_n \quad \text{and} \quad F_{q_n,n} \subset \cdots \subset F_{1,n} = H_n
\]
in (12) and (16) as dual normal bases.

**Proposition 18.** There exist dual normal bases \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_n \) of the space \( H_n \).

**Proof.** Let \( v'_1, \ldots, v'_n \) be a basis of \( H_n \) with
\[
\lambda(v'_1) \leq \cdots \leq \lambda(v'_n),
\]
(66) which is normal for the first family of subspaces in (65). We consider another filtration by subspaces
\[
E'_{1,n} \subset E'_{2,n} \subset \cdots \subset E'_{q,n} = H_n
\]
(note that \( q \) need not be equal to \( p_n \)). It is easy to see that there exists a nonsingular upper triangular matrix \( C \) (in the basis \( v'_1, \ldots, v'_n \)), such that the new basis \( v_1 = Cv'_1, \ldots, v_n = Cv'_n \) is normal for the filtration in (67) (compare with Section 1.2 in [2]). On the other hand, in view of (66) and since \( C \) is upper triangular, the new basis \( v_1, \ldots, v_n \) continues to be normal for the first family of subspaces in (65). We now consider the particular case of \( E'_{j,n} = F^\perp_{j,n} \) with \( j = 1, \ldots, q_n \). Then, \( v_1, \ldots, v_n \) is a basis of \( H_n \) which is normal simultaneously for the families of subspaces
\[
E_{1,n} \subset \cdots \subset E_{p_n,n} = H_n \quad \text{and} \quad F^\perp_{1,n} \subset \cdots \subset F^\perp_{q_n,n} = H_n.
\]
Then the (unique) dual basis \( w_1, \ldots, w_n \) of \( H_n \) is normal for the family of subspaces \( F_{i,n} \) with \( j = 1, \ldots, q_n \). □

The following result provides several alternative characterizations of Lyapunov regularity in terms of the regularity and Perron coefficients, and in terms of the values of the Lyapunov exponents \( \lambda \) and \( \mu \).

**Theorem 19.** The following properties are equivalent:

1. the equation in (10) is Lyapunov regular, i.e., \( \gamma(\lambda, \mu) = 0 \),
2. \( \gamma_n(\lambda, \mu) = 0 \) for every \( n \in \mathbb{N} \),
3. \( \pi_n(\lambda, \mu) = 0 \) for every \( n \in \mathbb{N} \).
4. for every \( n \in \mathbb{N} \), given dual normal bases \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_n \) of the space \( H_n \),

\[
\lambda(v_i) + \mu(w_i) = 0 \quad \text{for} \quad i = 1, \ldots, n, 
\]

(68)

5. for every \( n \),

\[
\lambda'_{i,n} + \mu'_{i,n} = 0 \quad \text{for} \quad i = 1, \ldots, n. 
\]

(69)

In addition, if (19) holds for some \( \delta > 0 \), and the equation in (10) is Lyapunov regular, then \( \pi(\lambda, \mu) = 0 \) and the property (20) holds.

**Proof.** By (60), we have \( \gamma_n(\lambda, \mu) \geq 0 \) for every \( n \in \mathbb{N} \), and the equivalence of the first two properties is immediate from the definition of the regularity coefficient. The fact that these are equivalent to the third property follows readily from the inequalities in (60).

Before proceeding we require an additional property.

**Lemma 2.** If \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_n \) are dual bases of the space \( H_n \), then \( \lambda(v_i) + \mu(w_i) \geq 0 \) for every \( i = 1, \ldots, n \).

**Proof of the lemma.** Let \( v(t) \) be a solution of \( v' = A(t)v \), and \( w(t) \) a solution of \( w' = -A(t)^*w \). We have

\[
\frac{d}{dt} \langle v(t), w(t) \rangle = \langle A(t)v(t), w(t) \rangle + \langle v(t), -A(t)^*w(t) \rangle = \langle A(t)v(t), w(t) \rangle - \langle A(t)v(t), w(t) \rangle = 0
\]

and hence,

\[
\langle v(t), w(t) \rangle = \langle v(0), w(0) \rangle \quad \text{for every} \quad t \geq 0.
\]

For each \( i \), let \( v_i(t) \) be the unique solution of (10) with \( v_0 = v_i \), and \( w_i(t) \) the unique solution of (14) with \( w_0 = w_i \). We obtain

\[
\|v_i(t)\| \cdot \|w_i(t)\| \geq 1
\]

for every \( t \geq 0 \), and hence, \( \lambda(v_i) + \mu(w_i) \geq 0 \) for every \( i \).

We now show that Lyapunov regularity implies each of the last two properties in the theorem. By Proposition 18 we can consider dual normal bases \( v_1, \ldots, v_n \) and
$w_1, \ldots, w_n$, and hence the numbers $\lambda(v_i)$ and $\mu(w_i)$ are respectively the values of the Lyapunov exponents $\lambda$ and $\mu$ on $H_n \setminus \{0\}$ counted with multiplicities, although possibly not ordered. Therefore,

$$0 \leq \sum_{i=1}^{n} (\lambda(v_i) + \mu(w_i)) \leq n \pi_n(\lambda, \mu). \quad (70)$$

If the equation in (10) is regular, we have $\pi_n(\lambda, \mu) = 0$ for every $n \in \mathbb{N}$, and thus (68) holds. Moreover, by the definition of $\pi_n(\lambda, \mu)$ we have $\lambda'_{i,n} + \mu'_{i,n} \leq 0$ for every $i$, and in view of (70) we conclude that (69) holds.

We now show that each of the last two properties yields regularity. It follows from (69) that $\pi_n(\lambda, \mu) = 0$ for every $n \in \mathbb{N}$, and thus the equation in (10) is regular. It remains to show that the last property yields regularity. In view of Proposition 18 we can write

$$\gamma_n(\lambda, \mu) \leq \min \max \{\lambda(v_i) + \mu(w_i) : 1 \leq i \leq n\} = 0,$$

where the minimum is taken over all dual normal bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ of the space $H_n$. Therefore, the equation is regular.

For the last statement, observe that using (61) we conclude that a regular equation has Perron coefficient $\pi(\lambda, \mu) = 0$. Furthermore, in a similar manner to that in the proof of Theorem 17, it follows from the monotonicity in (19) that given $k \in \mathbb{N}$, if $n$ is sufficiently large then $\lambda'_{i,n} = \lambda_i$ and $\mu'_{i,n} = \mu_i$ for $i \leq k$. It follows from (69) that $\lambda_i + \mu_i = 0$ for every $i \leq k$. The desired result follows now from the arbitrariness of $k$. \(\square\)

We can now establish the estimate for the constant $D_{\varepsilon,n}$ in Proposition 13 in the case of a regular equation.

**Proof of Proposition 13.** Due to Proposition 18 there exist dual normal bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ of the space $H_n$. Furthermore, by Theorem 19 the regularity implies that $\lambda(v_i) + \mu(w_i) = 0$ for every $i$, and hence

$$0 \leq \gamma_n(\lambda, \mu) \leq \max \{\lambda(v_i) + \mu(w_i) : i = 1, \ldots, n\} = 0.$$ 

Therefore, we can consider these bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ when we define $D_{\varepsilon,n}$ by the inequalities (46). Since these are normal bases we readily obtain the desired result. \(\square\)

### 5.3. Geometric consequences of regularity

We now present an alternative geometric characterization of regularity, expressed in terms of the existence of exponential growth rates of finite-dimensional volumes.
Given vectors $v_1, \ldots, v_m \in H$ we recall that the $m$-volume $\Gamma(v_1, \ldots, v_m)$ defined by these vectors is given by $|\det K|^{1/2}$, where $K$ is the $m \times m$ matrix with entries $k_{ij} = \langle v_i, v_j \rangle$ for each $i$ and $j$. With a slight abuse of notation, given $v \in H$ we denote by $v(t)$ the solution of (10) with $v(0) = v$. For a given continuous function $A(t)$ we consider also the new continuous function $B(t)$ given by Theorem 7 which is upper triangular for each $t \geq 0$.

**Theorem 20.** The following properties are equivalent:

1. the equation in (10) is Lyapunov regular, i.e., $\gamma(\lambda, \mu) = 0$;
2. for each $n \in \mathbb{N}$, and each normal basis $v_1, \ldots, v_n$ of $H_n$,
   $$\lim_{t \to +\infty} \frac{1}{t} \log \Gamma(v_1(t), \ldots, v_n(t)) = \sum_{i=1}^{p_n} \lambda_{i,n} = \sum_{j=1}^{n} \lambda'_{j,n};$$
3. given $n, m \in \mathbb{N}$ with $m \leq n$, and a normal basis $v_1, \ldots, v_n$ of $H_n$ the limit
   $$\lim_{t \to +\infty} \frac{1}{t} \log \Gamma(v_1(t), \ldots, v_m(t))$$
   exists;
4. given $n, m \in \mathbb{N}$ with $m \leq n$, and a normal basis $v_1, \ldots, v_n$ of $H_n$ for each $1 \leq i_1 < \cdots < i_m \leq n$ the limit
   $$\lim_{t \to +\infty} \frac{1}{t} \log \Gamma(v_{i_1}(t), \ldots, v_{i_m}(t))$$
   exists;
5. for each $n \in \mathbb{N}$,
   $$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \text{tr}(B(s)|H_n) \, ds = \sum_{i=1}^{p_n} \lambda_{i,n} = \sum_{j=1}^{n} \lambda'_{j,n};$$
6. for each $n \in \mathbb{N}$, the limit
   $$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \text{tr}(B(s)|H_n) \, ds$$
   exists;
7. for each $n \in \mathbb{N}$, the limit
   $$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle B(s)u_n, u_n \rangle \, ds$$
   exists.
Proof. We recall that by Theorem 7 the initial value problem (10) is equivalent to

$$x' = B(t)x, \quad x(0) = v_0, \quad (71)$$

with $B(t)$ upper triangular for each $t$, i.e., $\langle B(t)u_i, u_j \rangle = 0$ for each $t \geq 0$ whenever $i < j$, with the solutions $v(t)$ of (10) and $x(t)$ of (71) related by $v(t) = U(t)x(t)$ with $U(t)$ unitary for each $t \geq 0$. Similarly, the initial value problem (14) is equivalent to

$$y' = -B(t)^*y, \quad y(0) = w_0, \quad (72)$$

with the solutions $w(t)$ of (14) and $y(t)$ of (72) related by $w(t) = U(t)y(t)$ using the same operator $U(t)$ (see the proof of Theorem 15). Since the operator $U(t)$ is unitary for each $t$, the Lyapunov exponents for the equations in (71) and (72) coincide, respectively, with the Lyapunov exponents $\lambda$ and $\mu$ for the equations in (10) and (14). We continue to denote by $\lambda$ and $\mu$ the Lyapunov exponents of (71) and (72).

Furthermore, the regularity coefficient of the new pair of Eqs. ((71) and (72)) is the same at that for Eqs. (10) and (14).

In view of the above discussion, the equation in (71) is Lyapunov regular if and only if the same happens with (10). Furthermore, by Theorem 19, these equations are Lyapunov regular if and only if $\gamma_n(\lambda, \mu) = 0$ for every $n \in \mathbb{N}$. Since $B(t)$ is upper triangular with respect to the basis $u_1, u_2, \ldots$ of $H$, and for each $n$ the space $H_n$ is spanned by $u_1, \ldots, u_n$, we have $B(t)H_n \subset H_n$ for each $t \geq 0$ and each $n \in \mathbb{N}$. Therefore, the equation $x' = (B(t)|H_n)x$ is Lyapunov regular. This allows us to apply the finite-dimensional abstract theory of Lyapunov exponents to $B(t)|H_n$. In particular, it follows immediately from Theorem 1.3.1 in [2] that the following properties are equivalent:

1. $\gamma_n(\lambda, \mu) = 0$;
2. $\lim_{t \to +\infty} \frac{1}{t} \int_0^t \text{tr}(B(s)|H_n) \, ds = \sum_{i=1}^{p_n} \lambda_{i,n};$
3. given $n, m \in \mathbb{N}$ with $m \leq n$, and a normal basis $v_1, \ldots, v_n$ of $H_n$ the limit
   $$\lim_{t \to +\infty} \frac{1}{t} \log \Gamma(x_1(t), \ldots, x_m(t))$$
   exists, where $x_1(t), \ldots, x_n(t)$ are the solutions of (71) with $x_1(0) = v_1, \ldots, x_n(0) = v_n$.

Note that since $U(t)$ is unitary for each $t$, we have

$$\langle v_i(t), v_j(t) \rangle = \langle U(t)x_i(t), U(t)x_j(t) \rangle = \langle x_i(t), x_j(t) \rangle.$$

Therefore,
\[
\Gamma(v_1(t), \ldots, v_m(t)) = \Gamma(x_1(t), \ldots, x_m(t)).
\]

Furthermore, it is well-known that in the finite-dimensional setting the determinant of the monodromy operator satisfies
\[
\frac{\Gamma(v_1(t), \ldots, v_n(t))}{\Gamma(v_1, \ldots, v_n)} = \exp \int_0^t \text{tr}(B(s)|H_n) \, ds.
\] (73)

Furthermore,
\[
\int_0^t \langle B(s)u_n, u_n \rangle \, ds = \int_0^t \text{tr}(B(s)|H_n) \, ds - \int_0^t \text{tr}(B(s)|H_{n-1}) \, ds
\]
\[
= \log \frac{\Gamma(v_1(t), \ldots, v_n(t))/\Gamma(v_1, \ldots, v_n)}{\Gamma(v_1(t), \ldots, v_{n-1}(t))/\Gamma(v_1, \ldots, v_{n-1})}.
\] (74)

The desired statement can now be easily obtained by putting together the above results. □

We now briefly describe several geometric consequences of regularity.

**Theorem 21.** If Eq. (10) is Lyapunov regular, then the following properties hold:

1. for each \( n \in \mathbb{N}, \ i = 1, \ldots, p_n, \) and \( v \in E_{i,n} \setminus E_{i-1,n} \) we have
\[
\lim_{t \to +\infty} \frac{1}{t} \log \|v(t)\| = \lambda_{i,n}
\]
with uniform convergence for \( v \) on any subspace \( F \subset E_{i,n}, \) such that \( F \cap E_{i-1,n} = \{0\}; \)

2. for any two vectors \( v_1, v_2 \in H \) we have
\[
\lim_{t \to +\infty} \frac{1}{t} \log |\sin \angle(v_1(t), v_2(t))| = 0;
\]

3. for any \( n \in \mathbb{N} \) and any collection of vectors \( v_1, \ldots, v_m \in H \) the limit
\[
\lim_{t \to +\infty} \frac{1}{t} \log \Gamma(v_1(t), \ldots, v_m(t))
\]
exists.
Proof. We already saw in the proof of Theorem 20 that, since the equation in (10) is Lyapunov regular, for each \( n \in \mathbb{N} \) the equation defined by the finite-dimensional operator \( B(t)|H_n \) is also Lyapunov regular. Therefore, it follows from the finite-dimensional regularity theory (see [2, Chapter 1]; see also [11,1]) that for each \( n \in \mathbb{N} \):

1. if \( i = 1, \ldots, p_n \) and \( v \in E_{i,n} \setminus E_{i-1,n} \) then

\[
\lim_{t \to +\infty} \frac{1}{t} \log \|v(t)\| = \lambda_{i,n} ;
\]

(75)

2. if \( v_1, v_2 \in H_n \) then

\[
\lim_{t \to +\infty} \frac{1}{t} \log |\sin \angle (v_1(t), v_2(t))| = 0 ;
\]

3. if \( v_1, \ldots, v_m \in H_n \) then the limit

\[
\lim_{t \to +\infty} \frac{1}{t} \log \Gamma(v_1(t), \ldots, v_m(t))
\]

exists.

We note that the third statement is a consequence of the other two, since the volume \( \Gamma(v_1(t), \ldots, v_m(t)) \) can be written as a product of \( \prod_{j=1}^m \|v_j(t)\| \) and sines of angles between solutions. Furthermore, we have uniform convergence in (75) for \( v \) on any subspace \( F \subset E_{i,n} \), such that \( F \cap E_{i-1,n} = \{0\} \) (we refer to [3] for a detailed proof of the uniformity; although this paper only considers the case of discrete time the arguments can be repeated almost verbatim to obtain a proof for continuous time). We thus obtain the three statements in the theorem, observing for the last two that one can always find \( n \) such that \( v_1, \ldots, v_m \in H_n \).

We now discuss the geometric consequences of a positive regularity coefficient. To the best of our knowledge this result was not formulated before also in the finite-dimensional case.

Theorem 22. For each given \( n \in \mathbb{N} \), let \( v_1, \ldots, v_n \) be a normal basis of \( H_n \), such that \( v_1, \ldots, v_{n-1} \) is a normal basis of \( H_{n-1} \). Then we have the following properties:

1. 

\[
\limsup_{t \to +\infty} \frac{1}{t} \log \Gamma_n(t) = \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \text{tr}(B(s)|H_n) \, ds \leq a_n ,
\]

(76)

\[
\liminf_{t \to +\infty} \frac{1}{t} \log \Gamma_n(t) = \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \text{tr}(B(s)|H_n) \, ds \\
\geq a_n - n \pi_n(\lambda, \mu) \geq a_n - n \gamma_n(\lambda, \mu) ,
\]
where

\[ \Gamma_n(t) = \Gamma(v_1(t), \ldots, v_n(t)) \quad \text{and} \quad a_n = \sum_{i=1}^{n} \lambda_i, n = \sum_{j=1}^{n} \lambda_j', \]

2.

\[
\limsup_{t \to +\infty} \frac{1}{t} \log \rho_n(t) = \limsup_{t \to +\infty} \frac{1}{t} \int_0^t b_{nn}(s) \, ds \\
\leq \lambda(v_n) + (n-1) \pi_{n-1}(\lambda, \mu) \\
\leq \lambda(v_n) + (n-1) \gamma_{n-1}(\lambda, \mu),
\]

\[ (77) \]

\[
\liminf_{t \to +\infty} \frac{1}{t} \log \rho_n(t) = \liminf_{t \to +\infty} \frac{1}{t} \int_0^t b_{nn}(s) \, ds \\
\geq \lambda(v_n) - n \pi_n(\lambda, \mu) \geq \lambda(v_n) - n \gamma_n(\lambda, \mu),
\]

where \( b_{nn}(t) = \langle B(t)u_n, u_n \rangle \), and \( \rho_n(t) \) is the distance from \( v_n(t) \) to the space \( U(t)H_{n-1} \).

**Proof.** The equalities in (76) follow readily from (73). For the first inequality, note that \( \Gamma_n(t) \leq \prod_{j=1}^{n} \| v_j(t) \| \). Since \( v_1, \ldots, v_n \) is a normal basis of \( H_n \) we obtain

\[
\limsup_{t \to +\infty} \frac{1}{t} \log \Gamma_n(t) \leq \sum_{j=1}^{n} \lambda(v_j) = a_n.
\]

For the remaining inequalities in (76), note first that given a basis \( w_1, \ldots, w_n \) of \( H_n \) we have an analogous identity to (73), namely

\[
\frac{\Gamma(w_1(t), \ldots, w_n(t))}{\Gamma(w_1, \ldots, w_n)} = \exp \int_0^t \text{tr}(-B(s)|H_n)^{*} \, ds,
\]

\[ (78) \]

where \( w_1(t), \ldots, w_n(t) \) are the solutions of (14) with \( w_i(0) = w_i \) for each \( i = 1, \ldots, n \). By (73) and (78),

\[
\liminf_{t \to +\infty} \frac{1}{t} \log \Gamma_n(t) = \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \text{tr}(B(s)|H_n) \, ds \\
= - \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \text{tr}(-(B(s)|H_n)^{*}) \, ds \\
= - \limsup_{t \to +\infty} \frac{1}{t} \log \Gamma(w_1(t), \ldots, w_n(t)) \geq - \sum_{j=1}^{n} \mu(w_j).
\]
We now assume that \( w_1, \ldots, w_n \) is a normal basis (with respect to \( \mu \)). Then, using (60),

\[
\liminf_{t \to +\infty} \frac{1}{t} \log \Gamma_n(t) \geq a_n - \sum_{j=1}^{n} (\lambda(v_j) + \mu(w_j)) = a_n - \sum_{j=1}^{n} (\lambda_{j,n}' + \mu_{j,n}') \\
\geq a_n - n \pi_n(\lambda, \mu) \geq a_n - n \gamma_n(\lambda, \mu).
\]

This completes the proof of the first statement.

Observe now that

\[
\frac{\Gamma_n(t)}{\Gamma_{n-1}(t)/\Gamma_{n-1}(0)}.
\]

It follows readily from (74) that

\[
\rho_n(t) = \exp \int_0^t [\text{tr}(B(s)|H_n) - \text{tr}(B(s)|H_{n-1})] \, ds = \exp \int_0^t b_{nn}(s) \, ds,
\]

and we obtain the identities in (77). It follows from (79) and (76) that

\[
\limsup_{t \to +\infty} \frac{1}{t} \log \rho_n(t) \leq \limsup_{t \to +\infty} \frac{1}{t} \log \Gamma_n(t) - \liminf_{t \to +\infty} \frac{1}{t} \log \Gamma_{n-1}(t) \leq \sum_{j=1}^{n} \lambda(v_j) - \sum_{j=1}^{n-1} \lambda(v_j) + (n - 1) \pi_{n-1}(\lambda, \mu) = \lambda(v_n) + (n - 1) \pi_{n-1}(\lambda, \mu) \leq \lambda(v_n) + (n - 1) \gamma_{n-1}(\lambda, \mu),
\]

using (60) in the last inequality. Similarly, we obtain

\[
\liminf_{t \to +\infty} \frac{1}{t} \log \rho_n(t) \geq \liminf_{t \to +\infty} \frac{1}{t} \log \Gamma_n(t) - \limsup_{t \to +\infty} \frac{1}{t} \log \Gamma_{n-1}(t) \geq \sum_{j=1}^{n} \lambda(v_j) - n \pi_n(\lambda, \mu) - \sum_{j=1}^{n-1} \lambda(v_j) = \lambda(v_n) - n \pi_n(\lambda, \mu) \geq \lambda(v_n) - n \gamma_n(\lambda, \mu).
\]

This completes the proof. \( \Box \)

We note that there always exists a basis \( v_1, \ldots, v_n \) as in the statement of Theorem 22: given a normal basis \( v_1, \ldots, v_{n-1} \) of \( H_{n-1} \), we select any vector

\[
v_n \in (E_{k,n} \setminus E_{k-1,n}) \cap (H_n \setminus H_{n-1}),
\]

where \( k \leq p_n \) is the smallest integer, such that \( E_{k,n} \cap (H_n \setminus H_{n-1}) \neq \emptyset \).
The above results allow us to make precise the meaning of our condition (40). It follows readily from (76) and (77) that

\[
\limsup_{t \to +\infty} \frac{1}{t} \log \Gamma_n(t) - \liminf_{t \to +\infty} \frac{1}{t} \log \Gamma_n(t) \leq n \pi_n(\lambda, \mu) \leq n \gamma_n(\lambda, \mu)
\]

and

\[
\limsup_{t \to +\infty} \frac{1}{t} \log \rho_n(t) - \liminf_{t \to +\infty} \frac{1}{t} \log \rho_n(t) \leq (n - 1) \pi_{n-1}(\lambda, \mu) + n \pi_n(\lambda, \mu).
\]  

(80)

In particular, when Eq. (10) is regular, the limits

\[
\lim_{t \to +\infty} \frac{1}{t} \log \Gamma_n(t) \quad \text{and} \quad \lim_{t \to +\infty} \frac{1}{t} \log \rho_n(t)
\]

exist. We note that the existence of these limits is also a consequence of Theorems 20 and 21; for the second limit this can be obtained from the identity

\[
\rho_n(t) = \|v_n(t)\| \cdot |\sin \angle(v_n(t), U(t) H_{n-1})|,
\]

(81)

together with the first and second statements of Theorem 21 applied to the right-hand side of (81). This means that while for a regular equation the angles between solutions can vary at most subexponentially (in view of the second statement in Theorem 21), for a nonregular equation the estimate (80) shows that the angles are now allowed to vary with exponential speed. Nevertheless, this speed will be small compared to the Lyapunov exponents when the Perron coefficients \(\pi_n(\lambda, \mu)\) or the regularity coefficients \(\gamma_n(\lambda, \mu)\) are sufficiently small. This is precisely the meaning of our condition (40): it requires that the eventual variation of the angles has a sufficiently small speed when compared to the Lyapunov exponents.

Note that the estimates in (76) and (77) are given in terms of the Perron coefficients and of the regularity coefficients. We can also obtain analogous estimates for these coefficients in terms of the initial system, or, more precisely, in terms of the associated upper triangular operators \(B(t)\). Set

\[
\beta_i = \liminf_{t \to +\infty} \frac{1}{t} \int_0^t b_{ii} (s) \, ds \quad \text{and} \quad \bar{\beta}_i = \limsup_{t \to +\infty} \frac{1}{t} \int_0^t b_{ii} (s) \, ds.
\]

Theorem 23. We have

\[
\sup_{n \geq 1} \frac{1}{n^2} \sum_{i=1}^{\infty} (\bar{\beta}_i - \beta_i) \leq \gamma(\lambda, \mu) \leq \sum_{i=1}^{\infty} (\bar{\beta}_i - \beta_i).
\]  

(82)
In addition, if (19) holds for some $\delta > 0$, then

$$
\limsup_{n \to \infty} \frac{1}{2n} \sum_{i=1}^{n} \frac{\bar{\beta}_i - \beta_i}{i^2} \leq \pi(\lambda, \mu) \leq \sum_{i=1}^{\infty} (\bar{\beta}_i - \beta_i). \tag{83}
$$

**Proof.** It follows from results in [4] in the finite-dimensional setting that

$$
\gamma_n(\lambda, \mu) \leq \sum_{i=1}^{n} (\bar{\beta}_i - \beta_i).
$$

This readily gives the second inequality in (82), and thus, by Theorem 17, also the second inequality in (83) when (19) holds. By Theorem 22 (see (77)), for each $i \in \mathbb{N}$,

$$
\bar{\beta}_i - \beta_i \leq (i - 1)\gamma_{i-1}(\lambda, \mu) + i\gamma_i(\lambda, \mu)
\leq (2i - 1) \max\{\gamma_i(\lambda, \mu) : i = 1, \ldots, n\}. \tag{84}
$$

Summing over $i$ we obtain

$$
\sum_{i=1}^{n} (\bar{\beta}_i - \beta_i) \leq n^2 \max\{\gamma_i(\lambda, \mu) : i = 1, \ldots, n\}.
$$

This implies the first inequality in (82). To establish the first inequality in (83) note that by (84) and (60),

$$
\bar{\beta}_i - \beta_i \leq (i - 1)^2 \pi_{i-1}(\lambda, \mu) + i^2 \pi_i(\lambda, \mu)
\leq i^2 [\pi_{i-1}(\lambda, \mu) + \pi_i(\lambda, \mu)].
$$

Again by Theorem 17, we have $\pi(\lambda, \mu) = \lim_{n \to \infty} \pi_n(\lambda, \mu)$ when (19) holds, and thus,

$$
\pi(\lambda, \mu) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \pi_i(\lambda, \mu).
$$

Therefore,

$$
\limsup_{n \to \infty} \frac{1}{2n} \sum_{i=1}^{n} \frac{\bar{\beta}_i - \beta_i}{i^2} \leq \lim_{n \to \infty} \frac{1}{2n} \sum_{i=1}^{n} [\pi_{i-1}(\lambda, \mu) + \pi_i(\lambda, \mu)] = \pi(\lambda, \mu).
$$

This completes the proof. \qed
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References