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SMALL COMMUTATORS OF PIECEWISE LINEAR HOMEOMORPHISMS OF THE REAL LINE

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IN THIS PAPER we study the group $PL_c(\mathbf{R})$ of piecewise linear (PL) homeomorphisms of the real line $\mathbf{R}$ with compact support.

This group has been studied by many people. As for the homology of this group, Epstein showed that the group is a perfect group and hence it is a simple group [2]. The lower-dimensional homology of this group was determined by Greenberg [8]. In particular, his result says that the 2-dimensional homology is $\mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R}$ and it is easy to see that the canonical bilinear map $\mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R} \to \mathbf{R}$ is nothing but the discrete Godbillon–Vey class described in [6] and [4]. In [22], we determined all the homology group using Greenberg's description [8] of the classifying space for transversely PL foliations.

Our study on the group $PL_c(\mathbf{R})$ was motivated by an intention to use these results on the homology of $PL_c(\mathbf{R})$ to understand the homology of groups of Lipschitz homeomorphisms of $\mathbf{R}$. In particular, since the (discrete) Godbillon–Vey 2-cocycle for $PL_c(\mathbf{R})$ was completely understood, we tried to understand the Godbillon–Vey 2-cocycle for groups of Lipschitz homeomorphisms of $\mathbf{R}$ using approximations by elements of the group $PL_c(\mathbf{R})$.

The Godbillon–Vey invariant was first defined for codimension-1 C^2 foliations of closed oriented 3-manifolds [7]. This invariant is the only known non-trivial invariant for C^2 foliated cobordism and it varies continuously under the deformation of the foliation [14].

For transversely oriented codimension-1 C^1 foliations, the classifying space for them is contractible [18] and such foliations of closed oriented 3-manifolds are all cobordant. This fact has already been shown for transversely oriented Lipschitz foliations [11]. Hence the Godbillon–Vey invariant cannot be defined for codimension-1 C^1 or Lipschitz foliations. There are, however, a lot of classes of foliations between C^1 or Lipschitz and C^2. In fact, the Godbillon–Vey invariant was extended to the foliations of class $C^{1+\alpha}$ $(\alpha > \frac{1}{2})$ by Hurder and Katok [9], and to the transversely piecewise linear (or piecewise C^2) by Ghys and Sergiescu [6, 4]. We defined the Godbillon–Vey invariant for the foliations of class $C^{L,\mathcal{V}_\beta}$ with $(1 \le \beta < 2)$ [20, 23], which contain both the foliations of class $C^{1+1/\beta}(1/\beta > 1/2)$ and the transversely PL foliations.

Since each cobordism class of the foliations of class $C^{L,\mathcal{V}_\beta}$ $(1 \le \beta < 2)$ of closed oriented 3-manifolds contains a representative which is a 2-cycle for the group $\mathbf{G}_c^{L,\mathcal{V}_\beta}(\mathbf{R})$ of diffeomorphisms of class $C^{L,\mathcal{V}_\beta}$ of the real line $\mathbf{R}$ with compact support [12], the key problem is to understand the Godbillon–Vey invariant on 2-cycles of the group $\mathbf{G}_c^{L,\mathcal{V}_\beta}(\mathbf{R})$.

For a real number β $(\beta \ge 1)$, the group $\mathbf{G}_c^{L,\mathcal{V}_\beta}(\mathbf{R})$ of $C^{L,\mathcal{V}_\beta}$ diffeomorphisms of $\mathbf{R}$ with compact support is defined as follows. An element f of $\mathbf{G}_c^{L,\mathcal{V}_\beta}(\mathbf{R})$ is a Lipschitz homeomorphism with compact support such that $\log f'(x-0)$ is with bounded β-variation (see

Section 1). The β-variation to the $1/\beta$ power gives rise to a right invariant metric on the group $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$. The Godbillon–Vey 2-cocycle is defined for the group $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ $(1 \leq \beta < 2)$, and is continuous with respect to the topology induced by this metric [20]. Note that $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ contains both the group $\mathbf{G}_c^{1+1/\beta}(\mathbf{R})$ of $C^{1+1/\beta}$ diffeomorphisms of $\mathbf{R}$ with compact support and the group $PL_c(\mathbf{R})$, and the Godbillon–Vey 2-cocycle restricted to these groups coincides with the Godbillon–Vey 2-cocycle previously defined.

Now, the group $PL_c(\mathbf{R})$ is dense in $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ (in the topology of $\mathbf{G}_c^{L,\mathscr{V}_{\beta'}}(\mathbf{R})$ $(\beta' > \beta)$ as we show it in Section 6), and the 2-cycles of $PL_c(\mathbf{R})$ are known by Greenberg [8] as was mentioned. In order to understand a 2-cycle of the group $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$, we try to approximate the 2-cycle by a sequence of 2-cycles of the group $PL_c(\mathbf{R})$. In fact, we can approximate a stabilization of the 2-cycle, and in order to make the approximations, it is necessary to have controlled estimates on commutators.

A 2-cycle of $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ is geometrically represented by a $C^{L,\mathscr{V}_\beta}$ foliated $\mathbf{R}$-product with compact support over a closed oriented surface Σ, i.e. a $C^{L,\mathscr{V}_\beta}$ foliation of $\Sigma \times \mathbf{R}$ transverse to the fibers of the projection $\Sigma \times \mathbf{R} \to \Sigma$ which coincides with the product foliation with leaves $\Sigma \times \{*\}$ outside a compact set. This foliated $\mathbf{R}$-product is determined by the holonomy homomorphism $\pi_1(\Sigma) \to \mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$.

If we approximate the holonomies along the usual generators of $\pi_1(\Sigma)$ by PL homeomorphisms, then the product of commutators of them which was originally the identity becomes a PL homeomorphism close to the identity. If we can write it as a product of a fixed number of commutators of PL homeomorphisms close to the identity, then the PL approximations of the holonomies along the generators together with these PL homeomorphisms define an approximation of a stabilization of the original 2-cycle.

Thus, the problem in which we are interested is as follows. We consider the group $PL_c(\mathbf{R})$ with the metric induced from that of $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$. For a subinterval $[A, B]$ of $[0, 1]$, can we write any PL homeomorphisms with support in $[A, B]$ sufficiently close to the identity as a product (composition) of a fixed number of commutators of PL homeomorphisms with support in $[0, 1]$ close to the identity? Note that it is known (see Section 1) that any PL homeomorphism of $\mathbf{R}$ with compact support is written as a product of two commutators of PL homeomorphisms with compact support. But the known commutators are not small. We show in this paper that, with an assumption on the number of non-differentiable points, the answer to the above problem is yes.

More precisely, we show first the following theorem (see Section 3). A piecewise linear homeomorphism of $\mathbf{R}$ with compact support is said to be elementary if it has at most three non-differentiable points.

THEOREM A. *Let β be a real number not less than 1. There exist positive real numbers c and C satisfying the following conditions. Let ε be a positive real number such that $\varepsilon \leq c$. Let f be an elementary piecewise linear homeomorphism of $\mathbf{R}$ with support in $[\frac{1}{8}, \frac{7}{8}]$. Assume that*

$$|||\log f'|||_\beta \leqslant \varepsilon^2.$$

Then f is written as a product (composition) of three commutators of piecewise linear homeomorphisms of $\mathbf{R}$ as follows:

$$f = [g_1, g_2][g_3, g_4][g_5, g_6]$$

where the supports of g_i $(i = 1, \ldots, 6)$ are contained in $[0, 1]$ and

$$|||\log g_i'|||_\beta \leqslant C\varepsilon.$$

This theorem applies to a sequence of PL homeomorphisms with a bounded number of non-differentiable points converging to the identity. When we approximate an element of $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ by a sequence of PL homeomorphisms, we have to treat PL homeomorphisms with an increasing number of non-differentiable points. To treat such PL homeomorphisms, we show the following theorem (see Section 5).

THEOREM B. *Let β be a real number not less than* 1. *Let q be a positive integer and δ a positive real number. There exist positive real numbers c and C satisfying the following conditions. Let ε be a positive real number such that $\varepsilon \le c$. Let f be a piecewise linear homeomorphism of $\mathbf{R}$ with support in $[\frac{1}{4}, \frac{3}{4}]$ such that the number of the non-differentiable points of f is at most $4\varepsilon^{-q} + 2$ and*

$$|||\log f'|||_\beta \leqslant \delta\varepsilon^3.$$

Then

$$f = \prod_{i=1}^{16(q+1)} [g_{2i-1}, g_{2i}]$$

where the supports of g_j $(j = 1, \dots, 32(q+1))$ are contained in $[0, 1]$ *and*

$$|||\log(g_j)|||_\beta \le C\varepsilon.$$

Writing a diffeomorphism close to the identity as a product of a fixed number of commutators of diffeomorphisms close to the identity is important when we work on the 2- or higher-dimensional homology groups of Lipschitz homeomorphism groups. The reason is that we need to treat the cycles whose members are the countable juxtapositions of diffeomorphisms with disjoint supports. If we have a small commutator result, we can show many such cycles are homologous to zero. This idea was first applied to the C^∞ differentiable case to determine certain cobordism classes of foliations [17]. Here is another application. By using Theorem A, we can determine cobordism classes of transversely PL foliations of closed oriented 3-manifolds (see [21, Appendix] for the proof).

THEOREM C [21]. *The foliated cobordism class as foliations of class $C^{L,\mathscr{V}_\beta}$ $(1 \leqslant \beta < 2)$ of transversely oriented transversely piecewise linear foliations of closed oriented* 3-*manifolds is characterized by its* (*discrete*) *Godbillon–Vey class.*

Now we look at 2-cycles of $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$. A 2-cycle of $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ is represented by a $C^{L,\mathscr{V}_\beta}$ foliated $\mathbf{R}$-product $\mathscr{G}$ with compact support over a closed oriented surface Σ. A stabilization of $\mathscr{G}$ is a foliated $\mathbf{R}$-product over the connected sum $\Sigma \# \Sigma'$ with a closed oriented surface Σ' such that the holonomy homomorphism $\pi_1(\Sigma \# \Sigma') \to \mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ factors through the holonomy homomorphism $\pi_1(\Sigma) \to \mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ of $\mathscr{G}$. The 2-cycle represented by a stabilization is homologous to the original 2-cycle.

As for the stable approximation of a 2-cycle of the group $\mathbf{G}_c^{1+\alpha}(\mathbf{R})$ $(0 < \alpha \leqslant 1)$ by 2-cycles of $PL_c(\mathbf{R})$, using Theorem B, we show the following theorem (see Section 6).

THEOREM D. *Let $\mathscr{G}$ be a foliated $\mathbf{R}$-product of class $C^{1+\alpha}$ $(0 < \alpha \le 1)$ with compact support over the closed oriented surface Σ. Let β be a positive real number greater than $1/\alpha$. Then there is a sequence of PL-foliated $\mathbf{R}$-product $\mathscr{G}_k$ over the connected sum $\Sigma \# \Sigma'$ converging to the stabilization of $\mathscr{G}$ in the $C^{L,\mathscr{V}_\beta}$ topology. Here the meaning of convergence is that for any $\gamma \in \pi_1(\Sigma \# \Sigma')$, the holonomy along γ converges. In particular, if $1/\alpha < \beta < 2$, the Godbillon–Vey invariant $GV(\mathscr{G}_k)$ converges to $GV(\mathscr{G})$.*

Note that if $GV(\mathscr{G}) = 0$, then $\mathscr{G}_k$ in Theorem D can be chosen so that $GV(\mathscr{G}_k) = 0$. Then, as a corollary to Theorems C and D, we have the following characterization of the Godbillon–Vey invariant (see [23] for the proof).

THEOREM E [23]. *Let $\mathscr{F}$ be a codimension-1 transversely oriented foliation of class $C^{1+\alpha}$ ($\frac{1}{2} < \alpha \leq 1$) of a closed oriented 3-manifold M. The Godbillon–Vey invariant of $\mathscr{F}$ is zero if and only if $\mathscr{F}$ is foliated cobordant to a codimension-1 transversely oriented foliation $\mathscr{G}$ of class $C^{1+\alpha}$ of a closed oriented 3-manifold N and there exists a sequence $\mathscr{G}_k$ of codimension-1 null-cobordant foliations of class $C^{L,\mathscr{V}_{1/\alpha}}$ of N converging to $\mathscr{G}$ in the $C^{L,\mathscr{V}_\beta}$ topology ($1/\alpha < \beta < 2$). Here, $\mathscr{G}$ is a foliated S^1-product over a surface Σ and the meaning of convergence is that for any $\gamma \in \pi_1(\Sigma)$, the holonomy along γ converges.*

There are generalizations of Theorem D and E for 2-cycles of the group $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ and for foliations of class $C^{L,\mathscr{V}_\beta}$, respectively (see Section 6 and [23]).

We organize this paper as follows.

In Section 1, first we prepare notations to write down PL homeomorphisms. We also review the metric we consider on the group of PL homeomorphisms. Then we write an elementary PL homeomorphism as a commutator following [2]. We also formulate a version of Mather's trick [13] which is the main tool for us. Then we show that any PL homeomorphism of $\mathbf{R}$ with compact support is written as a product of two commutators of PL homeomorphisms with compact support. This implies that the group $PL_c(\mathbf{R})$ is uniformly perfect. This has been well known for specialists and can also be proved by purely algebraic considerations [1]. However, this argument is not sufficient to prove that an elementary PL homeomorphism which is close to the identity is written as a product of commutators of PL homeomorphisms close to the identity.

In Section 2, we establish two technical results. Lemma 2.2 expresses a small elementary PL homeomorphism with small support as a product of two small commutators. Lemma 2.3 improves on this, showing that a small elementary PL homeomorphism with small support can be written as a single commutator of small PL homeomorphisms. Lemma 2.4 extends this result to juxtapositions of small elementary PL homeomorphisms with small support.

In Section 3, we develop several techniques to obtain small commutators. We use Mather's trick and prove Theorem A (Theorem 3.2). Lemma 3.7 extends Theorem 3.2 to juxtapositions of elementary PL homeomorphisms. By Lemma 3.5, a juxtaposition of elementary PL homeomorphisms is written as a product of small commutators and a juxtaposition of elementary PL homeomorphisms with small support. The latter is written as a commutator of small PL homeomorphisms by Lemma 2.4 and we obtain Lemma 3.7. The proof of Theorem B is reduced to Lemma 3.7 by the argument of the following two sections.

Section 4 is the technical heart of this paper, and very rough going. In this section, we develop a technique to express a PL homeomorphism f as a composition $f = f_1[g_1, g_2]$, where f_1 is a juxtaposition of small PL homeomorphisms with smaller numbers of non-differentiable points and $[g_1, g_2]$ is a small commutator. By iterating this process, we can write a PL homeomorphism f as a composition of a fixed number of juxtapositions of small elementary PL homeomorphisms and a product of a fixed number of small commutators. The expression is obtained in two steps: Lemmas 4.1 and 4.2 express PL homeomorphism as a composition of PL homeomorphisms with controlled estimates on the norms and the numbers of non-differentiable points. Lemmas 4.4 and 4.5 use Mather's trick (reformulated in Lemma 4.3) to derive the form $f = f_1[g_1, g_2]$.

In Section 5, we prove Theorem B (Theorem 5.1). We show that if the number of non-differentiable points of a PL homeomorphism f is less than the $(-q)$th power of the distance from f to the identity, then f is written as a fixed number of small commutators. Our proof shows that this number depends linearly on q.

In Section 6, we prove Theorem D (Theorem 6.1). Lemmas 6.4 and 6.6 show that $PL_c(\mathbf{R})$ is dense in $\mathbf{G}_c^{1+1/\beta}(\mathbf{R})$ and $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ in the topology of $\mathbf{G}_c^{L,\mathscr{V}_{\beta'}}(\mathbf{R})$ $(\beta' > \beta)$, respectively. For a $C^{1+\alpha}$ foliated $\mathbf{R}$-product with compact support over a closed oriented surface Σ, we can approximate the holonomies along the usual generators of $\pi_1(\Sigma)$ by PL homeomorphisms, and then we can apply Theorem 5.1 to the product of commutators of PL approximations to show Theorem 6.1. We also prove Theorem 6.5, a generalization of Theorem 6.1 for 2-cycles of $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$, i.e. a 2-cycle of $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ is stably approximated by 2-cycles of $PL_c(\mathbf{R})$ in the topology of $\mathbf{G}_c^{L,\mathscr{V}_{\beta'}}(\mathbf{R})$ $(\beta' > \beta)$.

1. PRELIMINARIES

We first establish our notation for describing the piecewise linear (PL) homeomorphisms of $\mathbf{R}$ with compact support.

For closed intervals $[a_1, a_2]$ and $[b_1, b_2]$, let $L_{[b_1,b_2],[a_1,a_2]}$ be the affine map of $\mathbf{R}$ which sends $[a_1, a_2]$ onto $[b_1, b_2]$. For two finite increasing sequences $a_0 < \cdots < a_k$ and $b_0 < \cdots < b_k$ of real numbers such that $a_0 = b_0$ and $a_k = b_k$, let

$$PL\begin{pmatrix} a_0, \ldots, a_k \\ b_0, \ldots, b_k \end{pmatrix}$$

denote the piecewise linear (PL) homeomorphism of $\mathbf{R}$ which coincides with $L_{[b_{i-1},b_i],[a_{i-1},a_i]}$ on $[a_{i-1}, a_j]$ $(i = 1, \ldots, k)$ and with the identity on $(-\infty, a_0] \cup [a_k, \infty)$.

For a PL homeomorphism f, let $ND(f)$ denote the set of non-differentiable points of f and $NND(f)$, the number of non-differentiable points. Then for $PL\left(\begin{smallmatrix} a_0, \ldots, a_k \\ b_0, \ldots, b_k \end{smallmatrix}\right)$, $NND(f) \leqslant k + 1$. We call a PL homeomorphism f of $\mathbf{R}$ with compact support elementary, if it has at most three non-differentiable points, i.e. $NND(f) \leqslant 3$. By using the above notation, f is written as $PL\left(\begin{smallmatrix} a_0, a_1, a_2 \\ b_0, b_1, b_2 \end{smallmatrix}\right)$, for $a_0 < a_1 < a_2$ and $b_0 < b_1 < b_2$ such that $a_0 = b_0$ and $a_2 = b_2$. It is easy to see that any PL homeomorphism of $\mathbf{R}$ with compact support can be written as a composition of $(NND(f) - 2)$ elementary PL homeomorphisms.

In the group of PL homeomorphisms, we consider the topology induced by the β-norm $|||\cdot|||_\beta$ introduced in [20]. Let β be a real number not less than 1. For a function φ on $\mathbf{R}$ with compact support, we put

$$V_\beta(\varphi) = \sup \sum_{j=1}^{k} |\varphi(x_j) - \varphi(x_{j-1})|^\beta$$

where the supremum is taken over all finite subsets $\{x_0, \ldots, x_k\}$ $(x_0 < \cdots < x_k)$ of $\mathbf{R}$. We call it the β-variation of φ. The functions on $\mathbf{R}$ with compact support whose β-variations are bounded form a normed linear space $\mathscr{V}_\beta$ with respect to the following β-norm $|||\ |||_\beta$:

$$|||\varphi|||_\beta = V_\beta(\varphi)^{1/\beta}.$$

We use the following properties of β-variations frequently in this paper. Note that they are different from obvious consequences of the fact that $|||\ |||_\beta$ is a norm.

PROPOSITION 1.1. *Let $a_0 < \cdots < a_k$ be a finite increasing sequence of real numbers. Let φ be a function on* $\mathbf{R}$ *with compact support.*

(i) *If $\varphi = \sum \varphi_i$ and the support of φ_i is contained in (a_{i-1}, a_i), then*

$$\sum_{i=1}^{k} V_\beta(\varphi_i) \le V_\beta(\varphi).$$

(ii) *If $\varphi = \sum \varphi_i$ and the support of φ_i is contained in $[a_{i-1}, a_i]$, then*

$$V_\beta(\varphi) \le 2^{\beta-1} \sum_{i=1}^{k} V_\beta(\varphi_i).$$

Proof. (i) follows from the definition of V_β noting that the left-hand side can be seen as the supremum of the sum of $|\varphi(x_j) - \varphi(x_{j-1})|^\beta$ over special finite subsets containing $\{a_0, \ldots, a_k\}$. To show (ii), we use the inequality $(|a| + |b|)^\beta \le 2^{\beta-1}(|a|^\beta + |b|^\beta)$ and we replace $|\varphi(x_j) - \varphi(x_{j-1})|^\beta$ by $|\varphi(x_{j-1})|^\beta + |\varphi(x_j)|^\beta$ when $[x_{j-1}, x_j]$ contains some a_i. □

The right invariant metric on the group of PL homeomorphisms with compact support is defined as follows. For a PL homeomorphism f of $\mathbf{R}$ with compact support, let f' denote the left derivative of f. In other words, $f'(x)$ is the limit from the left of the almost everywhere defined derivative of f. For PL homeomorphisms f_1 and f_2 of $\mathbf{R}$ with compact support,

$$\operatorname{dist}(f_1, f_2) = |||\log(f_1 \circ f_2^{-1})'|||_\beta.$$

The group $PL_c(\mathbf{R})$ of PL homeomorphisms of $\mathbf{R}$ with compact support is not a topological group with respect to this topology. However, the "discrete" Godbillon–Vey cocycle is continuous with respect to this topology ($1 \le \beta < 2$). (The metric is of course the one induced from the metric defined on the group $\mathbf{G}_c^{L,\mathcal{V}_\beta}$ of Lipschitz homeomorphisms f with compact support such that $\log f'$ is contained in $\mathcal{V}_\beta$ and the Godbillon–Vey invariant is defined on $\mathbf{G}_c^{L,\mathcal{V}_\beta}$ ($\beta < 2$) and continuous with respect to the topology induced by the metric [20].) The metric we are considering in the group $PL_c(\mathbf{R})$ of PL homeomorphisms of $\mathbf{R}$ with compact support has the following properties.

PROPOSITION 1.2. (i) *For $x \in \mathbf{R}$, $|\log f'(x)| \le 2^{-1/\beta} |||\log f'|||_\beta$.*
(ii) *If $\beta' \le \beta$, then $|||\log f'|||_\beta \le |||\log f'|||_{\beta'}$.*
(iii) *For any $\beta (1 \le \beta)$, $|||\log(f_1 f_2)'|||_\beta \le |||\log f_1'|||_\beta + |||\log f_2'|||_\beta$.*
In other words, $\operatorname{dist}(f_1 \circ f_2, \mathrm{id}) \le \operatorname{dist}(f_1, \mathrm{id}) + \operatorname{dist}(f_2, \mathrm{id})$.
(iv) $|||\log f'|||_\beta$ *is invariant under the conjugation by an affine homeomorphism of* $\mathbf{R}$.

By the right invariance of the metric and (iii) above, the inversion and the composition is continuous at the identity. Such groups are called partial topological groups by Gardiner and Sullivan [3].

We begin the study of commutators by recalling a result from [2].

LEMMA 1.3. *For real numbers $0 < a < b < c < 1$, put*

$$f = PL\begin{pmatrix} 0, a, 1 \\ 0, b, 1 \end{pmatrix} \quad \text{and} \quad g = PL\begin{pmatrix} 0, c, 1 \\ 0, a, 1 \end{pmatrix}.$$

Then

$$[f, g] = fgf^{-1}g^{-1} = PL\begin{pmatrix} g(b), & a, & b \\ g(b), & fgf^{-1}(c), & b \end{pmatrix}$$

where

$$\frac{fgf^{-1}(c) - g(b)}{a - g(b)} = \frac{b(1-a)}{a(1-b)} > 1 \quad and \quad \frac{b - fgf^{-1}(c)}{b - a} = \frac{b(1-c)}{c(1-b)} < 1.$$

This lemma implies that any elementary PL homeomorphism with support in (0, 1) can be written as a commutator.

COROLLARY 1.4. *Any elementary PL homeomorphism h with support in* (0, 1) *can be written as a commutator with support in* [0, 1].

Proof. We can arrange $0 < a < b < c < 1$ so that the two derivatives of the elementary PL homeomorphism h coincide with those of $[f, g]$ or $[g, f]$, and then we can conjugate it by a PL homeomorphism which sends $[g(b), b]$ linearly to the support of h. □

If f and g of Lemma 1.3 are close to the identity, then the support $[g(b), b]$ of the commutator $[f, g]$ is also small. In order to write a PL homeomorphism close to the identity with big support as a product of small commutators, we need other technique. Here, we give the most useful device for us which is due to Mather [13].

PROPOSITION 1.5. *For a positive real number a, let T_a denote the translation on* $\mathbf{R}$ *by a. Let f_i $(i = 1, 2)$ be a PL homeomorphism with support in $[A, B] \subset (0, 1)$ such that $|f_i - \mathrm{id}| < a$. Let h be the PL homeomorphism defined by*

$$h(x) = (T_a f_2)^k (T_a f_1)^{-k}(x)$$

where k is a positive integer such that $(T_a f_1)^{-k}(x) < A$. Then $hT_a(x) = T_a h(x)$ if both x and h(x) belong to $[B, \infty)$. Moreover if h is the identity on $[B, B + a]$, then h is a PL homeomorphism with compact support and

$$f_2 = T_a^{-1} h T_a f_1 h^{-1} = f_1[(T_a f_1)^{-1}, h] = [T_a^{-1}, h] h f_1 h^{-1}.$$

In writing this, one can replace T_a by any PL homeomorphism with support in [0, 1] *which coincides with T_a on $[A, B]$.*

Proof. Since $T_a f_1 = T_a = T_a f_1$ on $(-\infty, A]$, h is well defined. Since $T_a f_2 = T_a = T_a f_1$ on $[B, \infty)$, $hT_a(x) = T_a h(x)$ if both x and $h(x)$ belong to $[B, \infty)$. If we choose a large integer k depending on x, we have

$$T_a f_2 h(x) = (T_a f_2)^{k+1} (T_a f_1)^{-k}(x) = (T_a f_2)^{k+1} (T_a f_1)^{-k-1} T_a f_1(x) = hT_a f_1(x).$$

Thus the proposition follows. □

As a corollary to this proposition, we can show that any PL homeomorphism with compact support is written as a composition of two commutators of PL homeomorphisms with compact support. This implies that the group $PL_c(\mathbf{R})$ of PL homeomorphisms of $\mathbf{R}$ with compact support is uniformly perfect. This has been well known for specialists. This also follows from [1].

THEOREM 1.6. *Let f be a PL homeomorphism of* $\mathbf{R}$ *with compact support. Then there exist four PL homeomorphisms f_1, f_2, f_3, f_4* of $\mathbf{R}$ *with compact support such that*

$$f = [f_1, f_2][f_3, f_4].$$

Sketch of the proof. Assume that f is a composition of elementary PL homeomorphisms f_i $(i = 1, \dots, n)$. We compare it with the juxtaposition of them, i.e. the product g of $(t)^{n-i} f_i(t)^{-n+i}$ $(i = 1, \dots, n)$, where t is an appropriate translation made support compact. Since tf and tg are conjugate by Proposition 1.5 and g is written as a commutator (of juxtapositions) by Corollary 1.4, Theorem 1.6 follows. □

The conclusion of Theorem 1.6 is not suitable for our purposes. The supports of the PL homeomorphism f_i appearing in the commutators $[f_1, f_2]$ $[f_3, f_4]$ may be arbitrarily large even if the support of the PL homeomorphism f is contained in a fixed interval $[A, B] \subset (0, 1)$. If we want the support to be small, then we have to use a conjugation by a PL homeomorphism with big norm. It is not very easy to write a small elementary PL homeomorphism $f = PL\left(\begin{smallmatrix} A, & a, & B \\ A, & b, & B \end{smallmatrix}\right)$ as a composition of small commutators with support in $[0, 1]$. In the next two sections we solve this problem for a small elementary PL homeomorphism f.

2. SMALL ELEMENTARY PL HOMEOMORPHISMS WITH SMALL SUPPORT

We show that a small elementary PL homeomorphism with small support can be written as a product of small commutators. In fact we write it as a single small commutator.

We fix positive real numbers $K \leq 2$ and $\varepsilon_0 \leq 1/2$, such that $|x| \leq \varepsilon_0$ implies

$$|e^x - 1| \leq K|x|, \qquad |\log(1 + x)| \leq K|x| \quad \text{and} \quad |(1 \pm x)^{\pm 1}| \leq K.$$

For example, we can take $K = 2$ and $\varepsilon_0 = \frac{1}{2}$ or K closer to 1 and ε_0 closer to 0.

We begin writing down a small PL translation as a commutator of small PL homeomorphisms.

LEMMA 2.1. *Let $[A, B]$ be a subinterval of $(0, 1)$. Let c, d and u be real numbers such that $A < c < d < d + u < B$. Put*

$$h = PL\begin{pmatrix} A, & c, & d, & B \\ A, & c + u, & d + u, & B \end{pmatrix}.$$

Then there are real numbers a and b such that $0 < a < b < 1$ and

$$h = [f, g] = fgf^{-1}g^{-1}$$

where

$$f = PL\begin{pmatrix} A, & a, & B \\ A, & b, & B \end{pmatrix} \quad \text{and} \quad g = PL\begin{pmatrix} 0, & A, & B, & 1 \\ 0, & c, & d, & 1 \end{pmatrix}.$$

Moreover, for a real number $\beta (1 \geq \beta)$, suppose that there are positive real numbers $\varepsilon_1, \varepsilon_2 \leq \varepsilon_0$ such that

$$|||\log h'|||_\beta \leq \varepsilon_1 \quad \text{and} \quad \max\left\{\frac{c - A}{A}, \frac{B - d}{1 - B}, \frac{c - A + B - d}{B - A}\right\} \leq \varepsilon_2.$$

Then

$$|||\log f'|||_\beta = |||\log h'|||_\beta \leq \varepsilon_1$$

and

$$|||\log g'|||_\beta \leq 2K\left(\frac{B - d}{1 - B} + \frac{c - A}{A} + \frac{c - A + B - d}{B - A}\right) \leq 2{\cdot}3{\cdot}K\varepsilon_2.$$

Proof. Put $a = c/(1 + c - d)$ and $b = (c + u)/(1 + c - d)$. Then the above f and g are well defined and a direct computation shows the lemma. Note that $g(a) = a$. The estimate for f follows from the fact that the derivatives of f coincide with those of h. The estimate for g follows from the following inequality:

$$|||\log g'|||_\beta \leq |||\log g'|||_1 \leq 2\log\left(1 + \frac{B-d}{1-B}\right) + 2\log\left(1 + \frac{c-A}{A}\right)$$
$$+ 2\log\left(1 + \frac{c - A + B - d}{B - A}\right). \qquad \square$$

Here the value of $\beta \geq 1$ is not important. In fact, if the number of non-differentiable points is bounded, the norms are equivalent. For example, we have for above h,

$$2^{1/\beta}\max\left\{\left|\log\left(1 + \frac{u}{c-A}\right)\right|, \left|\log\left(1 - \frac{u}{B-d}\right)\right|\right\} \leq |||\log h'|||_\beta \leq |||\log h'|||_1$$
$$= 2\left|\log\left(1 + \frac{u}{c-A}\right)\right| + 2\left|\log\left(1 - \frac{u}{B-d}\right)\right|.$$

Using Lemma 2.1, we can write an elementary PL homeomorphism with small support as a composition of two small commutators, because such a PL homeomorphism is a composition of two PL homeomorphisms satisfying Lemma 2.1.

LEMMA 2.2. *Let $[A, B]$ be a subinterval of $(0, 1)$. Let $f = PL\left(\begin{smallmatrix} A & a & B \\ A, & b, & B \end{smallmatrix}\right)$ be an elementary PL homeomorphism, where $A < a < b < B$. Suppose that there are positive real numbers $\varepsilon_1, \varepsilon_2 \leq \varepsilon_0$ such that*

$$|||\log f'|||_\beta \leq \varepsilon_1 \quad \textit{and} \quad \frac{4(B - A)}{\min\{A, 1 - B\}} \leq \varepsilon_2.$$

Then f is written as a composition of two commutators as follows:

$$f = [g_1, g_2]\,[g_3, g_4]$$

where the supports of g_1, g_2, g_3, g_4 are contained in $[0, 1]$ and

$$|||\log g_i'|||_\beta \leq \max\{2^2\varepsilon_1, 2^2K\varepsilon_2\} \quad \textit{for} \quad i = 1, 2, 3, 4.$$

If $[A, B] \subset [1/4, 3/4]$, $2^4(B - A) \leq \varepsilon \leq \varepsilon_0$ and $|||\log f'|||_\beta \leq \varepsilon \leq \varepsilon_0$, then $|||\log g_i'|||_\beta \leq 2^2K\varepsilon$.

We have the following improvement of Lemma 2.2, i.e. we can write a small elementary PL homeomorphism with small support as a single commutator.

To obtain Lemma 2.3, we would like to refine Lemma 1.3. Lemma 1.3 itself does not give a good way to write an elementary PL homeomorphism as a small commutator. We can see it as follows. The equation $[f, g] = h$ is equivalent to $fgf^{-1} = hg$. In Lemma 1.3, g is an elementary PL homeomorphism and hg is a PL homeomorphism with 4 non-differentiable points. The two non-differentiable points other than 0 and 1 correspond to one fundamental domain of g. Hence if h is very close to the identity then g is also close to the identity and the support of h becomes too small. We can get through this difficulty by taking a number of fundamental domains of g, and then f becomes complicated.

LEMMA 2.3. *Let $[A, B]$ be a subinterval of $(0, 1)$. Let $h = PL\left(\begin{smallmatrix} A & a & B \\ A, & b, & B \end{smallmatrix}\right)$ be an elementary PL homeomorphism, where $A < a < b < B$. Suppose that there are positive real numbers*

$\varepsilon_1, \varepsilon_2 \leq \varepsilon_0/2^2 3$ *such that*

$$|||\log h'|||_\beta \leq \varepsilon_1 \quad \textit{and} \quad \frac{B-A}{\min\{A, 1-B\}} \leq \varepsilon_2.$$

Then h is written as a commutator as follows:

$$h = [f, g]$$

where the supports of f and g are contained in $[0, 1]$ *and*

$$|||\log f'|||_\beta \leq 2^3 \cdot K^2(\varepsilon_1 + 2\varepsilon_2) \quad \textit{and} \quad |||\log g'|||_\beta = |||\log h'|||_\beta \leq \varepsilon_1.$$

If $[A, B] \subset [\frac{1}{4}, \frac{3}{4}]$, $2^2(B - A) \leq \varepsilon \leq \varepsilon_0/2^2 3$ *and* $|||\log h'|||_\beta \leq \varepsilon \leq \varepsilon_0/2^2 3$, *then* $h = [g_1, g_2]$, *where* $|||\log (g_i)'|||_\beta \leq 2^3 \cdot 3K^2 \varepsilon$ $(i = 1, 2)$.

Proof. We may assume that $b - A \geq (B - A)/2$, for otherwise $B - a \geq (B - A)/2$ and we consider the images under the outer automorphism which sends the homeomorphism f to the homeomorphism $x \mapsto 1 - f(1 - x)$.

Note that

$$\max\left\{\left|\log\frac{b-A}{a-A}\right|, \left|\log\frac{B-b}{B-a}\right|\right\} \leq 2^{-1/\beta}\varepsilon_1$$

implies

$$\frac{b-a}{\min\{a-A, B-b\}} \leq 2^{-1/\beta}K\varepsilon_1$$

hence

$$\frac{b-a}{B-A} \leq 2^{-1-1/\beta}K\varepsilon_1 \leq 2^{-1}K\varepsilon_1.$$

Let k be the minimum positive integer such that

$$1 + \frac{B-A}{k(b-a)} \leq \frac{1}{\varepsilon_2}.$$

Then we put g to be h^{-1} conjugated by the similarity transformation of ratio $1 + (B - A)/k(b - a)$ with center a. In other words, put

$$A' = a - \left(1 + \frac{B-A}{k(b-a)}\right)(a - A)$$

$$B' = a + \left(1 + \frac{B-A}{k(b-a)}\right)(B - a)$$

and

$$b' = a + \left(1 + \frac{B-A}{k(b-a)}\right)(b - a).$$

Then by $(B - A) \leq \varepsilon_2 \min\{A, 1 - B\}$,

$$A' \geq a - \frac{1}{\varepsilon_2}(B - A) \geq a - A \geq 0$$

and

$$B' \le a + \frac{1}{\varepsilon_2}(B - A) \le a + 1 - B \le 1$$

and g is written as follows:

$$g = PL\begin{pmatrix} A', & b', & B' \\ A', & a, & B' \end{pmatrix}.$$

We can write hg easily as follows:

$$hg = PL\begin{pmatrix} A', & g^{-1}(A), & g^{-1}(B), & B' \\ A', & A, & B, & B' \end{pmatrix}.$$

Note that by the choice of k,

$$(hg)^k(B) = A.$$

This implies that g and hg are conjugate by a PL homeomorphism. This is also a version of Mather's trick [13]. In fact, we can explicitly write down f such that $fgf^{-1} = hg$ as follows:

$$f = PL\begin{pmatrix} A', & g^k(a), & g^{k-1}(a), & \dots, & g(a), & a, & B' \\ A', & A, & (hg)^{k-1}(B), & \dots, & hg(B), & B, & B' \end{pmatrix}.$$

Then the derivative of f is estimated as follows:

$$\left|\log\frac{A - A'}{g^k(a) - A'}\right| \le \left|\log\left(1 + \frac{g^k(a) - A'}{A - A'}\right)\right|$$
$$= \left|\log\left(1 + \frac{k(b-a)}{(a-A)(B-A)}(g^k(a) - A)\right)\right|.$$

Here

$$|g^k(a) - A| \le |g^k(a) - a| + |a - A| \le k\frac{B-A}{k} + |a - A| \le 2(B - A).$$

Since k is the smallest integer for the inequality to define it,

$$\frac{(k-1)(b-a)}{B-A} \le \frac{\varepsilon_2}{1-\varepsilon_2} \le K\varepsilon_2.$$

Then $(b-a)/(B-A) \le 2^{-1}K\varepsilon_1$ implies

$$\frac{k(b-a)}{B-A} \le K(\varepsilon_1/2 + \varepsilon_2).$$

Since $b - a \le K\varepsilon_1(a - A)$,

$$a - A = b - A - (b - a) \ge b - A - K\varepsilon_1(a - A).$$

Since we assumed that $b - A \ge (B - A)/2$,

$$a - A \ge \frac{b - A}{1 + K\varepsilon_1} \ge \frac{(B-A)}{2K}.$$

Thus

$$\left|\frac{k(b-a)}{(a-A)(B-A)}(g^k(a) - A)\right| \le K^2(\varepsilon_1 + 2\varepsilon_2)$$

and we have

$$\left|\log \frac{A - A'}{g^k(a) - A'}\right| \leq K^3(\varepsilon_1 + 2\varepsilon_2).$$

Now for $j = 1, \dots, k$,

$$\begin{aligned}\left|\log \frac{(hg)^{j-1}(B) - (hg)^j(B)}{g^{j-1}(a) - g^j(a)}\right| &= \left|\log \frac{(B - A)/k}{(b' - a)\cdot\{(a - A)/(b - A)\}^j}\right| \\ &= \left|\log \frac{(B - A)/k}{(b - a)\cdot(1 + \frac{B-A}{k(b-a)})\cdot\{(a - A)/(b - A)\}^j}\right| \\ &= \left|\log\left(1 + \frac{k(b - a)}{B - A}\right)\cdot\{(a - A)/(b - A)\}^j\right| \\ &= \left|\log\left(1 + \frac{k(b - a)}{B - A}\right)\right| + j\left|\log\{(a - A)/(b - A)\}\right| \\ &\leq K^2(\varepsilon_1/2 + \varepsilon_2) + j\left|\log\left(1 + \frac{b - a}{b - A}\right)\right|.\end{aligned}$$

Since

$$\left|\frac{b - a}{b - A}\right| \leq \frac{2(b - a)}{B - A},$$

$$\begin{aligned}j\left|\log\left(1 + \frac{b - a}{b - A}\right)\right| &\leq j\cdot 2(b - a)\, K/(B - A) \\ &\leq k2(b - a)\, K/(B - A) \\ &\leq K^2(\varepsilon_1/2 + 2\varepsilon_2)\end{aligned}$$

and we have

$$\left|\log \frac{(hg)^{j-1}(B) - (hg)^j(B)}{g^{j-1}(a) - g^j(a)}\right| \leq 3K^2(\varepsilon_1/2 + \varepsilon_2).$$

Finally,

$$\left|\log \frac{B' - B}{B' - a}\right| = \left|\log\left(1 + \frac{k(b - a)}{B - A}\right)\right| \leq K^2(\varepsilon_1/2 + \varepsilon_2).$$

Note that

$$\log \frac{(hg)^{j-1}(B) - (hg)^j(B)}{g^{j-1}(a) - g^j(a)}$$

is positive and increasing in j. Hence

$$|||f|||_\beta \leq |||f|||_1 \leq (2K^3 + 3K^2 + K^2)(\varepsilon_1 + 2\varepsilon_2) \leq 2^3\cdot K^2(\varepsilon_1/2 + \varepsilon_2). \qquad \square$$

For a juxtaposition of elementary PL homeomorphisms with small support we have the following lemma.

LEMMA 2.4. *Let f be a PL homeomorphism with support in $(0, 1)$ such that $|||\log f'|||_\beta \leq \varepsilon_0/2^2 3$. Suppose that the support of f is contained in a disjoint union of N intervals* *of length l and the restriction of f to each interval is elementary. Suppose also that the distance between any two intervals is not smaller than a positive real number L and the distance between an interval and the boundary $\{0, 1\}$ is not smaller than $L/2$, where the positive real number L satisfies $l/L \leq \varepsilon_0/2^3 3$. Then f is written as a commutator*

$$f = [g_1, g_2]$$

where g_i $(i = 1, 2)$ are PL homeomorphisms with support in $[0, 1]$ such that

$$|||\log(g_i)'|||_\beta \leq 2^{5-2/\beta} K^2(|||\log f'|||_\beta + N^{1/\beta}(2^2 l/L))$$

or

$$|||\log(g_i)'|||_\beta \leq 2^3 K^2(|||\log f'|||_\beta + N(2^2 l/L))$$

Proof. Let $[a_k, a_k + l]$ $(k = 1, \ldots, N)$ be the intervals such that $f|[a_k, a_k + l]$ are elementary. We write each restriction $f|[a_k, a_k + l]$ as a commutator with support in $[a_k - L/2, a_k + l + L/2]$ as follows:

$$f|[a_k, a_k + l] = [g_1^{(k)}, g_2^{(k)}].$$

Then by Lemma 2.3,

$$|||\log(g_i^{(k)})'|||_\beta \leq 2^3 K^2(|||\log(f|[a_k, a_k + l])'|||_\beta + 2^2 l/L).$$

Let g_i be the composition of $g_i^{(k)}$. Then by Proposition 1.1,

$$\begin{aligned}
V_\beta(\log(g_i)') &\leq 2^{\beta-1} \sum V_\beta(\log(g_i^{(k)})') \\
&\leq 2^{\beta-1} \sum \{2^3 K^2(|||\log(f|[a_k, a_k + l])'|||_\beta + 2^2 l/L)\}^\beta \\
&\leq 2^{\beta-1} \sum \{2^3 K^2\}^\beta 2^{\beta-1}(V_\beta(\log(f|[a_k, a_k + l])') + (2^2 l/L)^\beta) \\
&\leq 2^{5\beta-2} K^{-2\beta}(V_\beta(\log f') + N(2^2 l/L)^\beta) \\
&\leq 2^{5\beta-2} K^{-2\beta}(|||\log f'|||_\beta + N^{1/\beta}(2^2 l/L))^\beta.
\end{aligned}$$

Hence we obtain the first estimate. The second estimate is just a sum of the norms of $\log(g_i^{(k)})'$. □

3. SMALL ELEMENTARY *PL* HOMEOMORPHISMS

In this section, we write a small elementary PL homeomorphism $f = PL\begin{pmatrix} A; & a; & B \\ A; & b; & B \end{pmatrix}$ as a composition of three small commutators with support in $[0, 1]$ without assuming that $B - A$ is small. To do this we combine Lemma 2.4 and the following lemma.

LEMMA 3.1. *Let $[A, B]$ be a subinterval of $(0, 1)$. For a positive real number $a \leq \min\{A, 1 - B, B - A\}\varepsilon_0/2^2$, let T_a denote the translation on $\mathbf{R}$ by a. Let ε be a positive real number such that $\varepsilon/a \leq \varepsilon_0$. Let f be a PL homeomorphism with support in $[A, B]$ such that $|f - \mathrm{id}| \leq \varepsilon$ and $|||\log f'|||_\beta \leq \varepsilon'$ for a positive real number ε'. Put $x_k = (T_a f)^k(A)$ $(k \geq 0)$. Suppose that $x_n = B$ and $ND(f) \subset \{x_k\}_{k=0,\ldots,n}$. Then f is written as a product of two commutators as follows:*

$$f = [g_1, g_2][g_3, g_4]$$

where g_i $(i = 1, \ldots, 4)$ are PL homeomorphisms with support in $[0, 1]$ such that

$$|||\log(g_i)'|||_\beta \leq \max\left\{2^3 K^2 \frac{a}{\min\{A, 1 - B, B - A\}} + 2^2 K \frac{\varepsilon}{a}, K\varepsilon' \frac{B - A}{a} + 2^2 K \frac{\varepsilon}{a}\right\}.$$

Proof. Note that the above n satisfies

$$n \le (B-A)/(a-\varepsilon) \le K(B-A)/a.$$

Let g be the PL homeomorphism defined by

$$g = PL\begin{pmatrix} x_0, & y_1, & y_{n-1}, & y_n \\ x_0, & y_1+d, & y_{n-1}+d, & y_n \end{pmatrix} = PL\begin{pmatrix} A, & y_1, & y_{n-1}, & B \\ A, & y_1+d, & y_{n-1}+d, & B \end{pmatrix}$$

where $d = (B-A-na)/(n-1)$ and $y_k = A + ka + (k-1)\,d\,(k = 1, \dots, n)$ (hence $y_n = x_n = B$). Let h be the PL homeomorphism defined by

$$h(x) = (T_a f)^n (T_a g)^{-n}(x) \quad \text{for } x \in [A, B]$$

and $h(x) = x$ elsewhere. It is easy to see that the above h satisfies $h|[x_n, x_{n+1}] = \mathrm{id}_{[x_n, x_{n+1}]}$, hence h is a PL homeomorphism with support in $[A, B]$. By Proposition 1.5,

$$f = g[(T_a g)^{-1}, h]$$

and we can replace T_a by t_a given by

$$t_a = PL\begin{pmatrix} 0, & A, & B, & 1 \\ 0, & A+a, & B+a, & 1 \end{pmatrix}$$

where

$$|||\log(t_a)'|||_\beta \le 2^2 Ka/\min\{A, 1-B\}.$$

For g we see that $x_1 = A + a$ and, for $1 \le k \le n$,

$$|x_k - A - ka| = \left| \sum_{i=0}^{k-1} \{x_{k+1} - (x_k + a)\} \right| \le \sum_{i=1}^{k-1} |T_a(f - \mathrm{id})x_k| \le (k-1)\varepsilon.$$

Hence $|d| = |(x_n - A - na)/(n-1)| \le \varepsilon$. Since $\varepsilon/a \le \varepsilon_0$, $|\log g'| \le K\varepsilon/a$. For $|||\log g'|||_\beta$, we see that

$$|||\log g'|||_\beta \le |||\log g'|||_1 \le 2^2 K\varepsilon/a.$$

For h, we see that

$$h|[A, B](x) = (T_a f)^n (T_a g)^{-n}(x) = (T_a f)^n (T_a)^{-n} (T_a)^n (T_a g)^{-n}(x).$$

The distance to the identity of

$$(T_a g)^n (T_a)^{-n} = PL\begin{pmatrix} A+a, & A+an, & A+a(n+1), & y_n+an \\ A+a, & y_n, & y_n+a, & y_n+an \end{pmatrix}$$

is the same as that of g, hence

$$|||\log((T_a g)^n (T_a)^{-n})'|||_\beta \le 2^2 K\varepsilon/a.$$

Since $|||\log f'|||_\beta \le \varepsilon'$,

$$|||\log((T_a f)^n (T_a)^{-n})'|||_\beta \le n|||\log f'|||_\beta \le n\varepsilon'.$$

Thus we obtain

$$|||\log h'|||_\beta \le n\varepsilon' + 2^2 K\varepsilon/a \le K\varepsilon'(B-A)/a + 2^2 K\varepsilon/a.$$

Now by Lemma 2.1, g is written as a commutator

$$g = [g_1, g_2],$$

where

$$|||\log(g_1)'|||_\beta = |||\log g'|||_\beta \leq 2^2K\varepsilon/a$$

and

$$\begin{aligned}|||\log(g_2)'|||_\beta &\leq 2K\left(\frac{a+d}{1-B}+\frac{a}{A}+\frac{2a+d}{B-A}\right)\\ &\leq 2K\left(\frac{a+\varepsilon}{1-B}+\frac{a}{A}+\frac{2a+\varepsilon}{B-A}\right)\\ &\leq 2^3K^2\max\left\{\frac{a}{1-B},\frac{a}{A},\frac{a}{B-A}\right\}.\end{aligned}$$

Thus we have

$$f=[g_1,g_2]\,[g_3,g_4]$$

where $g_3=(T_a g)^{-1}$ and $g_4=h$. These g_1, g_2, g_3, g_4 satisfy the desired estimates. □

Now we prove Theorem A, which says that an elementary PL homeomorphism $PL\left(\begin{smallmatrix}A; & a; & B\\ A; & b; & B\end{smallmatrix}\right)$ close to the identity is written as a product of three commutators of PL homeomorphisms close to the identity. This theorem is used in [21] to show Theorem C of the Introduction.

THEOREM 3.2. *Let $[A, B]$ be a subinterval of $(0, 1)$. Let ε be a positive and real number such that $\varepsilon \leq \min\{A, 1-B, 1/4\}\varepsilon_0/2^2 3$. Let $f = PL\left(\begin{smallmatrix}A; & a; & B\\ A; & b; & B\end{smallmatrix}\right)$ $(A<a<b<B)$ be an elementary PL homeomorphism such that*

$$|||\log f'|||_\beta \leq \varepsilon^2.$$

Then f is written as a composition of three commutators as follows:

$$f=[g_1,g_2]\,[g_3,g_4]\,[g_5,g_6]$$

where the supports of g_i $(i=1,\dots,6)$ are contained in $[0, 1]$ and

$$|||\log g_i'|||_\beta \leq |||\log g_i'|||_1 \leq 2^6K^3\varepsilon/\min\{A, 1-B\}.$$

Therefore, if $[A, B]\subset[1/4, 3/4]$ and $\varepsilon \leq \varepsilon_0/2^2 3$ and $|||\log f'|||_\beta \leq \varepsilon^2$, then $f=[g_1,g_2]\,[g_3,g_4]\,[g_5,g_6]$ with $|||\log g_i'|||_\beta \leq 2^8K^3\varepsilon$.

Proof. We may assume that $B-A\geq\frac{1}{2}$. For otherwise, we replace $[0,1]$ by $[0,1]\cap[A-(B-A)/2,\ B+(B-A)/2]$ using the conjugacy by an affine homeomorphism.

Let T_ε be the translation by ε. Since $|||\log f'|||_\beta\leq\varepsilon^2$, $|\log f'|\leq 2^{-1/\beta}\varepsilon^2(\leq\varepsilon_0)$, we see that $|f'-1|\leq 2^{-1/\beta}K\varepsilon^2\leq K\varepsilon^2$. Since the support of f is $[A, B]\subset[0, 1]$, this implies $|f-\mathrm{id}|\leq K\varepsilon^2/2\leq\varepsilon^2$. Now let us consider the subset $\{x_0,\dots,x_n\}$ of the orbit of the non-differentiable point a under $T_\varepsilon f$, where $x_i=(T_\varepsilon f)^i(x_0)$ $(i=1,\dots,n)$, $a=x_i$ for some i, $A-\varepsilon<x_0\leq A$ and $B\leq x_n<B+\varepsilon$. Then

$$(B-A)/(\varepsilon+\varepsilon^2)\leq n\leq(B-A)/(\varepsilon-\varepsilon^2)+2.$$

Now we put

$$\bar{f}=PL\begin{pmatrix}x_0, & x_1, & \dots, & x_i, & \dots, & x_{n-1}, & x_n\\ x_0, & x_2-\varepsilon, & \dots, & x_{i+1}-\varepsilon, & \dots, & x_n-\varepsilon, & x_n\end{pmatrix}=PL\begin{pmatrix}x_0, & x_1, & a, & x_{n-1}, & x_n\\ x_0, & x_2-\varepsilon, & b, & x_n-\varepsilon, & x_n\end{pmatrix}.$$

Then we see that $\bar{f}$ differs from f only near x_0 and x_n and

$$|||\log \bar{f}'|||_\beta \le |||\log \bar{f}'|||_\beta \le \varepsilon^2.$$

We apply Lemma 3.1 to this PL homeomorphism $\bar{f}$ to obtain

$$\bar{f} = [g_1, g_2][g_3, g_4],$$

Here $g_i (i = 1, \dots, 4)$ are PL homeomorphisms with support in $[0, 1]$ such that

$$|||\log(g_i)|||_\beta \le \max\left\{2^3K^2\frac{\varepsilon}{\min\{A-\varepsilon, 1-B-\varepsilon, B-A\}} + 2^2K\frac{\varepsilon^2}{\varepsilon}, K\varepsilon^2\frac{B-A}{\varepsilon} + 2^2K\frac{\varepsilon^2}{\varepsilon}\right\}$$
$$\le 3^2K^3\varepsilon/\min\{A, 1-B\}.$$

On the other hand, put $\hat{f} = (\bar{f})^{-1}f$. Then the support of $\hat{f}$ is contained in $[x_0, x_1] \cup [x_{n-1}, x_n]$. Note that $x_1 - x_0 = \varepsilon$ and $x_n - x_{n-1} \le \varepsilon$. Moreover

$$\hat{f}|[x_0, x_1] = PL\begin{pmatrix} x_0, & A, & x_1 \\ x_1, & (\bar{f})^{-1}(A), & x_1 \end{pmatrix} \quad \text{and} \quad \hat{f}|[x_{n-1}, x_n] = PL\begin{pmatrix} x_{n-1}, & B, & x_n \\ x_{n-1}, & (\bar{f})^{-1}(B), & x_n \end{pmatrix}$$

where

$$\frac{x_1 - (\bar{f})^{-1}(A)}{x_1 - A}\Big/\frac{(\bar{f})^{-1}(A) - x_0}{A - x_0} = \frac{b-A}{a-A} \quad \text{and} \quad \frac{(\bar{f})^{-1}(B) - x_{n-1}}{B - x_{n-1}}\Big/\frac{x_n - (\bar{f})^{-1}(B)}{x_n - B} = \frac{B-b}{B-a}.$$

Hence

$$\begin{aligned}
&V_\beta(\log(\hat{f}|[x_0, x_1])') + V_\beta(\log(\hat{f}|[x_{n-1}, x_n])') \\
&= \left|\log\frac{x_1 - (\bar{f})^{-1}(A)}{x_1 - A}\right|^\beta + \left|\log\frac{b-A}{a-A}\right|^\beta + \left|\log\frac{x_1 - (\bar{f})^{-1}(A)}{x_1 - A}\right|^\beta \\
&\quad + \left|\log\frac{(\bar{f})^{-1}(B) - x_{n-1}}{B - x_{n-1}}\right|^\beta + \left|\log\frac{B-b}{B-a}\right|^\beta + \left|\log\frac{x_n - (\bar{f})^{-1}(B)}{x_n - B}\right|^\beta \\
&\le 2\left|\log\frac{b-A}{a-A}\right|^\beta + 2\left|\log\frac{B-b}{B-a}\right|^\beta \\
&\le \left|\log\frac{b-A}{a-A}\right|^\beta + \left|\log\frac{B-b}{B-a}\Big/\frac{b-A}{a-A}\right|^\beta + \left|\log\frac{B-b}{B-a}\right|^\beta \\
&= V_\beta(\log f').
\end{aligned}$$

Hence by Proposition 1.1,

$$|||\log(\hat{f})'|||_\beta \le 2^{1-1/\beta}|||\log f'|||_\beta \le 2\varepsilon^2.$$

Now we apply Lemma 2.4 to $\hat{f}$ to obtain

$$\hat{f} = [g_5, g_6]$$

where by the second estimate of Lemma 2.4,

$$\begin{aligned}
|||g_i|||_\beta &\le 2^3K^2(|||\log(\hat{f})'|||_\beta + 2^3\varepsilon/\min\{A, 1-B\}) \\
&\le 2^3K^2(2\varepsilon^2 + 2^3\varepsilon/\min\{A, 1-B\}) \\
&\le 2^6K^3\varepsilon/\min\{A, 1-B\}.
\end{aligned}$$

Thus f is written as a product of three commutators:

$$f = \bar{f}\hat{f} = [g_1, g_2]\,[g_3, g_4]\,[g_5, g_6]$$

where g_i $(i = 1, \dots, 6)$ satisfy the desired estimates. □

The construction of $\bar{f}$ used in the above proof is very useful. We can generalize this as follows.

LEMMA 3.3. *Let f be a PL homeomorphism of $\mathbf{R}$ with support in* $[0, 1]$. *Let* $0 = a_0 < \cdots < a_k = 1$ *be a finite increasing sequence. Put*

$$\bar{f} = PL\begin{pmatrix} a_0, & a_1, & \dots, & a_{k-1}, & a_k \\ f(a_0), & f(a_1), & \dots, & f(a_{k-1}), & f(a_k) \end{pmatrix}.$$

Then $|||\log(\bar{f})'|||_\beta \le |||\log f'|||_\beta$.

Proof. Let $\{x_0, \dots, x_n\}$ $(x_0 < \cdots < x_n)$ be a finite subset of $\mathbf{R}$. In the sum

$$\sum_{j=1}^{n} |\log(\bar{f})'(x_j) - \log(\bar{f})'(x_{j-1})|^\beta$$

we may assume that $\log(\bar{f})'(x_j) - \log(\bar{f})'(x_{j-1})$ has alternating sign. For, if $\log(\bar{f})'(x_j) - \log(\bar{f})'(x_{j-1})$ and $\log(\bar{f})'(x_{j+1}) - \log(\bar{f})'(x_j)$ have the same sign,

$$\begin{aligned} &|\log(\bar{f})'(x_j) - \log(\bar{f})'(x_{j-1})|^\beta + |\log(\bar{f})'(x_{j+1}) - \log(\bar{f})'(x_j)|^\beta \\ &\qquad \le |\log(\bar{f})'(x_{j+1}) - \log(\bar{f})'(x_{j-1})|^\beta \end{aligned}$$

and we get rid of x_j in the following argument. Now if $\log(\bar{f})'(x_{j-1}) \le \log(\bar{f})'(x_j)$ and $\log(\bar{f})'(x_{j+1}) \le \log(\bar{f})'(x_j)$, then in the interval $[a_{i-1}, a_i]$ containing x_j, there exists a point y_j such that $\log(\bar{f})'(x_j) \le \log(\bar{f})'(y_j)$. In a similar way, if $\log(\bar{f})'(x_{j-1}) \ge \log(\bar{f})'(x_j)$ and $\log(\bar{f})'(x_{j+1}) \ge \log(\bar{f})'(x_j)$, then in the interval $[a_{i-1}, a_i]$ containing x_j, there exists a point y_i such that $\log(\bar{f})'(x_j) \ge \log(\bar{f})'(y_j)$. Thus

$$\sum_{j=1}^{n} |\log(\bar{f})'(x_j) - \log(\bar{f})'(x_{j-1})|^\beta \le \sum_{j=1}^{n} |\log \bar{f}(y_j) - \log \bar{f}(y_{j-1})|^\beta.$$

This implies $|||\log(\bar{f})'|||_\beta \le |||\log f'|||_\beta$. □

We generalize Lemma 3.1 to juxtapositions of PL homeomorphisms.

LEMMA 3.4. *Let* $[A, B]$ *be a subinterval of* $(0, 1)$ *containing* $[\frac{1}{4}, \frac{3}{4}]$. *For a positive real number* $a \le \min\{A, 1 - B\}\varepsilon_0/2^2$, *let* T_a *denote the translation on* $\mathbf{R}$ *by* a. *Let* ε *be a positive real number such that* $\varepsilon/a \le \varepsilon_0$. *Let f be a PL homeomorphism with support in* $[A, B]$ *such that* $|f - \mathrm{id}| \le \varepsilon$ *and* $|||\log f'|||_\beta \le \varepsilon'$ *for a positive real number* ε'. *Suppose that for a positive integer* q, *there are points* $x_{1,0} < x_{2,0} < \cdots < x_{r,0}$ *in* $[A, B]$ *and positive integers* $q_1, q_2, \dots, q_r$ *not greater than* q *satisfying the following conditions. If we put* $x_{k,j} = (T_a f)^j (x_{k,0})$ *for* $1 \le j \le q_k (\le q)$, *then*

$$x_{k,q_k} < x_{k+1,0} \quad \textit{and} \quad f|[x_{k,q_k}, x_{k+1,0}] = \mathrm{id}_{[x_{k,q_k}, x_{k+1,0}]} \quad \textit{for} \quad k = 1, \dots, r-1$$

and

$$ND(f) \subset \bigcup_{k=1}^{r} \{x_{k,0}, \dots, x_{k,q_k}\}.$$

Then f is written as a product of two commutators as follows:

$$f = [g_1, g_2][g_3, g_4]$$

where g_i $(i = 1, \ldots, 4)$ *are PL homeomorphisms with support in* $[0, 1]$ *such that*

$$|||\log (g_i)'|||_\beta \le \max\left\{2^3K^2\frac{a}{\min\{A, 1-B\}} + 2^2K\frac{\varepsilon}{a}, 2^{1-1/\beta}q\varepsilon' + 2^2K\frac{\varepsilon}{a}\right\}.$$

Proof. Let n be the positive integer such that

$$B \le (T_a f)^n(x_0) < B + a.$$

Let g be the PL homeomorphism defined by

$$g = PL\begin{pmatrix} 0, & y_1, & y_{n-1}, & y_n \\ 0, & y_1 + d, & y_{n-1} + d, & y_n \end{pmatrix}$$

where

$$d = \frac{1}{n-1}\sum_{k-1}^{r}\{x_{k,q_k} - x_{k,0} - q_k a\}.$$

Let h be the PL homeomorphism defined by

$$h(x) = (T_a f)^n (T_a g)^{-n}(x) \quad \text{for} \quad x \in [A, B]$$

and $h(x) = x$ elsewhere. Since $(T_a f)^{q_k}$ translates $[x_{k,0}, x_{k,1}] = [x_{k,0}, x_{k,0} + a]$ onto $[x_{k,q_k}, x_{k,q_k} + a]$, we see that $h(x)|[y_n, y_{n+1}] = \mathrm{id}_{[y_n, y_{n+1}]}$ and h is a PL homeomorphism with support in $[A, B]$. As in Lemma 3.1, by Proposition 1.5,

$$f = g[(T_a g)^{-1}, h]$$

and we can replace T_a by t_a given by

$$t_a = PL\begin{pmatrix} 0, & A, & B, & 1 \\ 0, & A + a, & B + a, & 1 \end{pmatrix}$$

where $|||\log (t_a)'|||_\beta \le 2^2Ka/\min\{A, 1 - B\}$.

For g, we have $x_{k,q_k} - x_{k,0} - q_k a \le (q_k - 1)\varepsilon$, hence

$$|d| \le \frac{1}{n-1}\sum (q_k - 1)\varepsilon \le \varepsilon$$

and

$$|||\log g'|||_\beta \le |||\log g'|||_1 \le 2^2K\varepsilon/a.$$

For h we have

$$h|[A, B](x) = (T_a f)^n (T_a g)^{-n}(x) = (T_a f)^n (T_a)^{-n} (T_a)^n (T_a g)^{-n}(x)$$

and we again have

$$|||\log ((T_a g)^n (T_a)^{-n})'|||_\beta \le 2^2K\varepsilon/a.$$

For $(T_a f)^n (T_a)^{-n}$, note that $\log((T_a f)^n)' = 0$ on $[x_{k,q_k}, x_{k+1,0}]$ and for $[x_{k,i}, x_{k,i+1}] \subset [x_{k,0}, x_{k,q_k}]$,

$$\log((T_a f)^n)'|[x_{k,i}, x_{k,i+1}] = \log((T_a f|[x_{k,0}, x_{k,q_k}])^q)'|[x_{k,i}, x_{k,i+1}].$$

Hence

$$|||\log((T_a f)^n (T_a)^{-n})'|[x_{k,0}+na, x_{k,q_k}+na]|||_\beta = |||\log((T_a f|[x_{k,0}, x_{k,q_k}])^q)'|||_\beta \leq q|||\log f'|[x_{k,0}, x_{k,q_k}]|||_\beta.$$

Thus by Proposition 1.1,

$$\begin{aligned} V_\beta(\log((T_a f)^n (T_a)^{-n})') &\leq 2^{\beta-1} \sum V_\beta(\log((T_a f)^n (T_a)^{-n})'|[x_{k,0}+na, x_{k,q_k}+na]) \\ &\leq 2^{\beta-1} \sum q^\beta V_\beta(\log f'|[x_{k,0}, x_{k,q_k}]) \\ &\leq 2^{\beta-1} q^\beta V_\beta(\log f'). \end{aligned}$$

We have

$$|||\log h'|||_\beta \leq 2^{1-1/\beta} q\varepsilon' + 2^2 K\varepsilon/a.$$

As in Lemma 3.1, by Lemma 2.1, g is written as a commutator:

$$g = [g_1, g_2]$$

where

$$|||\log(g_1)'|||_\beta = |||\log g'|||_\beta \leq 2^2 K\varepsilon/a$$

and

$$|||\log(g_1)'|||_\beta \leq 2^3 K^2 \max\left\{\frac{a}{1-B}, \frac{a}{A}, \frac{a}{B-A}\right\} \leq 2^3 K^2 a/\min\{A, 1-B\}.$$

Thus we have

$$f = [g_1, g_2][g_3, g_4]$$

where $g_3 = (T_a g)^{-1}$ and $g_4 = h$. These g_1, g_2, g_3, g_4 satisfy the desired estimates. □

Now we look at juxtapositions of elementary *PL* homeomorphisms.

LEMMA 3.5. *Let δ and ε be positive real numbers such that $2\delta\varepsilon \leq 1$ and $\varepsilon \leq \varepsilon_0/2^2 3$. Let f be a PL homeomorphism with support in $[A, B] \subset (0, 1)$ such that $|||\log f'|||_\beta \leq \delta\varepsilon^2$. Suppose that the support of f is contained in a disjoint union of N intervals of length l and the restriction of f to each interval is elementary. Suppose also that the distance between any two intervals is not smaller than a positive real number L, where the positive real number L satisfies $\varepsilon l/L \leq \varepsilon_0/2^3$ and $L/\min\{A, 1-B\} \leq \varepsilon_0/2^2$. Then*

$$f = [g_1, g_2][g_3, g_4][g_5, g_6] f_1$$

where the supports of g_i ($i = 1, \ldots, 6$) are contained in $[0, 1]$ and f_1 is a PL homeomorphism with support in $[A - \varepsilon l, B + L/2]$, the support of f_1 is contained in a disjoint union of $2N$ intervals of length $\varepsilon(1+\varepsilon)l$ and the restriction of f_1 to each interval is elementary. Moreover, the distance between any two intervals and the distance between an interval and the boundary $\{0, 1\}$ are not smaller than a positive real number $L/2 = \varepsilon(1+\varepsilon)l$. We have following estimates on the norms:

$$|||\log f_1'|||_\beta \leq 2^{-1/\beta} |||\log f'|||_\beta \leq 2^{1-1/\beta} \delta\varepsilon^2$$

and

$$|||\log g_i'|||_\beta \leq \max\{2^3 K^3 (l/L)\varepsilon + 2K^2\delta\varepsilon, 2{\cdot}3K^2\delta\varepsilon, 2K^2 L/\min\{A, 1-B\}\}.$$

Proof. Let $[A_k, B_k]$ be an interval where $f|[A_k, B_k]$ is elementary. Put

$$f|[A_k, B_k] = PL\begin{pmatrix} A_k, & a_k, & B_k \\ A_k, & b_k, & B_k \end{pmatrix}.$$

Since $|||\log f'|||_\beta \le \delta\varepsilon^2$, we have $|f' - 1| \le 2^{-1/\beta} K\delta\varepsilon^2$ and $|f - \mathrm{id}| \le 2^{-1-1/\beta} K\delta\varepsilon^2 l < \varepsilon l$. Let $\{x_{k,0}, \dots, x_{k,q_k}\}$ be the subset of the orbit of a_k under $T_{\varepsilon l} f$ which is contained in $(A_k - \varepsilon l, B_k + \varepsilon l)$. Then $A_k - \varepsilon l < x_{k,0} \le A_k$ and $B_k \le x_{k,q_k} < B_k + \varepsilon l$. Hence

$$q_k \le \frac{B_k - A_k}{\varepsilon l - 2^{-1} K\delta\varepsilon^2 l} + 2 \le \frac{1}{\varepsilon - 2^{-1} K\delta\varepsilon^2} + 2 \le 2(\varepsilon^{-1} + 1) \le 2K/\varepsilon.$$

We define

$$\bar{f}|[A_k - \varepsilon l, B_k + \varepsilon l] = PL\begin{pmatrix} x_{k,0}, & x_{k,1}, & a_k, & x_{k,q_k-1}, & x_{k,q_k} \\ x_{k,0}, & x_{k,2} - \varepsilon l, & b_k, & x_{k,q_k} - \varepsilon l, & x_{k,q_k} \end{pmatrix}.$$

Then by Lemma 3.4, we have

$$\bar{f} = [g_1, g_2]\,[g_3, g_4]$$

where

$$\begin{aligned} |||\log(g_i)'|||_\beta &\le \max\left\{2^3 K^2 \frac{\varepsilon l}{L - \varepsilon l} + 2^2 K \frac{2^{-1} K\delta\varepsilon^2 l}{\varepsilon l}, 2^{1-1/\beta}(2K/\varepsilon)\delta\varepsilon^2 + 2^2 K \frac{2^{-1} K\delta\varepsilon^2 l}{\varepsilon l}\right\} \\ &\le \max\{2^3 K^3 (l/L)\varepsilon + 2K^2\delta\varepsilon, 2\cdot 3K^2\delta\varepsilon\}. \end{aligned}$$

Now the support of $\hat{f} = (\bar{f})^{-1} f$ is contained in

$$\bigcup_k ([x_{k,0}, x_{k,1}] \cap [x_{k,q_k-1}, x_{k,q_k}]).$$

Here

$$x_{k,1} - x_{k,0} = \varepsilon l \quad \text{and} \quad x_{k,q_k} - x_{k,q_k-1} \le \varepsilon l(1 + 2^{-1/\beta} K\delta\varepsilon^2) \le \varepsilon l(1 + \varepsilon).$$

Put

$$h_1 = \hat{f}|\bigcup_k [x_{k,0}, x_{k,1}] \quad \text{and} \quad h_2 = \hat{f}|\bigcup_k [x_{k,q_k-1}, x_{k,q_k}].$$

We define f_1 by

$$f_1 = h_1\, T_{L/2 - \varepsilon(1+\varepsilon)l}\, h_2 (T_{L/2 - \varepsilon(1+\varepsilon)l})^{-1}.$$

Here we can replace $T_{L/2 - \varepsilon(1+\varepsilon)l}$ by $t_{L/2 - \varepsilon(1+\varepsilon)l}$ given by

$$t_{L/2 - \varepsilon(1+\varepsilon)l} = PL\begin{pmatrix} 0, & A - \varepsilon(1+\varepsilon)l, & B + \varepsilon(1+\varepsilon)l, & 1 \\ 0, & A + L/2 - 2\varepsilon(1+\varepsilon)l, & B + L/2, & 1 \end{pmatrix}.$$

Then f_1 satisfies the condition on the support. As in the proof of Theorem 3.2,

$$V_\beta(\log(\hat{f}|[x_{k,0}, x_{k,1}])') + V_\beta(\log(\hat{f}|[x_{k,q_k-1}, x_{k,q_k}])') \le V_\beta(\log(f|[A_k, B_k])')$$

and by Proposition 1.1,

$$\begin{aligned} V_\beta(\log(f_1)') &\leq 2^{\beta-1}\sum_k \{V_\beta(\log(\hat{f}|[x_{k,0}, x_{k,1}])') + V_\beta(\log(\hat{f}|[x_{k,q_k-1}, x_{k,q_k}])')\} \\ &\leq 2^{\beta-1}\sum_k V_\beta(\log(f|[A_k, B_k])') \\ &\leq 2^{\beta-1} V_\beta(\log f'). \end{aligned}$$

Hence $|||\log f_1'|||_\beta \leq 2^{1-1/\beta}|||\log f'|||_\beta$. We have

$$\hat{f} = h_1 h_2 = [g_5, g_6] f_1$$

where

$$[g_5 = h_2]$$

$$\begin{aligned} |||\log(g_5)'|||_\beta &\leq |||\log(f_1)'|||_\beta \leq 2^{1-1/\beta}|||\log f'|||_\beta \\ &\leq 2\delta\varepsilon^2 \end{aligned}$$

$$g_6 = t_{L/2-\varepsilon(1+\varepsilon)l}$$

$$\begin{aligned} |||\log(g_6)'|||_\beta &\leq 2^2K(L/2-\varepsilon(1+\varepsilon)l/\min\{A-\varepsilon(1+\varepsilon)l, 1-B-\varepsilon(1+\varepsilon)l\} \\ &\leq 2K^2L/\min\{A, 1-B\}. \end{aligned}$$

Thus we have

$$f = \bar{f}\hat{f} = [g_1, g_2][g_3, g_4][g_5, g_6] f_1$$

where these g_i satisfy the desired estimates. □

Remark. The ratio l/L becomes smaller for f_1. In fact,

$$\frac{\varepsilon(1+\varepsilon)l}{L/2-\varepsilon(1+\varepsilon)l} \leq K^2\varepsilon l/L.$$

COROLLARY 3.6. *Under the same assumption as that of Lemma* 3.5,

$$f = \left(\prod_{i=1}^{3s}[g_{2i-1}, g_{2i}]\right) f_s$$

where f_s is a PL homeomorphism with support in $[A-\varepsilon l/(1-\varepsilon(1+\varepsilon)), B+L]$, the support of f_s is contained in a disjoint union of $2^s N$ intervals of length $\{\varepsilon(1+\varepsilon)\}^s l$ and the restriction of f_s to each interval is elementary and the minimal length between the intervals is not smaller than

$$\frac{L}{2^s}\left\{1-2\varepsilon(1+\varepsilon)\frac{l}{L}\frac{1-(2\varepsilon(1+\varepsilon))^s}{1-2\varepsilon(1+\varepsilon)}\right\} \geq \frac{L}{2^s}\left\{1-2K^2\varepsilon(1+\varepsilon)\frac{l}{L}\right\} \geq L/2^{s+1}.$$

We have the following estimates on the norms:

$$|||\log f_s'|||_\beta \leq 2^{1-1/\beta}|||\log f'|||_\beta \leq 2^{1-1/\beta}\delta\varepsilon^2$$

and

$$|||\log g_i'|||_\beta \leq \max\{2^3K^3(l/L)\varepsilon + 2K^2\delta\varepsilon, 2\cdot 3K^2\delta\varepsilon, 2K^2L/\min\{A, 1-B\}\}.$$

Proof. For the estimate for $\log f_s'$, note that the sum of the β-variations V_β of the derivatives of the elementary PL homeomorphisms which are the restrictions of f_s is not greater than that of f_{s-1}, hence it is not greater than that of f. For the estimates for $\log g_i'$, by the remark above, the estimates become better as s becomes bigger. □

By using Lemma 2.4, we obtain the following lemma.

LEMMA 3.7. *Under the same assumption as that of Lemma* 3.5 *suppose also that s satisfies*

$$(5\varepsilon)^s (l/L) N^{1/\beta} \leq C\varepsilon$$

for some positive real number C, then

$$f = \left(\prod_{i=1}^{3s} [g_{2i-1}, g_{2i}] \right) [g_{6s+1}, g_{6s+2}]$$

where

$$|||\log g_i'|||_\beta \leq \max\left\{ 2^3 K^3 (l/L)\varepsilon + 2K^2 \delta\varepsilon, 2\cdot 3K^2 \delta\varepsilon, 2K^2 \frac{L}{\min\{A, 1-B\}}, 2^{5-2/\beta} K^2 (2\delta\varepsilon^2 + 2^3 C\varepsilon) \right\}.$$

Proof. By Lemma 2.4, f_s in Corollary 3.6 is written as a commutator:

$$f_s = [g_{6s+1}, g_{6s+2}]$$

where

$$|||\log (g_i)'|||_\beta \leq 2^{5-2/\beta} K^2 \left(|||\log f_s'|||_\beta + (2^s N)^{1/\beta} 2^2 \frac{\{\varepsilon(1+\varepsilon)\}^s l}{L/2^{s+1}} \right)$$
$$\leq 2^{5-2/\beta} K^2 (2^{1-1/\beta} \delta\varepsilon^2 + 2^3 (5\varepsilon)^s (l/L) N^{1/\beta}).$$

Here we used $2^{1+1/\beta}(1+\varepsilon) \leq 2^2(1+2^{-2}) \leq 5$. □

4. MODIFICATIONS IN *PL* HOMEOMORPHISMS

In this section, we develop a method to modify *PL* homeomorphisms to the form of juxtaposition of elementary *PL* homeomorphisms by multiplying small commutators.

First we study the decomposition of *PL* homeomorphisms into *PL* homeomorphisms with estimates on the norms and the numbers of non-differentiable points.

LEMMA 4.1. *Let f be a PL homeomorphism of* **R** *with support in* $[0,1]$ *such that* $NND(f) = N + 2$. *Let* $N_1, N_2, \ldots, N_r$ *be positive integers such that* $N_1 + N_2 + \cdots + N_r = N$. *Then there are PL homeomorphisms* $f_1, f_2, \ldots, f_r$ *of* **R** *with support in* $[0, 1]$ *such that*

(i) $f = f_1 \circ \cdots \circ f_r$,
(ii) $NND(f_i) = N_i + 2$, *and*
(iii) $|||\log f_i'|||_\beta \leq 2|||\log f'|||_\beta$ *for* $i = 1, \ldots, r$.

In other words, if f is a composition of N elementary PL homeomorphisms then f is written as a composition of PL homeomorphims f_i which are compositions of N_i elementary PL homeomorphisms with the above estimates.

Proof. The graphs of the *PL* homeomorphisms $f_i \circ \cdots \circ f_r$ $(i = 1, \ldots, r)$ are obtained from the graph of f by connecting several non-differentiable points by line segments. Then by Lemma 3.3, $|||\log (f_i \circ \cdots \circ f_r)'|||_\beta \leq |||\log f_s'|||_\beta$. This gives the desired estimates for f_i. □

The method of decomposition described in the above lemma does not give sufficiently good control on the norms of f_i. The following lemma yields better estimates for a PL homeomorphism f with $NND(f)$ large with respect to the number of PL homeomorphisms to be decomposed.

LEMMA 4.2. *Let f be a PL homeomorphism of $\mathbf{R}$ with support in $[0,1]$ such that $NND(f) = N + 2$. Let $N_0, N_1, \ldots, N_r$ be positive integers such that $N_0 + N_1 + \cdots + N_r = N$ and $N_0 \geq 2r$. Then there are PL homeomorphisms $f_0, f_1, \ldots, f_r$ of $\mathbf{R}$ with support in $[0,1]$ such that*

(i) $f = f_0 \circ \cdots \circ f_r$,
(ii) $NND(f_i) = N_i + 2$ *for* $i = 0, \ldots, r$,
(iii) $|||\log f_0'|||_\beta \leq |||\log f'|||_\beta$,
(iv) *the supports of* f_i $(i = 1, \ldots, k)$ *are contained in disjoint intervals of* $(0, 1)$, *and*
(v) $\sum_{i=1}^r V_\beta(\log f_i') \leq 2^\beta V_\beta(\log f')$.

Proof. Let f be given as

$$f = PL\begin{pmatrix} a_0, a_1, \ldots, a_N, a_{N+1} \\ b_0, b_1, \ldots, b_N, b_{N+1} \end{pmatrix}.$$

We define f_0 to be the PL homeomorphism such that the non-differentiable points are a_0, $a_1, a_{N_1+2}, a_{N_1+3}, a_{N_1+N_2+4}, a_{N_1+N_2+5}, \ldots, a_{N_1+\cdots+N_{r-1}+2(r-1)}, a_{N_1+\cdots+N_{r-1}+2(r-1)+1}, a_{N_1+\cdots+N_r+2r}, a_{N_1+\cdots+N_r+2r+1}, a_{N_1+\cdots+N_r+2r+2}, \ldots, a_N, a_{N+1}$ and the values of these points are those of f:

$$f_0 = PL\begin{pmatrix} a_0, a_1, a_{N_1+2}, a_{N_1+3}, a_{N_1+N_2+4}, a_{N_1+N_2+5}, \ldots, a_{N_1+\cdots+N_{r-1}+2(r-1)}, \\ b_0, b_1, b_{N_1+2}, b_{N_1+3}, b_{N_1+N_2+4}, b_{N_1+N_2+5}, \ldots, b_{N_1+\cdots+N_{r-1}+2(r-1)}, \end{pmatrix}$$

$$\begin{pmatrix} a_{N_1+\cdots+N_{r-1}+2(r-1)+1}, a_{N_1+\cdots+N_r+2r}, a_{N_1+\cdots+N_r+2r+1}, a_{N_1+\cdots+N_r+2r+2}, \ldots, a_N, a_{N+1} \\ b_{N_1+\cdots+N_{r-1}+2(r-1)+1}, b_{N_1+\cdots+N_r+2r}, b_{N_1+\cdots+N_r+2r+1}, b_{N_1+\cdots+N_r+2r+2}, \ldots, b_N, b_{N+1} \end{pmatrix}.$$

Then f_0 coincides with f on $[a_{N_1+\cdots+N_r+2r}, a_{N+1}]$ and the support of $f_0^{-1}f$ is

$$[a_1, a_{N_1+2}] \bigcup [a_{N_1+3}, a_{N_1+N_2+4}] \bigcup \cdots \bigcup [a_{N_1+\cdots+N_{r-1}+2(r-1)+1}, a_{N_1+\cdots+N_r+2r}]$$

and we put

$$f_i = f_0^{-1} f \,|\, [a_{N_1+\cdots+N_{i-1}+2(i-1)+1}, a_{N_1+\cdots+N_i+2i}].$$

Then by Lemma 3.3, we obtain $|||\log f_0'|||_\beta \leq |||\log f'|||_\beta$, hence $|||\log (f_0^{-1}f)'|||_\beta \leq 2|||\log f'|||_\beta$. Thus

$$\sum_{i=1}^r V_\beta(\log f_i') \leq V_\beta(\log (f_0^{-1}f)') \leq 2^\beta V_\beta(\log f'). \qquad \square$$

We try to write a PL homeomorphism f in a form $f = f_1[g_1, g_2]$, where f_1 is a juxtaposition of small PL homeomorphisms with smaller numbers of non-differentiable points, and $[g_1, g_2]$ is a small commutator. So the support of f_1 would be contained in a union of q intervals of length $1/q$ of the interval containing the support of f. We would like to arrange the number of non-differentiable points of the restriction of f_1 to each interval to be the $1/q$ of that of f. We use the following Lemma 4.3 when the number of non-differentiable points of the restriction to each interval is different to each other. The situation of Lemma 4.3 is that there are r boxes and the kth box has q small boxes and p_k homeomorphisms and that we exchange homeomorphisms between small boxes in each big box. This is an easy application of Proposition 1.5.

Lemma 4.3. *Let q, r and $p_1, \ldots, p_r$ be positive integers. For a positive real number a, let T_a denote the translation by a. Let $[A_k, A_k + qa]$ $(k = 1, \ldots, r)$ be disjoint subintervals of $[A, B] \subset (0, 1)$. Let $f_i^{(k)}$ $(k = 1, \ldots, r,\ i = 1, \ldots, p_k)$ be PL homeomorphisms with support in $[A_k, A_k + a]$. Let $j_m^{(k)}: \{1, \ldots, p_k\} \to \{0, \ldots, q-1\}$ $(k = 1, \ldots, r,\ m = 1, 2)$ be functions such that $j_m^{(k)}(i) \geq j_m^{(k)}(i')$ for $i \leq i'$. For $m = 1, 2$, put*

$$F_m = \prod_k T_a^{j_m^{(k)}(1)} f_1^{(k)} T_a^{-j_m^{(k)}(1)} \circ \cdots \circ T_a^{j_m^{(k)}(p_k)} f_{p_k}^{(k)} T_a^{-j_m^{(k)}(p_k)}.$$

Then

$$T_a F_2 = h T_a F_1 h^{-1}$$

where

$$h | I_k = (T_a F_2 | I_k)^q (T_a F_1 | I_k)^{-q} \quad \text{for} \quad I_k = [A_k, A_k + qa] \quad (k = 1, \ldots, r)$$

and h is the identity on $\mathbf{R} - \bigcup [A_k, A_k + qa]$. Here T_a can be replaced by any PL homeomorphism t_a which coincides with T_a on $[A, B]$. Moreover, we have

$$|||\log h'|||_\beta \leq 2^{2-2/\beta} q(|||\log (F_1)'|||_\beta + |||\log (F_2)'|||_\beta).$$

Proof. The first part of the lemma follows from Proposition 1.5. Since $I_k = [A_k, A_k + qa]$ are disjoint, by Proposition 1.1, we have

$$\begin{aligned}
V_\beta(\log h') &\leq \sum_{k=1}^{r} 2^{\beta-1} V_\beta(\log h' | I_k) \\
&= \sum_{k=1}^{r} 2^{\beta-1} (|||\log h' | I_k|||_\beta)^\beta \\
&= \sum_{k=1}^{r} 2^{\beta-1} (q|||\log F_1' | I_k|||_\beta + q|||\log F_2' | I_k|||_\beta)^\beta \\
&\leq \sum_{k=1}^{r} 2^{\beta-1} q^\beta 2^{\beta-1} (V_\beta(\log F_1' | I_k) + V_\beta(\log F_2' | I_k)) \\
&\leq 2^{2\beta-2} q^\beta (V_\beta(\log F_1') + V_\beta(\log F_2')) \\
&\leq 2^{2\beta-2} q(|||\log F_1'|||_\beta + |||\log F_2'|||_\beta)^\beta.
\end{aligned}$$

Thus we obtain the estimate. □

In the following two lemmas, we express a *PL* homeomorphism f in the form $f = f_1 [g_1, g_2]$, where f_1 is a juxtaposition of small *PL* homeomorphisms with smaller numbers of non-differentiable points, and $[g_1, g_2]$ is a small commutator.

Lemma 4.4. *Let p and q be positive integers. For a positive real number a, let T_a denote the translation by a. Let f be a PL homeomorphism with support in the disjoint union $\bigcup_k \bigcup_{i=1}^q (A_k + (i-1)a, A_k + ia)$ of intervals and $\bigcup [A_k, A_k + qa] \subset [A, B] \subset (0, 1)$. Suppose that $NND(f | [A_k, A_k + qa]) \leq qN + 2$ for $k = 1, \ldots, r$. Then there are PL homeomorphisms f_1, g_1*

and g_2 *such that*

$$f = f_1 [g_1, g_2]$$

and

$$NND(f_1 | [A_k + (i-1)a, A_k + ia]) \le N + 2 \quad (k = 1, \ldots, r; i = 1, \ldots, q).$$

Moreover, if $|||\log f'|||_\beta \le \varepsilon$, *then*

$$|||\log (f_1)'|||_\beta \le 2^{2-1/\beta} q\varepsilon$$

and

$$|||\log (g_i)'|||_\beta \le \max\{2^2 q(1 + 2^2 q)\varepsilon, 2^2 Ka/\min\{A, 1 - B\} + 2^2 q\varepsilon\} \quad (i = 1, 2).$$

Proof. Note that $f(A_k + ia) = A_k + ia$ $(i = 0, \ldots, q)$ and let $f | [A_k + (i-1)a, A_k + ia]$ be a composition of $\lambda_i^{(k)}$ elementary PL homeomorphisms. Then

$$\sum_{i=1}^{q} \lambda_i^{(k)} \le qN.$$

We can decompose $\lambda_i^{(k)}$ into a sum of q non-negative integers

$$\lambda_i^{(k)} = \mu_1^{(k,i)} + \cdots + \mu_q^{(k,i)}$$

and decompose N into a sum of q non-negative integers

$$N = \nu_1^{(k,i)} + \cdots + \nu_q^{(k,i)}$$

satisfying the condition that the sequences

$$\mu_1^{(k,q)}, \ldots, \mu_q^{(k,q)}, \mu_1^{(k,q-1)}, \ldots, \mu_q^{(k,q-1)}, \ldots, \mu_1^{(k,1)}, \ldots, \mu_q^{(k,1)}$$

and

$$\nu_1^{(k,q)}, \ldots, \nu_q^{(k,q)}, \nu_1^{(k,q-1)}, \ldots, \nu_q^{(k,q-1)}, \ldots, \nu_1^{(k,1)}, \ldots, \nu_q^{(k,1)}$$

coincide after deleting zero terms.

This decomposition can always be done.

Now, by Lemma 4.1, $f | [A_k + (i-1)a, A_k + ia]$ is decomposed as follows:

$$f | [A_k + (i-1)a, A_k + ia] = f_1^{(k,i)} \ldots f_q^{(k,i)}$$

where $f_j^{(k,i)}$ is a product of $\mu_j^{(k,i)}$ elementary PL homeomorphisms. By Lemma 4.1,

$$|||\log (f_j^{(k,i)})'|||_\beta \le 2|||\log (f | [A_k + (i-1)a, A_k + ia])'|||_\beta.$$

Note that by Proposition 1.1,

$$\sum_{k,i} V_\beta(\log (f | [A_k + (i-1)a, A_k + ia])') \le V_\beta(\log f').$$

We can write f in the form of Lemma 4.3:

$$f = \prod_k T_a^{q-1} (T_a^{-q+1} f_1^{(k,q)} T_a^{q-1}) T_a^{-q+1} \ldots T_a^{q-1} (T_a^{-q+1} f_q^{(k,q)} T_a^{q-1}) T_a^{-q+1} \ldots f_1^{(k,1)} \ldots f_q^{(k,1)}.$$

By the condition above the sequence

$$T_a^{-q+1} f_1^{(k,q)} T_a^{q-1}, \ldots, T_a^{-q+1} f_q^{(k,q)} T_a^{q-1}, \ldots, f_1^{(k,1)}, \ldots, f_q^{(k,1)}$$

can be regrouped into

$$g_1^{(k,q)}, \ldots, g_q^{(k,q)}, g_1^{(k,q-1)}, \ldots, g_q^{(k,q-1)}, \ldots, g_1^{(k,1)}, \ldots, g_q^{(k,1)}$$

by subtracting and adding the identities, where $g_j^{(k,q)}$ is a product of $\nu_j^{(k,q)}$ elementary PL homeomorphisms.

Put

$$f_1 = \prod_k T_a^{q-1} g_1^{(k,q)} T_a^{-q+1} \dots T_a^{q-1} g_q^{(k,q)} T_a^{-q+1} \dots g_1^{(k,1)} \dots g_q^{(k,1)}.$$

Then by Lemma 4.3, we have

$$T_a f_1 = h T_a f h^{-1},$$

and we can replace T_a by t_a defined by

$$t_a = PL\begin{pmatrix} 0, & A, & B, & 1 \\ 0, & A+a, & B+a, & 1 \end{pmatrix}$$

where $|||\log(t_a)'|||_\beta \le 2^2 Ka/\min\{A, 1-B\}$.

We have the following estimate on f_1:

$$\begin{aligned} V_\beta(\log(f_1)') &\le \sum_{k,i} 2^{\beta-1} V_\beta(\log(f_1 \mid [A_k + (i-1)a, A_k + ia])') \\ &\le \sum_{k,i} 2^{\beta-1} V_\beta\left(\log\left(\prod_i g_j^{(k,i)}\right)'\right) \\ &\le \sum_{k,i} 2^{\beta-1} \left(\sum_j |||\log(g_j^{(k,i)})'|||_\beta\right)^\beta \\ &\le \sum_{k,i} 2^{\beta-1} q^{\beta-1} \sum_j V_\beta(\log(g_j^{(k,i)})') \\ &\le 2^{\beta-1} q^{\beta-1} \sum_{k,i,j} V_\beta(\log(f_j^{(k,i)})') \\ &\le 2^{\beta-1} q^{\beta-1} \sum_{k,i,j} 2^\beta V_\beta(\log(f \mid [A_k + (i-1)a, A_k + ia])') \\ &\le 2^{2\beta-1} q^\beta \sum_{k,i} V_\beta(\log(f \mid [A_k + (i-1)a, A_k + ia])') \\ &\le 2^{2\beta-1} q^\beta V_\beta(\log f'). \end{aligned}$$

Thus

$$|||\log(f_1)'|||_\beta \le 2^{2-1/\beta} q |||\log f'|||_\beta.$$

We have

$$f = t_a^{-1} h^{-1} t_a f_1 h = f_1 [(t_a f_1)^{-1}, h^{-1}]$$

where

$$|||\log h'|||_\beta \le 2^{2-2/\beta} q(|||f'|||_\beta + |||f_1|||_\beta) \le 2^2 q(1 + 2^2 q)\varepsilon$$

and

$$|||\log(t_a f_1)'|||_\beta \le 2^2 Ka/\min\{A, 1-B\} + 2^2 q\varepsilon. \qquad \square$$

If the number of non-differentiable points is big with respect to the number of pieces to be decomposed, then we can use Lemma 4.2.

LEMMA 4.5. *Let p and q be positive integers. For a positive real number a, let T_a denote the translation by a. Let f be a PL homeomorphism with support in the disjoint union $\bigcup [A_k, A_k + qa]$ of intervals and $\bigcup [A_k, A_k + qa] \subset [A, B] \subset (0, 1)$. Suppose that $f(A_k + ia) = A_k + ia$ $(i = 0, \dots, q)$ and $NND(f \mid [A_k, A_k + qa]) \le qN + 2$ for $k = 1, \dots, r$,* *where $N \geq 4q$. Then there are PL homeomorphisms f_1, g_1 and g_2 such that*

$$f = f_1[g_1, g_2]$$

and

$$NND(f_1|[A_k + (i-1)a, A_k + ia]) \leq N + 2 \quad (k = 1, \ldots, r;\ i = 1, \ldots, q).$$

Moreover, if $|||\log f'|||_\beta \leq \varepsilon$, then

$$|||\log(f_1)'|||_\beta \leq (1 + 2^{2-1/\beta}3^{1-1/\beta})\varepsilon$$

and

$$|||\log(g_i)'|||_\beta \leq \max\{2^3 7q\varepsilon, 2^2 3\varepsilon + 2^2 Ka/\min\{A, 1 - B\}\} \quad (i = 1, 2).$$

Proof. Let $f|[A_k + (i-1)a, A_k + ia]$ be a composition of $\lambda_i^{(k)}$ elementary PL homeomorphisms. Then

$$\sum_{i=1}^{q} \lambda_i^{(k)} \leq qN.$$

We can decompose $\lambda_i^{(k)}$ into a sum of $q + 1$ non-negative integers

$$\lambda_i^{(k)} = \mu_1^{(k,i)} + \cdots + \mu_q^{(k,i)}$$

and decompose N into a sum of 4 non-negative integers

$$N = \nu_0^{(k,i)} + \nu_1^{(k,i)} + \nu_2^{(k,i)} + \nu_3^{(k,i)}$$

satisfying the following conditions:

1. If $\lambda_i^{(k)} \leq N$, then $\mu_0^{(k,i)} = \lambda_i^{(k)}$, $\mu_1^{(k,i)} = \cdots = \mu_q^{(k,i)} = 0$.
   If $\lambda_i^{(k)} > N$, then $\mu_0^{(k,i)} = 2q$.
2. The sequences

$$\mu_1^{(k,q)}, \ldots, \mu_q^{(k,q)}, \mu_1^{(k,q-1)}, \ldots, \mu_q^{(k,q-1)}, \ldots, \mu_1^{(k,1)}, \ldots, \mu_q^{(k,1)}$$

and

$$\nu_1^{(k,q)}, \nu_2^{(k,q)}, \nu_3^{(k,q)}, \nu_1^{(k,q-1)}, \nu_2^{(k,q-1)}, \nu_3^{(k,q-1)}, \ldots, \nu_1^{(k,1)}, \nu_2^{(k,1)}, \nu_3^{(k,1)}$$

coincide after deleting zero terms.

This decomposition can always be done by the following reason. Since $N \geq 4q$, if $\lambda_i^{(k)} > N$, then $\lambda_i^{(k)} - 2q \geq N - 2q \geq N/2$. We fill up the demands of the parts where $\lambda_i^{(k)} < N$ by the supplies of the part where $\lambda_i^{(k)} > N$ from the larger index i, then the demand of one part is supplied by at most 3 parts.

Now, by Lemma 4.2, $f|[A_k + (i-1)a, A_k + ia]$ is decomposed as follows:

$$f|[A_k + (i-1)a, A_k + ia] = f_0^{(k,i)} f_1^{(k,i)} \ldots f_q^{(k,i)}$$

where $f_j^{(k,i)}$ is a product of $\lambda_j^{(k,i)}$ elementary PL homeomorphisms. Note that the supports of $f_1^{(k,q)}, \ldots, f_q^{(k,q)}, \ldots, f_1^{(k,1)}, \ldots, f_q^{(k,1)}$ are contained in disjoint intervals.

First we decompose f as follows:

$$f = f_0 \circ F_1$$

where $f_0|[A_k + a(i-1), A_k + ai] = f_0^{(k,i)}$ and $F_1|[A_k + a(i-1), A_k + ai] = f_1^{(k,i)} \ldots f_q^{(k,i)}$. Then by Lemma 4.2,

$$|||\log(f_0)'|||_\beta \leq |||\log f'|||_\beta \quad \text{and} \quad \sum_{k,i,j} V_\beta(\log(f_j^{(k,i)})') \leq 2^\beta V_\beta(\log f').$$

We can also write F_1 in the form of Lemma 4.3:

$$F_1 = \prod_k T_a^{q-1}(T_a^{-q+1} f_1^{(k,q)} T_a^{q-1}) T_a^{-q+1} \dots T_a^{q-1}(T_a^{-q+1} f_q^{(k,q)} T_a^{q-1}) T_a^{-q+1} \dots f_1^{(k,1)} \dots f_q^{(k,1)}.$$

By the condition (2) the sequence

$$T_a^{-q+1} f_1^{(k,q)} T_a^{q-1}, \dots, T_a^{-q+1} f_q^{(k,q)} T_a^{q-1}, \dots, f_1^{(k,1)}, \dots, f_q^{(k,1)}$$

can be regrouped into

$$g_1^{(k,q)}, g_2^{(k,q)}, g_3^{(k,q)}, g_1^{(k,q-1)}, g_2^{(k,q-1)}, g_3^{(k,q-1)}, \dots, g_1^{(k,1)}, g_2^{(k,1)}, g_3^{(k,1)}$$

by subtracting and adding the identities, where $g_j^{(k,q)}$ is a product of $v_j^{(k,q)}$ elementary PL homeomorphisms.

Put

$$F_2 = T_a^{q-1} g_1^{(k,q)} T_a^{-q+1} T_a^{q-1} g_2^{(k,q)} T_a^{-q+1} T_a^{q-1} g_3^{(k,q)} T_a^{-q+1} \dots g_1^{(k,1)} g_2^{(k,1)} g_3^{(k,1)}.$$

Then by Lemma 4.3, we have

$$T_a F_2 = h T_a F_1 h^{-1}$$

and we can replace T_a by t_a defined by

$$t_a = PL\begin{pmatrix} 0, & A, & B, & 1 \\ 0, & A+a, & B+a, & 1 \end{pmatrix}$$

where $|||\log(t_a)'|||_\beta \le 2^2 Ka/\min\{A, 1-B\}$ as before.

We have the following estimate on F_2:

$$\begin{aligned}
V_\beta(\log(F_2)') &\le \sum_{k,i} 2^{\beta-1} V_\beta(\log(F_2 \mid [A_k + (i-1)a, A_k + ia])') \\
&\le \sum_{k,i} 2^{\beta-1} V_\beta(\log(g_1^{(k,i)} g_2^{(k,i)} g_3^{(k,i)})') \\
&\le \sum_{k,i}^{r} 2^{\beta-1} (|||\log(g_1^{(k,i)})'|||_\beta + |||\log(g_2^{(k,i)})'|||_\beta + |||\log(g_3^{(k,i)})'|||_\beta)^\beta \\
&\le \sum_{k,i} 2^{\beta-1} 3^{\beta-1} (V_\beta(\log(g_1^{(k,i)})') + V_\beta(\log(g_2^{(k,i)})') + V_\beta(\log(g_3^{(k,i)})')) \\
&\le 2^{\beta-1} 3^{\beta-1} \sum_{k,i,j} V_\beta(\log(f_j^{(k,i)})') \\
&\le 2^{\beta-1} 3^{\beta-1} 2^\beta V_\beta(\log f').
\end{aligned}$$

Thus

$$|||\log(F_2)'|||_\beta \le 2^{2-1/\beta} 3^{1-1/\beta} |||\log f'|||_\beta.$$

Since $|||\log(F_1)'|||_\beta \le 2|||\log f'|||_\beta \le 2\varepsilon$,

$$|||\log h'|||_\beta \le 2^{2-2/\beta} q(|||F_1|||_\beta + |||F_2|||_\beta) \le 2^3 7q\varepsilon.$$

Hence by putting $f_1 = f_0 F_2$,

$$f = f_0 F_1 = f_0 t_a^{-1} h^{-1} t_a F_2 h = f_1[(t_a F_2)^{-1}, h^{-1}].$$

Here

$$|||\log(f_1)'|||_\beta \le (1 + 2^{2-1/\beta})\varepsilon \quad \text{and} \quad |||\log(t_a F_2)'|||_\beta \le 2^2 Ka/\min\{A, 1-B\} + 2^2 3\varepsilon. \qquad \square$$

5. MAIN THEOREM

In this section, we prove Theorem B. Let K and ε_0 be the positive real numbers fixed in the beginning of Section 2.

THEOREM 5.1. *Let q be a positive integer. Let ε be a positive real number such that $\varepsilon \leq \min\{A,\ 1-B\}\,\varepsilon_0/2^3$. Let f be a PL homeomorphism of* $\mathbf{R}$ *with support in $[A, B]$ ($[\frac{1}{4}, \frac{3}{4}] \subset [A, B] \subset (0, 1)$) such that*

$$|||\log f'|||_\beta \leq \delta\varepsilon^3 \text{ and } NND(f) \leq 4\varepsilon^{-q} + 2$$

where $2^{2q+6}(1 + 2^{2-1/\beta}3^{1-1/\beta})^q \delta\varepsilon \leq 1$. Then

$$f = \prod_{i=1}^{16(q+1)} [g_{2i-1}, g_{2i}]$$

where the support of g_i ($i = 1, \ldots, 32(q+1)$) is contained in $[0, 1]$ and there exists a positive real number c depending only on q, A, B, and δ such that

$$|||\log g_i'|||_\beta \leq c\varepsilon.$$

The following Lemmas 5.2 and 5.3 reduce Theorem 5.1 to Lemma 3.7.

LEMMA 5.2. *Let ε be a positive real number such that $\varepsilon \leq \min\{A, 1-B\}\,\varepsilon_0/2^3$. Let f_m be a PL homeomorphism of* $\mathbf{R}$ *with support in $[A, B]$ ($[\frac{1}{4}, \frac{3}{4}] \subset [A, B] \subset (0, 1)$) satisfying the following conditions:*

1. *$|||\log f_m'|||_\beta \leq \delta_m\varepsilon^3$, where $\delta_m \geq 1$ and $2^6\delta_m\varepsilon \leq 1$,*
2. *the support of f_m is contained in a disjoint union of intervals of length $(3\varepsilon/4)^m l$ ($l = B - A$),*
3. *the restriction of f_m to each interval has at most $4\varepsilon^{m-q} + 2$ ($q - m \geq 2$) non-differentiable points, and*
4. *the minimal distance between the intervals is at least $(3\varepsilon/4)^m l/3$.*

Then f_m is written as follows:

$$f_m = [g_1, g_2][g_3, g_4][g_5, g_6][g_7, g_8] f_{m+1}.$$

Here

(i) *the supports of g_i ($i = 1, \ldots, 8$) are contained in $[0, 1]$,*

(ii) *the support of f_{m+1} is contained in a disjoint union of intervals of length $(3\varepsilon/4)^{m+1}$ which is contained in $[A - (3\varepsilon/4)^{m+1}, B + (3\varepsilon/4)^{m+1}]$,*

(iii) *the restriction of f_{m+1} to each interval has at most $4\varepsilon^{m-q+1} + 2$ non-differentiable points, and*

(iv) *the minimal distance between the intervals is at least $(3\varepsilon/4)^{m+1} l/3$.*

We have following estimates on the norms:

$$|||\log(f_{m+1})'|||_\beta \leq 2^2(1 + 2^{2-1/\beta}3^{1-1/\beta})\delta_m\varepsilon^3$$

and

$$|||\log g_i'|||_\beta \leq \max\left\{2^3K^3\frac{(\varepsilon/2)(3\varepsilon/4)^m l}{\min\{A,\ 1-B\}} + 2^2K^2\delta_m\varepsilon^2,\ 2^5 7K^2\delta_m\varepsilon^2\right\}.$$

Proof. Put $x_i = (T_a f)^i(A)$ ($i = 0, \ldots, i$), where

$$a = (\varepsilon/2)(3\varepsilon/4)^m l$$

and $B \le x_n < B + a$. Let $\bar{f}$ be the PL homeomorphism given by

$$\bar{f} = PL\begin{pmatrix} x_0, & x_1, & \dots, & x_{n-1}, & x_n \\ f(x_0), & f(x_1), & \dots, & f(x_{n-1}), & f(x_n) \end{pmatrix}.$$

Then by Lemma 3.3, we have

$$|||\log(\bar{f})'|||_\beta \le |||\log f'|||_\beta \le \delta_m \varepsilon^3.$$

Put $\hat{f} = (\bar{f})^{-1} f$, then we have $|||\log(\hat{f})'|||_\beta \le 2\delta_m \varepsilon^3$. The supports of $\bar{f}$ and $\hat{f}$ are contained in a disjoint union of intervals I_k of length

$$(3\varepsilon/4)^m l + 2a \le (1+\varepsilon)(3\varepsilon/4)^m l \le K(3\varepsilon/4)^m l = 2aK/\varepsilon.$$

The minimal distance between the intervals is at least

$$(3\varepsilon/4)^m l/3 - 2a \ge (1/(3\varepsilon) - 1)2a \ge 2^5 3a.$$

By Lemma 3.4, $\bar{f}$ is written as a product of two commutators:

$$\bar{f} = [g_1, g_2]\,[g_3, g_4].$$

Since $\log f' \le 2^{-1/\beta} |||\log f'|||_\beta \le 2^{-1/\beta}\delta_m \varepsilon^3$,

$$|\bar{f} - \mathrm{id}| \le |f - \mathrm{id}| \le 2^{-1-1/\beta} K\delta_m \varepsilon^3 (3\varepsilon/4)^m l \le K\delta_m \varepsilon^2 a.$$

Hence, if $I_k = [x_{i_k}, x_{i_k + p_k}]$, then

$$p_k \le \frac{2Ka/\varepsilon}{a - |f - \mathrm{id}|} \le \frac{2K/\varepsilon}{1 - K\delta_m \varepsilon^2} \le 2K^2/\varepsilon.$$

Here we used $2\delta_m \varepsilon \le 1$. Hence by Lemma 3.4, we obtain the following estimates:

$$\begin{aligned} |||\log(g_i)'|||_\beta &\le \max\Big\{2^3 K^2 \frac{a}{\min\{A - a,\ 1 - B - a\}} + 2^2 K \frac{|f - \mathrm{id}|}{a}, \\ &\qquad 2^{1-1/\beta} q\, |||\log(\bar{f})'|||_\beta + 2^2 K \frac{|f - \mathrm{id}|}{a}\Big\} \\ &\le \max\Big\{2^3 K^3 \frac{(\varepsilon/2)(3\varepsilon/4)^m l}{\min\{A,\ 1 - B\}} + 2^2 K \frac{K\delta_m \varepsilon^2 a}{a}, \\ &\qquad 2^{1-1/\beta}(2K^2/\varepsilon)\, 2\delta_m \varepsilon^3 + 2^2 K \frac{K\delta_m \varepsilon^2 a}{a}\Big\} \\ &\le \max\Big\{2^3 K^3 \frac{(\varepsilon/2)(3\varepsilon/4)^m l}{\min\{A,\ 1 - B\}} + 2^2 K^2 \delta_m \varepsilon^2, 2^2 3K^2 \delta_m \varepsilon^2\Big\}. \end{aligned}$$

The restriction of $\hat{f}$ to each interval $I_k = [x_{i_k}, x_{i_k + p_k}]$ is a product of at most $4\varepsilon^{m-q} + 2K\varepsilon^{-1}$ elementary PL homeomorphisms and we decompose $\hat{f}$ as follows. Write

$$\hat{f} = h_1 h_2 = h_2 h_1$$

where the support of h_i is contained in $\bigcup_{j=0}^{[n/2]} [x_{2j+i-1}, x_{2j+i}]$ $(i = 1, 2)$. Put

$$h_3 = (T_a)^{-1} h_2 T_a h_1.$$

Then

$$\hat{f} = h_2 h_1 = [h_2, (T_a)^{-1}] h_3.$$

Here we replace T_a by t_a defined by

$$t_a = PL\begin{pmatrix} 0, & A-2a, & B+a, & 1 \\ 0, & A-a, & B+2a, & 1 \end{pmatrix}$$

where

$$|||\log(t_a)'|||_\beta \le 2^2 Ka/\min\{A-2a,\ 1-B-a\} \le 2^2 K^2 a/\min\{A,\ 1-B\}.$$

For h_i ($i = 1, 2$), we have

$$|||\log(h_i)'|||_\beta \le |||\log(\hat{f})'|||_\beta \le 2\delta_m \varepsilon^3.$$

Hence

$$|||\log(h_3)'|||_\beta \le 2^2 \delta_m \varepsilon^3.$$

Note that the restriction of h_3 to each interval $I_k = [x_{i_k}, x_{i_k+p_k}]$ is a product of at most $4\varepsilon^{m-q} + 2K\varepsilon^{-1}$ elementary PL homeomorphisms.

In $I_k = [x_{i_k}, x_{i_k+p_k}]$, we already know that $p_k \le 2K^2/\varepsilon$. For $0 \le j \le p_k$, we see that

$$\begin{aligned} |x_{i_k+j} - x_{i_k} - ja| &\le \sum_{i=1}^{j-1} |T_a(f-\mathrm{id})x_i| \\ &\le (j-1)K\delta_m\varepsilon^2 a \\ &\le (2K^2/\varepsilon)K\delta_m\varepsilon^2 a \\ &\le 2K^3\delta_m\varepsilon a \\ &\le 2^{-2}a. \end{aligned}$$

Here we used $2^6\delta_m\varepsilon \le 1$.

Now we apply Lemma 4.5 to T_{2a} and $(h_3)^{-1}$, and we obtain PL homeomorphisms g_7 and g_8 such that

$$h_3 = [g_7, g_8]f_{m+1}$$

where the support of f_{m+1} is contained in a disjoint union of intervals of length $a + 2\cdot 2^{-2}a = (3\varepsilon/4)^{m+1}$ which is contained in $[A - (3\varepsilon/4)^{m+1}, B + (3\varepsilon/4)^{m+1}]$ and the minimal distance between the intervals is at least $a - 2\cdot 2^{-2}a = (3\varepsilon/4)^{m+1}\, l/3$.

We may assume q in Lemma 4.5 is not smaller than $\varepsilon^{-1} + 2$. In fact, for $\varepsilon^{-1} + 2 \le j \le \varepsilon^{-1} + 3$,

$$\begin{aligned} |x_{i_k+2j} - x_{i_k} - 2ja| &\le 2(\varepsilon^{-1} + 3)K\delta_m\varepsilon^2 a \\ &\le 2K^2\delta_m\varepsilon a \\ &\le 2^{-3}a \end{aligned}$$

and $2ja \le (3\varepsilon/4)^m l + 6a$. Hence

$$x_{i_k+1} - x_{i_k+2j} \ge 2^3 5a - 6a - a \ge 33a.$$

Note also that q in Lemma 4.5 is not greater than $K^2\varepsilon^{-1}$.

Since $4\varepsilon^{m-q} + 2K^2\varepsilon^{-1} \le 4\varepsilon^{m-q+1}(\varepsilon^{-1} + 2)$, the restriction of f_{m+1} to each interval is a product of at most $4\varepsilon^{m-q+1}$ elementary PL homeomorphisms. For f_{m+1}, g_7 and g_8, we have

$$|||\log(f_{m+1})'|||_\beta \le (1 + 2^{2-1/\beta}3^{1-1/\beta})2^2\delta_m\varepsilon^3$$

and

$$|||\log(g_i)'|||_\beta \le \max\left\{2^3 7 K^2 \varepsilon^{-1} 2^2 \delta_m \varepsilon^3,\ 2^2 3 \cdot 2^2 \delta_m \varepsilon^3 + 2^2 K^2 \frac{(\varepsilon/2)\ (3\varepsilon/4)^m l}{\min\{A,\ 1-B\}}\right\}$$

$$\le \max\left\{2^5 7 K^2 \delta_m \varepsilon^2, 2^4 3 \delta_m \varepsilon^3 + 2^3 K \frac{(\varepsilon/2)\ (3\varepsilon/4)^m l}{\min\{A,\ 1-B\}}\right\} \quad (i = 7, 8). \qquad \square$$

LEMMA 5.3. *Let ε be a positive real number such that $\varepsilon \le \min\{A, 1-B\}\varepsilon_0/2^3$. Let f_{q-1} be a PL homeomorphism of $\mathbf{R}$ with support in $[A, B]$ satisfying the following conditions:*

1. $|||\log f'_{q-1}|||_\beta \le \delta_{q-1}\varepsilon^3$, *where $\delta_{q-1} \ge 1$ and $2^6\delta_{q-1}\varepsilon \le 1$,*
2. *the support of f_{q-1} is contained in a disjoint union of intervals of length $(3\varepsilon/4)^{q-1} l$ $(l = B - A)$,*
3. *the restriction of f_{q-1} to each interval has at most $4\varepsilon^{-1}$ non-differentiable points, and*
4. *the minimal distance between the intervals is at least $(3\varepsilon/4)^{q-1} l/3$.*

Then f_{q-1} is written as follows:

$$f_{q-1} = [g_1, g_2][g_3, g_4][g_5, g_6][g_7, g_8] f_q.$$

Here

(i) *the supports of g_i $(i = 1, \ldots, 8)$ are contained in $[0, 1]$,*
(ii) *the support of f_q is contained in a disjoint union of intervals of length $(3\varepsilon/4)^q$,*
(iii) *the restriction of f_q to each interval is a product of 4 elementary PL homeomorphisms,*
(iv) *the minimal distance between the intervals is at least $(3\varepsilon/4)^q l/3$.*

We have following estimates on the norms:

$$|||\log(f_q)'|||_\beta \le 2^{2-1/\beta} K^2 \delta_{q-1}\varepsilon^2$$

and

$$|||\log g'_i|||_\beta \le \max\left\{2^4 K^5 \delta_{q-1}\varepsilon,\ 2^3 K \frac{(\varepsilon/2)(3\varepsilon/4)^{q-1} l}{\min\{A,\ 1-B\}}\right\}.$$

Proof. We go through as in the proof of Lemma 5.2 and we use Lemma 4.4 instead of Lemma 4.5. $\square$

Proof of Theorem 5.1. For $m = 0, \ldots, q-2$ we use Lemma 5.2, where we put $\delta_m = 2^{2m}(1 + 2^{2-1/\beta}3^{1-1/\beta})^m\delta$. Then since $2^6\delta_{q-2}\varepsilon \le 1$, we have

$$f = \prod_{i=1}^{4(q-1)} [g_{2i-1}, g_{2i}] f_{q-1}.$$

Here, for $\delta_{q-1} = 2^{2(q-1)}(1 + 2^{2-1/\beta}3^{1-1/\beta})^{q-1}\delta$,

$$|||\log(f_{q-1})'|||_\beta \le \delta_{2-1}\varepsilon^2$$

and

$$|||\log g'_i|||_\beta \le \max\left\{2^3 K^3 \frac{(\varepsilon/2) l}{\min\{A,\ 1-B\}} + 2^2 K^2 \delta_{q-1}\varepsilon^2,\ 2^5 7 K^2 \delta_{q-1}\varepsilon^2\right\} \quad (1 \le i \le 8q-8).$$

Now by Lemma 5.3, since $2^6 \delta_{q-1} \varepsilon \leq 1$, we have

$$f = \prod_{i=1}^{4q} [g_{2i-1}, g_{2i}] f_q.$$

Here we have

$$|||\log (f_q)'|||_\beta \leq 2^{2-1/\beta} K^2 \delta_{q-1} \varepsilon^2$$

and

$$|||\log g_i'|||_\beta \leq \max \left\{ 2^4 K^5 \delta_{q-1} \varepsilon, 2^3 K \frac{(\varepsilon/2)(3\varepsilon/4)^{q-1} l}{\min\{A,\ 1-B\}} \right\} \quad (8q-7 \leq i \leq 8q).$$

We can decompose $f_q = h_1 \dots h_4$ so that h_j ($j = 1, 2, 3, 4$) is a juxtaposition of at most $(3\varepsilon/4)^{-q}$ elementary PL homeomorphisms with support in a disjoint union of intervals of length $(3\varepsilon/4)^q$ and the minimal distance between the intervals is at least $(3\varepsilon/4)^q l/3$. Now we apply Lemma 3.7 for h_j. Here for $\delta_q = 2^{2-1/\beta} K^2 \delta_{q-1}$,

$$|||\log (h_j)'|||_\beta \leq \delta_q \varepsilon^2.$$

Note that

$$(5\varepsilon)^{q+1}(3)((3\varepsilon/4)^{-q})^{1/\beta} \leq 5^{q+1}(3)(4/3)^q \varepsilon \leq 7^q \cdot 5 \cdot 3\varepsilon.$$

Then, since $2\delta_q \varepsilon \leq 1$, h_j is written as follows:

$$h_j = \prod_{i=1}^{3(q+1)+1} [g_{2i-1}^{(j)}, g_{2i}^{(j)}]$$

where

$$|||\log (g_i^{(j)})'|||_\beta \leq \max \left\{ 2^3 3 K^3 \varepsilon + 2K^2 \delta_q \varepsilon,\ 2 \cdot 3K^2 \delta_q \varepsilon, 2K^2 \frac{(3\varepsilon/4)^q/3}{\min\{A,\ 1-B\}}, \right.$$
$$\left. 2^{5-2/\beta} K^2 (2\delta_q \varepsilon^2 + 2^3 7^q 5 \cdot 3\varepsilon) \right\}.$$

Hence we have

$$f = \prod_{i=1}^{16(q+1)} [g_{2i-1}, g_{2i}]$$

and we can find a positive real number c depending only on q, A, B and δ such that

$$|||\log g_i'|||_\beta \leq c\varepsilon.$$

6. STABLE APPROXIMATIONS

In this section, we stably approximate a 2-cycle of $\mathbf{G}_c^{1+\alpha}(\mathbf{R})$ or $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ by 2-cycles of $PL_c(\mathbf{R})$.

A 2-cycle of $\mathbf{G}_c^{1+\alpha}(\mathbf{R})$ or $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ is geometrically represented by a $C^{1+\alpha}$ or $C^{L,\mathscr{V}_\beta}$ foliated $\mathbf{R}$-product with compact support over a closed oriented surface Σ, i.e. a foliation of $\Sigma \times \mathbf{R}$ transverse to the fibers of the projection $\Sigma \times \mathbf{R} \to \Sigma$ which coincides with the product foliation with leaves $\Sigma \times \{*\}$ outside a compact set. This foliated $\mathbf{R}$-product is determined by the holonomy homomorphism $\pi_1(\Sigma) \to \mathbf{G}_c^{1+\alpha}(\mathbf{R})$ or $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$.

If we fix generators of $\pi_1(\Sigma)$, the topology of the space of foliated **R**-products is given so that a sequence of foliated **R**-products converges if the holonomies along the generators converge. For $C^{1+\alpha}$ foliated **R**-products, this topology does not depend on the choice of the generators of $\pi_1(\Sigma)$ and in fact this topology is the same as the one given so that a sequence of foliated **R**-products converges if for any $\gamma \in \pi_1(\Sigma)$, the holonomy along γ converges. The reason is that $\mathbf{G}_c^{1+\alpha}(\mathbf{R})$ is a topological group, and this topology is the usual topology of a representation space. But this is not the case for $C^{L,\mathscr{V}_\beta}$ foliated **R**-products because $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ is not a topological group. However, as we show in Lemma 6.2, the group operations of $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ are continuous at the elements of the subgroup $\mathbf{G}_c^{1+\alpha}(\mathbf{R})$ for $\alpha > 1/\beta$ and to approximate a $C^{1+\alpha}$ foliated **R**-product by $C^{L,\mathscr{V}_\beta}$ foliated **R**-products, we can use the usual notion of convergence.

For a $C^{1+\alpha}$ foliated **R**-product $\mathscr{F}$ with compact support over a closed oriented surface Σ_N of genus N, a stabilization of $\mathscr{F}$ is a foliated **R**-product over the connected sum $\Sigma_N \# \Sigma_M$ such that the holonomy homomorphism $\pi_1(\Sigma_N \# \Sigma_M) \to \mathbf{G}_c^{1+\alpha}(\mathbf{R})$ factors through the holonomy homeomorphism $\pi_1(\Sigma_N) \to \mathbf{G}_c^{1+\alpha}(\mathbf{R})$ of $\mathscr{F}$, i.e. the holonomy of the stabilization coincides with that of $\mathscr{F}$ for the first $2N$ usual generators and is the identity for the last $2M$ usual generators. In other words, the stabilization is the ★-sum $\mathscr{F}\bigstar\mathscr{P}$ of $\mathscr{F}$ and the trivial foliated **R**-product $\mathscr{P}$ over Σ_M. The ★-sum is similar to that described in [10], and $\mathscr{F}\bigstar\mathscr{P}$ is obtained from the disjoint union of foliated products $\mathscr{F}$ and $\mathscr{P}$ with a tubular neighbourhood of a fiber of each foliated product deleted by identifying along the boundary. The 2-cycle represented by a stabilization is homologous to the original 2-cycle.

As we show in Lemma 6.4, $PL_c(\mathbf{R})$ is dense in $\mathbf{G}_c^{1+\alpha}(\mathbf{R})$ in the topology of $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ ($\beta > 1/\alpha$). So we can only show that a $C^{1+\alpha}$-foliated **R**-product ($0 < \alpha \leq 1$) with compact support over a surface is stably approximated by a PL foliated **R**-product with compact support with respect to the topology of $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ ($\beta > 1/\alpha$).

For a $C^{1+\alpha}$-foliated **R**-product over the surface Σ_N, if we approximate the holonomies along the usual generators of $\pi_1(\Sigma)$ by PL homeomorphisms, then Lemma 6.2 and Corollary 6.3 show that the product of commutators of them which was originally the identity becomes a PL homeomorphism close to the identity, and we can apply Theorem 5.1 to it. Then the PL approximations of the holonomies along the generators together with these PL homeomorphisms defines an approximation of a stabilization of the original 2-cycle. This is the idea of the proof of Theorem D.

For $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$, we also show in Lemma 4.6 that $PL_c(\mathbf{R})$ is dense in $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ in the topology of $\mathbf{G}_c^{L,\mathscr{V}_{\beta'}}(\mathbf{R})$ ($\beta' \geq \beta$). For $C^{L,\mathscr{V}_\beta}$ foliated **R**-products over the surface Σ_N, we need to fix the generators of $\pi_1(\Sigma_N)$ to fix a topology of the space of foliated products. For a $C^{L,\mathscr{V}_\beta}$ foliated **R**-product over the surface Σ_N, even if we approximate the holonomies along the usual generators of $\pi_1(\Sigma_N)$ by PL homeomorphisms, the product of commutators of them might not be close to the identity. Hence we need to modify the notion of stabilization. This is done in Theorem 6.5.

Now we restate Theorem D.

THEOREM 6.1. *Let $\mathscr{F}$ be a foliated* **R***-product with support in $[\frac{1}{4}, \frac{3}{4}]$ over the closed surface Σ_N of genus N of class $C^{1+\alpha}$ ($0 < \alpha \leq 1$). Let β be a positive real number greater than $1/\alpha$. Then there are a positive integer M and a family of PL foliated* **R***-products $\mathscr{F}_k$ with support in* [0, 1] *over $\Sigma_N \# \Sigma_M$ such that $\mathscr{F}_k$ converges to the stabilization $\mathscr{F}\bigstar\mathscr{P}$ of $\mathscr{F}$ in the $C^{L,\mathscr{V}_\beta}$ topology. Here the meaning of convergence is that for any $\gamma \in \pi_1(\Sigma_N \# \Sigma_M)$, the holonomy along γ converges. In particular, if $1/\alpha < \beta < 2$, the Godbillon–Vey invariant $GV(\mathscr{F}_k)$ converges to $GV(\mathscr{F})$.*

In the following lemma, we show that for $\beta > 1/\alpha$, the product (composition) in $\mathbf{G}_c^{L,\mathscr{V}_\beta}((0, 1))$ is continuous at the elements of $\mathbf{G}_c^{1+\alpha}((0, 1))$. Hence if the holonomies along generators of $\pi_1(\Sigma_N \# \Sigma_M)$ converge to elements of $\mathbf{G}_c^{1+\alpha}((0, 1))$, then for any $\gamma \in \pi_1(\Sigma_N \# \Sigma_M)$, the holonomy along γ converges.

LEMMA 6.2. *Let α and β be positive real numbers such that $1 \leq 1/\alpha < \beta$.*

(i) *The composition $(f_1, f_2) \to f_1 \circ f_2$ in $\mathbf{G}_c^{L,\mathscr{V}_\beta}$ is continuous at (f_1, f_2) with f_1 of class $C^{1+\alpha}$. Moreover, for $(g_1, g_2) \in \mathbf{G}_c^{L,\mathscr{V}_\beta}((0, 1)) \times \mathbf{G}_c^{L,\mathscr{V}_\beta}((0, 1))$ such that $|||\log (f_2(g_2)^{-1})'|||_\beta \leq \varepsilon_0/2$, we have*

$$|||\log ((g_1 \circ g_2)(f_1 \circ f_2)^{-1})'|||_\beta \leq |||\log g_1' - \log f_1'|||_\beta$$
$$+ 3K^\alpha \|\log f_1'\|_{C^\alpha}(|||\log g_2' - \log f_2'|||_\beta)^{\alpha - 1/\beta} + |||\log g_2' - \log f_2'|||_\beta$$

where

$$\|\log f_1'\|_{C^\alpha} = \sup \frac{|\log f_1'(x) - \log f_1'(y)|}{|x - y|^\alpha}.$$

(ii) *The inversion $f \mapsto f^{-1}$ in $\mathbf{G}_c^{L,\mathscr{V}_\beta}$ is continuous at f of class $C^{1+\alpha}$. Moreover, for $g \in \mathbf{G}_c^{L,\mathscr{V}_\beta}((0, 1))$ such that $|||\log(fg^{-1})'|||_\beta \leq \varepsilon_0/2$, we have*

$$|||\log (g^{-1} f)'|||_\beta \leq |||\log g' - \log f'|||_\beta + 3K^\alpha \|\log f' \circ f^{-1}\|_{C^\alpha}(|||\log (fg^{-1})'|||_\beta)^{\alpha - 1/\beta}.$$

Proof. We follow the argument in Proposition 4.6 of [20]. First note that

$$|||\log ((g_1 \circ g_2)(f_1 \circ f_2)^{-1})'|||_\beta = |||\log (g_1 \circ g_2)' - \log (f_1 \circ f_2)'|||_\beta$$

and

$$\begin{aligned}|(\log (g_1 \circ g_2)' - \log (f_1 \circ f_2)')(x)| \leq{} & |\log g_1'(g_2(x)) - \log f_1'(g_2(x))| \\ & + |\log f_1'(g_2(x)) - \log f_1'(f_2(x))| \\ & + |\log g_2'(x) - \log f_2'(x)|.\end{aligned}$$

Put $\varphi = \log f_1'$ and $F = f_2(g_2)^{-1}$ and $C = \|\log f_1'\|_{C^\alpha}$. Then

$$|||\log f_1' \circ g_2 - \log f_1' \circ f_2|||_\beta = |||\varphi \circ g_2 - \varphi \circ F \circ g_2|||_\beta = |||\varphi - \varphi \circ F|||_\beta.$$

Let $A = \{x_1, \ldots, x_k\}$ be a finite subset of $\mathbf{R}$. Since $\|F - \mathrm{id}\|_{C^0} \leq K2^{-1-1/\beta}|||\log F'|||_\beta$,

$$|(\varphi \circ F(x_i) - \varphi(x_i)) - (\varphi \circ F(x_{i-1}) - \varphi(x_{i-1}))|$$

is always smaller than $2C(K2^{-1-1/\beta}|||\log F'|||_\beta)^\alpha$. For those x_{i-1} and x_i such that $|x_i - x_{i-1}| \geq |||\log F'|||_\beta$, we use this estimate and we obtain an estimate for the sum over such x_{i-1} and x_i:

$$\begin{aligned}&\sum |(\varphi \circ F(x_i) - \varphi(x_i)) - (\varphi \circ F(x_{i-1}) - \varphi(x_{i-1}))|^\beta \\ &\leq \frac{1}{|||\log F'|||_\beta} C^\beta K^{\alpha\beta} 2^{\beta - \alpha\beta - 1} (|||\log F'|||_\beta)^{\alpha\beta} \\ &= 2^{\beta - \alpha\beta - 1} K^{\alpha\beta} C^\beta (|||\log F'|||_\beta)^{\alpha\beta - 1}.\end{aligned}$$

Now, since $\|F' - 1\|_{C^0} \leq K2^{-1/\beta}|||\log F'|||_\beta$,

$$|F'| \leq 1 + K2^{-1/\beta}|||\log F'|||_\beta \leq K.$$

For those x_{i-1} and x_i such that $|x_i - x_{i-1}| \leq |||\log F'|||_\beta$, we obtain $|F(x_i) - F(x_{i-1})| \leq K|x_i - x_{i-1}|$, and

$$|(\varphi \circ F(x_i) - \varphi(x_i)) - (\varphi \circ F(x_{i-1}) - \varphi(x_{i-1}))| \leq CK^\alpha |x_i - x_{i-1}|^\alpha.$$

The sum for such x_{i-1} and x_i is estimated by

$$\begin{aligned}\sum |(\varphi \circ f(x_i) - \varphi(x_i)) - (\varphi \circ f(x_{i-1}) - \varphi(x_{i-1}))|^\beta &\leq \sum C^\beta K^{\alpha\beta} |x_i - x_{i-1}|^{\alpha\beta} \\ &\leq \sum C^\beta K^{\alpha\beta} (|||\log F'|||_\beta)^{\alpha\beta - 1} |x_i - x_{i-1}| \\ &\leq C^\beta K^{\alpha\beta} (|||\log F'|||_\beta)^{\alpha\beta - 1}.\end{aligned}$$

Thus we obtain

$$V_\beta(\varphi \circ F - \varphi) \leq (2^{\beta - \alpha\beta - 1} + 1) C^\beta K^{\alpha\beta} (|||\log F'|||_\beta)^{\alpha\beta - 1}.$$

Hence

$$|||\varphi \circ F - \varphi|||_\beta \leq 3K^\alpha C (|||\log F'|||_\beta)^{\alpha - 1/\beta}.$$

For the inversion, we have

$$\begin{aligned}|\log (g^{-1})' - \log (f^{-1})'| &= | - \log g' \circ g^{-1} + \log f' \circ f^{-1}| \\ &\leq |\log g' \circ g^{-1} - \log f' \circ g^{-1}| \\ &\quad + |\log f' \circ g^{-1} - \log f' \circ f^{-1}|.\end{aligned}$$

Here for the second term, by the argument above,

$$\begin{aligned}|||\log f' \circ g^{-1} - \log f' \circ f^{-1}|||_\beta &= |||\log f' \circ f^{-1} \circ (fg^{-1}) - \log f' \circ f^{-1}|||_\beta \\ &\leq 3K^\alpha \|\log f' \circ f^{-1}\|_{C^\alpha} (|||\log (fg^{-1})'|||_\beta)^{\alpha - 1/\beta}. \qquad \square\end{aligned}$$

COROLLARY 6.3. *Let α and β be positive real numbers such that $1 \leq 1/\alpha < \beta$. If f is of class $C^{1+\alpha}$, then conjugation on $\mathbf{G}_c^{L,\mathscr{V}_\beta}((0, 1))$ by f is continuous. If $|||\log (g_1(g_2)^{-1})'|||_\beta \leq \varepsilon_0/2$, then*

$$\begin{aligned}&|||\log(fg_1 f^{-1})' - \log(fg_2 f^{-1})'|||_\beta = |||\log(fg_1)' - \log(fg_2)'|||_\beta \\ &\leq |||\log(g_1)' - \log(g_2)'|||_\beta + 3K^\alpha \|\log f'\|_{C^\alpha} (|||\log(g_1)' - \log(g_2)'|||_\beta)^{\alpha - 1/\beta}.\end{aligned}$$

Remark. The fact that $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ is not a topological group can be seen as follows. Put

$$f_1 = PL\begin{pmatrix}0, & 2, & 6 \\ 0, & 4, & 6\end{pmatrix}$$

and for a real number a $(0 < |a| < 1)$, put

$$f_2 = PL\begin{pmatrix}0, & 1, & 4, & 6 \\ 0, & 1 + a, & 4 + a, & 6\end{pmatrix}.$$

Then

$$f_1 f_2 (f_1)^{-1} = PL\begin{pmatrix}0, & 2, & 4 - 2a, & 4, & 5, & 6 \\ 0, & 2 + a, & 4, & 4 + a/2, & 5 + a/2, & 6\end{pmatrix}.$$

Since the interval $[4 - 2a, 4]$ is mapped onto $[4, 4 + a/2]$,

$$|||\log(f_1 f_2 (f_1)^{-1})'|||_\beta \geq \log 4$$

and $f_1 f_2$ does not converge to f_1 as a tends to zero (and f_2 tends to the identity).

LEMMA 6.4. *Let α and β be positive real numbers such that $1 \le 1/\alpha < \beta$. Let f be a $C^{1+\alpha}$ diffeomorphism of $\mathbf{R}$ with support in $[0, 1]$. Let $A = \{a_0, \ldots, a_k\}$ be a finite subset of $[0, 1]$ $(0 = a_0 < \cdots < a_k = 1)$ with mesh not greater than ε. Put*

$$\bar{f} = PL\begin{pmatrix} a_0, \ldots, & a_i, \ldots, & a_k \\ f(a_0), \ldots, & f(a_i), \ldots, & f(a_k) \end{pmatrix}.$$

Then

$$V_\beta(\log(\bar{f}f^{-1})') \le (k+1)2^\beta C^\beta \varepsilon^{\alpha\beta} + C^\beta \varepsilon^{\alpha\beta - 1}$$

where $C = \|\log f'\|_{C^\alpha}$. In particular, if

$$A = \{i/n\}_{i=0,\ldots,n} \cup f^{-1}(\{i/n\}_{i=0,\ldots,n})$$

then we have

$$|||\log(f(\bar{f})^{-1})'|||_\beta \le 7\|\log f'\|_{C^\alpha}(1/n)^{\alpha - 1/\beta}$$

and

$$|||\log((\bar{f})^{-1}f)'|||_\beta \le 7\|\log(f^{-1})'\|_{C^\alpha}(1/n)^{\alpha - 1/\beta}.$$

Proof. Note that

$$|||\log(f(\bar{f})^{-1})'|||_\beta = |||\log f' - \log \bar{f}'|||_\beta.$$

Let $y_k \in (a_{i-1}, a_i)$ be a point where $(\bar{f})'(y_i) = f'(y_i)$. Then if $x \in (a_{i-1}, a_i]$,

$$\begin{aligned} |\log f'(x) - \log(\bar{f})'(x)| &\le |\log f'(x) - \log f'(y_i)| \\ &\le \|\log f'\|_{C^\alpha} |x - y_i|^\alpha \\ &\le \|\log f'\|_{C^\alpha} \varepsilon^\alpha. \end{aligned}$$

Here note that $\log(\bar{f})'$ is the derivative to the left. Hence if $x_{j-1} \le a_i < x_j$, then we have

$$|(\log f'(x_j) - \log \bar{f}'(x_j)) - (\log f'(x_{j-1}) - \log \bar{f}'(x_{j-1}))| \le 2C\varepsilon^\alpha$$

and the sum for such x_{j-1} and x_j is estimated by

$$\sum |(\log f'(x_j) - \log \bar{f}'(x_j)) - (\log f'(x_{j-1}) - \log \bar{f}'(x_{j-1}))|^\beta \le (k+1)2^\beta C^\beta \varepsilon^{\alpha\beta}.$$

If $a_{i-1} < x_{j-1} < x_j \le a_i$, then we have

$$\begin{aligned} |(\log f'(x_j) - \log \bar{f}'(x_j)) - (\log f'(x_{j-1}) - \log \bar{f}'(x_{j-1}))| &= |\log f'(x_j) - \log f'(x_{j-1})| \\ &\le C|x_j - x_{j-1}|^\alpha \end{aligned}$$

and the sum for such x_{j-1} and x_j is estimated by

$$\begin{aligned} \sum |(\log f'(x_j) - \log \bar{f}'(x_j)) - (\log f'(x_{j-1}) - \log \bar{f}'(x_{j-1}))|^\beta &\le \sum C^\beta |x_j - x_{j-1}|^{\alpha\beta} \\ &\le \sum C^\beta \varepsilon^{\alpha\beta - 1} |x_j - x_{j-1}| \\ &\le C^\beta \varepsilon^{\alpha\beta - 1}. \end{aligned}$$

Thus

$$V_\beta(\log(\bar{f}f^{-1})') \le (k+1)2^\beta C^\beta \varepsilon^{\alpha\beta} + C^\beta \varepsilon^{\alpha\beta - 1}.$$

□

Proof of Theorem 6.1. Let $h:\pi_1(\Sigma_N)\to \mathbf{G}_c^{1+\alpha}(\mathbf{R})$ denote the holonomy homomorphism of the $\mathbf{R}$-product $\mathscr{F}$. For the usual generators $\gamma_1,\dots,\gamma_{2N}$ of $\pi_1(\Sigma_N)$, take $\overline{h(\gamma_i)}$ with respect to

$$\{i/n\}_{i=0,\dots,n} \cup (h(\gamma_i))^{-1}(\{i/n\}_{i=0,\dots,n}).$$

Let C be a positive real number bigger than the C^α norms of $\log(h(\gamma))'$, where γ are products of at most $4N$ members of $\{\gamma_1, \gamma_1^{-1},\dots,\gamma_{2N}, \gamma_{2N}^{-1}\}$. Then by Lemma 6.4, we have

$$|||\log(h(\gamma_i)(\overline{h(\gamma_i)})^{-1})'|||_\beta \le 7C(1/n)^{\alpha-1/\beta}$$

and

$$|||\log(h(\gamma_i)^{-1}\overline{h(\gamma_i)})'|||_\beta \le 7C(1/n)^{\alpha-1/\beta}.$$

Since

$$\begin{aligned} g_1\dots g_N(h_1\dots h_N)^{-1} = (g_1h_1^{-1})&(h_1g_2h_2^{-1}h_1^{-1})\\ &\times((h_1h_2)g_3h_3^{-1}(h_1h_2)^{-1})\dots\\ &\times((h_1\dots h_{N-1})g_Nh_N^{-1}(h_1\dots h_{N-1})^{-1}\end{aligned}$$

and

$$\begin{aligned} H &= [\overline{h(\gamma_1)}, \overline{h(\gamma_2)}]\dots[\overline{h(\gamma_{2N-1})}, \overline{h(\gamma_{2N})}]\\ &= [\overline{h(\gamma_1)}, \overline{h(\gamma_2)}]\dots[\overline{h(\gamma_{2N-1})}, \overline{h(\gamma_{2N})}]([h(\gamma_1), h(\gamma_2)]\dots[h(\gamma_{2N-1}), h(\gamma_{2N})])^{-1},\end{aligned}$$

we obtain, by Corollary 6.3,

$$|||\log H'||| \le 4N\{7C(1/n)^{\alpha-1/\beta} + 3K^\alpha C(7C)^{\alpha-1/\beta}(1/n)^{(\alpha-1/\beta)^2}\}.$$

Note that

$$NND(H)\le 8Nn.$$

Let k be a positive integer such that $(\alpha-1/\beta)^2 \ge 3/(k-1)$. Then if n is sufficiently large, for some positive real number δ, we have

$$|||\log H'|||_\beta \le \delta((1/n)^{1/(k-1)})^3$$

and

$$NND(H) \le ((1/n)^{1/(k-1)})^{-k}.$$

Thus by Theorem 5.1 this element can be written as a product of $16(k+1)$ small commutators of PL homeomorphisms g_i with support in $[0, 1]$ such that

$$|||\log g_i'|||_\beta \le c(1/n)^{1/(k-1)}$$

where c depends only on k and δ.

Now we put $M = 16(k+1)$ and define $\mathscr{F}_n$ to be the foliated $\mathbf{R}$ product with the holonomy homomorphism $\bar{h}:\pi_1(\Sigma_{N+M})\to PL_c(\mathbf{R})$ such that

$$\bar{h}(\gamma_i) = \begin{cases} \overline{h(\gamma_i)} & \text{for } i=1,\dots,2N\\ g_{2N+2M+1-i} & \text{for } i = 2N+1,\dots,2N+2M.\end{cases}$$

Then $\mathscr{G}_n$ is the desired foliated $\mathbf{R}$-product. □

We can generalize Theorem 6.1 to the foliated products of class $C_c^{L,\mathscr{V}_\beta}(\beta \geq 1)$.

Since the group $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ is not a topological group as is remarked after Corollary 6.3, we should be careful. First for a $C^{L,\mathscr{V}_\beta}$-foliated $\mathbf{R}$-product, even if we approximate the holonomies along the usual generators of $\pi_1(\Sigma)$ by PL homeomorphisms, the product of commutators of them might not be close to the identity. Secondly, the convergence of holonomies along the generators of $\pi_1(\Sigma)$ does not imply the convergence of the holonomy along an arbitrary element $\gamma \in \pi_1(\Sigma)$.

In order to give the meaning of approximation of foliated products of class $C^{L,\mathscr{V}_\beta}$, we fix a cell decomposition of the base surface by triangles which has only one vertex. With respect to such a cell decomposition we can say two foliated products are near if the holonomies along the edges of this cell decomposition are near.

The theorem is as follows.

THEOREM 6.5. *Let $\mathscr{G}$ be a foliated $\mathbf{R}$-product of class $C^{L,\mathscr{V}_\beta}(\beta \geq 1)$ with support in $[\frac{1}{4}, \frac{3}{4}]$ over a closed surface Σ_N of genus N with a cell decomposition by triangles with one vertex. Let β' be a positive real number greater than β. Then there exist a positive integer M, a closed surface Σ_{N+M} of genus $N+M$ with a cell decomposition by triangles with one vertex, a simplicial map $s: \Sigma_{N+M} \to \Sigma_N$ of degree 1 and a family of PL-foliated $\mathbf{R}$-products $\mathscr{G}_k$ with support in $[0, 1]$ over Σ_{N+M} such that $\mathscr{G}_k$ converges to the induced foliated product $s^*\mathscr{G}$ in the $C^{L,\mathscr{V}_{\beta'}}$ topology. Here the meaning of the convergence is that the holonomies along the edges of the cell decomposition converge. In particular, if $\beta < \beta' < 2$, the Godbillon–Vey invariant $GV(\mathscr{G}_k)$ converges to $GV(\mathscr{G})$.*

We need the following lemma which gives an approximation of an element of $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ by elements of $PL_c(\mathbf{R})$ and which replaces Lemma 6.4. We use the notations in [20].

LEMMA 6.6. *Let f be a $C^{L,\mathscr{V}_\beta}$ diffeomorphism of $\mathbf{R}$ with support in $[0, 1]$. There exists a finite subset B of $[0, 1]$ such that $B = \{b_0, \ldots, b_{k'}\}$ $(0 = b_0 < \cdots < b_{k'} = 1)$ with k' not greater than $(1 + s_\beta(\log f') + 3^{\beta-1} V_\beta(\log f'))/\varepsilon + 1$, and, for any finite subset $A = \{a_0, \ldots, a_k\}$ $(0 = a_0 < \cdots < a_k = 1)$ containing B and for*

$$\bar{f} = PL\begin{pmatrix} a_0, \ldots, & a_i, \ldots, & a_k \\ f(a_0), \ldots, & f(a_i), \ldots, & f(a_k) \end{pmatrix},$$

$$V_{\beta'}(\log(f(\bar{f})^{-1})') \leq (2 + s_\beta(\log f') + 3^{\beta-1} V_\beta(\log f'))2^{\beta'+1}\varepsilon^{\beta'/\beta-1}.$$

In particular, if B' is the set defined for f^{-1} and

$$A = B \cup f^{-1}(B')$$

then we have

$$|||\log(f(\bar{f})^{-1})'|||_{\beta'} \leq (2 + s_\beta(\log f') + 3^{\beta-1} V_\beta(\log f'))2^{\beta'+1}\varepsilon^{\beta'/\beta-1}$$

and

$$|||\log((\bar{f})^{-1} f)'|||_{\beta'} \leq (2 + s_\beta(\log f') + 3^{\beta-1} V_\beta(\log f'))2^{\beta'+1}\varepsilon^{\beta'/\beta-1}.$$

Proof. Let f be an element of $\mathbf{G}_c^{L,\mathscr{V}_\beta}(\mathbf{R})$ with support in $[0, 1]$. Then by Proposition 2.12 in [20], there is a map

$$p_\beta: [0, 1 + s_\beta(\log f')] \to [0, 1]$$

such that $(\log f')_{p_\beta}$ is continuous and

$$V_\beta((\log f')_{p_\beta}) \leq 3^{\beta-1} V_\beta(\log f')$$

by Proposition 2.12 in [20]. Then by Proposition 2.3 in [20], we have a map

$$h: [0, 1 + s_\beta(\log f') + 3^{\beta-1} V_\beta(\log f')] \to [0, 1 + s_\beta(\log f')]$$

such that $(\log f')_{p_\beta} \circ h$ is $1/\beta$ Hölder and

$$\| (\log f')_{p_\beta} \circ h \|_{C^{1/\beta}} \leq 1.$$

Now let ε be a positive real number. Let k'' be the greatest integer smaller than $(1 + s_\beta(\log f') + 3^{\beta-1} V_\beta(\log f'))/\varepsilon + 1$. Then $(p_\beta \circ h)$ $(\{j\varepsilon; j = 0, \ldots, k''\})$ contains $\{x \in [0, 1]; |\Delta(\log f')\ (x)|^\beta \geq \varepsilon\}$. Let $B = \{b_0, \ldots, b_{k'}\}$ $(b_0 < \cdots < b_{k'})$ be the image $(p_\beta \circ h)$ $(\{j\varepsilon; j = 0, \ldots, k''\})$. Let $A = \{a_0, \ldots, a_k\}$ $(0 = a_0 < \cdots < a_k = 1)$ be a finite subset containing B and let $\bar{f}$ be the PL homeomorphism defined by

$$\bar{f} = PL\begin{pmatrix} a_0, \ldots, & a_i, \ldots, & a_k \\ f(a_0), \ldots, & f(a_i), \ldots, & f(a_k) \end{pmatrix}.$$

We always have

$$\min_{[a_{i-1}, a_i]} \log f' \leq \log \frac{f(a_i) - f(a_{i-1})}{a^i - a_{i-1}} \leq \max_{[a_{i-1}, a_i]} \log f'.$$

Since A contains B, $[a_{i-1}, a_i] \subset (p_\beta \circ h)$ $([(i'-1)\varepsilon, i'\varepsilon])$ for some i' and we have

$$\left(\max_{[a_{i-1}, a_i]} - \min_{[a_{i-1}, a_i]}\right)(\log f') = \left(\max_{[(i'-1)\varepsilon, i'\varepsilon]} - \min_{[(i'-1)\varepsilon, i'\varepsilon]}\right)((\log f')_{p_\beta} \circ h) \leq \varepsilon^{1/\beta}.$$

Hence

$$|\log f'(x) - \log(\bar{f})'(x)| \leq \varepsilon^{1/\beta}.$$

Now we look at the β-variation of $\log(f(\bar{f})^{-1})'$. Note that

$$|||\log(f(\bar{f})^{-1})'|||_\beta = |||\log f' - \log \bar{f}'|||_\beta.$$

Here note that $\log(\bar{f})'$ is the derivative to the left. Let $\{x_1, \ldots, x_m\}$ $(x_1 < \cdots < x_m)$ be a finite subset of $\mathbf{R}$. If $x_{j-1} \leq a_i < x_j$, then we have

$$|(\log f'(x_j) - \log \bar{f}'(x_j)) - (\log f'(x_{j-1}) - \log \bar{f}'(x_{j-1}))| \leq 2\varepsilon^{1/\beta}$$

and the sum for such x_{j-1} and x_j is estimated by

$$\begin{aligned} \sum |(\log f'(x_j) - \log \bar{f}'(x_j)) - (\log f'(x_{j-1}) - \log \bar{f}'(x_{j-1}))|^{\beta'} &\leq (k'' + 1)2^{\beta'}\varepsilon^{\beta'/\beta} \\ &\leq (2 + s_\beta(\log f') + 3^{\beta-1} V_\beta(\log f'))2^{\beta'}\varepsilon^{\beta'/\beta - 1}. \end{aligned}$$

If $a_{i-1} < x_{j-1} < x_j \leq a_i$, then we take points $y_i \in [(i'-1)\varepsilon, i'\varepsilon]$ such that $(p_\beta \circ h)(y_j) = x_i$ and we have

$$\begin{aligned} |(\log f'(x_j) - \log \bar{f}'(x_j)) - (\log f'(x_{j-1}) - \log \bar{f}'(x_{j-1}))| &= |\log f'(x_j) - \log f'(x_{j-1})| \\ &= |((\log f')_{p_\beta} \circ h)(y_i) - ((\log f')_{p_\beta} \circ h)(y_{j-1}) \\ &\leq |y_j - y_{j-1}|^{1/\beta} \end{aligned}$$

and the sum for such x_{j-1} and x_j is estimated by

$$\begin{aligned}\sum |(\log f'(x_j) - \log \bar{f}'(x_j)) - (\log f'(x_{j-1}) - \log \bar{f}'(x_{j-1}))|^{\beta'} &\leq \sum |y_j - y_{j-1}|^{\beta'/\beta} \\ &\leq \sum \varepsilon^{\beta'/\beta - 1} |y_j - y_{j-1}| \\ &\leq (1 + s_\beta(\log f') + 3^{\beta-1} V_\beta(\log f'))\varepsilon^{\beta'/\beta - 1}.\end{aligned}$$

Thus

$$V_\beta(\log (f(\bar{f})^{-1})') \leq (2 + s_\beta(\log f') + 3^{\beta-1} V_\beta(\log f'))2^{\beta'+1} \varepsilon^{\beta'/\beta - 1}. \qquad \square$$

Proof of Theorem 6.5. We fix a triangulation with one vertex of Σ_N. The triangulation of Σ_N has $6N - 3$ edges. Let C be a positive real number greater than $2 + s_\beta(\log h(\gamma)') + 3^{\beta-1} V_\beta(\log h(\gamma)')$ for any γ which is the edge of the triangulation. For each 2-simplex of Σ_N, we may assume that the orientations of its edges are given so that the holonomies along the three edges γ_0, γ_1 and γ_2 satisfy

$$h(\gamma_1) = h(\gamma_2) \circ h(\gamma_0).$$

Then we take the sets B_0, B_1 and B_2 given in Lemma 6.6 for $h(\gamma_0)^{-1}$, $h(\gamma_1)^{-1}$ and $h(\gamma_2)^{-1}$, respectively, and put

$$A = h(\gamma_0)^{-1}(B_0) \cup h(\gamma_1)^{-1}(B_1) \cup h(\gamma_1)^{-1}(B_2) = \{a_0, \ldots, a_k\}.$$

We define $\bar{f}_0$, $\bar{f}_1$ and $\bar{f}_2$ by

$$\bar{f}_0 = PL\begin{pmatrix} a_0, \ldots, & a_i, \ldots, & a_k \\ h(\gamma_0)(a_0), \ldots, & h(\gamma_0)(a_i), \ldots, & h(\gamma_0)(a_k)\end{pmatrix}$$

$$\bar{f}_1 = PL\begin{pmatrix} a_0, \ldots, & a_i, \ldots, & a_k \\ h(\gamma_1)(a_0), \ldots, & h(\gamma_1)(a_i), \ldots, & h(\gamma_1)(a_k)\end{pmatrix}$$

and

$$\bar{f}_2 = PL\begin{pmatrix} h(\gamma_0)(a_0), \ldots, h(\gamma_0)(a_i), \ldots, h(\gamma_0)(a_k) \\ h(\gamma_1)(a_0), \ldots, h(\gamma_1)(a_i), \ldots, h(\gamma_1)(a_k)\end{pmatrix}$$

respectively. Then we have $\bar{f}_1 = \bar{f}_2 \circ \bar{f}_0$ and by Lemma 6.6,

$$|||\log ((\bar{f}_i)^{-1} h(\gamma_i))'|||_{\beta'} \leq C2^{\beta'+1} \varepsilon^{\beta'/\beta} \quad \text{for } i = 0, 1, 2.$$

Now for each edge γ, there are two adjacent 2-simplices σ and σ'. When we approximate $h(\gamma)$ by PL homeomorphisms, we obtain two PL homeomorphisms $\bar{f}_\sigma$ and $\bar{f}_{\sigma'}$ which might be different because the sets A_σ and $A_{\sigma'}$ might be different. However the PL homeomorphism

$$(\bar{f}_\sigma)^{-1} \bar{f}_{\sigma'} = ((\bar{f}_\sigma)^{-1} h(\gamma))((\bar{f}_{\sigma'})^{-1} h(\gamma))^{-1}$$

is estimated as follows:

$$|||\log ((\bar{f}_\sigma)^{-1} \bar{f}_{\sigma'})'|||_{\beta'} \leq C2^{\beta'+2} \varepsilon^{\beta'/\beta - 1}.$$

If $\beta'/\beta - 1 \geq 3/(k-1)$ and ε is sufficiently small, then we have

$$|||\log ((\bar{f}_\sigma)^{-1} \bar{f}_{\sigma'})'|||_{\beta'} \leq (\varepsilon^{1/(k-1)})^3$$

and

$$NND((\bar{f}_\sigma)^{-1}\bar{f}_{\sigma'})' \leq 2C/\varepsilon \leq (\varepsilon^{1/(k-1)})^{-k}.$$

Thus by Theorem 5.1 this element is written as a product of $16(k+1)$ small commutators.

We construct a closed oriented surface $\Sigma_{N+16(k+1)(6N-3)}$ with a triangulation with one vertex as follows. Let $\Sigma_{16(k+1),1}$ be a closed oriented surface of genus $16(k+1)$ with a disk deleted. Consider a triangulation with one vertex of $\Sigma_{16(k+1),1}$. The boundary of $\Sigma_{16(k+1),1}$ consists of the vertex and one edge.

First we cut the given closed oriented surface Σ_N along the interiors of the edges, and to each resulted pair of edges we paste the two edges (∂_1 and ∂_2) of a 2-simplex. Now we paste a copy of $\Sigma_{16(k+1),1}$ to the third edge (∂_0) of the 2-simplex and we obtain $\Sigma_{N+16(k+1)(6N-3)}$ with a triangulation with one vertex.

The simplicial map is defined so that each pasted $\Sigma_{16(k+1),1}$ is mapped to the vertex and the added 2-simplices are mapped to the edges. Hence the induced foliated product $s^*\mathscr{G}$ is trivial over each pasted $\Sigma_{16(k+1),1}$. Let $\mathscr{G}_k$ be the foliated product over $\Sigma_{N+16(k+1)(6N-3)}$ defined by $\bar{f}_0, \bar{f}_1$ and $\bar{f}_2$ over the original simplices, $(\bar{f}_\sigma)^{-1}\bar{f}_{\sigma'}$, $\bar{f}_{\sigma'}$ and, $\bar{f}_\sigma$ over the added simplices and the homomorphism given by Theorem 5.1 over the pasted-in $\Sigma_{16(k+1),1}$. Then $\mathscr{G}_k$ converges to $s^*\mathscr{G}$ in the C^{L,γ_β} topology. □

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REFERENCES

1. R. K. Dennis and L. N. Vaserstein: Commutators in linear groups, *K-theory* **2** (1989), 761–767.
2. D. B. A. Epstein: The simplicity of certain groups of homeomorphisms, *Compositio Math.* **22** (1970), 165–173.
3. F. Gardiner and D. Sullivan: Symmetric and quasisymmetric structures on a closed curve, *Amer. J. Math.* **114** (1992), 683–736.
4. E. Ghys: Sur l'invariance topologique de la classe de Godbillon–Vey, *Ann. Inst. Fourier* **37** (1987), 59–76.
5. E. Ghys: L'invariant de Godbillon–Vey, Seminaire Bourbaki, exposé N°·706, 1988/89.
6. E. Ghys and V. Sergiescu: Sur un groupe remarquable de difféomorphismes du cercle, *Comment. Math. Helv.* **62** (1987), 185–239.
7. C. Godbillon and J. Vey: Un invariant des feuilletages de codimension 1, *C. R. Acad. Sci. Paris* **273** (1971), 92–95.
8. P. Greenberg: Classifying spaces for foliations with isolated singularities, *Trans. Amer. Math. Soc.* **304** (1987), 417–429.
9. S. Hurder and A. Katok: Differentiability, rigidity and Godbillon–Vey classes for Anosov flows, *Publ. Math. IHES* **72** (1990), 5–61.
10. H. B. Lawson Jr., Foliations, *Bull. Amer. Math. Soc.* **80** (1974), 369–418.
11. J. Mather: The vanishing of the homology of certain groups of homeomorphisms, *Topology* **10** (1971), 297–298.
12. J. Mather: Integrability in codimension 1, *Comm. Math. Helv.* **48** (1973), 195–233.
13. J. Mather: Commutators of diffeomorphisms I, II and III, *Comm. Math. Helv.* **49** (1974), 512–528, **50** (1975), 33–40 and **60** (1985), 122–124.
14. W. Thurston: Noncobordant foliations of S^3, *Bull. Amer. Math. Soc.* **78** (1972), 551–514.
15. W. Thurston: Foliations and groups of diffeomorphism, *Bull. Amer. Math. Soc.* **80** (1974), 304–307.
16. W. Thurston: Existence of codimension-one foliations, *Ann. Math.* **104** (1976), 249–269.
17. T. Tsuboi: On 2-cycles of B Diff(S^1) which are represented by foliated S^1-bundles over T^2, *Ann. Inst. Fourier* **31** **(2)** (1981), 1–59.
18. T. Tsuboi: On the foliated products of class C^1, *Ann. Math.* **130** (1989), 227–271.
19. T. Tsuboi: On the Hurder–Katok extension of the Godbillon–Vey invariant, *J. Fac. Sci., Univ. of Tokyo*, Sec. IA **37** (1990), 255–262.
20. T. Tsuboi: Area functional and Godbillon–Vey cocycles, *Ann. Inst. Fourier* **42** (1992), 421–447.
21. T. Tsuboi: Homological and dynamical study on certain groups of Lipschitz homeomorphisms of the circle, to appear in *J. Math. Soc. Japan.*

22. T. TSUBOI: Rationality of piecewise linear foliations and homology of the group of piecewise linear homeomorphisms, *l'Enseign. Math.* **38** (1992), 329–344.
23. T. TSUBOI: The Godbillon–Vey invariant and the foliated cobordism group, *Proc. Japan Acad.* **68** Ser. A (1992), 85–90.

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