Nonoscillatory solutions of second-order nonlinear neutral delay equations✩

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Abstract

In this paper we study nonoscillatory solutions of a class of second-order nonlinear neutral delay differential equations with positive and negative coefficients. Some sufficient conditions for existence of nonoscillatory solutions are obtained.

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Keywords: Nonoscillation; Neutral equations

1. Introduction

Consider the second-order nonlinear neutral delay differential equations with positive and negative coefficients

\[ (r(t)(x(t) + P(t)x(t - \tau)))' + Q_1(t)f(x(t - \sigma_1)) - Q_2(t)g(x(t - \sigma_2)) = 0, \quad (E) \]

where \( t_0, \tau > 0, \sigma_1, \sigma_2 \geq 0, P, Q_1, Q_2, r \in C([t_0, \infty), R), f, g \in C(R, R). \) Throughout this paper we assume that

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(C₁) f and g satisfy local Lipschitz condition and xf(x) > 0, xg(x) > 0 for x ≠ 0.

(C₂) r(t) > 0, Q₁(t) ≥ 0, ∫[∞]₀ R(t)Q₁(t) dt < ∞, i = 1, 2, where R(t) = ∫₀ᵗ [1 / r(s)] ds.

(C₃) aQ₁(t) − Q₂(t) is eventually nonnegative for every a > 0.

Let u ∈ C([t₀ − ρ, ∞), R), where ρ = max{τ, σ₁, σ₂}, be a given function and let y₀ be a given constant. Using the method of steps, Eq. (E) has a unique solution x ∈ C([t₀ − ρ, ∞), R) in the sense that both x(t) + P(t)x(t − τ) and r(t)(x(t) + P(t)x(t − τ))' are continuously differentiable for t ≥ t₀, x(t) satisfies the Eq. (E) and

\[ x(s) = u(s) \quad \text{for } s \in [t₀ − ρ, t₀], \]

\[ \left[ x(t) + P(t)x(t − τ) \right]'_{t=t₀} = y₀. \]

For further questions concerning existence and uniqueness of solutions of neutral delay differential equations, see Hale [1].

A solution of Eq. (E) is called oscillatory if it has arbitrarily large zeros, and otherwise it is nonoscillatory.

We observe that the oscillatory and asymptotic behaviour of solutions for second-order neutral and nonneutral delay differential equations have been studied in many papers, e.g., [2–12]. The second-order neutral equation (E) received much less attention, which is due mainly to the technical difficulties arising in its analysis. See [2,3,5] for reviews of this theory.

This paper was motivated by recent paper [6], where the authors give a criterion for the existence of nonoscillatory solution of second-order linear neutral delay equation

\[ \frac{d^2}{dt^2} \left[ x(t) + P(t)x(t − τ) \right] + Q₁(t)x(t − σ₁) − Q₂(t)x(t − σ₂) = 0, \]  

(E₁)

where P ∈ R, τ ∈ (0, ∞), σ₁, σ₂ ∈ [0, ∞), and Q₁, Q₂ ∈ C([t₀, ∞), R⁺). The purpose of this paper is to present some new criteria for the existence of nonoscillatory solution of (E), which extend results in [6,7].

2. Main results

Theorem 1. Suppose that conditions (C₁)–(C₃) hold and there exists a constant P₀ such that

\[ |P(t)| ≤ P₀ < \frac{1}{2}, \quad \text{eventually.} \]  

(1)

Then (E) has a nonoscillatory solution.

Proof. Choose constants N₁ ≥ M₁ > 0 such that

\[ \frac{1}{1 − P₀} < N₁ ≤ \frac{1 − M₁}{P₀} < \frac{1}{P₀}. \]

(2)

Let X be the set of all continuous and bounded functions on [t₀, ∞) with the sup norm. Set

\[ A₁ = \{ x ∈ X : M₁ ≤ x(t) ≤ N₁, \ t ≥ t₀ \}. \]
Let $L_f(A_1), L_g(A_1)$ denote Lipschitz constants of functions $f, g$ on the set $A_1$, respectively, and $L_1 = \max\{L_f(A_1), L_g(A_1)\}, \alpha_1 = \max_{x \in A_1}\{f(x)\}, \beta_1 = \min_{x \in A_1}\{f(x)\}$, $\alpha_2 = \max_{x \in A_1}\{g(x)\}, \beta_2 = \min_{x \in A_1}\{g(x)\}$.

Choose a $t_1 > t_0 + \rho, \rho = \max\{\tau, \sigma_1, \sigma_2\}$, sufficiently large such that

\[ a Q_1(t) - Q_2(t) \geq 0 \quad \text{for } t \geq t_1 \text{ and } a > 0, \]
\[ |P(t)| \leq P_0 < \frac{1}{2} \quad \text{for } t \geq t_1, \]
\[
\int_{t_1}^{\infty} R(s)\left[Q_1(s) + Q_2(s)\right] ds < \frac{1 - P_0}{L_1}, \tag{3}
\]
\[
0 \leq \int_{t_1}^{\infty} R(s)\left[\alpha_1 Q_1(s) - \beta_2 Q_2(s)\right] ds \leq (1 - P_0)N_1 - 1, \tag{4}
\]
and
\[
\int_{t_1}^{\infty} R(s)\left[\beta_1 Q_1(s) - \alpha_2 Q_2(s)\right] ds \geq 0. \tag{5}
\]

Define a mapping $T_1 : A_1 \to X$ as follows:

\[
(T_1 x)(t) = \begin{cases} 
1 - P(t)x(t - \tau) \\
+ R(t) \int_{t_1}^{\infty} R(s)\left[Q_1(s)f(x(s - \sigma_1)) + Q_2(s)g(x(s - \sigma_2))\right] ds \\
+ \int_{t}^{t_1} R(s)\left[Q_1(s)f(x(s - \sigma_1)) - Q_2(s)g(x(s - \sigma_2))\right] ds, & t \geq t_1, \\
(x(t_1)), & 0 \leq t < t_1.
\end{cases}
\]

Clearly, $T_1x$ is continuous. For every $x \in A_1$ and $t \geq t_1$ using (1) and (4), we get

\[
(T_1 x)(t) \leq 1 + P_0N_1 + \int_{t_1}^{\infty} R(s)\left[\alpha_1 Q_1(s) - \beta_2 Q_2(s)\right] ds \leq N_1, \quad t \geq t_1.
\]

On the other hand, in view of (1), (2) and (5) we have

\[
(T_2 x)(t) \geq - P_0N_1 \geq M_1, \quad t \geq t_1.
\]

Thus we proved that $T_1 A_1 \subset A_1$. Since $A_1$ is a bounded, closed and convex subset of $X$, we have to prove that $T_1$ is a contraction mapping on $A_1$ to apply the contraction principle.

Now, for $x_1, x_2 \in A_1$ and $t \geq t_1$, in view of (3) we have

\[
\left|(T_1 x_1)(t) - (T_1 x_2)(t)\right|
\leq P_0\left|x_1(t - \tau) - x_2(t - \tau)\right| + R(t) \int_{t}^{\infty} Q_1(s)\left|f(x_1(s - \sigma_1)) - f(x_2(s - \sigma_1))\right| ds
\]
\[
+ R(t) \int_{t}^{\infty} Q_2(s)\left|g(x_1(s - \sigma_2)) - g(x_2(s - \sigma_2))\right| ds
\]
\[ + \int_{t_1}^t R(s)Q_1(s) \left| f(x_1(s - \sigma_1)) - f(x_2(s - \sigma_1)) \right| ds \]
\[ + \int_{t_1}^t R(s)Q_2(s) \left| g(x_1(s - \sigma_2)) - g(x_2(s - \sigma_2)) \right| ds \]
\[ \leq P_0 \| x_1 - x_2 \| + L_1 \| x_1 - x_2 \| \]
\[ \times \left\{ \int_{t_1}^\infty R(s) \left[ Q_1(s) + Q_2(s) \right] ds + \int_{t_1}^t R(s) \left[ Q_1(s) + Q_2(s) \right] ds \right\} \]
\[ = \| x_1 - x_2 \| \left\{ P_0 + L_1 \int_{t_1}^\infty R(s) \left[ Q_1(s) + Q_2(s) \right] ds \right\} \]
\[ = q_0 \| x_1 - x_2 \|, \]
where we used sup norm. This immediately implies that
\[ \| T_1 x_1 - T_1 x_2 \| \leq q_0 \| x_1 - x_2 \|, \]
where in view of (3), \( q_0 < 1 \), which proves that \( T_1 \) is a contraction mapping. Consequently, \( T_1 \) has the unique fixed point \( x \), which is obviously a positive solution of Eq. (E). This completes the proof of Theorem 1.

Theorem 2. Suppose that conditions (C_1)–(C_3) hold, and if one of the following two conditions is satisfied:

(i) \( P(t) \geq 0 \) eventually and \( 0 < P_1 < 1 \), \hspace{1cm} (6)
(ii) \( P(t) \leq 0 \) eventually, and \( -1 < P_2 < 0 \), \hspace{1cm} (7)

where \( P_1 = \lim_{t \to \infty} \sup P(t), \ P_2 = \lim_{t \to \infty} \inf P(t) \), then (E) has a nonoscillatory solution.

Proof. Suppose (6) holds. Choose constants \( N_2 \geq M_2 > 0 \) such that
\[ 1 - P_1 < N_2 \leq \frac{4}{3P_1 + 1} \left[ (1 - P_1) - M_2 \right]. \] \hspace{1cm} (8)

Let \( X \) be the set as in Theorem 1. Set
\[ A_2 = \left\{ x \in X : M_2 \leq x(t) \leq N_2, \ t \geq t_0 \right\} \]

Define
\[ L_2 = \max \{ L_f(A_2), L_g(A_2) \}, \quad \alpha_1 = \max_{x \in A_2} \{ f(x) \}, \]
\[ \beta_1 = \min_{x \in A_2} \{ f(x) \}, \quad \alpha_2 = \max_{x \in A_2} \{ g(x) \}, \quad \beta_2 = \min_{x \in A_2} \{ g(x) \}, \]
\[ \text{where } L_f(A_2), L_g(A_2) \text{ are Lipschitz constants of functions } f, g \text{ on the set } A_2, \text{ respectively.} \]
Choose a \( t_2 > t_0 + \rho \) sufficiently large such that

\[
0 \leq P(t) < \frac{1 + 3P_1}{4} \quad \text{for } t \geq t_2,
\]

\[
\int_{t_2}^{\infty} R(s)[Q_1(s) + Q_2(s)] \, ds < \frac{3(1 - P_1)}{4L_2},
\]

\[
0 \leq \int_{t_2}^{\infty} R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)] \, ds \leq N_2 - (1 - P_1),
\]

and

\[
\int_{t_2}^{\infty} R(s)[\beta_1 Q_1(s) - \alpha_2 Q_2(s)] \, ds \geq 0.
\]

Define a mapping \( T_2 : A_2 \to X \) as follows:

\[
(T_2x)(t) = \begin{cases} 
1 - P_1 - P(t)x(t - \tau) \\
+ R(t) \int_{t}^{\infty} [Q_1(s)f(x(s - \sigma_1)) - Q_2(s)g(x(s - \sigma_2))] \, ds \\
+ \int_{t_2}^{t} R(s)[Q_1(s)f(x(s - \sigma_1)) - Q_2(s)g(x(s - \sigma_2))] \, ds \\
(T_2x)(t_2), \quad t_0 \leq t < t_2.
\end{cases}
\]

Clearly, \( T_2x \) is continuous. For every \( x \in A_2 \) and \( t \geq t_2 \), using (C3) and (11), we get

\[
(T_2x)(t) = 1 - P_1 + \int_{t}^{\infty} R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)] \, ds \\
+ \int_{t_2}^{t} R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)] \, ds \\
\leq 1 - P_1 + \int_{t}^{\infty} R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)] \, ds \\
+ \int_{t_2}^{t} R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)] \, ds \\
= 1 - P_1 + \int_{t_2}^{\infty} R(s)[\alpha_1 Q_1(s) - \beta_2 Q_2(s)] \, ds \leq N_2, \quad t \geq t_2.
\]

Furthermore, in view of (8) and (9) we have
\[(T_2x)(t) \geq 1 - P_1 - \frac{1 + 3P_1}{4} N_2 + R(t) \int_t^\infty \left[ \beta_1 Q_1(s) - \alpha_2 Q_2(s) \right] ds \]
\[+ \int_{t_2}^t R(s) \left[ \beta_1 Q_1(s) - \alpha_2 Q_2(s) \right] ds \]
\[\geq 1 - P_1 - \frac{1 + 3P_1}{4} \frac{4}{1 + 3P_1} [(1 - P_1) - M_2] = M_2, \quad t \geq t_2.\]
Thus we proved that \(T_2 A_2 \subset A_2\). Since \(A_2\) is a bounded, closed and convex subset of \(X\), we have to prove that \(T_2\) is a contraction mapping on \(A_2\) to apply the contraction principle. Now for \(x_1, x_2 \in A_2\) and \(t \geq t_2\) we have
\[
\left| (T_2x_1)(t) - (T_2x_2)(t) \right| \\
\leq P_1 \left| x_1(t - \tau) - x_2(t - \tau) \right| \\
+ R(t) \int_t^\infty Q_1(s) \left| f(x_1(s - \sigma_1)) - f(x_2(s - \sigma_1)) \right| ds \\
+ R(t) \int_t^\infty Q_2(s) \left| g(x_1(s - \sigma_2)) - g(x_2(s - \sigma_2)) \right| ds \\
+ \int_{t_2}^t R(s) Q_1(s) \left| f(x_1(s - \sigma_1)) - f(x_2(s - \sigma_1)) \right| ds \\
+ \int_{t_2}^t R(s) Q_2(s) \left| g(x_1(s - \sigma_2)) - g(x_2(s - \sigma_2)) \right| ds \\
\leq \|x_1 - x_2\| + L_2 \|x_1 - x_2\| \\
\times \left\{ \int_t^\infty R(s) [Q_1(s) + Q_2(s)] ds + \int_{t_2}^t R(s) [Q_1(s) + Q_2(s)] ds \right\} \\
= \|x_1 - x_2\| \left\{ P_1 + L_2 \int_t^\infty R(s) [Q_1(s) + Q_2(s)] ds \right\} \\
< \|x_1 - x_2\| \left\{ P_1 + L_2 \frac{3(1 - P_1)}{4L_2} \right\} = \frac{3 + P_1}{4} \|x_1 - x_2\| = q_1 \|x_1 - x_2\|,
\]
where we used sup norm. This immediately implies that
\[\|T_2x_1 - T_2x_2\| \leq q_1 \|x_1 - x_2\|,
\]
where in view of (6), \(q_1 < 1\), which proves that \(T_2\) is a contraction mapping. Consequently, \(T_2\) has the unique fixed point \(x\), which is obviously a positive solution of (E).
(ii) Suppose (7) holds. Choose constants $N_3 \geq M_3 > 0$ such that

$$0 < M_3 < 1 + P_2 \quad \text{and} \quad N_3 > \frac{4}{3}.$$

Set

$$A_3 = \{ x \in X : M_3 \leq x(t) \leq N_3, \; t \geq t_0 \}.$$

Define $L_3, \alpha_1, \beta_1, \alpha_2, \beta_2$ as in Theorem 1 with $A_3$ instead of $A_1$. Choose a $t_3 > t_0 + \rho$ sufficiently large such that

$$-1 < \frac{3P_2 - 1}{4} \leq P(t) \leq 0 \quad \text{for} \; t \geq t_3,$$

$$\int_{t_3}^{\infty} R(s) [Q_1(s) + Q_2(s)] \; ds < \frac{3(1 + P_2)}{4L_3},$$

$$0 \leq \int_{t_3}^{\infty} R(s) \left[ \alpha_1 Q_1(s) - \beta_2 Q_2(s) \right] \; ds \leq (1 + P_2) \left( \frac{3}{4} N_3 - 1 \right),$$

and

$$\int_{t_3}^{\infty} R(s) \left[ \beta_1 Q_1(s) - \alpha_2 Q_2(s) \right] \; ds > 0.$$

Define a mapping $T_3 : A_3 \rightarrow X$ as follows:

$$(T_3 x)(t) = \begin{cases} 1 + P_2 - P(t)(x(t) - \tau) 
+ \int_{t_3}^{t} R(s) \left[ Q_1(s) f(x(s - \sigma_1)) - Q_2(s) g(x(s - \sigma_2)) \right] \; ds, & t \geq t_3, \\
(T_3 x)(t_0), & t_0 \leq t < t_3. \end{cases}$$

Clearly, $T_3 x$ is continuous. For every $x \in A_3$ and $t \geq t_3$, using (13) and (15), we get

$$(T_3 x)(t) \geq 1 + P_2 - \frac{3P_2 - 1}{4} N_3 + \int_{t_3}^{\infty} R(s) \left[ \alpha_1 Q_1(s) - \beta_2 Q_2(s) \right] \; ds$$

$$\geq 1 + P_2 - \frac{3P_2 - 1}{4} N_3 + (1 + P_2) \left( \frac{3}{4} N_3 - 1 \right) = N_3.$$

Furthermore, in view of (16) we have

$$(T_3 x)(t) \geq 1 + P_2 + R(t) \int_{t}^{\infty} \left[ \beta_1 Q_1(s) - \alpha_2 Q_2(s) \right] \; ds$$

$$+ \int_{t_3}^{t} R(s) \left[ \beta_1 Q_1(s) - \alpha_2 Q_2(s) \right] \; ds$$

$$\geq 1 + P_2 > M_3.$$
Thus, we proved that \( T_3A_3 \subset A_3 \). Since \( A_3 \) is a bounded, closed and convex subset of \( X \), we have to prove that \( T_3 \) is a contraction mapping on \( A_3 \) to apply the contraction principle.

Now for \( x_1, x_2 \in A_3 \) and \( t \geq t_3 \), in view of (14) we have

\[
\left| (T_3x_1)(t) - (T_3x_2)(t) \right| \\
\leq -P_2 \| x_1 - x_2 \| + L_3 \| x_1 - x_2 \| \int_{t_3}^\infty R(s) \left[ Q_1(s) + Q_2(s) \right] ds \\
\leq \| x_1 - x_2 \| \left\{ -P_2 + \frac{3 + (1 + P_2)}{4} \right\} \\
= \frac{3 - P_2}{4} \| x_1 - x_2 \| = q_2 \| x_1 - x_2 \|,
\]

where we used sup norm. This immediately implies that

\[
\| T_3x_1 - T_3x_2 \| \leq q_2 \| x_1 - x_2 \|,
\]

where in view of (7), \( q_2 < 1 \). This proves that \( T_3 \) is a contraction mapping. Consequently, \( T_3 \) has the unique fixed point \( x \), which is obviously a positive solution of (E). This completes the proof of Theorem 2.

**THEOREM 3.** Suppose that conditions (C_1)–(C_3) hold, and if one of the following two conditions is satisfied:

(i) \( P(t) > 1 \) eventually, and \( 1 < P_2 \leq P_1 < P_2^2 < +\infty \),

(ii) \( P(t) < -1 \) eventually, and \(-\infty < P_2 \leq P_1 < -1 \),

where \( P_1 \) and \( P_2 \) are defined as in Theorem 2. Then (E) has a nonoscillatory solution.

**Proof.** (i) Suppose that (17) holds. Set \( 0 < \varepsilon < P_2 - 1 \) be sufficiently small such that

\[
1 < \frac{P_2 - \varepsilon}{P_1 + \varepsilon} < P_1 + \varepsilon < (P_2 - \varepsilon)^2.
\]

Choose constants \( N_4 \geq M_4 > 0 \) such that

\[
\frac{1}{P_2 - \varepsilon} < N_4 < \frac{P_2 - \varepsilon}{P_1 + \varepsilon}
\]

and

\[
0 < M_4 \leq \frac{1}{P_1 + \varepsilon} - \frac{1}{P_2 - \varepsilon} N_4.
\]

Let \( X \) be the set as in Theorem 1. Set

\[
A_4 = \{ x \in X : M_4 \leq x(t) \leq N_4, \ t \geq t_0 \}.
\]
Choose a \( t_4 > t_0 + \rho \) sufficiently large such that
\[
P_2 - \varepsilon \leq P(t) \leq P_1 + \varepsilon \quad \text{for } t \geq t_4,
\]
\[
\int_{t_4}^{\infty} R(s) \left[ Q_1(s) + Q_2(s) \right] ds < \frac{P_1 + P_2}{L_4(P_1 + \varepsilon)},
\]
\[
0 \leq \int_{t_4}^{\infty} R(s) \left[ \alpha_1 Q_1(s) - \beta_2 Q_2(s) \right] ds \leq (P_2 - \varepsilon) N_4 - 1,
\]
and
\[
\int_{t_4}^{\infty} R(s) \left[ \beta_1 Q_1(s) - \alpha_2 Q_2(s) \right] ds \geq 0,
\]
where \( \alpha_1, \beta_1, \alpha_2, \beta_2, L_4 \) are defined as in Theorem 1, but with \( A_4 \) instead of \( A_1 \).

Define a mapping \( T_4 : A_4 \to X \) as follows:
\[
(T_4 x)(t) = \begin{cases} 
1 & t \geq t_4, \\
1 & \text{otherwise},
\end{cases}
\]
where \( t + \tau \geq t_0 + \max\{\sigma_1, \sigma_2\} \). Clearly, \( T_4 x \) is continuous. For every \( x \in A_4 \) and \( t \geq t_4 \), using (25), we get
\[
(T_4 x)(t) \leq 1 + \frac{1}{P_2 - \varepsilon} \int_{t_4}^{\infty} R(s) \left[ \alpha_1 Q_1(s) - \beta_2 Q_2(s) \right] ds
\leq 1 + \frac{1}{P_2 - \varepsilon} [(P_2 - \varepsilon) N_4 - 1] = N_4.
\]
Furthermore, in view of (21) and (26) we have
\[
T_4 x(t) \geq \frac{1}{P_1 + \varepsilon} - \frac{1}{P_2 - \varepsilon} N_4 + \frac{1}{P_1 + \varepsilon} R(t + \tau) \int_{t + \tau}^{\infty} \left[ \beta_1 Q_1(s) - \alpha_2 Q_2(s) \right] ds
\geq \frac{1}{P_1 + \varepsilon} \int_{t_4}^{t + \tau} R(s) \left[ \beta_1 Q_1(s) - \alpha_2 Q_2(s) \right] ds \geq M_4.
\]
Thus, we proved that \( T_4 A_4 \subset A_4 \). Since \( A_4 \) is a bounded, closed and convex subset of \( X \), we have to prove that \( T_4 \) is a contraction mapping on \( A_4 \) to apply the contraction principle.

Now, for \( x_1, x_2 \in A_4 \) and \( t \geq t_4 \), in view of (24) we have
\[ |(T_4x_1)(t) - (T_4x_2)(t)| \leq \frac{1}{P_1 + \varepsilon} \|x_1 - x_2\| \]
\[ + \frac{L_4}{P_2 - \varepsilon} \|x_1 - x_2\| \int_{t_4}^{\infty} R(s) \left[ Q_1(s) + Q_2(s) \right] ds \]
\[ \leq \|x_1 - x_2\| \left\{ - \frac{1}{P_1 + \varepsilon} + \frac{1}{P_2 - \varepsilon} \left( 1 + \frac{P_2 - \varepsilon}{P_1 + \varepsilon} \right) \right\} \]
\[ = \frac{1}{P_2 - \varepsilon} \|x_1 - x_2\| = q_3 \|x_1 - x_2\|, \]
where we used sup norm. This immediately implies that
\[ \|T_4x_1 - T_4x_2\| \leq q_3 \|x_1 - x_2\|. \]

In view of (20), \( q_3 < 1 \), which proves that \( T_4 \) is a contraction mapping. Consequently, \( T_4 \) has the unique fixed point \( x \), which is obviously a positive solution of (E).

(ii) Suppose that (18) holds. Set \( 0 < \delta < - (1 + P_2) \) be sufficiently small such that
\[ P_2 - \delta < P_1 + \delta < -1. \] (27)

Choose constants \( N_5 \geq M_5 > 0 \) such that
\[ M_5 < \frac{-1}{1 + P_2 - \delta} < \frac{-1}{1 + P_1 + \delta} < N_5. \] (28)

Let \( X \) be the set as in Theorem 1. Set
\[ A_5 = \{ x \in X : M_5 \leq x(t) \leq N_5, t \geq t_0 \}, \]
choose a \( t_5 > t_0 + \rho \) sufficiently large such that (C3) holds and
\[ P_2 - \delta < P(t) < P_1 + \delta \] for \( t \geq t_5 \), (29)
\[ \int_{t_5}^{\infty} R(s) \left[ Q_1(s) + Q_2(s) \right] ds < \frac{1 + P_1 + \delta}{L_5}, \] (30)
\[ 0 \leq \int_{t_5}^{\infty} R(s) \left[ \alpha_1 Q_1(s) - \beta_2 Q_2(s) \right] ds \leq \frac{P_1 + \delta}{P_2 - \delta} [1 + M_5(1 + P_2 - \delta)], \] (31)
\[ \int_{t_5}^{\infty} R(s) \left[ \beta_1 Q_1(s) - \alpha_2 Q_2(s) \right] ds \geq 0, \] (32)
where \( \alpha_1, \beta_1, \alpha_2, \beta_2, L_5 \) are defined as in Theorem 1 with \( A_5 \) instead of \( A_1 \).

Define a mapping \( T_5 : A_5 \to X \) as follows:
\[ (T_5x)(t) = \begin{cases} \frac{1 - x(t + \tau)}{P(t + \tau)} \frac{x(t + \tau)}{P(t + \tau)} & \text{for } t \geq t_5, \\
+ \frac{R(t + \tau)}{P(t + \tau)} \int_{t + \tau}^{\infty} [Q_1(s)f(x(s - \sigma_1)) - Q_2(s)g(x(s - \sigma_2))] ds \\
+ \frac{1}{P(t + \tau)} \int_{t_5}^{t + \tau} R(s) [Q_1(s)f(x(s - \sigma_1)) \\
- Q_2(s)g(x(s - \sigma_2))] ds, & t \geq t_5, \\
(T_5x)(t_5), & t_0 \leq t < t_5, \end{cases} \]
where \( t + \tau \geq t_0 + \max\{\sigma_1, \sigma_2\} \). Clearly, \( T_5 x \) is continuous. For every \( x \in A_5 \) and \( t \geq t_5 \), using (C3) and (32), we get

\[
(T_5 x)(t) \leq -\frac{1}{P_1 + \delta} + \frac{1}{P_1 + \delta} N_5 + \frac{R(t + \tau)}{P_2 - \delta} \int_{t+\tau}^{\infty} \left[ \beta_1 Q_1(s) - \alpha_2 Q_2(s) \right] ds
\]

\[
+ \frac{1}{P_2 - \delta} \int_{t_5}^{t+\tau} R(s) \left[ \beta_1 Q_1(s) - \alpha_2 Q_2(s) \right] ds
\]

\[
\leq -\frac{1}{P_1 + \delta} + \frac{1}{P_1 + \delta} N_5 < N_5.
\]

Furthermore, in view of (28) and (31) we have

\[
(T_5 x)(t) \geq -\frac{1}{P_2 - \delta} + \frac{1}{P_2 - \delta} M_5 + \frac{1}{P_1 + \delta} \int_{t_5}^{\infty} R(s) \left[ \alpha_1 Q_1(s) - \beta_2 Q_2(s) \right] ds
\]

\[
\geq -\frac{1}{P_2 - \delta} - \frac{1}{P_2 - \delta} M_5 + \frac{1}{P_1 + \delta} \left[ 1 - M_5 (1 + P_2 - \delta) \right] = M_5.
\]

Thus, we proved that \( T_5 A_5 \subset A_5 \). Since \( A_5 \) is a bounded, closed and convex subset of \( X \), we have to prove that \( T_5 \) is a contraction mapping on \( A_5 \) to apply the contraction principle.

Now for \( x_1, x_2 \in A_5 \) and \( t \geq t_5 \), in view of (30) we get

\[
| (T_5 x_1)(t) - (T_5 x_2)(t) | \leq \frac{1}{P_1 + \delta} | x_1(t + \tau) - x_2(t + \tau) | + \frac{R(t + \tau)}{P_2 - \delta} \int_{t_5}^{t+\tau} Q_1(s) \left[ f(x_1(s - \sigma_1)) - f(x_2(s - \sigma_1)) \right] ds
\]

\[
+ \frac{R(t + \tau)}{P(t + \tau)} \int_{t_5}^{t+\tau} Q_2(s) \left[ g(x_1(s - \sigma_2)) - g(x_2(s - \sigma_2)) \right] ds
\]

\[
+ \frac{1}{P(t + \tau)} \int_{t_5}^{t+\tau} R(s) Q_1(s) \left[ f(x_1(s - \sigma_1)) - f(x_2(s - \sigma_1)) \right] ds
\]

\[
+ \frac{1}{P(t + \tau)} \int_{t_5}^{t+\tau} R(s) Q_2(s) \left[ g(x_1(s - \sigma_2)) - g(x_2(s - \sigma_2)) \right] ds
\]

\[
\leq -\frac{1}{P_1 + \delta} \| x_1 - x_2 \| - \frac{L_5}{P_2 - \delta} \| x_1 - x_2 \|
\]

\[
\subseteq \frac{1}{P_1 + \delta} \| x_1 - x_2 \| - \frac{L_5}{P_2 - \delta} \| x_1 - x_2 \|
\]
\[
\times \left\{ \int_{t+\tau}^{\infty} R(s) \left[ Q_1(s) + Q_2(s) \right] ds + \int_{t+\tau}^{\infty} R(s) \left[ Q_1(s) + Q_2(s) \right] ds \right\}
\]

\[
< \| x_1 - x_2 \| \left\{ \frac{-1}{P_1 + \delta} - \frac{L_5}{P_2 - \delta} \int_{t}^{\infty} R(s) \left[ Q_1(s) + Q_2(s) \right] ds \right\}
\]

\[
\leq \| x_1 - x_2 \| \left\{ \frac{-1}{P_1 + \delta} + \frac{1}{P_2 - \delta} \right\}
\]

\[
= q_4 \| x_1 - x_2 \|,
\]

where we used sup norm. This immediately implies that

\[
\| T_5 x_1 - T_5 x_2 \| \leq q_4 \| x_1 - x_2 \|,
\]

where in view of (27), \( q_4 < 1 \). which proves that \( T_5 \) is a contraction mapping. Consequently, \( T_5 \) has the unique fixed point \( x \), which is obviously a positive solution of (E). This completes the proof of Theorem 2. \( \square \)

**Remark.** If \( f(x(t)) = g(x(t)) = x(t) \), \( r(t) = 1 \) and \( P(t) = P = \text{const} \), then Theorems 2 and 3 improve the theorem of Kulenovic and Hadziomerspahic [6].

**References**


