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Approximability results for the maximum and minimum maximal induced matching problems

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Abstract

An *induced matching* M of a graph G is a set of pairwise non-adjacent edges such that their end-vertices induce a 1-regular subgraph. An induced matching M is *maximal* if no other induced matching contains M. The Minimum Maximal Induced Matching problem asks for a minimum maximal induced matching in a given graph. The problem is known to be NP-complete. Here we show that, if P $\neq$ NP, for any $\varepsilon > 0$, this problem cannot be approximated within a factor of $n^{1-\varepsilon}$ in polynomial time, where n denotes the number of vertices in the input graph. The result holds even if the graph in question is restricted to being bipartite. For the related problem of finding an induced matching of maximum size (Maximum Induced Matching), it is shown that, if P $\neq$ NP, for any $\varepsilon > 0$, the problem cannot be approximated within a factor of $n^{1/2-\varepsilon}$ in polynomial time. Moreover, we show that Maximum Induced Matching is NP-complete for planar line graphs of planar bipartite graphs.

MSC: 68W25; 68Q17; 05C69; 05C70

Keywords: Combinatorial optimization; Induced matching; Bipartite graph; Approximability; NP-completeness

1. Introduction

We consider only finite, undirected graphs without loops and multiple edges and use standard graph-theoretic terminology; see for example Bondy and Murty [3]. For concepts related to approximability, we follow Ausiello et al. [1].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a graph G, a subset $M \subseteq E(G)$ of edges is called an *induced matching* if (i) set M is a matching in G (a set of pairwise non-adjacent edges) and (ii) there is no edge in

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$E(G) \setminus M$ connecting two edges of M. In other words, the degrees of all vertices of the subgraph of G induced by the end-vertices of edges of M are equal to 1, i.e., the subgraph is 1-regular. An induced matching M is *maximal* if no other induced matching in G contains M. Let $IM(G)$ be the set of all maximal induced matchings in G. We define *minimum maximal* and *maximum induced matching numbers*, respectively, as follows:

$$\sigma(G) = \min\{|M| : M \in IM(G)\}$$

and

$$\Sigma(G) = \max\{|M| : M \in IM(G)\}.$$

A *minimum maximal induced matching* is a maximal induced matching that contains $\sigma(G)$ edges and a *maximum induced matching* is an induced matching that contains $\Sigma(G)$ edges.

Induced matchings, also known as 2-*separated matchings*, see Stockmeyer and Vazirani [35], and *strong matchings*, see Fricke and Laskar [17], have received much attention lately because of the growing number of applications. The importance of induced matchings has been recently demonstrated by Golumbic and Lewenstein [20] in connection with applications for secure communication channels, VLSI design and network flow problems. It is interesting to note that there is an immediate connection between the maximum size of an induced matching and the irredundancy number of a graph; see Golumbic and Laskar [19] for details. Finally, finding largest induced matchings is connected with the important problem of finding a strong edge-colouring considered by Erdös [15], Steger and Yu [34], Liu and Zhou [28] (finding largest induced matchings can be used as one of the heuristics for strong edge-colouring).

In this paper, we investigate the problem of approximating the size of a minimum maximal (maximum) induced matching of a given graph and consider some complexity results for induced matching problems.

First, we introduce some definitions. A set $S \subseteq V(G)$ is called *independent* if no two vertices in S are adjacent. An independent set S is *maximal* if no other independent set in G contains S. A set $D \subseteq V(G)$ is a *dominating* set if each vertex in $V(G) \setminus D$ is adjacent to some vertex in D. An *independent dominating set* is a vertex subset that is both independent and dominating.

Consider the following decision problem.

MINIMUM MAXIMAL INDUCED MATCHING
Instance: A graph G and an integer k.
Question: Is $\sigma(G) \leqslant k$?

Recently, Orlovich and Zverovich [30] have proved that MINIMUM MAXIMAL INDUCED MATCHING is an NP-complete problem even for bi-size matched graphs. A graph G is *bi-size matched* if there exists $k \geqslant 1$ such that $|M| \in \{k, k+1\}$ for every maximal induced matching M in G.

Cameron [6] observed that induced matchings in a graph G correspond to independent sets in the square of the line graph of G. Maximal induced matchings in G correspond to maximal independent sets in the square of the line graph of G. On the other hand, any maximal independent set is an independent dominating set, and vice versa [2]. Thus, a minimum maximal induced matching in a graph G corresponds to a minimum independent dominating set in the square of the line graph of G.

For many classes $\mathscr{G}$ of graphs (e.g., chordal and weakly chordal graphs, circular-arc graphs, and AT-free graphs), it is proved that if a graph G is in the class $\mathscr{G}$, then the square of the line graph of G is also in $\mathscr{G}$; see Cameron and Cameron et al. [6,8], Golumbic and Laskar [19], Chang [9], and Cameron [7]. Since finding the minimum independent dominating set for the most above mentioned classes can be done in polynomial time (see Farber [16], Chang [10], Broersma et al. [5]), we can obtain polynomial time algorithms for the MINIMUM MAXIMAL INDUCED MATCHING problem for chordal graphs, circular-arc graphs, and AT-free graphs. Notice that the latter class contains trapezoid graphs, interval graphs, permutation graphs, and co-comparability graphs. For definitions of these graph classes see Brandstädt et al. [4].

Recall that an algorithm is an $f(n)$-*approximation algorithm* for a minimization (respectively maximization) problem if for each instance x of the problem of size n, it returns a solution y of value $m(x, y)$ such that $m(x, y)/\mathrm{opt}(x) \leqslant f(n)$ (respectively $\mathrm{opt}(x)/m(x, y) \leqslant f(n)$), where $\mathrm{opt}(x)$ is the value of the optimum solution of x. An algorithm is a *constant approximation algorithm* if $f(n)$ is a constant. If an NP optimization problem (i.e., its decision version is in NP) admits a polynomial time $f(n)$-approximation algorithm we say that it is *approximable within a factor of* $f(n)$.

In Section 2, we show that MINIMUM MAXIMAL INDUCED MATCHING is NP-complete for bipartite graphs even of maximum degree 4. Note that the NP-completeness of the problem for bipartite graphs can be derived also as part of a general framework for establishing NP-completeness results for a family of graph parameters in bipartite and chordal graphs; see the McRae PhD thesis [32]. In Section 3, we turn to approximability results by showing that, if P $\neq$ NP, MINIMUM MAXIMAL INDUCED MATCHING cannot be approximated for bipartite graphs in polynomial time within a factor of $n^{1-\varepsilon}$ for any constant $\varepsilon > 0$, where n is the number of vertices in the input graph. (Note that in Sections 3 and 4 we use the notation p for the number of vertices in the input graph.) As an immediate consequence we obtain that there is no polynomial time constant approximation algorithm for the MINIMUM MAXIMAL INDUCED MATCHING problem even when restricted to bipartite graphs, unless P = NP.

Consider now the maximum induced matching decision problem.

MAXIMUM INDUCED MATCHING
Instance: A graph G and an integer k.
Question: Is $\Sigma(G) \geqslant k$?

Stockmeyer and Vazirani [35] proved that finding a maximum induced matching is NP-hard even for bipartite graphs of maximum degree 4. An alternate proof for bipartite graphs was presented later by Cameron [6]. This result was improved by Ko and Shepherd [26]: the NP-completeness was shown for planar bipartite graphs of maximum degree 4 with one side of the bipartition consisting only of degree 2 vertices and with each cycle having length $\equiv 0 (\mathrm{mod}\ 2^k)$ for any $k \geqslant 1$. Further strengthening was done by Orlovich and Zverovich [30] for some classes of planar bipartite graphs of maximum degree 3 which are defined by finite lists of forbidden subgraphs. Additionally, Kobler and Rotics [27] proved that the MAXIMUM INDUCED MATCHING problem is NP-complete within the classes of Hamiltonian graphs and line graphs. Recently, Orlovich and Zverovich [31] generalized the result of Kobler and Rotics showing the NP-completeness even if the input is restricted to Hamiltonian line graphs of *well-matched* graphs (i.e., graphs with equal minimum maximal and maximum induced matching numbers).

In Section 2, we show that the MAXIMUM INDUCED MATCHING problem is NP-complete for planar line graphs of planar bipartite graphs with maximum degree 4. Thereby, the MAXIMUM INDUCED MATCHING problem is NP-complete for $\{K_{1,3}, K_4 - e, C_{2n+1} : n \geqslant 2\}$-free graphs. Here $K_{1,3}$ is a claw (a star on four vertices), $K_4 - e$ is a graph obtained from the complete graph K_4 by deleting an edge, and C_{2n+1} is a simple cycle on $2n + 1$ vertices.

Duckworth et al. [13] present an approximation algorithm with asymptotic performance ratio $d - 1$ for the MAXIMUM INDUCED MATCHING problem in d-regular graphs, for each $d \geqslant 3$. Some inapproximability results for the problem of finding a maximum induced matching in bounded degree graphs are given in [11]. Gotthilf and Lewenstein [21] give a $(0.75d + 0.15)$-approximation algorithm for the MAXIMUM INDUCED MATCHING problem in d-regular graphs. It is interesting whether there exists a constant approximation algorithm for maximum induced matching in general graphs. We give the negative answer. In fact, in Section 4 we show that, for any $\varepsilon > 0$, the problem of finding a maximum induced matching in a given graph with n vertices is not approximable in polynomial time within a factor of $n^{1/2-\varepsilon}$, unless P = NP.

2. NP-completeness

An interesting special case of the MINIMUM MAXIMAL INDUCED MATCHING problem arises when the input graph is bipartite. We prove that this special case is NP-complete by presenting a polynomial time reduction from the well-known NP-complete problem 3-SATISFIABILITY, abbreviated as 3-SAT (Cook [12]; see also Garey and Johnson [18]).

3-SAT
Instance: A collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses over a set $X = \{x_1, x_2, \ldots, x_n\}$ of $0 - 1$ variables such that $|c_j| = 3$ for $j = 1, 2, \ldots, m$.
Question: Is there a truth assignment for X that satisfies all the clauses in C?

Theorem 1. MINIMUM MAXIMAL INDUCED MATCHING *is* NP-*complete for bipartite graphs and this remains valid even for bipartite graphs with maximum degree* 4.

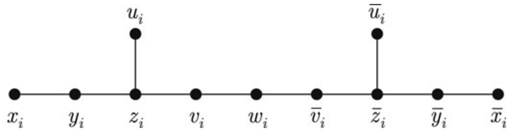

Fig. 1. Graph F_i.

Proof. Clearly the problem is in NP. To show that it is NP-complete, we establish a polynomial time reduction from 3-SAT. Let $C = \{c_1, c_2, \dots, c_m\}$ and $X = \{x_1, x_2, \dots, x_n\}$ be an instance of 3-SAT. We construct a bipartite graph G in the following way:

- For each variable x_i, we construct a graph F_i as follows. First, take a path on nine vertices $P^i = (x_i, y_i, z_i, v_i, w_i, \overline{v}_i, \overline{z}_i, \overline{y}_i, \overline{x}_i)$, where the end-vertices x_i and $\overline{x}_i$ of P^i are called *literal* vertices. Then add new vertices u_i and $\overline{u}_i$ and join u_i to z_i and $\overline{u}_i$ to $\overline{z}_i$, respectively. See Fig. 1 for the graph F_i.
- For each clause c_j, we construct a graph H_j on two vertices c_j and d_j with the unique edge $c_j d_j$, where c_j is called the *clause* vertex.
- Edges connecting $V(F_i)$ and $V(H_j)$ are defined as follows: a clause vertex c_j is connected to the three literal vertices corresponding to literals in the clause c_j.

It is easy to see that the graph G can be constructed in polynomial time in $m = |C|$ and $n = |X|$. Also, G is clearly a bipartite graph. A bipartition of G is given by sets

$$V_1 = \{x_i, \overline{x}_i, z_i, \overline{z}_i, w_i : i = 1, 2, \dots, n\} \cup \{d_j : j = 1, 2, \dots, m\},$$
$$V_2 = \{y_i, \overline{y}_i, u_i, \overline{u}_i, v_i, \overline{v}_i : i = 1, 2, \dots, n\} \cup \{c_j : j = 1, 2, \dots, m\}.$$

To complete the proof, it now suffices to show the following.

Claim 1. *There exists a satisfying truth assignment for C if and only if G has an induced matching of size $k = 2n$.*

Proof. First, suppose that there exists a truth assignment satisfying C. We construct an induced matching M in G as follows. If x_i is assigned the value 1, then include the edges $x_i y_i$ and $\overline{v}_i \overline{z}_i$ into M; otherwise, the edges $\overline{x}_i \overline{y}_i$ and $v_i z_i$ are included into M. Clearly, M is a maximal induced matching of size $2n$.

Conversely, suppose that M is an induced matching with $|M| = 2n$. Note that each maximal induced matching in G contains at least two edges of each subgraph F_i (in particular, the edges $u_i z_i$ and $\overline{u}_i \overline{z}_i$ are introduced into graph F_i in order to provide this property). Since $|M| = 2n$, we have $|M \cap E(F_i)| = 2$ for each $i = 1, 2, \dots, n$, and no other edge of G is in M. Since the edge $c_j d_j$ is not in M, then either c_j is adjacent to x_i and $x_i y_i \in M$, or c_j is adjacent to $\overline{x}_i$ and $\overline{x}_i \overline{y}_i \in M$. It follows that we can define a satisfying truth assignment to C setting a literal to be true if and only if the corresponding edge is in M. The edge corresponding to a literal x_i (respectively, $\overline{x}_i$) is $x_i y_i$ (respectively, $\overline{x}_i \overline{y}_i$). Note that the condition $|M| = 2n$ implies that M cannot contain both $x_i y_i$ and $\overline{x}_i \overline{y}_i$, since otherwise $v_i w_i \in M$ or $\overline{v}_i w_i \in M$. This completes the proof of Claim 1 and shows that Theorem 1 is valid for bipartite graphs. □

Let us show that Minimum Maximal Induced Matching remains NP-complete for bipartite graphs with maximum degree 4. For demonstrating this, we can use the following modified version of 3-SAT in the proof above: every clause contains at most three literals and every variable in C appears, negated or not, at most three times. Note that this modified version of 3-SAT is also NP-complete; see Garey and Johnson [18]. Then for the degrees of vertices of graph G, we have $\deg x_i \leqslant 4$ and $\deg c_j \leqslant 4$ for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$ (rather than $\deg x_i \leqslant m+1$ and $\deg c_j = 4$). This completes the proof of Theorem 1. □

Remark 1. As pointed out by one of the referees, the NP-completeness of Minimum Maximal Induced Matching can be shown also for planar bipartite graphs of maximum degree 4 by noting that Planar Exact Cover by 3-Sets is NP-complete even if each element of finite set X (see formulation of the problem below) occurs in two or three subsets of X [14], and using a reduction of McRae [32] to Minimum Maximal Induced Matching which preserves planarity.

Turning now to the Maximum Induced Matching problem we show that it is NP-complete for planar line graphs of planar bipartite graphs. We denote by $G(U)$ the subgraph of G induced by a subset of vertices $U \subseteq V(G)$.

Let us consider the following decision problem.

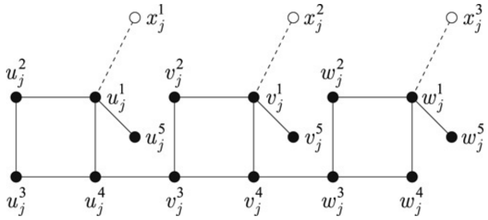

Fig. 2. Graph H_j.

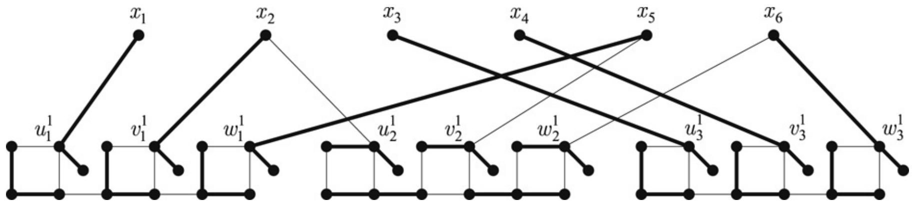

Fig. 3. An example of graph G.

PARTITION INTO ISOMORPHIC SUBGRAPHS
Instance: Graphs G and H with $|V(G)| = q|V(H)|$ for some positive integer q.
Question: Is there a partition $V_1 \cup V_2 \cup \ldots \cup V_q = V(G)$ such that $G(V_i)$ contains a subgraph isomorphic to H for all $i = 1, 2, \ldots, q$?

It is well known that this problem is NP-complete for any fixed H that contains a connected component of three or more vertices (Kirkpatrick and Hell [24,25]; see also Garey and Johnson [18]). Consider a special case of PARTITION INTO ISOMORPHIC SUBGRAPHS when H is a graph P_3: the problem PARTITION INTO SUBGRAPHS ISOMORPHIC TO P_3. Recall that $P_3 = (u, v, w)$ is a 3-path, i.e., the graph with edge set $\{uv, vw\}$. Vertex v is the central vertex and u, w are the end-vertices of this 3-path.

We will show that the problem PARTITION INTO SUBGRAPHS ISOMORPHIC TO P_3 is NP-complete for bipartite graphs. Notice that this result follows from the proof given by Kirkpatrick and Hell [24,25] on NP-completeness of the problem for general graphs. We offer another proof which can be directly used for proving NP-completeness of the problem for planar bipartite graphs of maximum degree 4 (see Remark 2 below).

Lemma 1. PARTITION INTO SUBGRAPHS ISOMORPHIC TO P_3 *is an* NP-*complete problem for bipartite graphs.*

Proof. We shall use a polynomial time transformation from the following NP-complete problem; see Garey and Johnson [18].

EXACT COVER BY 3-SETS (X3C)
Instance: A finite set $X = \{x_1, x_2, \ldots, x_{3q}\}$ of cardinality $3q$, for some positive integer q, and a set $C = \{C_1, C_2, \ldots, C_m\}$ of 3-element subsets of X.
Question: Does C contain an exact cover for X, that is, a subset $C' \subseteq C$ such that every element of X occurs in exactly one 3-element subset of C'?

Given an instance of X3C, we construct the following graph G. For each element $x_i \in X$, we create a vertex x_i in $V(G)$. For each 3-element subset $C_j = \{x_j^1, x_j^2, x_j^3\}$ in C, we create a graph H_j on fifteen vertices u_j^k, v_j^k, w_j^k, where $1 \leqslant k \leqslant 5$, as shown in Fig. 2. Then we add communication edges $x_j^1 u_j^1$, $x_j^2 v_j^1$ and $x_j^3 w_j^1$ between the vertices u_j^1, v_j^1, w_j^1 and the three vertices corresponding to the 3-element subset $C_j = \{x_j^1, x_j^2, x_j^3\}$ (see dashed edges in Fig. 2).

An example of graph G for X3C with $q = 2$, $m = 3$, $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $C = \{\{x_1, x_2, x_5\}, \{x_2, x_5, x_6\}, \{x_3, x_4, x_6\}\}$ is shown in Fig. 3. One of the partitions of G into P_3 is shown with thick lines.

Note that G is a bipartite graph. A bipartition of G is given by sets

$$V_1 = \{x_i : i = 1, 2, \ldots, 3q\} \cup \{u_j^k, v_j^k, w_j^k : j = 1, 2, \ldots, m;\ k = 2, 4, 5\},$$

$$V_2 = \{u_j^k, v_j^k, w_j^k : j = 1, 2, \ldots, m;\ k = 1, 3\}.$$

To complete the proof, it now suffices to show the following.

Claim 2. *The set C contains an exact cover for X if and only if the bipartite graph G has a partition into P_3.*

Proof. Suppose the set C has an exact cover C' for X. Then we can construct the following partition of G into P_3. If $C_j \in C'$ then 3-paths

$$(x_j^1, u_j^1, u_j^5), \quad (x_j^2, v_j^1, v_j^5), \quad (x_j^3, w_j^1, w_j^5), \quad (u_j^2, u_j^3, u_j^4), \quad (v_j^2, v_j^3, v_j^4), \quad (w_j^2, w_j^3, w_j^4)$$

belong to the partition of G; if $C_j \notin C'$ then 3-paths

$$(u_j^2, u_j^1, u_j^5), \quad (v_j^2, v_j^1, v_j^5), \quad (w_j^2, w_j^1, w_j^5), \quad (u_j^3, u_j^4, v_j^3), \quad (v_j^4, w_j^3, w_j^4)$$

are in this partition.

Conversely, suppose graph G has a partition into P_3. Note that a vertex x_i, $i = 1, 2, \ldots 3q$, of graph G cannot be a central vertex of any 3-path of the partition into P_3. Indeed, if x_i is a central vertex of some 3-path, e.g. 3-path (u_j^1, x_j^1, y), then the vertex u_j^5 cannot belong to any 3-path of the partition. The idea of the proof is to show that (i) if a communication edge for any subgraph H_j of G belongs to one of the 3-paths of the partition into P_3, then the other two communication edges associated with H_j are also included into some 3-paths of this partition, and (ii) if a communication edge for a subgraph H_j of G does not belong to a 3-path of the partition into P_3, then the other two communication edges associated with H_j are not included into 3-paths of this partition.

Then an exact cover C' for X can be constructed from the partition of G into P_3 as follows. For each $j = 1, 2, \ldots, m$, consider the communication edge $x_j^1 u_j^1$ and include the 3-element subset $\{x_j^1, x_j^2, x_j^3\}$ into C' in case of (i). For an example of Fig. 3, the exact cover C' includes 3-element subsets $\{x_1, x_2, x_5\}$ and $\{x_3, x_4, x_6\}$.

Let Q be some partition of G into P_3 and $E(Q)$ be an edge set of 3-paths of Q. To complete the proof of Claim 2, we have to consider the six following cases for the communication edges associated with each subgraph H_j: (1) $x_j^1 u_j^1 \in E(Q)$, (2) $x_j^2 v_j^1 \in E(Q)$, (3) $x_j^3 w_j^1 \in E(Q)$, (4) $x_j^1 u_j^1 \notin E(Q)$, (5) $x_j^2 v_j^1 \notin E(Q)$, (6) $x_j^3 w_j^1 \notin E(Q)$ (cases 1–3 correspond to the situation (i), and cases (4)–(6) to the situation (ii), respectively).

We will prove only cases 1 and 4 (proofs of the other cases can be done in the similar manner and are omitted).

CASE 1. If $x_j^1 u_j^1 \in E(Q)$, then $u_j^1 u_j^5 \in E(Q)$ and (x_j^1, u_j^1, u_j^5) is a 3-path of Q. Hence (u_j^2, u_j^3, u_j^4) is also a 3-path of Q. First we show that the vertex v_j^3 cannot be an end-vertex of any 3-path of Q. Indeed, if (v_j^3, v_j^2, v_j^1) is a 3-path of Q then vertex v_j^5 cannot be covered by Q. If (v_j^3, v_j^4, w_j^3) is a 3-path of Q then the vertex w_j^2 is an end-vertex of 3-path of Q, and either w_j^4 or w_j^5 cannot be covered by Q. So, v_j^3 is a central vertex of 3-path (v_j^2, v_j^3, v_j^4) and, as a consequence, (x_j^2, v_j^1, v_j^5) is in Q. The vertex w_j^3 cannot be an end-vertex of any 3-path of Q. Since otherwise w_j^5 cannot be covered by Q. Hence (w_j^2, w_j^3, w_j^4) and (x_j^3, w_j^1, w_j^5) are in Q. Thus, $x_j^1 u_j^1 \in E(Q)$ implies $x_j^2 v_j^1 \in E(Q)$ and $x_j^3 w_j^1 \in E(Q)$.

CASE 4. The vertex u_j^5 is not an end-vertex of (u_j^4, u_j^1, u_j^5) since otherwise the vertices u_j^2 and u_j^3 are not covered by Q. Therefore, 3-paths (u_j^2, u_j^1, u_j^5) and (u_j^3, u_j^4, v_j^3) are in Q. The vertex v_j^5 is not an end-vertex of (v_j^4, v_j^1, v_j^5) or (x_j^2, v_j^1, v_j^5) since otherwise v_j^2 cannot be covered by Q. Thus, 3-path (v_j^2, v_j^1, v_j^5) is in Q and $x_j^2 v_j^1 \notin E(Q)$. If (v_j^4, w_j^3, w_j^4) is in Q, then (w_j^2, w_j^1, w_j^5) is also in Q and $x_j^3 w_j^1 \notin E(Q)$. If (v_j^4, w_j^3, w_j^2) is in Q, then (w_j^4, w_j^1, w_j^5) is also in Q and $x_j^3 w_j^1 \notin E(Q)$. Thus, $x_j^1 u_j^1 \notin E(Q)$ implies $x_j^2 v_j^1 \notin E(Q)$ and $x_j^3 w_j^1 \notin E(Q)$. □

The proof of Lemma 1 is complete.

Remark 2. Note that PARTITION INTO SUBGRAPHS ISOMORPHIC TO P_3 remains NP-complete for planar bipartite graphs of maximum degree 4. For demonstrating this, instead of the reduction from X3C we can use, without changing the scheme of the proof of Lemma 1, the reduction from PLANAR EXACT COVER BY 3-SETS, which is NP-complete even if each element of X appears in at most three 3-element subsets of X [14].

This lemma together with Remark 2 yields our final conclusion.

Theorem 2. MAXIMUM INDUCED MATCHING *is* NP-*complete for planar line graphs of planar bipartite graphs with maximum degree* 4.

Proof. Let G be a planar bipartite graph of maximum degree 4 with $|V(G)| = 3q$ for some positive integer q. Kobler and Rotics [27] remark that an induced matching in line graph $L(H)$ of an arbitrary graph H corresponds precisely to a set of vertex-disjoint 3-paths (not necessarily induced) in H. Thus, G has a partition into P_3's if and only if $L(G)$ has an induced matching of size at least q. Sedláček [33] proves that the line graph $L(H)$ of planar graph H is planar if and only if the maximum degree of H is 4 and every vertex of degree 4 is a cut-vertex. Note that the vertices of degree 4 in graph G from Lemma 1 are cut-vertices. Now the conclusion of the theorem follows from Lemma 1 and Remark 2. □

Remark 3. As pointed out by one of the referees, the NP-completeness of MAXIMUM INDUCED MATCHING can be shown also for line graphs of bipartite graphs with maximum degree 3 by using the fact that PARTITION INTO SUBGRAPHS ISOMORPHIC TO P_3 is NP-complete for bipartite graphs of maximum degree 3 [29].

Remark 4. Theorem 2 strengthens the result of Kobler and Rotics [27] on NP-completeness of the MAXIMUM INDUCED MATCHING problem for line graphs. Obviously, Theorem 2 holds for the class of line graphs of bipartite graphs. This class can be characterized in terms of forbidden induced subgraphs: graph G is a line graph of a bipartite graph if and only if G does not contain $K_{1,3}$, $K_4 - e$ and C_{2n+1} ($n \geqslant 2$) as induced subgraphs; see Harary and Holzmann [22]. Thus, Theorem 2 shows that the MAXIMUM INDUCED MATCHING problem is NP-complete for $\{K_{1,3}, K_4 - e, C_{2n+1} : n \geqslant 2\}$-free graphs.

3. Hardness of approximating MINIMUM MAXIMAL INDUCED MATCHING

In this section, we show that the MINIMUM MAXIMAL INDUCED MATCHING problem is hard to approximate within bipartite graphs. First we prove the following fact.

Lemma 2. *For each instance (C, X) of 3-SAT with a set C of m clauses and a set X of n variables, and for each integer $t \geqslant 1$, there exists a bipartite graph $G = G_{(C,X),t}$ on $3n+2tn(n+m)$ vertices such that the following property holds for the minimum maximal induced matching number:*

$$\sigma(G)\begin{cases}\leqslant n, & \text{if } C \text{ is satisfiable},\\ >tn, & \text{if } C \text{ is not satisfiable}.\end{cases}$$

Proof. Let $C = \{c_1, c_2, \dots, c_m\}$ and $X = \{x_1, x_2, \dots, x_n\}$ be an instance of 3-SAT, and let $t \geqslant 1$ be an integer. We construct a graph $G_{(C,X),t}$ with vertex set $C' \cup D \cup X' \cup Y \cup U \cup V$, where $X' = \{x_i, \overline{x}_i : i = 1, 2, \dots, n\}$, $Y = \{y_i : i = 1, 2, \dots, n\}$,

$$C' = \{c_{j,k} : j = 1, 2, \dots, m,\ k = 1, 2, \dots, tn\},$$
$$D = \{d_{j,k} : j = 1, 2, \dots, m,\ k = 1, 2, \dots, tn\},$$
$$U = \{u_{i,k} : i = 1, 2, \dots, n,\ k = 1, 2, \dots, tn\},$$
$$V = \{v_{i,k} : i = 1, 2, \dots, n,\ k = 1, 2, \dots, tn\}$$

are disjoint sets. The edges of $G_{(C,X),t}$ are such that:

- The set $X' \cup Y$ induces n 3-paths $P^i = (x_i, y_i, \overline{x}_i)$ with edges $x_i y_i$ and $y_i \overline{x}_i$.
- The set $C' \cup D$ induces a matching $\{c_{j,k} d_{j,k} : j = 1, 2, \dots, m,\ k = 1, 2, \dots, tn\}$.
- The set $U \cup V$ induces a matching $\{u_{i,k} v_{i,k} : i = 1, 2, \dots, n,\ k = 1, 2, \dots, tn\}$.
- For each variable x_i (or, rather, vertex y_i), introduce tn edges $y_i u_{i,k}$, $k = 1, 2, \dots, tn$, between Y and U in G.
- For each clause $c_j = (l_j^1 \vee l_j^2 \vee l_j^3)$, introduce three edges $c_{j,k} l_j^1$, $c_{j,k} l_j^2$, $c_{j,k} l_j^3$, $k = 1, 2, \dots, tn$, between C' and X' in G.

The graph G associated with an instance (C, X) of 3-SAT, where $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $C = \{c_1 = (x_1 \vee x_2 \vee x_3), c_2 = (\overline{x}_2 \vee \overline{x}_3 \vee x_4), c_3 = (\overline{x}_1 \vee x_3 \vee x_5), c_4 = (\overline{x}_3 \vee \overline{x}_4 \vee x_5)\}$, and integer $t = 1$ is shown in Fig. 4.

It is easy to see that graph $G_{(C,X),t}$ is bipartite with parts $C' \cup Y \cup V$ and $D \cup X' \cup U$.

Now, suppose that there exists a truth assignment ϕ satisfying C. We construct an induced matching $M \subseteq \{x_i y_i, y_i \overline{x}_i : i = 1, 2, \dots, n\}$ choosing the n edges that correspond to true literals under ϕ. That is, if $\phi(x_i) = 1$,

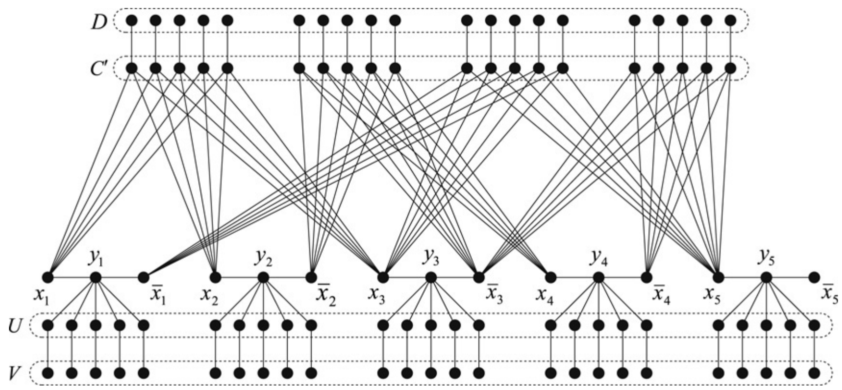

Fig. 4. An illustration to the construction.

the edge $x_i y_i$ is included in M, otherwise $y_i \overline{x}_i \in M$. It is straightforward to verify that M is a maximal induced matching of size n in G. In particular, $\sigma(G) \leqslant n$.

On the other hand, suppose that C is not satisfiable. Assume for a contradiction that $\sigma(G) \leqslant tn$. Let M be a minimum maximal induced matching in G. If $c_{j,k}d_{j,k} \in M$ for some $c_{j,k} \in C'$ and $d_{j,k} \in D$, then $c_{j,k}d_{j,k} \in M$ for all k $(1 \leqslant k \leqslant tn)$ by maximality of M. It is easy to see that in any case M contains at least one edge different from $c_{j,k}d_{j,k}$, $1 \leqslant k \leqslant tn$, so that $|M| > tn$, a contradiction.

Similarly if $c_{j,k}x_i \in M$ for some $c_{j,k} \in C'$ and $x_i \in X'$ then $u_{i,k}v_{i,k} \in M$ for all k $(1 \leqslant k \leqslant tn)$ by maximality of M, so that $|M| > tn$, a contradiction. A similar argument suffices if $c_{j,k}\overline{x}_i \in M$ for some $c_{j,k} \in C'$ and $\overline{x}_i \in X'$.

Hence M contains no edge incident to a vertex in $C' \cup D$. For each j $(1 \leqslant j \leqslant m)$, we claim that either $x_i y_i \in M$ for some $x_i \in X'$ from clause c_j or $\overline{x}_i y_i \in M$ for some $\overline{x}_i \in X'$ from c_j. For otherwise $c_{j,k}d_{j,k} \in M$ for all k $(1 \leqslant k \leqslant tn)$ by maximality of M, a contradiction. We thus construct a truth assignment ϕ by setting $\phi(x_i) = 1$ if $x_i y_i \in M$, and $\phi(x_i) = 0$ otherwise. Similarly to the proof of the first part of the lemma, ϕ is a satisfying truth assignment and hence C is satisfiable, a contradiction. Thus $\sigma(G) > tn$ as required. □

We are now in a position to present the following theorem.

Theorem 3. *Assuming that* P $\neq$ NP, MINIMUM MAXIMAL INDUCED MATCHING *for bipartite graphs cannot be approximated in polynomial time within a factor of $p^{1-\varepsilon}$ for any constant $\varepsilon > 0$, where p denotes the number of vertices in the input graph.*

Proof. For a constant $\varepsilon > 0$, we define $k = \max\{2, \lceil 3/\varepsilon \rceil\}$. Given an instance (C, X) to 3-SAT with $|C| = m$ and $|X| = n$, we set $t = n^{k-2}$. Now we construct the graph $G = G_{(C,X),t}$ as in the proof of Lemma 2. First we show the following.

Claim 3. *Approximating $\sigma(G)$ for $G = G_{(C,X),t}$ within a factor of n^{k-2} is* NP-*hard.*

Proof. Suppose that there exists a polynomial time n^{k-2}-approximation algorithm ALG for MINIMUM MAXIMAL INDUCED MATCHING within the class of graphs $G = G_{(C,X),t}$. Then we can use ALG to solve 3-SAT in polynomial time. Applying ALG to G produces an induced matching M. If C is satisfiable, then $\sigma(G) \leqslant n$ by Lemma 2, and therefore $|M| \leqslant n^{k-2}\sigma(G) \leqslant n^{k-1}$. If C is not satisfiable, $|M| \geqslant \sigma(G) > n^{k-1}$ by Lemma 2.

Thus, comparing the size of the matching found by ALG with n^{k-1} resolves the satisfiability of C in polynomial time, implying the result. □

Now we estimate $t = n^{k-2}$ in terms of $p = |V(G)| = 3n + 2n^{k-1}(n+m)$. We may assume that $n \geqslant 5$ and $n \geqslant m$. Indeed, 3-SAT remains NP-complete under these additional restrictions. We have $p \geqslant n^k$. Using the assumption $n \geqslant m$, we obtain

$$n^{k-2} = \frac{p-3n}{2n(n+m)} \geqslant \frac{p-3n}{4n^2} \geqslant \frac{p-3n}{4p^{2/k}}.$$

Since $k \geqslant 2$ and $n \geqslant 5$, we have $p \geqslant 15n$ and therefore

$$\frac{p-3n}{4p^{2/k}} \geqslant \frac{1}{5}p^{1-2/k} \geqslant p^{1-3/k}.$$

According to Claim 3, approximating $\sigma(G)$ within a factor of $p^{1-\varepsilon}$ is NP-hard, since $n^{k-2} \geqslant p^{1-3/k} \geqslant p^{1-\varepsilon}$. The proof of Theorem 3 is complete. □

4. Hardness of approximating Maximum Induced Matching

In this section, we show that the Maximum Induced Matching problem is hard to approximate for arbitrary graphs by a reduction from the Stable Set problem.

Recall that a *stable* or *independent* set in a graph G is a set of pairwise non-adjacent vertices. The *stability number* of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of a stable set in G. A stable set $S \subseteq V(G)$ is *maximum* if $|S| = \alpha(G)$.

Stable Set
Instance: A graph G and an integer k.
Question: Is there a stable set S in G with $|S| \geqslant k$?

Håstad [23] proved that Stable Set cannot be approximated in polynomial time within a factor of $|V(G)|^{1-\varepsilon}$ for each constant $\varepsilon > 0$, unless NP = ZPP. Here, ZPP denotes the class of languages decidable by a random expected polynomial time algorithm that makes no errors. In view of the recent paper by Zuckerman [36], "unless NP = ZPP" in the above approximability result for Stable Set can be changed to "unless P = NP" and we can pass to the following theorem.

Theorem 4. *Assuming that* P ≠ NP, Maximum Induced Matching *cannot be approximated in polynomial time within a factor of* $p^{1/2-\varepsilon}$ *for any constant* $\varepsilon > 0$*, where* p *is the number of vertices in the input graph.*

Proof. We construct a polynomial time reduction from Stable Set problem for arbitrary graphs. Given an instance (G, k) to Stable Set with $V(G) = \{v_1, v_2, \ldots, v_n\}$, we construct a new graph H by adding to G a new vertex u_i and making it adjacent to v_i, $i = 1, 2, \ldots, n$. The following property was mentioned (without proof) by Ko and Shepherd [26].

Claim 4. *The size of a maximum induced matching of* H *is equal to the size of a maximum stable set of* G*, i.e.,* $\Sigma(H) = \alpha(G)$.

Proof. If M is a maximum induced matching in H, then at least one vertex w_e of each edge $e \in M$ belongs to $V(G)$. The set $S = \{w_e : e \in M\}$ is stable in G, so $\Sigma(H) = |S| \leqslant \alpha(G)$. Conversely, if S is a maximum stable set of G, then the set $M = \{v_i u_i : v_i \in S\}$ is an induced matching in H, and therefore $\Sigma(H) \geqslant |M| = |S| = \alpha(G)$. □

Since $p = |V(H)| = 2|V(G)|$ and, unless P = NP, Stable Set cannot be approximated in polynomial time within a factor of $|V(G)|^{1-\varepsilon}$ for each constant $\varepsilon > 0$, we have that the Maximum Induced Matching problem cannot be approximated in polynomial time within a factor of $(p/2)^{1-\varepsilon}$ and therefore within a factor of $p^{1/2-\varepsilon}$. □

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